# TAYLOR COEFFICIENTS OF $t$-MOTIVIC MULTIPLE ZETA VALUES AND EXPLICIT FORMULAE

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Abstract. For each positive characteristic multiple zeta value (defined by Thakur [T04]), the first and third authors in [CM17] constructed a $t$-module such that a certain coordinate of a logarithmic vector of a specified algebraic point is a rational multiple of that multiple zeta value. The main result in this paper gives explicit formulae for all of the coordinates of this logarithmic vector in terms of Taylor coefficients of $t$-motivic multiple zeta values and $t$-motivic Carlitz multiple star polylogarithms.

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# 1. Introduction

1.1. **Classical theory.** The theory of generalized logarithms for commutative algebraic groups defined over number fields at algebraic points has a long history. Many developments came about because of the proposal of Hilbert's famous seventh problem in 1900, which is related to the linear independence of two logarithms at algebraic numbers over the field $\overline{\mathbb{Q}}$ of algebraic numbers. Gelfond and Schneider provided an affirmative answer to Hilbert's problem in the 1930's, then Baker generalized this with his celebrated theorem on linear forms in logarithms from the 1960's. In the following years, the analogue of Baker's theorem for elliptic curves was successfully developed by Coates, Lang, Masser, Bertand. Lang was the first to use the language of group varieties to interprate a reformulation of the Gelfond-Schneider theorem and other classical results. For more details, see [BW07, W02]. One of the most significant results and greatest breakthroughs in classical transcendence theory is the following analytic subgroup theorem achieved by Wüstholz.

**Theorem 1.1.1** (Wüstholz [W89]). *Let $G$ be a commutative algebraic group defined over $\overline{\mathbb{Q}}$, and let $\exp_G : \operatorname{Lie} G(\mathbb{C}) \to G(\mathbb{C})$ be the exponential map when regarding $G(\mathbb{C})$ as a Lie group. Let $\mathbf{u} \in \operatorname{Lie} G(\mathbb{C})$ satisfy $\exp_G(\mathbf{u}) \in G(\overline{\mathbb{Q}})$, and let $V_{\mathbf{u}} \subset \operatorname{Lie} G(\mathbb{C})$ be the smallest linear subspace that contains $\mathbf{u}$ and that is defined over $\overline{\mathbb{Q}}$. Then we have*

$$V_{\mathbf{u}} = \operatorname{Lie} H(\mathbb{C})$$

*for some algebraic subgroup $H$ of $G$ defined over $\overline{\mathbb{Q}}$.*

The spirit of the theorem above is to assert that the $\overline{\mathbb{Q}}$-linear relations among the coordinates of the generalized logarithm $\mathbf{u}$ arise from the defining equations of $\operatorname{Lie} H$. All the results mentioned above are consequences of Wüstholz's theory as they can be formulated as questions concerning generalized logarithms defined in terms of suitable commutative algebraic groups. So, attempting to relate certain special values which one desires to study to the coordinates of generalized logarithms fitting into Wüstholz's theory is a fruitful line of study.

Classical real-valued multiple zeta values are generalizations of the special values of the Riemann $\zeta$-function at positive integers. They occur as periods of mixed Tate motives and have many interesting connections between different research areas (see [An04, BGF18, Zh16]). It is a natural question to ask whether one can relate any classical multiple zeta values to certain coordinates of generalized logarithms fitting into Wüstholz's analytic subgroup theorem. The answer is still unknown. In the function fields case, we have a positive answer of the analogous question above from [CM17], which shows that any positive characteristic multiple zeta value can be realized as a coordinate of the logarithm of a certain $t$-module at an algebraic point. The purpose of this paper is to give explicit formulae for all the coordinates of such a logarithmic vector in terms of Taylor coefficients of the $t$-motivic multiple zeta value in question and $t$-motivic Carlitz multiple star polylogarithms.

1.2. **The main result.** Let $A := \mathbb{F}_q[\theta]$ be the polynomial ring in the variable $\theta$ over a finite field $\mathbb{F}_q$ with quotient field $K = \mathbb{F}_q(\theta)$. Let $K_\infty$ be the completion of $K$ with respect to the normal absolute value $|\cdot|_\infty$ associated to the infinite place $\infty$, and let $\mathbb{C}_\infty$ be the $\infty$-adic completion of a fixed algebraic closure of $K_\infty$. Let $\overline{K}$ be the algebraic closure of $K$ inside $\mathbb{C}_\infty$. Thakur [T04] defined positive characteristic multiple zeta values associated to $A$ as follows. For any $r$-tuple of positive integers $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$, define

$$\zeta_A(\mathfrak{s}) := \sum \frac{1}{a_1^{s_1} \cdots a_r^{s_r}} \in K_\infty, \tag{1.2.1}$$

where the sum is over all monic polynomials $a_1, \ldots, a_r$ in $A$ with the restriction $\deg_\theta a_1 > \deg_\theta a_2 > \cdots > \deg_\theta a_r$. The weight and the depth of the presentation $\zeta_A(\mathfrak{s})$ are defined by $\mathrm{wt}(\mathfrak{s}) := \sum_{i=1}^{r} s_i$ and $\mathrm{dep}(\mathfrak{s}) := r$ respectively. When $r = 1$, the special values above are called Carlitz zeta values, due to Carlitz [Ca35], and they play the analogue of special values of Riemann $\zeta$-function at positive integers.

In the world of function fields in positive characteristic, the Carlitz module $\mathbf{C}$ defined in Sec. 2.2 plays the analogue of the multiplicative group $\mathbb{G}_m$ in the classical case, and the Carlitz logarithm $\mathrm{Log}_{\mathbf{C}}$ is analogous to the classical logarithm. In [Ca35], Carlitz proved the interesting and important identity $\zeta_A(1) = \mathrm{Log}_{\mathbf{C}}(1)$, whose classical counterpart does not exist. Carlitz's formula reveals the fact that positive characteristic zeta values have a close connection with logarithms. Anderson and Thakur generalized this connection to all Carlitz zeta values in [AT90], and gave a beautiful interpretation in terms of tensor powers of the Carlitz module. More precisely, for each positive integer $s$ they constructed a vector $Z_s \in \mathrm{Lie}\,\mathbf{C}^{\otimes s}(\mathbb{C}_\infty)$, which is the Lie algebra of the $s$-th tensor power of the Carlitz module (see Sec. 2.2 for the definition), so that $\Gamma_s \zeta_A(s)$ occurs as the last coordinate of $Z_s$ and so that $Z_s$ is mapped to an integral point of $\mathbf{C}^{\otimes s}$ under the exponential map $\mathrm{Exp}_{\mathbf{C}^{\otimes s}}$. Here $\Gamma_s \in A$ is the Carlitz factorial defined in (5.1.1).

The first and third authors of this paper generalized Anderson-Thakur's work to arbitrary depth MZV's in [CM17], where the case of Eulerian MZV's was previously established in [C16]. In [CM17], they showed that for any $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$ one can construct a uniformizable $t$-module $G_\mathfrak{s}$ and a vector $Z_\mathfrak{s} \in \mathrm{Lie}\,G_\mathfrak{s}(\mathbb{C}_\infty)$ so that $\Gamma_\mathfrak{s} \zeta_A(\mathfrak{s})$ occurs as the $\mathrm{wt}(\mathfrak{s})$-th coordinate of $Z_\mathfrak{s}$ and $Z_\mathfrak{s}$ is mapped to an integral point of $G_\mathfrak{s}$ under the exponential map $\mathrm{Exp}_{G_\mathfrak{s}}$, where $\Gamma_\mathfrak{s} := \Gamma_{s_1} \cdots \Gamma_{s_r}$. This result, combined with Yu's sub-$t$-module theorem [Yu97] — which is the function field analogue of Wüstholz's analytic subgroup theorem — is a key ingredient in verifying a function field analogue of Furusho's conjecture, which asserts that the $p$-adic MZV's satisfy the same linear relations that the corresponding real-valued MZV's satisfy. Thus, it is a natural and interesting question to ask what the other coordinates of $Z_\mathfrak{s}$ are. The main result of this paper is to give explicit formulae for them.

Let $t$ be a new variable. One core object in this paper is the $t$-motivic multiple zeta value $\zeta_A^{\mathrm{mot}}(\mathfrak{s}) \in \mathbb{C}_\infty[\![t]\!]$ given in Definition 5.1.4. Our motivation for the definition of $\zeta_A^{\mathrm{mot}}(\mathfrak{s}) \in \mathbb{C}_\infty[\![t]\!]$ comes from two perspectives. The first is from the point of view of periods. In [AT09], for each index $\mathfrak{s} \in \mathbb{N}^r$ Anderson and Thakur construct a $t$-motive $M_\mathfrak{s}$ together with a system of Frobenius difference equations $\Psi_\mathfrak{s}^{(-1)} = \Phi_\mathfrak{s} \Psi_\mathfrak{s}$ so that $\Gamma_\mathfrak{s} \zeta_A(\mathfrak{s})/\tilde{\pi}^{\mathrm{wt}(\mathfrak{s})}$ occurs as an entry of the period matrix $\Psi_\mathfrak{s}|_{t=\theta}$ of $M_\mathfrak{s}$, where $\Phi_\mathfrak{s}$ is a matrix of size $r+1$ with entires in $\overline{K}[t]$, where $\Psi_\mathfrak{s}$ is an invertiable matrix of size $r+1$ with entries in $\mathbb{C}_\infty[\![t]\!]$, and where $\tilde{\pi}$ is a fundamental period of the Carlitz module $\mathbf{C}$. Here the terminology of $t$-motive is in the sense of [P08]. The value $\Gamma_\mathfrak{s} \zeta_A(\mathfrak{s})/\tilde{\pi}^{\mathrm{wt}(\mathfrak{s})}$ is the specialization at $t = \theta$ of a certain power series $\Omega^{\mathrm{wt}(\mathfrak{s})} \cdot \zeta_A^{\mathrm{mot}}(\mathfrak{s}) \in \mathbb{C}_\infty[\![t]\!]$, which is an entry of $\Psi_\mathfrak{s}$ (see [AT09, §2.5] for details). Papanikolas [P08] showed that $Z_{\Psi_\mathfrak{s}} := \mathrm{Spec}\,\overline{K}(t)[\Psi_\mathfrak{s}, \det \Psi_\mathfrak{s}^{-1}]$ is a torsor for the algebraic group $\Gamma_{M_\mathfrak{s}} \times_{\mathbb{F}_q(t)} \overline{K}(t)$, which arises from the base change of the fundamental group of the Tannakian category generated by $M_\mathfrak{s}$, the so-called $t$-motivic Galois group of $M_\mathfrak{s}$. Since $\zeta_A^{\mathrm{mot}}(\mathfrak{s})$ occurs in the affine coordinate ring of $Z_{\Psi_\mathfrak{s}}$, which we regard as a period torsor for $\Gamma_{M_\mathfrak{s}}$, it can be viewed as a $t$-motivic period (cf. [Br14, Def. 4.1]).

The second perspective is from the point of view of deformations. In [P08], Papanikolas constructed a deformation series of the Carlitz logarithm, which he connected with periods of certain $t$-motives. Let $\mathbb{L}_0 := 1$ and $\mathbb{L}_i := (t - \theta^q) \cdots (t - \theta^{q^i})$ for $i = 1, 2, 3, \cdots$. Papanikolas's deformation series is defined by

$$\mathfrak{Li}_{1,u}(t) := \sum_{i=0}^{\infty} \frac{u^{q^i}}{\mathbb{L}_i} \in \mathbb{C}_\infty[\![t]\!]$$

for $u \in \mathbb{C}_\infty$ with $|u|_\infty < q^{\frac{q}{q-1}}$, and one has that $\mathfrak{Li}_{1,u}(\theta) = \mathfrak{Li}_{1,u}(t)|_{t=\theta} = \mathrm{Log}_{\mathbf{C}}(u)$. The first author and Yu then extended $\mathfrak{Li}_{1,u}(t)$ to a deformation $\mathfrak{Li}_{n,u}(t)$ of $n$-th Carlitz polylogarithm at $u \in \mathbb{C}_\infty$ with $|u|_\infty < q^{\frac{nq}{q-1}}$ in [CY07] in order to connect $\zeta_A(n)$ to periods of certain $t$-motives. This inspired us to use Anderson-Thakur polynomials (see Sec. 5.1.1) to define the $t$-motivic MZV's $\zeta_A^{\mathrm{mot}}(\mathfrak{s})$ in Definition 5.1.4 as a deformation of function field multiple zeta values $\zeta_A(\mathfrak{s})$. Indeed, from the interpolation formula of Anderson-Thakur [AT90, AT09], one has that $\zeta_A^{\mathrm{mot}}(\mathfrak{s})|_{t=\theta} = \Gamma_{\mathfrak{s}}\zeta_A(\mathfrak{s})$. In [C14, CM17], the first and third authors give a formula that expresses $\zeta_A(\mathfrak{s})$ as a linear combination of Carlitz multiple star polylogarithms (abbreviated as CMSPL's — see (3.2.4) for the definition) generalizing the work of Anderson-Thakur for Carlitz zeta values in [AT90]. In this paper, we define $t$-motivic CMSPL's in (3.2.4) that are deformation series of CMSPL's, and show that the identity for $\zeta_A(\mathfrak{s})$ mentioned above can be deformed as an identity for $\zeta_A^{\mathrm{mot}}(\mathfrak{s})$ in Lemma 5.2.4. So our definition of $\zeta_A^{\mathrm{mot}}(\mathfrak{s})$ seems quite appropriate from the $t$-motivic aspect.

In many number theory settings, Taylor coefficients of important functions have interesting arithmetic interpretations. Two well known classical examples are the celebrated class number formula and the Gross-Zagier formula. In the frame work of $t$-modules, such a phenomenon first occurs in the paper of Anderson and Thakur [AT90]. There, they show that for any positive integer $s$, the kernel of $\mathrm{Exp}_{\mathbf{C}^{\otimes s}}$ is a free $A$-module of rank one, and the first $s$ Taylor coefficients of the Anderson-Thakur function $\omega_s$ (see [AT90, Sec. 2.5]) give the coordinates of the generator of $\mathrm{Ker}\,\mathrm{Exp}_{\mathbf{C}^{\otimes s}}$ (see also [Ma18, Lem. 8.3]). A similar phenomenon occurs for the more general Drinfeld $\mathbf{A}$-modules in [G17a, Thm. 6.7].

This paper studies Taylor coefficients of the series expansion of $\zeta_A^{\mathrm{mot}}(\mathfrak{s})$ at $t = \theta$, which are given by hyperderivatives of $\zeta_A^{\mathrm{mot}}(\mathfrak{s})$ evaluated at $t = \theta$. For any $r$-tuple of positive integers $\mathfrak{s} = (s_1, \ldots, s_r)$, our main result stated in Theorem 6.2.1 shows that the first $\mathrm{wt}(\mathfrak{s})$ Taylor coefficients of the series expansion of $\zeta_A^{\mathrm{mot}}(\mathfrak{s})$ at $t = \theta$ give the first $\mathrm{wt}(\mathfrak{s})$ coordinates of $Z_{\mathfrak{s}}$, and that the other coordinates of $Z_{\mathfrak{s}}$ are given explicitly in terms of Taylor coefficients of the expansion of $t$-motivic CMSPL's at $t = \theta$. In particular, when $r = 1$ the coordinates of $Z_s$ are given by the first $s$ Taylor coefficients of $\zeta_A^{\mathrm{mot}}(s)$ at $t = \theta$. We mention that in [Pp] Papanikolas establishes a log-algebraicity theorem for $\mathbf{C}^{\otimes s}$, which is applied to derive an explicit formula for $Z_s$ also in terms of hyperderivatives, but his formula is different from ours. As our results reveal the importance of Taylor coefficients of $t$-motivic MZV's, it will be interesting to investigate their connections with other research topics. This would require additional time and we hope to tackle this project in the near future.

1.3. **Organization of this paper.** In Section 2 we review some basic theories of Anderson, particularly how to contstruct the associated $t$-module from a dual $t$-motive using his theory of $t$-frames. One primary tool that we adapt to prove our main result is Theorem 2.4.4, in which Anderson provides a dual t-motivic description of the exponential maps. We review the definition of hyperderivatives in Section 3, and then analyze $\delta_0 \circ \iota(\mathbf{g})$ (a vector of power series coming from the $t$-frames theory) in Theorem 2.4.4 to establish the general expression for the logarithmic vector in Theorem 3.4.1. The central topic of Section 4 is $t$-motivic CMSPL's. This extends the study of [CM19], wherein the first and third authors, for each CMSPL evaluated at an algebraic point, construct a $t$-module together with a special point so that up to a sign, the CMSPL occurs as the $\mathrm{wt}(\mathfrak{s})$-th coordinate of the logarithm of that $t$-module evaluated at this special point. In §4 of this paper, we use Theorem 3.4.1 to give explicit formulae for all coordinates of this logarithm in terms of Taylor coefficients of $t$-motivic CMSPL's in Theorem 4.2.2. In Section 5, we first define $t$-motivic MZV's and then derive a deformation identity for them in Lemma 5.2.3. In section 6, we first review fiber coproducts of dual $t$-motives and review the construction of $G_{\mathfrak{s}}$ and $Z_{\mathfrak{s}}$ mentioned in the introduction. With the results from Theorem 4.2.2 and Lemma 5.2.3, we use techniques developed in [CM17] to derive an explicit formula for $Z_{\mathfrak{s}}$ in Theorem 6.2.1. Finally, in Theorem 6.3.4 we further relate any monomial of MZV's to a coordinate of the logarithm of a certain $t$-module at a special point, and give a description of the other coordinates of this logarithmic vector explicitly in terms of Taylor coefficients of $t$-motivic CMSPL's.

## 2. Anderson's theory revisited

### 2.1. Notation and Frobenius twistings.

**Table of Symbols 2.1.1.** We use the following symbols throughout this paper.

| | | |
|---|---|---|
| $\mathbb{N}$ | $=$ | *the set of positive integers.* |
| $q$ | $=$ | *a power of a prime number $p$.* |
| $\mathbb{F}_q$ | $=$ | *the finite field of $q$ elements.* |
| $A$ | $=$ | $\mathbb{F}_q[\theta]$, *the polynomial ring in the variable $\theta$ over $\mathbb{F}_q$.* |
| $A_+$ | $=$ | *the set of monic polynomials in $A$.* |
| $K$ | $=$ | $\mathbb{F}_q(\theta)$, *the quotient field of $A$.* |
| $\mathrm{ord}_\infty$ | $=$ | *the normalized valuation of $K$ at the infinite place for which* $\mathrm{ord}_\infty(1/\theta)=1$. |
| $\deg$ | $=$ | $-\mathrm{ord}_\infty$, *the degree function on $K$.* |
| $\lvert\cdot\rvert_\infty$ | $=$ | $q^{\deg(\cdot)}$, *an absolute value on $K$.* |
| $K_\infty$ | $=$ | $\mathbb{F}_q(\!(1/\theta)\!)$, *the completion of $K$ at the infinite place.* |
| $\mathbb{C}_\infty$ | $=$ | $\widehat{\overline{K}}_\infty$, *the completion of an algebraic closure of $K_\infty$.* |
| $\overline{K}$ | $=$ | *the algebraic closure of $K$ in $\mathbb{C}_\infty$.* |
| $\widetilde{\Lambda}$ | $=$ | $(\lambda_r,\ldots,\lambda_1)$ *for any $r$-tuple $\Lambda=(\lambda_1,\ldots,\lambda_r)$ of elements of a nonempty set.* |
| $\mathrm{wt}(\mathfrak{s})$ | $=$ | $\sum_{i=1}^r s_i$ *for an index $\mathfrak{s}=(s_1,\ldots,s_r)\in\mathbb{N}^r$.* |
| $\mathrm{dep}(\mathfrak{s})$ | $=$ | $r$ *for an index $\mathfrak{s}=(s_1,\ldots,s_r)\in\mathbb{N}^r$.* |

Let $t$ be a new variable. We denote by $\mathbb{C}_\infty[\![t]\!]$ the ring of formal power series in the variable $t$ with coefficients in $\mathbb{C}_\infty$, and denote by $\mathbb{C}_\infty(\!(t)\!)$ the quotient field of $\mathbb{C}_\infty[\![t]\!]$. We define $\mathbb{T}_\theta$ to be the subring of $\mathbb{C}_\infty[\![t]\!]$ consisting of power series convergent on $|t|_\infty\leq|\theta|_\infty$ – for more details see (3.1.1). Finally for any integer $n$, we define the $n$th fold Frobenius twist on $\mathbb{C}_\infty(\!(t)\!)$:

$$\begin{array}{ccc}\mathbb{C}_\infty(\!(t)\!) & \to & \mathbb{C}_\infty(\!(t)\!)\\ f:=\sum a_it^i & \mapsto & f^{(n)}:=\sum a_i^{q^n}t^i.\end{array}$$

We then extend these Frobenius twists to matrices over $\mathbb{C}_\infty(\!(t)\!)$, i.e., for any matrix $B=(b_{ij})\in\mathrm{Mat}_{\ell\times m}(\mathbb{C}_\infty(\!(t)\!))$, we define

$$B^{(n)}:=(b_{ij}^{(n)}).$$

For any $A$-subalgebra $R$ of $\mathbb{C}_\infty$, we define the ring of twisted polynomials

$$\mathrm{Mat}_d(R)[\tau]:=\left\{\sum_{i=0}^{\infty}\alpha_i\tau^i|\alpha_i\in\mathrm{Mat}_d(R)\ \forall i\text{ and }\alpha_i=0\text{ for }i\gg0\right\}$$

subject to the relation: for $\alpha, \beta \in \mathrm{Mat}_d(R)$,

$$\alpha\tau^i \cdot \beta\tau^j := \alpha\beta^{(i)}\tau^{i+j}.$$

We put $R[\tau] := \mathrm{Mat}_1(R)[\tau]$ and indeed we have the natural identity $\mathrm{Mat}_d(R[\tau]) = \mathrm{Mat}_d(R)[\tau]$. For any $\phi = \sum_{i=0}^{\infty} \alpha_i\tau^i \in \mathrm{Mat}_d(R)[\tau]$, we put

$$\partial\phi := \alpha_0.$$

2.2. **Background on Anderson $t$-modules.** Fix a positive integer $d$ and an $A$-subalgebra $R$ of $\mathbb{C}_\infty$ with quotient field $F$. Let $\rho$ be an $\mathbb{F}_q$-linear ring homomorphism

$$\rho : \mathbb{F}_q[t] \to \mathrm{Mat}_d(R)[\tau]$$

so that $\partial\rho(t) - \theta I_d$ is a nilpotent matrix. We have the natural identification

$$\mathrm{Mat}_d(R)[\tau] \cong \mathrm{End}_{\mathbb{F}_q}\left(\mathbb{G}_{a/R}^d\right),$$

where the latter is the ring of $\mathbb{F}_q$-linear endomorphisms over $R$ of the algebraic group scheme $\mathbb{G}_{a/R}^d$ and $\tau$ is identified with the Frobenius operator that acts on $\mathbb{G}_{a/R}^d$ by raising each coordinate to the $q$-th power. Thus, the map $\rho$ gives rise to an $\mathbb{F}_q[t]$-module structure on $\mathbb{G}_{a/R}^d(R')$ for any $R$-algebra $R'$. By an $d$-dimensional $t$-module defined over $R$, we mean the pair $G = (\mathbb{G}_{a/R}^d, \rho)$, which has underlying group scheme $\mathbb{G}_{a/R}^d$, with an $\mathbb{F}_q[t]$-module structure via $\rho$. Note that since $\rho$ is an $\mathbb{F}_q$-linear ring homomorphism, $\rho$ is uniquely determined by $\rho(t)$. A sub-$t$-module of $G$ is a connected algebraic subgroup of $G$ that is invariant under the $\mathbb{F}_q[t]$-action.

A basic example is the $n$-th tensor power of the Carlitz module denoted by $\mathbf{C}^{\otimes n} := (\mathbb{G}_{a/A}^n, [\cdot]_n)$ for $n \in \mathbb{N}$ introduced by Anderson and Thakur [AT90], where $[\cdot]_n : \mathbb{F}_q[t] \to \mathrm{Mat}_n(A)[\tau]$ is the $\mathbb{F}_q$-linear ring homomorphism give by

$$[t]_n = \theta I_n + N_n + E_n\tau$$

with

$$N_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ & 0 & 1 & \ddots & \vdots \\ & & \ddots & \ddots & 0 \\ & & & \ddots & 1 \\ & & & & 0 \end{pmatrix} \in \mathrm{Mat}_n(\mathbb{F}_q) \text{ and } E_n = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix} \in \mathrm{Mat}_n(\mathbb{F}_q).$$

When $n = 1$, $\mathbf{C} := \mathbf{C}^{\otimes 1}$ is the so-called Carlitz $\mathbb{F}_q[t]$-module.

Fix a $t$-module $G = (\mathbb{G}_{a/R}^d, \rho)$ as above. Anderson [A86] showed that one has an exponential map $\mathrm{Exp}_G$, that is an entire $\mathbb{F}_q$-linear map

$$\mathrm{Exp}_G : \mathrm{Lie}\, G(\mathbb{C}_\infty) \to G(\mathbb{C}_\infty)$$

satisfying the functional equation

$$\mathrm{Exp}_G \circ \partial\rho(a) = \rho(a) \circ \mathrm{Exp}_G \ \ \forall a \in \mathbb{F}_q[t].$$

As a $d$-variable vector-valued power series, it is expressed as

$$\mathrm{Exp}_G(\mathbf{z}) = \sum_{i=0}^{\infty} e_i \mathbf{z}^{(i)},$$

where $e_0 = I_d$ and $e_i \in \mathrm{Mat}_d(F)$ for all $i$ and $\mathbf{z} = (z_1, \ldots, z_d)^{\mathrm{tr}}$ with $\mathbf{z}^{(i)} := (z_1^{q^i}, \ldots, z_d^{q^i})^{\mathrm{tr}}$. As vector-valued power series, the formal inverse of $\mathrm{Exp}_G$ is called the logarithm of $G$ and is denoted by $\mathrm{Log}_G$. That is, we have the identities

$$\mathrm{Log}_G \circ \mathrm{Exp}_G = \text{identity} = \mathrm{Exp}_G \circ \mathrm{Log}_G$$

and we find that $\mathrm{Log}_G$ satisfies the functional equation

$$\mathrm{Log}_G \circ \rho(a) = \partial\rho(a) \circ \mathrm{Log}_G \ \ \forall a \in \mathbb{F}_q[t].$$

We note that in general, $\mathrm{Log}_G$ is not an entire function on $G(\mathbb{C}_\infty)$ but converges on a certain open subset.

By a morphism $\phi$ of two $t$-modules $G_1 = (\mathbb{G}_{a/R}^{d_1}, \rho_1)$ and $G_2 = (\mathbb{G}_{a/R}^{d_2}, \rho_2)$ over $R$, we mean that $\phi : G_1 \to G_2$ is a morphism of algebraic group schemes over $R$ and that it commutes with the $\mathbb{F}_q[t]$-actions. By writing $\phi = \sum_{i=0}^{\infty} \alpha_i \tau^i \in \mathrm{Mat}_{d_2 \times d_1}(R)[\tau]$, we note that the differential of $\phi$ at the origin is identified with

$$\partial\phi := \alpha_0 : \mathrm{Lie}\, G_1 \to \mathrm{Lie}\, G_2. \tag{2.2.1}$$

The exponential maps of $t$-modules are functorial in the sense that one has the following commutative diagram:

$$\begin{array}{ccc} G_1 & \xrightarrow{\phi} & G_2 \\ \uparrow{\scriptstyle \mathrm{Exp}_{G_1}} & & \uparrow{\scriptstyle \mathrm{Exp}_{G_2}} \\ \mathrm{Lie}\, G_1 & \xrightarrow{\partial\phi} & \mathrm{Lie}\, G_2. \end{array}$$

2.3. **From Anderson dual $t$-motives to Anderson $t$-modules.** In what follows, we take an algebraically closed subfield $\mathbb{K}$ of $\mathbb{C}_\infty$ containing $K$. For example, $\mathbb{K}$ can be $\overline{K}$ or $\mathbb{C}_\infty$. Let $\mathbb{K}[t, \sigma] := \mathbb{K}[t][\sigma]$ be the ring obtained by joining the non-commutative variable $\sigma$ to the polynomial ring $\mathbb{K}[t]$ subject to the relation

$$\sigma f = f^{(-1)}\sigma \text{ for } f \in \mathbb{K}[t].$$

Note that $\mathbb{K}[t, \sigma]$ contains the two subrings $\mathbb{K}[t]$ and $\mathbb{K}[\sigma]$, but the latter is non-commutative.

2.3.1. *Frobenius modules and dual $t$-motives.* We follow [CPY19] to adapt the terminology of Frobenius modules.

**Definition 2.3.1.** *A Frobenius module over $\mathbb{K}$ is a left $\mathbb{K}[t, \sigma]$-module that is free of finite rank over $\mathbb{K}[t]$.*

The most basic example of a Frobenius module is the trivial module $\mathbf{1}$, which has underlying module $\mathbb{K}[t]$, on which $\sigma$ acts by

$$\sigma f := f^{(-1)} \ \forall f \in \mathbf{1}.$$

Another important example is the $n$-th tensor power of the Carlitz $t$-motive $C^{\otimes n}$ for $n \in \mathbb{N}$. Here, $\mathbb{K}[t]$ is the underlying module of $C^{\otimes n}$, on which the action of $\sigma$ is given by

$$\sigma f := (t - \theta)^n f^{(-1)} \ \forall f \in C^{\otimes n}.$$

One core object that we study in this paper is the $t$-module arising from the following Frobenius module $M$. Fix a positive integer $r$ and an $r$-tuple of positive integers $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$ and an $r$-tuple of polynomials $\mathfrak{Q} = (Q_1, \ldots, Q_r) \in \mathbb{K}[t]^r$. Let $M$ be a free left $\mathbb{K}[t]$-module of rank $r+1$ with a fixed basis $\{m_1,\dots,m_{r+1}\}$ and put $\mathbf{m} := (m_1,\dots,m_{r+1})^{\mathrm{tr}} \in \mathrm{Mat}_{(r+1)\times 1}(M)$. We define the following matrix

(2.3.2)
$$\Phi := \begin{pmatrix} (t-\theta)^{s_1+\cdots+s_r} & 0 & 0 & \cdots & 0 \\ Q_1^{(-1)}(t-\theta)^{s_1+\cdots+s_r} & (t-\theta)^{s_2+\cdots+s_r} & 0 & \cdots & 0 \\ 0 & Q_2^{(-1)}(t-\theta)^{s_2+\cdots+s_r} & \ddots & & \vdots \\ \vdots & & \ddots & (t-\theta)^{s_r} & 0 \\ 0 & \cdots & 0 & Q_r^{(-1)}(t-\theta)^{s_r} & 1 \end{pmatrix} \in \mathrm{Mat}_{(r+1)}(\mathbb{K}[t]),$$

then define a left $\mathbb{K}[t,\sigma]$-module structure on $M$ by setting

$$\sigma\mathbf{m} := \Phi\mathbf{m}.$$

It follows from the definition that $M$ is a Frobenius module. We mention that this $M$ was first studied by Anderson-Thakur [AT09] in order to give a period interpretation for the multiple zeta value $\zeta_A(\mathfrak{s})$ (defined in (1.2.1)) when restricting $Q_i$ to the Anderson-Thakur polynomial $H_{s_i-1}$. It was then revisited in [C14, CPY19] for studying Carlitz multiple polylogarithms when restricting $Q_i$ to certain algebraic elements over $K$.

We let $M'$ be the Frobenius submodule of $M$ which is the free $\mathbb{K}[t]$-submodule of rank $r$ spanned by the basis $\{m_1,\dots,m_r\}$, where the action of $\sigma$ on

$$\mathbf{m}' := (m_1,\dots,m_r)^{\mathrm{tr}} \in \mathrm{Mat}_{r\times 1}(M) \tag{2.3.3}$$

is represented by the matrix

(2.3.4)
$$\Phi' := \begin{pmatrix} (t-\theta)^{s_1+\cdots+s_r} & & & \\ Q_1^{(-1)}(t-\theta)^{s_1+\cdots+s_r} & (t-\theta)^{s_2+\cdots+s_r} & & \\ & \ddots & \ddots & \\ & & Q_{r-1}^{(-1)}(t-\theta)^{s_{r-1}+s_r} & (t-\theta)^{s_r} \end{pmatrix} \in \mathrm{Mat}_r(\mathbb{K}[t]).$$

Note that $\Phi'$ is the square matrix of size $r$ cut from the upper left square of $\Phi$. Following [CM19], for each $1\le i\le r$ we put

$$d_i := s_i + \cdots + s_r. \tag{2.3.5}$$

One observes that $M'$ possesses the following properties (cf. [CPY19]):

- $M'$ is free of rank $r$ over $\mathbb{K}[t]$.
- $M'$ is free of rank $d := d_1+\cdots+d_r$ over $\mathbb{K}[\sigma]$.
- $(t-\theta)^n M' \subset \sigma M'$ for all integers $n \ge d_1$ (see the proof of Proposition 2.4.2).

Note that a natural $\mathbb{K}[\sigma]$-basis of $M'$ is given by

$$\left\{(t-\theta)^{d_1-1}m_1, (t-\theta)^{d_1-2}m_1, \dots, m_1, \dots, (t-\theta)^{d_r-1}m_r, (t-\theta)^{d_r-2}m_r, \dots, m_r\right\} \tag{2.3.6}$$

and we label this basis as $\{e_1,\dots,e_d\}$. Note further that $M'$ is a dual $t$-motive in the sense of [ABP04, Sec. 4.4.1].

**Definition 2.3.7.** *A dual $t$-motive is a left $\mathbb{K}[t,\sigma]$-module $\mathcal{M}$ with the following three properties.*

- *$\mathcal{M}$ is free of finite rank over $\mathbb{K}[t]$.*
- *$\mathcal{M}$ is free of finite rank over $\mathbb{K}[\sigma]$.*
- *$(t-\theta)^n\mathcal{M} \subset \sigma\mathcal{M}$ for all $n \gg 0$.*

*Remark* 2.3.8. Note that the basis (2.3.6) involves powers of $(t-\theta)$, which is a uniformizer at $\theta$, multiplied by $m_i$. This is very convenient, because it lends itself to calculations involving Taylor series centered at $\theta$ and hyperderivatives as explained in §3.3. In more general settings, this convenience does not necessarily occur. For instance, for Drinfeld $\mathbf{A}$-modules, where $\mathbf{A}$ is the ring of regular functions of an elliptic curve, one finds a $\mathbb{C}_\infty[\sigma]$-basis of more complicated functions which necessitate new techniques not involving Taylor series (see [G17a, Prop. 3.3]). It would be interesting to see how the present techniques extend to a more general setting.

2.3.2. *The $t$-module associated to $M'$.* In this section, we quickly review Anderson's theory of $t$-frames, which allows one to construct the $t$-module $G := (\mathbb{G}_{a/\mathbb{K}}^d, \rho)$ which is associated to the Frobenius module $M'$. Since $M'$ is free over $\mathbb{K}[t]$ with basis $\{m_1, \ldots, m_r\}$, we can identify $\mathrm{Mat}_{1\times r}(\mathbb{K}[t])$ with $M'$:

$$\begin{array}{rcl} \mathrm{Mat}_{1\times r}(\mathbb{K}[t]) & \to & M' \\ (a_1, \ldots, a_r) & \mapsto & a_1 m_1 + \cdots + a_r m_r. \end{array}$$

As $M'$ is also free over $\mathbb{K}[\sigma]$ with basis $\{e_1, \ldots, e_d\}$, we can identify $M'$ with $\mathrm{Mat}_{1\times d}(\mathbb{K}[\sigma])$:

$$\begin{array}{rcl} M' & \to & \mathrm{Mat}_{1\times d}(\mathbb{K}[\sigma]) \\ b_1 e_1 + \cdots + b_d e_d & \mapsto & (b_1, \ldots, b_d). \end{array}$$

Compositing the two maps above, we have the following identification

$$\begin{array}{rrcl} \iota : & \mathrm{Mat}_{1\times r}(\mathbb{K}[t]) & \to & \mathrm{Mat}_{1\times d}(\mathbb{K}[\sigma]) \\ & (a_1, \ldots, a_r) & \mapsto & (b_1, \ldots, b_d) \end{array}$$

by expressing elements of $M'$ in terms of the fixed $\mathbb{K}[t]$-basis and $\mathbb{K}[\sigma]$-basis above.

We remark that if $x \in M'$ can be written as $x = a_1 m_1 + \cdots + a_r m_r$ as above and $\mathbf{m}' := (m_1, \ldots, m_r)^{\mathrm{tr}} \in \mathrm{Mat}_{r\times 1}(M')$, then we have the equation

$$\sigma x = \sigma(a_1, \ldots, a_r)\mathbf{m}' = (a_1, \ldots, a_r)^{(-1)}\Phi'\mathbf{m}',$$

and thus under the identification $\mathrm{Mat}_{1\times r}(\mathbb{K}[t]) \to M'$ the action of $\sigma$ on $\mathrm{Mat}_{1\times r}(\mathbb{K}[t])$ is given by

$$\sigma(a_1, \ldots, a_r) = (a_1, \ldots, a_r)^{(-1)}\Phi'. \tag{2.3.9}$$

We similarly observe that the action of $\sigma$ on $\mathrm{Mat}_{1\times d}(\mathbb{K}[\sigma])$ is the diagonal action.

Next, we define maps $\delta_0, \delta_1 : \mathrm{Mat}_{1\times d}(\mathbb{K}[\sigma]) = \mathrm{Mat}_{1\times d}(\mathbb{K})[\sigma] \to \mathbb{K}^d$ by

$$\delta_0\left(\sum_{i\geq 0} \mathbf{c}_i\sigma^i\right) := \mathbf{c}_0^{\mathrm{tr}} \tag{2.3.10}$$

$$\delta_1\left(\sum_{i\geq 0} \mathbf{c}_i\sigma^i\right) = \delta_1\left(\sum_{i\geq 0} \sigma^i\mathbf{c}_i^{(i)}\right) := \sum_{i\geq 0}\left(\mathbf{c}_i^{(i)}\right)^{\mathrm{tr}}. \tag{2.3.11}$$

Note that in [CPY19] we denote by $\Delta : M' \twoheadrightarrow \mathbb{K}^d$ the $\mathbb{F}_q$-linear composite map of $M' \cong \mathrm{Mat}_{1\times d}(\mathbb{K}[\sigma])$ and $\delta_1$. We then have $\operatorname{Ker}\Delta = (\sigma-1)M'$, and so the identification as $\mathbb{F}_q$-vector spaces

$$M'/(\sigma-1)M' \cong \mathbb{K}^d.$$

Since we have an $\mathbb{F}_q[t]$-module structure on $M'/(\sigma-1)M'$, we can equip a left $\mathbb{F}_q[t]$-module structure on $\mathbb{K}^d$ via the identification above. As the $\mathbb{K}$-valued points of $\mathbb{G}_{a/\mathbb{K}}^d$ is $\mathbb{K}^d$ which is

Zariski dense inside the algebraic group $\mathbb{G}_{a/\mathbb{K}}^d$, the $\mathbb{F}_q[t]$-module structure on $\mathbb{K}^d$ above gives rise to an $\mathbb{F}_q$-linear ring homomorphism

$$\rho : \mathbb{F}_q[t] \to \mathrm{Mat}_d(\mathbb{K})[\tau].$$

The $t$-module associated to $M'$ is defined to be $G := (\mathbb{G}_{a/\mathbb{K}}^d, \rho)$. For more details on this construction, see [HJ16, §5.2].

2.4. **One crucial result of Anderson.** Now we take the field $\mathbb{K} = \mathbb{C}_\infty$, and consider the $\mathbb{F}_q$-linear map

$$\delta_0 \circ \iota : \mathrm{Mat}_{1\times r}(\mathbb{K}[t]) \to \mathbb{K}^d.$$

In order to give an explicit description of the above map, we make the following definition first.

**Definition 2.4.1.** *Given a function $f(t) \in \mathbb{T}_\theta$ which has a Taylor series centered at $\theta$ that is given by*

$$f(t) = \sum_{n=0}^{\infty} a_n (t-\theta)^n,$$

*we define the $k$th jet of $f$ at the point $\theta$ to be the polynomial*

$$J_\theta^k(f) = a_k(t-\theta)^k + \cdots + a_1(t-\theta) + a_0 \in \mathbb{C}_\infty[t].$$

Note that by (2.4.3), power series in $\mathbb{T}_\theta$ always have such a Taylor series as given above, and so our definition is natural. We now describe the explicit formula for the map $\delta_0 \circ \iota$.

**Proposition 2.4.2.** *For a vector $\mathbf{a} = (a_1, \ldots, a_r) \in \mathrm{Mat}_{1\times r}(\mathbb{K}[t])$, for each coordinate $a_i$, we form the $d_i - 1$ jet at $\theta$ and label these as*

$$J_\theta^{d_i-1}(a_i) = c_{i,1}(t-\theta)^{d_i-1} + c_{i,2}(t-\theta)^{d_i-2} + \cdots + c_{i,d_i}.$$

*Then the map $\delta_0 \circ \iota$ is given by*

$$\delta_0 \circ \iota(\mathbf{a}) = \left(\begin{array}{c} c_{1,1} \\ c_{1,2} \\ \vdots \\ c_{1,d_1} \\ c_{2,1} \\ c_{2,2} \\ \vdots \\ c_{2,d_2} \\ \vdots \\ c_{r,1} \\ c_{r,2} \\ \vdots \\ c_{r,d_r} \end{array}\right) \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} d_1 \\ \left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} d_2 \\ \vdots \\ \left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} d_r \end{array}.$$

*Proof.* By the definition of $\delta_0$, it suffices to show for each $i$ with $1 \le i \le r$ that, $(t-\theta)^N m_i \in \sigma M'$ for all $N \ge d_i$. We prove this assertion by induction on $i$. For $i = 1$, the claim holds since we have

$$(t-\theta)^N m_1 = (t-\theta)^{N-d_1} \cdot (t-\theta)^{d_1} m_1 = (t-\theta)^{N-d_1} \cdot \sigma m_1 = \sigma (t-\theta^q)^{N-d_1} m_1 \in \sigma M'.$$

Let $i \geq 2$, and assume that there exists $m \in M'$ such that $(t-\theta)^{d_{i-1}} m_{i-1} = \sigma m$. Then we have

$$\begin{aligned}(t-\theta)^N m_i &= (t-\theta)^{N-d_i} \cdot (t-\theta)^{d_i} m_i = (t-\theta)^{N-d_i} \left(\sigma m_i - Q_{i-1}^{(-1)} (t-\theta)^{d_{i-1}} m_{i-1}\right) \\ &= (t-\theta)^{N-d_i} \left(\sigma m_i - Q_{i-1}^{(-1)} \sigma m\right) = \sigma (t-\theta^q)^{N-d_i} \left(m_i - Q_{i-1} m\right) \in \sigma M'.\end{aligned}$$

□

In other words, the map $\delta_0 \circ \iota$ factors through the following map still denoted by $\delta_0 \circ \iota$:

$$\mathbb{K}[t]/\left((t-\theta)^{d_1}\right) \times \cdots \times \mathbb{K}[t]/\left((t-\theta)^{d_r}\right) \to \mathbb{K}^d.$$

Anderson gives a theorem (see [HJ16] and [NP18]) that states that there exists a unique extension of the composition $\delta_0 \circ \iota$ to vectors over the Tate algebra $\mathbb{T}_\theta$ and he calls this extension

$$\widehat{\delta_0 \circ \iota} : \mathrm{Mat}_{1\times r}(\mathbb{T}_\theta) \to \mathbb{K}^d.$$

With the above analysis of $\delta_0 \circ \iota$ we can see concretely that the procedure for calculating the extended map $\widehat{\delta_0 \circ \iota}$ is the same as for the original. Namely, $\widehat{\delta_0 \circ \iota}$ is given by composing the following maps

$$\mathrm{Mat}_{1\times r}(\mathbb{T}_\theta) \hookrightarrow \mathrm{Mat}_{1\times r}(\mathbb{K}[\![t-\theta]\!]) \twoheadrightarrow \prod_{i=1}^r \mathbb{K}[\![t-\theta]\!]/\left((t-\theta)^{d_i}\right) \cong \prod_{i=1}^r \mathbb{K}[t]/\left((t-\theta)^{d_i}\right) \xrightarrow{\delta_0 \circ \iota} \mathbb{K}^d,$$

where the first embedding is via the following natural embedding componentwise

$$\begin{array}{cccc} \eta : & \mathbb{T}_\theta & \hookrightarrow & \mathbb{K}[\![t-\theta]\!] \\ & \displaystyle\sum_{i=0}^\infty b_i t^i & \mapsto & \displaystyle\eta\Big(\sum_{i=0}^\infty b_i t^i\Big) := \sum_{i=0}^\infty \left( \sum_{j=i}^\infty \binom{j}{i} b_j \theta^{j-i} \right) (t-\theta)^i. \end{array} \tag{2.4.3}$$

Note that $\eta$ is injective. Indeed, assume that $\eta(\sum_{i=0}^\infty b_i t^i) = 0$, and fix $i_0 \geq 0$. Since we have $\sum_{j=i_0}^\infty \binom{j}{i_0} b_j \theta^{j-i_0} = 0$ and the absolute value $|\cdot|_\infty$ is non-archimedean, there exists $i_1 > i_0$ such that

$$|b_{i_0}|_\infty \leq \left| \binom{i_1}{i_0} b_{i_1} \theta^{i_1 - i_0} \right|_\infty$$

and so

$$|b_{i_0} \theta^{i_0}|_\infty \leq \left| \binom{i_1}{i_0} b_{i_1} \theta^{i_1} \right|_\infty \leq |b_{i_1} \theta^{i_1}|_\infty.$$

Repeating this argument, we can take $i_0 < i_1 < i_2 < \cdots$ such that $|b_{i_0}\theta^{i_0}|_\infty \leq |b_{i_1}\theta^{i_1}|_\infty \leq |b_{i_2}\theta^{i_2}|_\infty \leq \cdots$. Since $|b_i \theta^i|_\infty \to 0$, we have $b_{i_0} = 0$. Therefore, $\eta$ is injective. Moreover, we can verify that the image of $\eta$ is

$$\left\{ \sum_{i=0}^\infty b_i (t-\theta)^i \in \mathbb{K}[\![t-\theta]\!] \;\middle|\; \left|b_i \theta^i\right|_\infty \to 0 \right\},$$

and we can construct $\eta^{-1}$ directly.

Since it does not cause any confusion, we will drop the hat notation and just use $\delta_0 \circ \iota$ for the extension. We now state Anderson's theorem from his unpublished notes, which one can find the statement and its proof in [HJ16, Cor. 5.20] and [NP18].

**Theorem 2.4.4** (Anderson)**.** *For all* $\mathbf{g} \in \mathrm{Mat}_{1\times r}(\mathbb{T}_\theta)$ *and* $\mathbf{w} \in \mathrm{Mat}_{1\times r}(\mathbb{C}_\infty[t])$ *satisfying the functional equation*

$$\mathbf{g}^{(-1)}\Phi' - \mathbf{g} = \mathbf{w} \tag{2.4.5}$$

*one has*

$$\mathrm{Exp}_G\left(\delta_0 \circ \iota(\mathbf{g} + \mathbf{w})\right) = \delta_1 \circ \iota(\mathbf{w}). \tag{2.4.6}$$

## 3. General formulae

In this section, we fix an algebraically closed subfield $\mathbb{K}$ of $\mathbb{C}_\infty$ containing $K$. Let $r$ be a positive integer, and fix an index $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$ with $d_i$ defined in (2.3.5). Fix $r-1$ polynomials $Q_1, \ldots, Q_{r-1} \in \mathbb{K}[t]$ and let $\Phi' \in \mathrm{Mat}_r(\mathbb{K}[t])$ be the matrix (2.3.4) defined using $Q_1, \ldots, Q_{r-1}$, and let $G$ be the $t$-module over $\mathbb{K}$ associated to the dual $t$-motive $M'$ defined by $\Phi'$ in Sec. 2.3.2.

3.1. **Constructions of g and its convergnece.** The aim of this subsection is to construct a solution $\mathbf{g} \in \mathrm{Mat}_{1\times r}(\mathbb{T}_\theta)$ of the difference equation (2.4.5) under certain conditions.

3.1.1. *Gauss norms.* First, we define a seminorm on $\mathrm{Mat}_{\ell,m}(\mathbb{C}_\infty)$ for $B = (b_{i,j}) \in \mathrm{Mat}_{\ell,m}(\mathbb{C}_\infty)$ by setting

$$\|B\| = \max_{i,j}\left\{|b_{i,j}|_\infty\right\}.$$

Note that the seminorm is only submultiplicative, i.e. for matrices $B \in \mathrm{Mat}_{k,\ell}(\mathbb{C}_\infty)$ and $C \in \mathrm{Mat}_{\ell,m}(\mathbb{C}_\infty)$

$$\|BC\| \le \|B\| \cdot \|C\|.$$

The above inequality also gives

$$\|B^{-1}\| \ge (\|B\|)^{-1}$$

for any invertible matrix $B$ with entries in $\mathbb{C}_\infty$. We also have identities and inequalities for $\alpha \in \mathbb{C}_\infty$ and $B, C \in \mathrm{Mat}_{\ell,m}(\mathbb{C}_\infty)$

$$\|\alpha B\| = |\alpha|_\infty \cdot \|B\|, \quad \|B + C\| \le \max\left\{\|B\|, \|C\|\right\}.$$

Then, for $\alpha \in \mathbb{C}_\infty^\times$ we define the Tate algebra

$$\mathbb{T}_\alpha = \left\{ \sum_{i=0}^{\infty} b_i t^i \in \mathbb{C}_\infty[\![t]\!] \;\middle|\; \left|b_i \alpha^i\right|_\infty \to 0 \right\}. \tag{3.1.1}$$

If $|\alpha|_\infty \ge 1$, then $\mathbb{T}_\alpha$ is stable under the action $f \mapsto f^{(n)}$ for each $n \ge 0$. Define the Gauss norm $\|\cdot\|_\alpha$ on $\mathbb{T}_\alpha$ by putting

$$\|f\|_\alpha := \max_i \left\{|b_i \alpha^i|_\infty\right\}$$

for $f = \sum_{i\ge 0} b_i t^i \in \mathbb{T}_\alpha$. We then extend the Gauss norm to $\mathrm{Mat}_{\ell,m}(\mathbb{T}_\alpha)$ by setting

$$\|\mathbf{h}\|_\alpha = \max_{i,j}\left\{\|h_{ij}\|_\alpha\right\}$$

for $\mathbf{h} = (h_{ij}) \in \mathrm{Mat}_{\ell,m}(\mathbb{T}_\alpha)$. We mention that $\|\mathbf{h}\|_\alpha$ coincides with $\|\mathbf{h}\|$ when $\mathbf{h} \in \mathrm{Mat}_{\ell\times m}(\mathbb{C}_\infty)$. Then for $\mathbf{h} \in \mathrm{Mat}_{k,\ell}(\mathbb{T}_\alpha)$ and $\mathbf{k} \in \mathrm{Mat}_{\ell,m}(\mathbb{T}_\alpha)$

$$\|\mathbf{hk}\|_\alpha \le \|\mathbf{h}\|_\alpha \cdot \|\mathbf{k}\|_\alpha.$$

Since $\mathbb{T}_\alpha \to \mathbb{T}_1; T \mapsto \alpha T$ is an isomorphism of normed algebras and $\mathbb{T}_1$ is complete (cf. [BGR84, Sec. 1.4, Prop. 3]), $\mathrm{Mat}_{\ell,m}(\mathbb{T}_\alpha)$ is complete under the Gauss norm $\|\cdot\|_\alpha$.

3.1.2. *Definition of* $\mathbf{g}$ *and its convergence.* We fix a vector $\mathbf{w} = (w_1, \dots, w_r) \in \mathrm{Mat}_{1\times r}(\mathbb{K}[t])$ with

$$w_i = c_{i,1}(t-\theta)^{d_i-1} + c_{i,2}(t-\theta)^{d_i-2} + \cdots + c_{i,d_i}, \quad c_{i,j} \in \mathbb{K}, \quad 1 \le i \le r,$$

then define

$$\mathbf{g} = \mathbf{w}^{(1)}(\Phi'^{-1})^{(1)} + \mathbf{w}^{(2)}(\Phi'^{-1})^{(2)}(\Phi'^{-1})^{(1)} + \mathbf{w}^{(3)}(\Phi'^{-1})^{(3)}(\Phi'^{-1})^{(2)}(\Phi'^{-1})^{(1)} + \cdots . \tag{3.1.2}$$

A quick check shows that the general term $\mathbf{w}^{(n)}(\Phi'^{-1})^{(n)} \cdots (\Phi'^{-1})^{(1)}$ is in $\mathrm{Mat}_{1\times r}(\mathbb{T}_\theta)$ and under certain hypothesis on the polynomials $Q_i$'s and $\mathbf{w}$, it is shown to converge to zero under the Gauss norm $\|\cdot\|_\theta$ in the following proposition.

**Proposition 3.1.3.** *Assume that* $\|Q_i\|_1 \le q^{\frac{s_i q}{q-1}}$ *for each* $1 \le i \le r-1$. *Let* $\mathbf{w} = (w_1, \dots, w_r) \in \mathrm{Mat}_{1\times r}(\mathbb{K}[t])$ *be given above with the condition that* $|c_{i,j}|_\infty < q^{j+\frac{d_i}{q-1}}$ *for each* $1 \le i \le r$ *and* $1 \le j \le d_i$. *Then the formal series* $\mathbf{g}$ *defined in* (3.1.2) *converges in* $\mathrm{Mat}_{1\times r}(\mathbb{T}_\theta)$.

*Proof.* Let $\mu_i := \deg_t Q_i$. It is clear that $\|Q_i^{(n)}\|_\theta \le q^{\frac{s_i q^{n+1}}{q-1}+\mu_i}$ for each $n \ge 0$. For each $n \ge 1$, we have $\dfrac{1}{t-\theta^{q^n}} \in \mathbb{T}_\theta$ and $\left\| \dfrac{1}{t-\theta^{q^n}} \right\|_\theta = q^{-q^n}$. From the definition of $\Phi'$ immediately following (2.3.4), we calculate that

$$\Phi'^{-1} = \begin{pmatrix} \frac{1}{(t-\theta)^{d_1}} & 0 & 0 & \cdots & 0 \\ \frac{-Q_1^{(-1)}}{(t-\theta)^{d_2}} & \frac{1}{(t-\theta)^{d_2}} & 0 & \cdots & 0 \\ \frac{(Q_1Q_2)^{(-1)}}{(t-\theta)^{d_3}} & \frac{-Q_2^{(-1)}}{(t-\theta)^{d_3}} & \frac{1}{(t-\theta)^{d_3}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{(-1)^{r-1}(Q_1\dots Q_{r-1})^{(-1)}}{(t-\theta)^{d_r}} & \frac{(-1)^{r-2}(Q_2\dots Q_{r-1})^{(-1)}}{(t-\theta)^{d_r}} & \frac{(-1)^{r-3}(Q_3\dots Q_{r-1})^{(-1)}}{(t-\theta)^{d_r}} & \cdots & \frac{1}{(t-\theta)^{d_r}} \end{pmatrix} \in \mathrm{Mat}_r(\mathbb{K}(t)). \tag{3.1.4}$$

So for each $1 \le \ell \le r$, the $\ell$-th component of $\mathbf{w}^{(n)}(\Phi'^{-1})^{(n)}(\Phi'^{-1})^{(n-1)} \cdots (\Phi'^{-1})^{(1)}$ is

$$\begin{aligned} &\sum_{i=\ell}^{r} w_i^{(n)} \cdot (-1)^{i-\ell} \sum_{\ell=k_0\le k_1\le\cdots\le k_n=i} \frac{(Q_{k_0}\cdots Q_{k_1-1})(Q_{k_1}\cdots Q_{k_2-1})^{(1)}\cdots(Q_{k_{n-1}}\cdots Q_{k_n-1})^{(n-1)}}{(t-\theta^q)^{d_{k_1}}(t-\theta^{q^2})^{d_{k_2}}\cdots(t-\theta^{q^n})^{d_{k_n}}} \\ &= \sum_{\substack{\ell\le i\le r \\ 1\le j\le d_i}} (-1)^{i-\ell} c_{i,j}^{q^n} \sum_{\ell=k_0\le k_1\le\cdots\le k_n=i} \frac{(Q_{k_0}\cdots Q_{k_1-1})(Q_{k_1}\cdots Q_{k_2-1})^{(1)}\cdots(Q_{k_{n-1}}\cdots Q_{k_n-1})^{(n-1)}}{(t-\theta^q)^{d_{k_1}}(t-\theta^{q^2})^{d_{k_2}}\cdots(t-\theta^{q^{n-1}})^{d_{k_{n-1}}}(t-\theta^{q^n})^{j}}. \end{aligned}$$

Then we calculate the Gauss norm for the general term:

$$\left\| \frac{(Q_{k_0}\cdots Q_{k_1-1})(Q_{k_1}\cdots Q_{k_2-1})^{(1)}\cdots(Q_{k_{n-1}}\cdots Q_{k_n-1})^{(n-1)}}{(t-\theta^q)^{d_{k_1}}(t-\theta^{q^2})^{d_{k_2}}\cdots(t-\theta^{q^{n-1}})^{d_{k_{n-1}}}(t-\theta^{q^n})^{j}} \right\|_\theta = q^{\alpha}$$

where $\alpha$ satisfies

$$\begin{aligned}\alpha &\le \sum_{a=1}^{n}\sum_{b=k_{a-1}}^{k_a-1}\left(\frac{s_b q^a}{q-1}+\mu_b\right)-\sum_{a=1}^{n-1} d_{k_a}q^a - jq^n\\ &= (\mu_{k_0}+\cdots+\mu_{k_n-1})+\sum_{a=1}^{n}\frac{d_{k_{a-1}}-d_{k_a}}{q-1}q^a-\sum_{a=1}^{n-1}d_{k_a}q^a-jq^n\\ &= (\mu_\ell+\cdots+\mu_{i-1})+\frac{d_{k_0}q}{q-1}+\sum_{a=1}^{n-1}d_{k_a}\left(\frac{q^{a+1}-q^a}{q-1}-q^a\right)-\frac{d_{k_n}q^n}{q-1}-jq^n\\ &= (\mu_\ell+\cdots+\mu_{i-1})+\frac{d_\ell q}{q-1}-\left(j+\frac{d_i}{q-1}\right)q^n.\end{aligned}$$

Therefore we have

$$\left\|\mathbf{w}^{(n)}(\Phi'^{-1})^{(n)}(\Phi'^{-1})^{(n-1)}\cdots(\Phi'^{-1})^{(1)}\right\|_\theta \le \max_{\substack{1\le\ell\le i\le r\\ 1\le j\le d_i}}\left\{q^{(\mu_\ell+\cdots+\mu_{i-1})+\frac{d_\ell q}{q-1}}\left(\frac{|c_{i,j}|}{q^{j+\frac{d_i}{q-1}}}\right)^{q^n}\right\},$$

which goes to 0 as $(n\to\infty)$. Since $\mathbb{T}_\theta$ is complete with respect to $\|\cdot\|_\theta$, we have the desired result.

□

*Remark* 3.1.5. Note that the condition on $|c_{i,j}|_\infty$ coincides with [CM17, Prop. 4.2.2].

*Remark* 3.1.6. Note that from the definition one sees that $\mathbf{g}$ satisfies the difference equation

$$\mathbf{g}^{(-1)}\Phi' - \mathbf{g} = \mathbf{w}.$$

3.2. **$t$-motivic Carlitz multiple star polylogarithms.** Carlitz multiple polylogarithms (CMPL's) were introduced by the first author in [C14] and are generalizations of the Carlitz polylogarithm of Anderson and Thakur from [AT90]. Carlitz multiple star polylogarithms (CMSPL's) were introduced by the first and third authors in [CM17] in order to connect MZV's with logarithms of $t$-modules.

Let $L_0=1$ and for $i\ge 1$ let $L_i=(\theta-\theta^q)\dots(\theta-\theta^{q^i})\in A$. We also define deformations $\mathbb{L}_0=1$ and $\mathbb{L}_i=(t-\theta^q)\dots(t-\theta^{q^i})\in A[t]$, so that $\mathbb{L}_i(\theta)=L_i$. For $\mathfrak{s}=(s_1,\dots,s_r)\in\mathbb{N}^r$, we define the associated CMPL and CMSPL by

$$\mathrm{Li}_{\mathfrak{s}}(z_1,\dots,z_r)=\sum_{i_1>\cdots>i_r\ge 0}\frac{z_1^{q^{i_1}}\dots z_r^{q^{i_r}}}{L_{i_1}^{s_1}\dots L_{i_r}^{s_r}}\in K[\![z_1,\dots,z_r]\!] \tag{3.2.1}$$

and

$$\mathrm{Li}^\star_{\mathfrak{s}}(z_1,\dots,z_r)=\sum_{i_1\ge\cdots\ge i_r\ge 0}\frac{z_1^{q^{i_1}}\dots z_r^{q^{i_r}}}{L_{i_1}^{s_1}\dots L_{i_r}^{s_r}}\in K[\![z_1,\dots,z_r]\!]. \tag{3.2.2}$$

Also the $t$-motivic CMPL and $t$-motivic CMSPL are defined by

$$\mathfrak{Li}_{\mathfrak{s}}(t;z_1,\dots,z_r)=\sum_{i_1>\cdots>i_r\ge 0}\frac{z_1^{q^{i_1}}\dots z_r^{q^{i_r}}}{\mathbb{L}_{i_1}^{s_1}\dots \mathbb{L}_{i_r}^{s_r}}\in K[\![t,z_1,\dots,z_r]\!] \tag{3.2.3}$$

and

$$\mathfrak{Li}^\star_{\mathfrak{s}}(t;z_1,\dots,z_r)=\sum_{i_1\ge\cdots\ge i_r\ge 0}\frac{z_1^{q^{i_1}}\dots z_r^{q^{i_r}}}{\mathbb{L}_{i_1}^{s_1}\dots \mathbb{L}_{i_r}^{s_r}}\in K[\![t,z_1,\dots,z_r]\!], \tag{3.2.4}$$

and observe that $\mathfrak{Li}_{\mathfrak{s}}\,|_{t=\theta}=\mathrm{Li}_{\mathfrak{s}}$ and $\mathfrak{Li}^{\star}_{\mathfrak{s}}|_{t=\theta}=\mathrm{Li}^{\star}_{\mathfrak{s}}$.

Given $\mathbf{u}=(u_1,\dots,u_r)^{\mathrm{tr}}\in\mathbb{K}^r$, we note that $\dfrac{u_1^{q^{i_1}}\cdots u_r^{q^{i_r}}}{\mathbb{L}_{i_1}^{s_1}\cdots\mathbb{L}_{i_r}^{s_r}}\in\mathbb{T}_\theta$, and if for each $2\le i\le r$,

$$|u_1|_\infty<q^{\frac{s_1q}{q-1}}\text{ and }|u_i|_\infty\le q^{\frac{s_iq}{q-1}} \tag{3.2.5}$$

then $\left\|\dfrac{u_1^{q^{i_1}}\cdots u_r^{q^{i_r}}}{\mathbb{L}_{i_1}^{s_1}\cdots\mathbb{L}_{i_r}^{s_r}}\right\|_\theta\to 0,\ (i_1\to\infty)$. Thus $\mathfrak{Li}_{\mathfrak{s}}(t;u_1,\dots,u_r)$ and $\mathfrak{Li}^{\star}_{\mathfrak{s}}(t;u_1,\dots,u_r)$ converge in $\mathbb{T}_\theta$ when specializating $z_i=u_i$ for each $i$. In this situation, we simplify the notation by putting

$$\mathfrak{Li}_{\mathfrak{s},\mathbf{u}}(t):=\mathfrak{Li}_{\mathfrak{s}}(t;u_1,\dots,u_r)\in\mathbb{K}[\![t]\!]\text{ and }\mathfrak{Li}^{\star}_{\mathfrak{s},\mathbf{u}}(t):=\mathfrak{Li}^{\star}_{\mathfrak{s}}(t;u_1,\dots,u_r)\in\mathbb{K}[\![t]\!].$$

When $r=1$ and $s=1$, the series $\mathfrak{Li}_{1,u}(t)$ for $u\in\overline{K}^{\times}$ with $|u|_\infty<|\theta|_\theta^{\frac{q}{q-1}}$ was first introduced by Papanikolas [P08] to relate the Carlitz logarithm at $u$ to a period of certain $t$-motive. It was then generalized by the first author and Yu [CY07] in the case of $r=1$ and $s>1$ to study Carlitz zeta values. The general series $\mathfrak{Li}_{\mathfrak{s},\mathbf{u}}$ was further studied by the first author [C14] to relate the CMPL's at algebraic points to periods of certain $t$-motives. These series play an essential role when applying the ABP-criterion [ABP04] for the CMPL's at algebraic points in question. See also [CPY19, M17].

3.3. **Hyperderivatives.** For any non-negative integer $n$, we define the $n$th hyperderivative (with respect to $t$) $\partial_t^n:\mathbb{C}_\infty(\!(t)\!)\to\mathbb{C}_\infty(\!(t)\!)$ by

$$\partial_t^n\left(\sum_{i=i_0}^{\infty}a_it^i\right):=\sum_{i=i_0}^{\infty}\binom{i}{n}a_it^{i-n},$$

where $\binom{i}{n}$ refers to the usual binomial coefficient, but modulo $p$. From the definition one sees that $\partial_t^n$ is a $\mathbb{C}_\infty$-linear operator and that $\partial_t^0$ is the identity map. We further note that the hyperderivatives satisfy the product rule: for $n\in\mathbb{N}$ and $f,g\in\mathbb{C}_\infty(\!(t)\!)$,

$$\partial_t^n\,(fg)=\sum_{i=0}^{n}\partial_t^i(f)\cdot\partial_t^{n-i}(g). \tag{3.3.1}$$

In this paper, we are interested in Taylor coefficients of the series expansion of $f\in\mathbb{T}_\theta$ at $t=\theta$, and the following proposition shows that such Taylor coefficients are expressed as the hyperderivatives of $f$ evaluated at $t=\theta$ (cf. [Pp, Lem. 2.4.1] in the case of rational functions).

**Proposition 3.3.2.** *For any $f\in\mathbb{T}_\theta$, we write $\eta(f)=\sum_{i=0}^{\infty}a_i(t-\theta)^i$. Then for any non-negative integer $n$ we have*

$$a_n=\partial_t^n(f)|_{t=\theta}.$$

*Proof.* We define the $n$th hyperderivative $\bar\partial_t^n$ on $\mathbb{C}_\infty[\![t-\theta]\!]$ by

$$\bar\partial_t^n\left(\sum_{i=0}^{\infty}a_i(t-\theta)^i\right):=\sum_{i=0}^{\infty}\binom{i}{n}a_i(t-\theta)^{i-n}.$$

Then we claim that one has the following commutative diagram

$$\begin{tikzcd}
\mathbb{T}_\theta \arrow[r, hook, "\eta"] \arrow[d, "\partial_t^n"'] & \mathbb{C}_\infty[\![t-\theta]\!] \arrow[d, "\bar{\partial}_t^n"] \\
\mathbb{T}_\theta \arrow[r, hook, "\eta"] \arrow[dr, "t=\theta"'] & \mathbb{C}_\infty[\![t-\theta]\!] \arrow[d, "t=\theta"] \\
& \mathbb{C}_\infty,
\end{tikzcd}$$

i.e., $\bar{\partial}_t^n(\eta(f)) = \eta(\partial_t^n(f))$ and $f|_{t=\theta} = \eta(f)|_{t=\theta}$(= constant term) for each $f \in \mathbb{T}_\theta$.

To prove the claim above, we first fix an $f(t) = \sum_{i=0}^\infty b_i t^i \in \mathbb{T}_\theta$. Then we have

$$\begin{aligned}
\bar{\partial}_t^n(\eta(f)) &= \bar{\partial}_t^n\left(\sum_{i=0}^\infty\left(\sum_{j=i}^\infty\binom{j}{i}b_j\theta^{j-i}\right)(t-\theta)^i\right) = \sum_{i=n}^\infty\binom{i}{n}\left(\sum_{j=i}^\infty\binom{j}{i}b_j\theta^{j-i}\right)(t-\theta)^{i-n} \\
&= \sum_{i=0}^\infty\binom{i+n}{n}\left(\sum_{j=i}^\infty\binom{j+n}{i+n}b_{j+n}\theta^{j-i}\right)(t-\theta)^i \\
&= \sum_{i=0}^\infty\left(\sum_{j=i}^\infty\binom{j}{i}\binom{j+n}{n}b_{j+n}\theta^{j-i}\right)(t-\theta)^i = \eta\left(\sum_{i=0}^\infty\binom{i+n}{n}b_{i+n}t^i\right) \\
&= \eta\left(\sum_{i=n}^\infty\binom{i}{n}b_it^{i-n}\right) = \eta(\partial_t^n(f)),
\end{aligned}$$

where we use

$$\binom{i+n}{n}\binom{j+n}{i+n} = \binom{j}{i}\binom{j+n}{n}$$

in the fourth equality. We also have

$$\eta(f)|_{t=\theta} = \sum_{i=0}^\infty\left(\sum_{j=i}^\infty\binom{j}{i}b_j\theta^{j-i}\right)(t-\theta)^i\Bigg|_{t=\theta} = \sum_{j=0}^\infty\binom{j}{0}b_j\theta^j = f|_{t=\theta}.$$

Then the desired result follows from the above commutativities:

$$\partial_t^n(f)|_{t=\theta} = \eta(\partial_t^n(f))|_{t=\theta} = \bar{\partial}_t^n(\eta(f))|_{t=\theta} = \sum_{i=n}^\infty\binom{i}{n}a_i(t-\theta)^{i-n}\Bigg|_{t=\theta} = a_n.$$

□

*Remark* 3.3.3. Since under the hypothesis (3.2.5) the above series $\mathfrak{Li}_{\mathfrak{s},\mathbf{u}}$ and $\mathfrak{Li}^\star_{\mathfrak{s},\mathbf{u}}$ are in $\mathbb{T}_\theta$, the hyperderivatives $\partial_t^j\,\mathfrak{Li}_{\mathfrak{s},\mathbf{u}}$ and $\partial_t^j\mathfrak{Li}^\star_{\mathfrak{s},\mathbf{u}}$ are still in $\mathbb{T}_\theta$ for every positive integer $j$.

*Remark* 3.3.4. Using the proposition above, we can rewrite $\delta_0 \circ \iota(\mathbf{a})$ in Proposition 2.4.2 as

$$\delta_0 \circ \iota(\mathbf{a}) = \left(\begin{array}{c} (\partial_t^{d_1-1}a_1)|_{t=\theta} \\ \vdots \\ (\partial_t^{1}a_1)|_{t=\theta} \\ a_1(\theta) \\ (\partial_t^{d_2-1}a_1)|_{t=\theta} \\ \vdots \\ (\partial_t^{1}a_2)|_{t=\theta} \\ a_2(\theta) \\ \vdots \\ (\partial_t^{d_r-1}a_r)|_{t=\theta} \\ \vdots \\ (\partial_t^{1}a_r)|_{t=\theta} \\ a_r(\theta) \end{array}\right) \begin{array}{l} \left.\begin{array}{c} \vphantom{(\partial_t^{d_1-1}a_1)|_{t=\theta}} \\ \vphantom{\vdots} \\ \vphantom{(\partial_t^{1}a_1)|_{t=\theta}} \\ \vphantom{a_1(\theta)} \end{array}\right\} d_1 \\ \left.\begin{array}{c} \vphantom{(\partial_t^{d_2-1}a_1)|_{t=\theta}} \\ \vphantom{\vdots} \\ \vphantom{(\partial_t^{1}a_2)|_{t=\theta}} \\ \vphantom{a_2(\theta)} \end{array}\right\} d_2 \\ \quad\vdots \\ \left.\begin{array}{c} \vphantom{(\partial_t^{d_r-1}a_r)|_{t=\theta}} \\ \vphantom{\vdots} \\ \vphantom{(\partial_t^{1}a_r)|_{t=\theta}} \\ \vphantom{a_r(\theta)} \end{array}\right\} d_r \end{array}. \tag{3.3.5}$$

Moreover, Theorem 2.4.4 can be reformulated via hyperderivatives as the following. Let the notation be as given in Theorem 2.4.4 and put $\mathbf{a} := \mathbf{g} + \mathbf{w}$. Then we have $\mathrm{Exp}_G(\delta_0 \circ \iota(\mathbf{a})) = \delta_1 \circ \iota(\mathbf{w})$, where $\delta_0 \circ \iota(\mathbf{a})$ is given in (3.3.5).

For convenience, we extend these hyperderivatives to operators on vectors with entries in $\mathbb{C}_\infty(\!(t)\!)$. Precisely, for a positive integer $m$ and for $g_1, \ldots, g_n \in \mathbb{C}_\infty(\!(t)\!)$ we define

$$\partial_t^m[g_1, \ldots, g_n] = \begin{pmatrix} \partial_t^{m-1}(g_1) & \ldots & \partial_t^{m-1}(g_n) \\ \vdots & & \vdots \\ \partial_t^{1}(g_1) & \ldots & \partial_t^{1}(g_n) \\ g_1 & \ldots & g_n \end{pmatrix} \in \mathrm{Mat}_{m\times n}(\mathbb{C}_\infty(\!(t)\!)), \tag{3.3.6}$$

and for $h \in \mathbb{C}_\infty(\!(t)\!)$ define

$$d_t^m[h] = \begin{pmatrix} h & \partial_t^1(h) & \partial_t^2(h) & \ldots & \partial_t^{m-1}(h) \\ 0 & h & \partial_t^1(h) & \ldots & \partial_t^{m-2}(h) \\ 0 & 0 & h & \ldots & \partial_t^{m-3}(h) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & h \end{pmatrix} \in \mathrm{Mat}_m(\mathbb{C}_\infty(\!(t)\!)). \tag{3.3.7}$$

These matrices are all defined in [Pp, §2.5] and are called $\partial$-matrices and $d$-matrices, respectively. We collect several facts about these matrices which are proved there.

**Proposition 3.3.8.** *Let $m$ be a positive integer. For any $h, g_1, \ldots, g_n \in \mathbb{C}_\infty(\!(t)\!)$, the following hold.*

(1) *The $d$-matrices are multiplicative,*

$$d_t^m[g_1]d_t^m[g_2] = d_t^m[g_1 g_2].$$

(2) *We can combine $d$-matrices and $\partial$-matrices as follows,*

$$d_t^m[h]\partial_t^m[g_1, \ldots, g_n] = \partial_t^m[hg_1, \ldots, hg_n].$$

(3) *Viewed as maps, $d_t^m[\cdot] : \mathbb{C}_\infty(\!(t)\!) \to \mathrm{Mat}_m(\mathbb{C}_\infty(\!(t)\!))$ and $\partial_t^m[\cdot] : \mathrm{Mat}_{1\times n}(\mathbb{C}_\infty(\!(t)\!)) \to \mathrm{Mat}_{m\times n}(\mathbb{C}_\infty(\!(t)\!))$ are $\mathbb{C}_\infty$-linear injections of vector spaces.*

3.4. **The formulae.** We continue the notation given at the beginning of this section.

3.4.1. *The set up.* Put

$$\Theta_{i,j} := (-1)^{j-i}\frac{(Q_i\cdots Q_{j-1})^{(-1)}}{(t-\theta)^{d_j}}\ (i\leq j) \text{ and } \mathbf{x}_j := \big((t-\theta)^{d_j-1},(t-\theta)^{d_j-2},\ldots,1\big)$$

and define

$$\Theta := \begin{pmatrix}\Theta_{1,1} & \cdots & \Theta_{1,r}\\ & \ddots & \vdots\\ & & \Theta_{r,r}\end{pmatrix} \text{ and } X := \begin{pmatrix}\mathbf{x}_1 & & \\ & \ddots & \\ & & \mathbf{x}_r\end{pmatrix}.$$

Note that

$$\Theta = (\Phi'^{-1})^{\mathrm{tr}} \in \mathrm{Mat}_r(\mathbb{K}(t)), \quad X\in \mathrm{Mat}_{r\times d}(A[t]).$$

We also define an operator

$$D := \begin{pmatrix}\partial_t^{d_1} & & \\ & \ddots & \\ & & \partial_t^{d_r}\end{pmatrix} :\ \mathrm{Mat}_{r\times d}(\mathbb{K}(t)) \to \mathrm{Mat}_d(\mathbb{K}(t))$$

by mapping

$$\begin{pmatrix}\mathbf{a}_1\\ \vdots\\ \mathbf{a}_r\end{pmatrix}\mapsto \begin{pmatrix}\partial_t^{d_1}[\mathbf{a}_1]\\ \vdots\\ \partial_t^{d_r}[\mathbf{a}_r]\end{pmatrix},$$

where $\mathbf{a}_i\in \mathrm{Mat}_{1\times d}(\mathbb{K}(t))$ for each $i$.

3.4.2. *The result.* We continue the notation as above. The main result of this subsection is the following identity.

**Theorem 3.4.1.** *Assume that $\|Q_i\|_1 \leq q^{\frac{s_i q}{q-1}}$ for each $1\leq i\leq r-1$. Let $c_{i,j}\in\mathbb{K}$ satisfy the condition that $|c_{i,j}|_\infty < q^{j+\frac{d_i}{q-1}}$ for each $1\leq i\leq r$ and $1\leq j\leq d_i$. Let $G$ be the $t$-module defined in §2.3.2. Then the following identity holds:*

$$\mathrm{Exp}_G\left(\sum_{n\geq 0} D(\Theta^{(1)}\Theta^{(2)}\cdots\Theta^{(n)}X^{(n)})\big|_{t=\theta}\begin{pmatrix}c_{1,1}\\ \vdots\\ c_{r,d_r}\end{pmatrix}^{(n)}\right) = \begin{pmatrix}c_{1,1}\\ \vdots\\ c_{r,d_r}\end{pmatrix}.$$

*Proof.* Firstly, we remark that $\delta_0\circ\iota : \mathrm{Mat}_{1\times r}(\mathbb{T}_\theta)\to\mathbb{K}^d$ is continuous. Indeed, let $\mathbf{f} = (f_1,\ldots,f_r)\in \mathrm{Mat}_{1\times r}(\mathbb{T}_\theta)$ with $f_i = \sum_{j=0}^\infty b_{i,j}t^j$. Since by Proposition 2.4.2 the $(d_1+\cdots+d_{i-1}+j)$-th coordinate of $\delta_0\circ\iota(\mathbf{f})$ is

$$\sum_{\ell\geq d_i-j}\binom{\ell}{d_i-j} b_{i,\ell}\theta^{\ell-d_i+j}$$

for each $1\leq i\leq r$ and $1\leq j\leq d_i$, we have

$$\|\delta_0\circ\iota(\mathbf{f})\| \leq \max_{\substack{1\leq i\leq r\\ 1\leq j\leq d_i\\ \ell\geq d_i-j}}\left\{\left|b_{i,\ell}\theta^{\ell-d_i+j}\right|_\infty\right\} \leq \max_{\substack{1\leq i\leq r\\ 1\leq j\leq d_i\\ \ell\geq d_i-j}}\left\{\left|b_{i,\ell}\theta^{\ell}\right|_\infty\right\} = \max_{\substack{1\leq i\leq r\\ \ell\geq 0}}\left\{\left|b_{i,\ell}\theta^{\ell}\right|_\infty\right\} = \|\mathbf{f}\|_\theta.$$

Therefore, $\delta_0\circ\iota$ is continuous.

Now we consider

$$\delta_0\circ\iota\left(\mathbf{w}^{(n)}(\Phi'^{-1})^{(n)}(\Phi'^{-1})^{(n-1)}\cdots(\Phi'^{-1})^{(1)}\right)$$

with

$$\mathbf{w}=(w_1,\dots,w_r),\quad w_j=\sum_{\ell=1}^{d_j}c_{j,\ell}(t-\theta)^{d_j-\ell}.$$

We compute

$$(\Phi'^{-1})^{(n)}(\Phi'^{-1})^{(n-1)}\cdots(\Phi'^{-1})^{(1)}=(\phi_{ij}^{<n>})_{i,j},$$

where

$$\phi_{i,j}^{<n>}=\begin{cases}(-1)^{i-j}\displaystyle\sum_{j=k_0\le k_1\le\cdots\le k_n=i}\frac{(Q_{k_0}\cdots Q_{k_1-1})(Q_{k_1}\cdots Q_{k_2-1})^{(1)}\cdots(Q_{k_{n-1}}\cdots Q_{k_n-1})^{(n-1)}}{(t-\theta^q)^{d_{k_1}}(t-\theta^{q^2})^{d_{k_2}}\cdots(t-\theta^{q^n})^{d_{k_n}}} & (i\ge j)\\ 0 & (i<j)\end{cases}.$$

Putting

$$(c_1^{<n>}(t),\dots,c_r^{<n>}(t)):=\mathbf{w}^{(n)}(\Phi'^{-1})^{(n)}(\Phi'^{-1})^{(n-1)}\cdots(\Phi'^{-1})^{(1)},\tag{3.4.2}$$

then we have that

$$\begin{aligned}c_i^{<n>}(t)&=\sum_{j=i}^{r}\sum_{\ell=1}^{d_j}c_{j,\ell}^{q^n}(t-\theta^{q^n})^{d_j-\ell}\phi_{j,i}^{<n>}\\&=\sum_{j=i}^{r}\left((t-\theta^{q^n})^{d_j-1}\phi_{j,i}^{<n>},(t-\theta^{q^n})^{d_j-2}\phi_{j,i}^{<n>},\dots,\phi_{j,i}^{<n>}\right)\begin{pmatrix}c_{j,1}\\ \vdots\\ c_{j,d_j}\end{pmatrix}^{(n)}\end{aligned}$$

and hence

$$\begin{aligned}\mathbf{d}_i^{<n>}&:=\left.\begin{pmatrix}\partial_t^{d_i-1}c_i^{<n>}\\ \vdots\\ \partial_t^1c_i^{<n>}\\ c_i^{<n>}\end{pmatrix}\right|_{t=\theta}\\&=\sum_{j=i}^{r}\partial_t^{d_i}\left[(t-\theta^{q^n})^{d_j-1}\phi_{j,i}^{<n>},(t-\theta^{q^n})^{d_j-2}\phi_{j,i}^{<n>},\dots,\phi_{j,i}^{<n>}\right]\Big|_{t=\theta}\begin{pmatrix}c_{j,1}\\ \vdots\\ c_{j,d_j}\end{pmatrix}^{(n)}\\&=\sum_{j=i}^{r}\partial_t^{d_i}\left[\phi_{j,i}^{<n>}\left((t-\theta^{q^n})^{d_j-1},(t-\theta^{q^n})^{d_j-2},\dots,1\right)\right]\Big|_{t=\theta}\begin{pmatrix}c_{j,1}\\ \vdots\\ c_{j,d_j}\end{pmatrix}^{(n)}\qquad(3.4.3)\\&=\sum_{j=i}^{r}\Upsilon_{i,j}^{<n>}\Big|_{t=\theta}\begin{pmatrix}c_{j,1}\\ \vdots\\ c_{j,d_j}\end{pmatrix}^{(n)}\\&=\left(0,\dots,0,\Upsilon_{i,i}^{<n>},\Upsilon_{i,i+1}^{<n>},\dots,\Upsilon_{i,r}^{<n>}\right)\Big|_{t=\theta}\begin{pmatrix}c_{1,1}\\ \vdots\\ c_{r,d_r}\end{pmatrix}^{(n)},\end{aligned}$$

where

$$\begin{aligned}\Upsilon_{i,j}^{<n>} &:= \partial_t^{d_i}\left[\phi_{j,i}^{<n>}\left((t-\theta^{q^n})^{d_j-1},(t-\theta^{q^n})^{d_j-2},\ldots,1\right)\right]\\ &= \partial_t^{d_i}\left[\sum_{i=k_0\leq k_1\leq\cdots\leq k_n=j}(-1)^{k_1-k_0}\frac{Q_{k_0}\cdots Q_{k_1-1}}{(t-\theta^q)^{d_{k_1}}}\cdot(-1)^{k_2-k_1}\frac{(Q_{k_1}\cdots Q_{k_2-1})^{(1)}}{(t-\theta^{q^2})^{d_{k_2}}}\cdot\cdots\right.\\ &\qquad\left.\cdots(-1)^{k_n-k_{n-1}}\frac{(Q_{k_{n-1}}\cdots Q_{k_n-1})^{(n-1)}}{(t-\theta^{q^n})^{d_{k_n}}}\left((t-\theta^{q^n})^{d_j-1},(t-\theta^{q^n})^{d_j-2},\ldots,1\right)\right].\end{aligned}$$

Note that the second equality of (3.4.3) comes from the definition of $\partial$-matrices (3.3.6), and the last two equalities are from definitions.

Based on the definition (3.4.2), by Remark 3.3.4 we have that

$$\delta_0\circ\iota\left(\mathbf{w}^{(n)}(\Phi'^{-1})^{(n)}(\Phi'^{-1})^{(n-1)}\cdots(\Phi'^{-1})^{(1)}\right)=\begin{pmatrix}\mathbf{d}_1^{<n>}\\ \vdots\\ \mathbf{d}_r^{<n>}\end{pmatrix},$$

and hence we derive the following from the calculations above:

$$\delta_0\circ\iota\left(\mathbf{w}^{(n)}(\Phi'^{-1})^{(n)}(\Phi'^{-1})^{(n-1)}\cdots(\Phi'^{-1})^{(1)}\right)=\begin{pmatrix}\mathbf{d}_1^{<n>}\\ \vdots\\ \mathbf{d}_r^{<n>}\end{pmatrix}=\left.\begin{pmatrix}\Upsilon_{1,1}^{<n>}&\cdots&\Upsilon_{1,r}^{<n>}\\ &\ddots&\vdots\\ &&\Upsilon_{r,r}^{<n>}\end{pmatrix}\right|_{t=\theta}\begin{pmatrix}c_{1,1}\\ \vdots\\ c_{r,d_r}\end{pmatrix}^{(n)}.$$

Recalling the definitions of $\Theta,\{\mathbf{x}_1,\cdots,\mathbf{x}_r\},X$ and $D$ given in Sec. 3.4.1, we have

$$\Upsilon_{i,j}^{<n>}=\partial_t^{d_i}\left[(\Theta^{(1)}\cdots\Theta^{(n)})_{ij}\cdot\mathbf{x}_j^{(n)}\right],$$

whence we have

$$\delta_0\circ\iota\left(\mathbf{w}^{(n)}(\Phi'^{-1})^{(n)}(\Phi'^{-1})^{(n-1)}\cdots(\Phi'^{-1})^{(1)}\right)=D(\Theta^{(1)}\Theta^{(2)}\cdots\Theta^{(n)}X^{(n)})\big|_{t=\theta}\begin{pmatrix}c_{1,1}\\ \vdots\\ c_{r,d_r}\end{pmatrix}^{(n)}$$

for each $n\geq 1$. This equality also holds when $n=0$. It follows by Theorem 2.4.4, Proposition 3.1.3 and Remark 3.1.6 that

$$\operatorname{Exp}_G\left(\sum_{n\geq 0}D(\Theta^{(1)}\Theta^{(2)}\cdots\Theta^{(n)}X^{(n)})\big|_{t=\theta}\begin{pmatrix}c_{1,1}\\ \vdots\\ c_{r,d_r}\end{pmatrix}^{(n)}\right)=\begin{pmatrix}c_{1,1}\\ \vdots\\ c_{r,d_r}\end{pmatrix},\tag{3.4.4}$$

since from the definition (2.3.11) of $\delta_1$ for the given $\mathbf{w}$ one has

$$\delta_1\circ\iota(\mathbf{w})=\begin{pmatrix}c_{1,1}\\ \vdots\\ c_{r,d_r}\end{pmatrix}.$$

□

*Remark* 3.4.5. Note that the shtuka function for the Carlitz module is $f=t-\theta\in A[t]$ and that, in general, the shtuka function caries a great deal of arithmetic information (see [T04, §7.7-8.2]). For example, for the $n$th tensor power of the Carlitz module, one defines the associated dual $t$-motive as $\mathbb{K}[t]$ with $\sigma$-action given by

$$\sigma x=f^n x^{(-1)},\quad x\in\mathbb{K}[t].$$

Further, one can express the coefficients of the exponential and logarithm function in terms of the reciprocal of the shtuka function. In the Carlitz module, for example, one has

$$\log_C(z) = \sum_{i\geq 0} \frac{z^{q^i}}{f^{(1)}\dots f^{(i)}}\Big|_{t=\theta}. \tag{3.4.6}$$

Note that the $\sigma$-action on our dual $t$-motive $M'$ is defined by the matrix $\Phi'$ (2.3.9), and that $\Theta = (\Phi'^{-1})^{\mathrm{tr}}$ which invites one to make the natural comparison between our formula (3.4.4) and the formula (3.4.6). This leads us to view the matrix $\Phi'$ as a matrix analogue of the shtuka function and motivates some of the constructions in this paper. There are several other arithmetic applications of the shtuka function (see [GP16], [G17b] and [ANT17]), and it would be interesting to study how they apply in this setting.

## 4. Hyperderivatives of $t$-motivic Carlitz multiple star polylogarithms

We continue with the notation as given in the previous section, but we restrict the $Q_i$'s to be in $\mathbb{K}$ in this section, i.e., we let $Q_i = u_i \in \mathbb{K}$ for $i = 1, \dots, r$ and set

$$\mathbf{u} = (u_1, \dots, u_r).$$

4.1. **$t$-modules associated to CMSPL's.** We note that the $t$-module $G = (\mathbb{G}_a^d, \rho)$ associated to the dual $t$-motive $M'$ in Sec. 2.3.2 can be explicitly written down. We will use this explicit description of $G$ heavily going forward, so we take a moment to recall it from [CM19]. Recall that $d := d_1 + \dots + d_r$. Let $B$ be a $d \times d$-matrix of the form

$$\left(\begin{array}{c|c|c} B[11] & \cdots & B[1r] \\ \hline \vdots & & \vdots \\ \hline B[r1] & \cdots & B[rr] \end{array}\right),$$

where $B[\ell m]$ is a $d_\ell \times d_m$-matrix for each $\ell$ and $m$ and we call $B[\ell m]$ the $(\ell, m)$-th block sub-matrix of $B$.

For $1 \leq \ell \leq m \leq r$, we define the following matrices:

$$N_\ell := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ & 0 & 1 & \ddots & \vdots \\ & & \ddots & \ddots & 0 \\ & & & \ddots & 1 \\ & & & & 0 \end{pmatrix} \in \mathrm{Mat}_{d_\ell}(\mathbb{F}_q), \tag{4.1.1}$$

$$N := \begin{pmatrix} N_1 & & & \\ & N_2 & & \\ & & \ddots & \\ & & & N_r \end{pmatrix} \in \mathrm{Mat}_d(\mathbb{F}_q), \tag{4.1.2}$$

$$E[\ell m] := \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix} \in \mathrm{Mat}_{d_\ell \times d_m}(\mathbb{K}) \quad (\text{if } \ell = m),$$

$$E[\ell m] := \begin{pmatrix} 0 & & \cdots & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & & \ddots & \vdots \\ (-1)^{m-\ell}\prod_{e=\ell}^{m-1} u_e & & 0 & \cdots & 0 \end{pmatrix} \in \mathrm{Mat}_{d_\ell \times d_m}(\mathbb{K}) \quad (\text{if } \ell < m),$$

$$E := \begin{pmatrix} E[11] & E[12] & \cdots & E[1r] \\ & E[22] & \ddots & \vdots \\ & & \ddots & E[r-1,r] \\ & & & E[rr] \end{pmatrix} \in \mathrm{Mat}_d(\mathbb{K}).$$

We then define the $t$-module $G = G_{\mathfrak{s},\mathbf{u}} := (\mathbb{G}_a^d, \rho)$ by

$$\rho(t) = \theta I_d + N + E\tau \in \mathrm{Mat}_d(\mathbb{K}[\tau]), \tag{4.1.3}$$

and note that $G$ depends only on $u_1, \ldots, u_{r-1}$. Finally, we define the special point

$$\mathbf{v} := \mathbf{v}_{\mathfrak{s},\mathbf{u}} := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ (-1)^{r-1}u_1 \cdots u_r \\ 0 \\ \vdots \\ 0 \\ (-1)^{r-2}u_2 \cdots u_r \\ \vdots \\ 0 \\ \vdots \\ 0 \\ u_r \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} 0 \\ \vdots \\ 0 \\ 0 \end{matrix}}\right\} d_1 \\ \left.\vphantom{\begin{matrix} 0 \\ \vdots \\ 0 \\ 0 \end{matrix}}\right\} d_2 \\ \vdots \\ \left.\vphantom{\begin{matrix} 0 \\ \vdots \\ 0 \\ 0 \end{matrix}}\right\} d_r \end{matrix} \in G(\mathbb{K}). \tag{4.1.4}$$

It is not hard to see that either the $t$-module $G$ is $\mathbf{C}^{\otimes d_1}$ if $r = 1$, or that $G$ is an iterated extension of the tensor powers of the Carlitz module if $r > 1$. Finally, we note that if we take $\mathbf{u} = (u_1, \ldots, u_r) \in A^r$, then the $t$-module $G$ is defined over $A$ in the sense that $\rho(t) \in \mathrm{Mat}_d(A[\tau])$, and $\mathbf{v} \in G(A)$.

*Remark* 4.1.5. Let $\mathcal{F}$ be the category of Frobenius modules with morphisms given by left $\mathbb{K}[t,\sigma]$-module homomorphisms. One then sees that $M$ is an extension of $\mathbf{1}$ by $M'$, i.e., $M \in \mathrm{Ext}^1_{\mathcal{F}}(\mathbf{1}, M')$. We can equip an $\mathbb{F}_q[t]$-module structure on $\mathrm{Ext}^1_{\mathcal{F}}(\mathbf{1}, M')$ and have the following isomorphisms as $\mathbb{F}_q[t]$-modules due to Anderson (see [CPY19]):

$$\mathrm{Ext}^1_{\mathcal{F}}(\mathbf{1}, M') \cong M'/(\sigma - 1)M' \cong G(\mathbb{K}).$$

The special point $\mathbf{v}$ is the image of $M$ under the composite of the isomorphisms above. For details, see [CPY19, CM19].

**Proposition 4.1.6.** *Let $(G, \rho)$ be the $t$-module defined as above. Then for every polynomial $b(t) \in \mathbb{F}_q[t]$, $\partial\rho(b)$ is a block diagonal matrix with*

$$d_t^{d_i}[b(t)]|_{t=\theta} = \begin{pmatrix} b(\theta) & (\partial_t^1 b)\,|_{t=\theta} & \cdots & \left(\partial_t^{d_i-1} b\right)|_{t=\theta} \\ & \ddots & \ddots & \vdots \\ & & \ddots & (\partial_t^1 b)\,|_{t=\theta} \\ & & & b(\theta) \end{pmatrix} \in \mathrm{Mat}_{d_i}(\mathbb{K})$$

*located at the $i$th block along the diagonal for $i = 1, \ldots, r$.*

*Proof.* Note that by definition of $\rho$ we have

$$\partial\rho(t) = \begin{pmatrix} \theta I_{d_1} + N_1 & & \\ & \ddots & \\ & & \theta I_{d_r} + N_r \end{pmatrix} = \begin{pmatrix} \partial\mathbf{C}^{\otimes d_1}(t) & & \\ & \ddots & \\ & & \partial\mathbf{C}^{\otimes d_r}(t) \end{pmatrix}$$

and also

$$\partial\mathbf{C}^{\otimes d_i}(t) = d_t^{d_i}[t]|_{t=\theta}$$

for each $1 \leq i \leq r$. Since $\partial\rho(\cdot)$ is an $\mathbb{F}_q$-linear ring homomorphism on $\mathbb{F}_q[t]$ and by Proposition 3.3.8 so is $d_t^{d_i}[\cdot]$ for all $1 \leq i \leq r$, the desired result follows. $\square$

The following lemma is a generalization of Yu's last coordinate logarithms theory [Yu91, Thm. 2.3] to our $t$-module $G$, and it will be used in the proof of the formulae given in the next subsection.

**Lemma 4.1.7.** *Fixing $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$ and $\mathbf{u} \in \overline{K}^r$, let $G$ be the $t$-module over $\overline{K}$ defined in (4.1.3). For each $1 \leq i \leq r$, we let $d_i := s_i + \cdots + s_r$. Suppose that we have two vectors*

$$Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_r \end{pmatrix} \in \operatorname{Lie} G(\mathbb{C}_\infty) \text{ and } V = \begin{pmatrix} V_1 \\ \vdots \\ V_r \end{pmatrix} \in \operatorname{Lie} G(\mathbb{C}_\infty)$$

*with $Y_i, V_i \in \mathrm{Mat}_{d_i \times 1}(\mathbb{C}_\infty)$ for each $i$ so that*

- $\mathrm{Exp}_G(Y) \in G(\overline{K})$ *and* $\mathrm{Exp}_G(V) \in G(\overline{K})$;
- *the last coordinate of $Y_i$ equals the last coordinate of $V_i$ for all $1 \leq i \leq r$.*

*Then we have that $Y = V$.*

*Proof.* We first note that when $r = 1$, $G = \mathbf{C}^{\otimes d_1}$, otherwise from the explicit definition $G$ is an iterated extension of certain tensor powers of the Carlitz module. We can also see this directly as follows, by using the dual $t$-motive $M'$ whose $\sigma$-action on the basis $\mathbf{m}'$ from (2.3.3) is given by $\Phi'$ (2.3.4). Note first that the $t$-module which corresponds to $M'$ is $G$. Then, we put $G_1 := \mathbf{C}^{\otimes d_1}$ and $G_r := G$ and for each $1 \leq i \leq r$, we let $\Phi'_i$ be the square matrix of size $i$ cut off from the upper left square of $\Phi'$, let $M'_i$ be the dual $t$-motive whose $\sigma$-action on a fixed $\mathbb{C}_\infty[t]$-basis is given by $\Phi'_i$ and let $G_i$ be the $t$-module associated to $M'_i$ as per Sec. 2.3. For each $2 \leq i \leq r$ we have the exact sequence of left $\mathbb{C}_\infty[t, \sigma]$-modules

$$0 \longrightarrow M'_{i-1} \hookrightarrow M'_i \twoheadrightarrow C^{\otimes d_i} \longrightarrow 0\,. \tag{4.1.8}$$

Recall that $M'_i/(\sigma - 1)M'_i \cong G_i(\mathbb{C}_\infty)$ and $C^{\otimes d_i}/(\sigma - 1)C^{\otimes d_i} \cong \mathbf{C}^{\otimes d_i}(\mathbb{C}_\infty)$ as $\mathbb{F}_q[t]$-modules. Since the $\mathbb{F}_q[t]$-linear map $(\sigma - 1) : \mathbb{C}^{\otimes d_i} \to \mathbb{C}^{\otimes d_i}$ is injective, the snake lemma combined with the two previously mentioned facts show the following exact sequence of $t$-modules

$$0 \longrightarrow G_{i-1} \xhookrightarrow{\iota_i} G_i \xrightarrow{\pi_i} \mathbf{C}^{\otimes d_i} \longrightarrow 0$$

induced from (4.1.8), where $\pi_i$ is the projection map onto the last $d_i$ coordinates which also equals $\partial\pi_i$ (cf. proof of [CPY19, Prop. 6.1.1]).

We prove the lemma by induction on the depth $r$. When $r = 1$, we consider

$$Y - V = \begin{pmatrix} * \\ \vdots \\ 0 \end{pmatrix},$$

which is mapped to an algebraic point of $\mathbf{C}^{\otimes d_1}$ via $\operatorname{Exp}_{\mathbf{C}^{\otimes d_1}}$ by the hypotheses on $Y$ and $V$. Since the last coordinate of $Y - V$ is zero, we have $Y = V$ by [Yu91, Thm. 2.3].

Suppose that the result is valid for all $r \leq n-1$ for a positive integer $n \geq 2$. Now we consider the case when $r = n$. In this case, we consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & G_{r-1} \arrow[r, hook, "\iota_r"] & G \arrow[r, two heads, "\pi_r"] & \mathbf{C}^{\otimes d_r} \arrow[r] & 0 \\
0 \arrow[r] & \operatorname{Lie} G_{r-1} \arrow[r, hook, "\partial\iota_r"] \arrow[u, "\operatorname{Exp}_{G_{r-1}}"] & \operatorname{Lie} G \arrow[r, two heads, "\partial\pi_r"] \arrow[u, "\operatorname{Exp}_G"] & \operatorname{Lie} \mathbf{C}^{\otimes d_r} \arrow[r] \arrow[u, "\operatorname{Exp}_{\mathbf{C}^{\otimes d_r}}"'] & 0.
\end{tikzcd}$$

Note that

$$\partial\pi_r\,(Y - V) = Y_r - V_r = \begin{pmatrix} * \\ \vdots \\ 0 \end{pmatrix},$$

which is mapped to an algebraic point of $\mathbf{C}^{\otimes d_r}$ via $\operatorname{Exp}_{\mathbf{C}^{\otimes d_r}}$. It follows again by [Yu91, Thm. 2.3] that $Y_r = V_r$. Hence the $Y - V$ are of the form

$$Y - V = \begin{pmatrix} Y_1 - V_1 \\ \vdots \\ Y_{r-1} - V_{r-1} \\ \mathbf{0} \end{pmatrix} \in \operatorname{Ker} \partial\pi_r = \operatorname{Lie} G_{r-1}.$$

Using the commutative diagram above, by the hypotheses on $Y$ and $V$, the vector $Y - V$ is mapped to an algebraic point of $G_{r-1}$ via $\operatorname{Exp}_{G_{r-1}}$. Then, since $G_{r-1}$ is defined using the index $(s_1, \ldots, s_{r-2}, s_{r-1} + s_r)$ of depth $r - 1$ and since the last coordinates of $Y_i$ and $V_i$ are the same for all $1 \leq i \leq r-1$, by induction hypothesis we obtain that $Y_i = V_i$ for all $1 \leq i \leq r-1$. Combining this with $Y_r = V_r$, the desired equality $Y = V$ follows. ☐

4.2. **The explicit formulae.** In [CM17], CMSPL's are related to certain coordinates of the logarithm of $G$ evaluated at $\mathbf{v}$ under certain assumptions on the absolute values of the coordinates of $\mathbf{u}$. We first recall the result [CM17, Thm. 4.2.3].

**Theorem 4.2.1** (C.-M.)**.** *Given any $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$, we let $\mathbf{u} = (u_1, \ldots, u_r) \in \mathbb{C}_\infty^r$ with $|u_i|_\infty \leq q^{\frac{s_i q}{q-1}}$ for each $1 \leq i \leq r-1$ and $|u_r|_\infty < q^{\frac{s_r q}{q-1}}$. Let $G$ and $\mathbf{v}$ be defined in (4.1.3) and (4.1.4) respectively using $\mathfrak{s}$ and $\mathbf{u}$. Then $\operatorname{Log}_G$ converges $\infty$-adically at $\mathbf{v}$ and we have* *the formula*

$$\operatorname{Log}_G(\mathbf{v}) = \begin{pmatrix} * \\ \vdots \\ * \\ (-1)^{r-1}\operatorname{Li}^\star_{(s_r,\dots,s_1)}(u_r,\dots,u_1) \\ * \\ \vdots \\ * \\ (-1)^{r-2}\operatorname{Li}^\star_{(s_r,\dots,s_2)}(u_r,\dots,u_2) \\ \vdots \\ * \\ \vdots \\ * \\ \operatorname{Li}^\star_{s_r}(u_r) \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} * \\ \vdots \\ * \\ x \end{matrix}}\right\} d_1 \\ \left.\vphantom{\begin{matrix} * \\ \vdots \\ * \\ x \end{matrix}}\right\} d_2 \\ \vdots \\ \left.\vphantom{\begin{matrix} * \\ \vdots \\ * \\ x \end{matrix}}\right\} d_r \end{matrix} \in \operatorname{Lie} G(\mathbb{C}_\infty).$$

*In particular, the $(s_1+\cdots+s_r)$th coordinate of* $\operatorname{Log}_G(\mathbf{v})$ *is* $(-1)^{\operatorname{dep}(\mathfrak{s})-1}\operatorname{Li}^\star_{\tilde{\mathfrak{s}}}(\widetilde{\mathbf{u}})$.

The primary result in this section is to give explicit formulae for the (previously unknown) $*$-coordinates in the theorem above.

**Theorem 4.2.2.** *Let the notation and assumptions be given as in Theorem 4.2.1. For each $1 \le i \le r$, we let $d_i := s_i + \cdots + s_r$ and set $d := d_1 + \cdots + d_r$. Define*

$$\mathbf{Y}_{\mathfrak{s},\mathbf{u}} := \begin{pmatrix} Y_1 \\ \vdots \\ Y_r \end{pmatrix} \in \operatorname{Mat}_{d\times 1}(\mathbb{C}_\infty),$$

*where for each $1 \le i \le r$, $Y_i$ is given by*

$$\begin{aligned} Y_i &= \begin{pmatrix} (-1)^{r-i}\left(\partial_t^{d_i-1}\mathfrak{Li}^\star_{(s_r,\dots,s_i),(u_r,\dots,u_i)}(t)\right)|_{t=\theta} \\ (-1)^{r-i}\left(\partial_t^{d_i-2}\mathfrak{Li}^\star_{(s_r,\dots,s_i),(u_r,\dots,u_i)}(t)\right)|_{t=\theta} \\ \vdots \\ (-1)^{r-i}\left(\partial_t^{0}\mathfrak{Li}^\star_{(s_r,\dots,s_i),(u_r,\dots,u_i)}(t)\right)|_{t=\theta} \end{pmatrix} \\ &= \begin{pmatrix} (-1)^{r-i}\left(\partial_t^{d_i-1}\mathfrak{Li}^\star_{(s_r,\dots,s_i),(u_r,\dots,u_i)}(t)\right)|_{t=\theta} \\ (-1)^{r-i}\left(\partial_t^{d_i-2}\mathfrak{Li}^\star_{(s_r,\dots,s_i),(u_r,\dots,u_i)}(t)\right)|_{t=\theta} \\ \vdots \\ (-1)^{r-i}\operatorname{Li}^\star_{(s_r,\dots,s_i)}(u_r,\dots,u_i) \end{pmatrix} \in \operatorname{Mat}_{d_i\times 1}(\mathbb{C}_\infty). \end{aligned}$$

*Then we get the following formula for the logarithm evaluated at* $\mathbf{v}$

$$\operatorname{Log}_G(\mathbf{v}) = \mathbf{Y}_{\mathfrak{s},\mathbf{u}} \in \operatorname{Lie} G(\mathbb{C}_\infty).$$

*Proof.* Since $\operatorname{Log}_G(\mathbf{v})$ and $\mathbf{Y}_{\mathfrak{s},\mathbf{u}}$ are continuous on $\mathbf{u}$ (see [CM19, Equation 3.2.4]), and $\overline{K}$ is dense in $\mathbb{C}_\infty$, we may assume that $\mathbf{u} \in \overline{K}^r$. Our starting point is Theorem 3.4.1. We calculate that

$$D(\Theta^{(1)}\Theta^{(2)}\cdots\Theta^{(n)}X^{(n)})\mathbf{v}^{(n)} = \begin{pmatrix} \beta_{n,1} \\ \vdots \\ \beta_{n,r} \end{pmatrix},$$

where $\beta_{n,i} \in \mathrm{Mat}_{d_i\times 1}(\mathbb{C}_\infty(t))$ for $1 \leq i \leq r$ is given by

$$\begin{aligned}
\beta_{n,i} &= \sum_{j=i}^{r} \partial_t^{d_i}\left[(-1)^{j-i}\sum_{i=k_0\leq\cdots\leq k_n=j}\frac{(u_{k_0}\cdots u_{k_1-1})\cdots(u_{k_{n-1}}\cdots u_{k_n-1})^{q^{n-1}}}{(t-\theta^q)^{d_{k_1}}\cdots(t-\theta^{q^n})^{d_{k_n}}}\cdot(-1)^{r-j}(u_j\cdots u_r)^{q^n}\right]\\
&= (-1)^{r-i}\partial_t^{d_i}\left[\sum_{i=k_0\leq\cdots\leq k_n\leq r}\frac{(u_{k_0}\cdots u_{k_1-1})(u_{k_1}\cdots u_{k_2-1})^q\cdots(u_{k_{n-1}}\cdots u_{k_n-1})^{q^{n-1}}(u_{k_n}\cdots u_r)^{q^n}}{\mathbb{L}_n^{d_{k_n}}(t-\theta)^{d_{k_n}-d_{k_0}}\prod_{0\leq a\leq n-1}(t-\theta^{q^a})^{d_{k_a}-d_{k_n}}}\right]\\
&= (-1)^{r-i}\partial_t^{d_i}\left[\sum_{i=k_0\leq\cdots\leq k_n\leq r}\frac{(u_{k_0}\cdots u_{k_1-1})(u_{k_1}\cdots u_{k_2-1})^q\cdots(u_{k_{n-1}}\cdots u_{k_n-1})^{q^{n-1}}(u_{k_n}\cdots u_r)^{q^n}}{\mathbb{L}_n^{d_{k_n}}(t-\theta)^{d_{k_n}-d_{k_0}}\prod_{0\leq a\leq b\leq n-1}(t-\theta^{q^a})^{d_{k_b}-d_{k_{b+1}}}}\right]\\
&= (-1)^{r-i}\partial_t^{d_i}\left[\sum_{i=k_0\leq\cdots\leq k_n\leq r}\frac{(u_{k_0}\cdots u_{k_1-1})(u_{k_1}\cdots u_{k_2-1})^q\cdots(u_{k_{n-1}}\cdots u_{k_n-1})^{q^{n-1}}(u_{k_n}\cdots u_r)^{q^n}}{\mathbb{L}_n^{d_{k_n}}(t-\theta)^{d_{k_n}-d_{k_0}}\prod_{0\leq b\leq n-1}((t-\theta)\mathbb{L}_b)^{d_{k_b}-d_{k_{b+1}}}}\right]\\
&= (-1)^{r-i}\partial_t^{d_i}\left[\sum_{i=k_0\leq\cdots\leq k_n\leq r}\frac{(u_{k_0}\cdots u_{k_1-1})(u_{k_1}\cdots u_{k_2-1})^q\cdots(u_{k_{n-1}}\cdots u_{k_n-1})^{q^{n-1}}(u_{k_n}\cdots u_r)^{q^n}}{\mathbb{L}_0^{s_{k_0}+\cdots+s_{k_1-1}}\mathbb{L}_1^{s_{k_1}+\cdots+s_{k_2-1}}\cdots\mathbb{L}_{n-1}^{s_{k_{n-1}}+\cdots+s_{k_n-1}}\mathbb{L}_n^{s_{k_n}+\cdots+s_r}}\right].
\end{aligned}$$

Note that for $n \geq 0$ and $1 \leq i \leq r$, we have a bijective map of sets

$$\{(k_0,\ldots,k_n) \mid i = k_0 \leq \cdots \leq k_n \leq r)\} \to \{(m_i,\ldots,m_r) \mid 0 \leq m_i \leq \cdots \leq m_r = n\}$$

given by $m_j = \max\{\ell | k_\ell \leq j\}$ and is explained by the following table.

| $j$ | $k_0 = i$ | $\cdots$ | $k_1-1$ | $k_1$ | $\cdots$ | $k_2-1$ | $k_2$ | $\cdots$ | $k_3-1$ | $k_3$ | $\cdots$ | $k_n-1$ | $k_n$ | $\cdots$ | $r$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $m_j$ | 0 | $\cdots$ | 0 | 1 | $\cdots$ | 1 | 2 | $\cdots$ | 2 | 3 | $\cdots$ | $n-1$ | $n$ | $\cdots$ | $n$ |

The converse of the map is given by $k_\ell = i + \#\{j | m_j < \ell\}$.

Therefore we can express $\beta_{n,i}$ as the following:

$$\beta_{n,i} = (-1)^{r-i}\partial_t^{d_i}\left[\sum_{0\leq m_i\leq\cdots\leq m_r=n}\frac{u_i^{q^{m_i}}\cdots u_r^{q^{m_r}}}{\mathbb{L}_{m_i}^{s_i}\cdots\mathbb{L}_{m_r}^{s_r}}\right].$$

Thus, summing $\beta_{n,i}$ over $n \geq 0$ gives

$$\sum_{n=0}^{\infty}\beta_{n,i} = (-1)^{r-i}\partial_t^{d_i}\left[\sum_{0\leq m_i\leq\cdots\leq m_r}\frac{u_i^{q^{m_i}}\ldots u_r^{q^{m_r}}}{\mathbb{L}_{m_i}^{s_i}\ldots\mathbb{L}_{m_r}^{s_r}}\right] = (-1)^{r-i}\partial_t^{d_i}\left[\mathfrak{Li}^{\star}_{(s_r,\ldots,s_i)}(t;u_r,\ldots,u_i)\right].$$

Note that the first equality comes from the continuity of of the map $\partial_t^{d_i}[\cdot] : \mathbb{T}_\theta \to \mathrm{Mat}_{d_i\times 1}(\mathbb{T}_\theta)$, which is clear from the definition of $\partial_t^{d_i}[\cdot]$. To finish the proof, we note that evaluating at $t = \theta$ gives

$$\sum_{n=0}^{\infty} D(\Theta^{(1)}\Theta^{(2)}\cdots\Theta^{(n)}X^{(n)})\big|_{t=\theta}\mathbf{v}^{(n)} = \sum_{n=0}^{\infty}\begin{pmatrix}\beta_{n,1}\\ \vdots\\ \beta_{n,r}\end{pmatrix}\Bigg|_{t=\theta} = \begin{pmatrix}Y_1\\ \vdots\\ Y_r\end{pmatrix}$$

and so by Theorem 3.4.1 we have

$$\mathrm{Exp}_G \begin{pmatrix} Y_1 \\ \vdots \\ Y_r \end{pmatrix} = \mathbf{v}.$$

Then the desired result follows from Lemma 4.1.7 since $\mathrm{Exp}_G(\mathrm{Log}_G(\mathbf{v})) = \mathbf{v}$ and that the $(d_1 + \cdots + d_i)$th coordinates of $\mathrm{Log}_G(\mathbf{v})$ and $\mathbf{Y}_{\mathfrak{s},\mathbf{u}}$ coincide with $(-1)^{r-i}\,\mathrm{Li}^{\star}_{(s_r,\ldots,s_i)}(u_r,\ldots,u_i)$ for each $i$. □

*Remark* 4.2.3. In the case of $r = 1$, the formulae above are due to Papanikolas. See [Pp, Prop. 4.3.6] and also [Pp, (4.3.1)].

**Corollary 4.2.4.** *Let notation and assumptions be given in Theorem 4.2.1 and Theorem 4.2.2. Then for any polynomial $b \in \mathbb{F}_q[t]$, we have*

$$\partial\rho(b)\,\mathrm{Log}_G(\mathbf{v}) = \partial\rho(b)\mathbf{Y}_{\mathfrak{s},\mathbf{u}} = \begin{pmatrix} \partial\mathbf{C}^{\otimes d_1}(b)\,(Y_1) \\ \vdots \\ \partial\mathbf{C}^{\otimes d_r}(b)\,(Y_r) \end{pmatrix} \in \mathrm{Mat}_d(\mathbb{C}_\infty),$$

*where for each $1 \le i \le r$, $\partial\mathbf{C}^{\otimes d_i}(b)\,(Y_i)$ is explicitly given by the following formula*

$$\begin{aligned} \partial\mathbf{C}^{\otimes d_i}(b)\,(Y_i) \quad &= \begin{pmatrix} (-1)^{r-i}\left(\partial_t^{d_i-1} b\mathfrak{Li}^{\star}_{(s_r,\ldots,s_i),(u_r,\ldots,u_i)}(t)\right)|_{t=\theta} \\ (-1)^{r-i}\left(\partial_t^{d_i-2} b\mathfrak{Li}^{\star}_{(s_r,\ldots,s_i),(u_r,\ldots,u_i)}(t)\right)|_{t=\theta} \\ \vdots \\ (-1)^{r-i}\left(\partial_t^{0} b\mathfrak{Li}^{\star}_{(s_r,\ldots,s_i),(u_r,\ldots,u_i)}(t)\right)|_{t=\theta} \end{pmatrix} \\ &= \begin{pmatrix} (-1)^{r-i}\left(\partial_t^{d_i-1} b\mathfrak{Li}^{\star}_{(s_r,\ldots,s_i),(u_r,\ldots,u_i)}(t)\right)|_{t=\theta} \\ (-1)^{r-i}\left(\partial_t^{d_i-2} b\mathfrak{Li}^{\star}_{(s_r,\ldots,s_i),(u_r,\ldots,u_i)}(t)\right)|_{t=\theta} \\ \vdots \\ (-1)^{r-i} b(\theta)\,\mathrm{Li}^{\star}_{(s_r,\ldots,s_i)}(u_r,\ldots,u_i) \end{pmatrix} \in \mathrm{Mat}_{d_i\times 1}(\mathbb{C}_\infty). \end{aligned}$$

*Proof.* The proof follows from Proposition 3.3.8(2) together with Theorem 4.2.2 and Proposition 4.1.6. □

## 5. $t$-MOTIVIC MZV'S

In [T04], Thakur defined the $\infty$-adic MZV's that are generalizations of Carlitz zeta values. For any index $\mathfrak{s} = (s_1,\ldots,s_r) \in \mathbb{N}^r$,

$$\zeta_A(\mathfrak{s}) := \sum \frac{1}{a_1^{s_1}\cdots a_r^{s_r}} \in K_\infty,$$

where $a_1,\ldots,a_r$ run over all elements of $A_+$ for which $|a_1|_\infty > \cdots > |a_r|_\infty$. Similarly as in the classical theory, the weight and the depth of the presentation $\zeta_A(\mathfrak{s})$ are defined as $\mathrm{wt}(\mathfrak{s})$ and $\mathrm{dep}(\mathfrak{s})$ respectively. In [CM17], the quantity $\Gamma_{\mathfrak{s}}\zeta_A(\mathfrak{s})$ is expressed as a linear combination of CMSPL's at certain integral points, and the purpose of this section is to deform this identity to an identity of power series.

### 5.1. **Definition of $t$-motivic MZV's.**

5.1.1. *Anderson-Thakur polynomials.* For any non-negative integer $n$, we let $H_n(t) \in A[t]$ be the Anderson-Thakur polynomial defined in [AT90, AT09], but we follow the notation given in [C14, CPY19]. Namely, we first let $x, y$ be two independent variables and put $G_0(y) := 1$ and define the polynomials $G_n(y) \in \mathbb{F}_q[t, y]$ for positive integers $n$ by

$$G_n(y) := \prod_{i=1}^{n} (t^{q^n} - y^{q^i}).$$

Put $D_0 := 1$ and $D_i := \prod_{j=0}^{i-1}(\theta^{q^i} - \theta^{q^j}) \in A$ for $i \in \mathbb{N}$. For any non-negative integer $n$, recall that the Carlitz factorial is defined by

$$\Gamma_{n+1} := \prod_{i=0}^{\infty} D_i^{n_i} \in A \tag{5.1.1}$$

where the $n_i \in \mathbb{Z}_{\geq 0}$ are given by writing the base $q$-expansion $n = \sum_{i=0}^{\infty} n_i q^i$ for $0 \leq n_i \leq q-1$. We then define the sequence of Anderson-Thakur polynomials [AT90] $H_n(t) \in A[t]$ by the following generating function identity,

$$\left(1 - \sum_{i=0}^{\infty} \frac{G_i(\theta)}{D_i|_{\theta=t}} x^{q^i}\right)^{-1} = \sum_{n=0}^{\infty} \frac{H_n(t)}{\Gamma_{n+1}|_{\theta=t}} x^n.$$

For any non-negative integer $d$, we denote by $A_{d,+}$ the set of elements of degree $d$ in $A_+$. Anderson and Thakur showed in [AT90] that for any positive integer $s$,

$$\|H_{s-1}(t)\|_1 < |\theta|_\infty^{\frac{sq}{q-1}}, \tag{5.1.2}$$

and that the interpolation formula holds for every $d \in \mathbb{Z}_{\geq 0}$:

$$\frac{H_{s-1}^{(d)}(\theta)}{L_d^s} = \Gamma_s \cdot \sum_{a \in A_{d,+}} \frac{1}{a^s}. \tag{5.1.3}$$

5.1.2. *The definition of t-motivic MZV's.* Recall that $\mathbb{L}_0 := 1$ and $\mathbb{L}_i := (t - \theta^q) \cdots (t - \theta^{q^i})$ for any positive integer $i$. We now define $t$-motivic multiple zeta values.

**Definition 5.1.4.** *For any index $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$, we define its associated $t$-motivic multiple zeta value by the following series*

$$\zeta_A^{\mathrm{mot}}(\mathfrak{s}) := \sum_{i_1 > \cdots > i_r \geq 0} \frac{H_{s_1-1}^{(i_1)} \cdots H_{s_r-1}^{(i_r)}}{\mathbb{L}_{i_1}^{s_1} \cdots \mathbb{L}_{i_r}^{s_r}} \in \mathbb{T}_\theta.$$

We fix a fundamental period $\tilde{\pi}$ of the Carlitz $\mathbb{F}_q[t]$-module $\mathbf{C}$, i.e., $\operatorname{Ker} \operatorname{Exp}_{\mathbf{C}} = A \cdot \tilde{\pi}$. Put

$$\Omega(t) := (-\theta)^{\frac{-q}{q-1}} \prod_{i=1}^{\infty} \left(1 - \frac{t}{\theta^{q^i}}\right) \in \mathbb{C}_\infty[\![t]\!],$$

where $(-\theta)^{\frac{1}{q-1}}$ is a suitable choice of $(q-1)$-st root of $-\theta$ so that $\frac{1}{\Omega(\theta)} = \tilde{\pi}$ (see [ABP04, AT09]). We note that $\Omega$ satisifies the functional equation

$$\Omega^{(1)} = \Omega/(t - \theta^q),$$

and hence

$$\Omega^{s_1+\cdots+s_r} \cdot \zeta_A^{\mathrm{mot}}(\mathfrak{s}) = \sum_{i_1 > \cdots > i_r \geq 0} (\Omega^{s_1} H_{s_1-1})^{(i_1)} \cdots (\Omega^{s_r} H_{s_r-1})^{(i_r)},$$

which was studied in [AT09].

*Remark* 5.1.5. By the Anderson-Thakur's interpolation formula (5.1.3) we have that

$$\zeta_A^{\mathrm{mot}}(\mathfrak{s})|_{t=\theta}=\Gamma_{\mathfrak{s}}\zeta_A(\mathfrak{s}), \tag{5.1.6}$$

where $\Gamma_{\mathfrak{s}}:=\Gamma_{s_1}\cdots\Gamma_{s_r}$. Note that $\Omega^{s_1+\cdots+s_r}\zeta_A^{\mathrm{mot}}(\mathfrak{s})$ is an entire power series (see [AT09, CPY19]). Since $\Omega$ is entire on $\mathbb{C}_\infty$ with simple zeros at $t=\theta^q,\theta^{q^2},\cdots$, the series $\zeta_A^{\mathrm{mot}}(\mathfrak{s})$ lies in $\mathbb{T}_\theta$ and hence $\partial_t^j\zeta_A^{\mathrm{mot}}(\mathfrak{s})\in\mathbb{T}_\theta$ for every positive integer $j$.

5.2. **Explicit formulae for $t$-motivic MZV's.** For any index $\mathfrak{s}=(s_1,\ldots,s_r)\in\mathbb{N}^r$, combining Anderson-Thakur's work on the interpolation formula with [CM17, Thm. 5.2.5] we can express $\zeta_A(\mathfrak{s})$ as

$$\Gamma_{\mathfrak{s}}\zeta_A(\mathfrak{s})=\sum_{\ell=1}^{T_{\mathfrak{s}}}b_\ell(\theta)\cdot(-1)^{\mathrm{dep}(\mathfrak{s}_\ell)-1}\operatorname{Li}^\star_{\mathfrak{s}_\ell}(\mathbf{u}_\ell), \tag{5.2.1}$$

for some number $T_{\mathfrak{s}}\in\mathbb{N}$, explicit coefficients $b_\ell(t)\in\mathbb{F}_q[t]$, explicit indexes $\mathfrak{s}_\ell\in\mathbb{N}^{\mathrm{dep}(\mathfrak{s}_\ell)}$ with $\mathrm{dep}(\mathfrak{s}_\ell)\le\mathrm{dep}(\mathfrak{s})$ and $\mathrm{wt}(\mathfrak{s}_\ell)=\mathrm{wt}(\mathfrak{s})$ and explicit integral points $\mathbf{u}_\ell\in A^{\mathrm{dep}(\mathfrak{s}_\ell)}$.

*Remark* 5.2.2. We mention in this remark that each value $\operatorname{Li}^\star_{\mathfrak{s}_\ell}(\mathbf{u}_\ell)$ occurring in (5.2.1) is non-vanishing (when $\mathbf{u}_\ell\in(A\setminus\{0\})^{\mathrm{dep}(\mathfrak{s}_\ell)}$). We simply argue as follows. For any index $\mathfrak{s}=(s_1,\ldots,s_r)\in\mathbb{N}^r$, we put

$$\mathbb{D}'_{\mathfrak{s}}:=\left\{(z_1,\ldots,z_r)\in\mathbb{C}_\infty^r;|z_i|_\infty<q^{\frac{s_iq}{q-1}}\text{ for }i=1,\ldots,r\right\}$$

and

$$\mathbb{D}''_{\mathfrak{s}}:=\left\{(z_1,\ldots,z_r)\in\mathbb{C}_\infty^r:|z_1|_\infty<q^{\frac{s_1q}{q-1}}\text{ and }|z_i|_\infty\le q^{\frac{s_iq}{q-1}}\text{ for }i=2,\ldots,r\right\}.$$

Then for any $\mathbf{u}=(u_1,\ldots,u_r)\in\mathbb{D}''_{\mathfrak{s}}$, by [C14, Rem. 5.1.4] the absolute value of the general terms of $\operatorname{Li}^\star_{\mathfrak{s}}(\mathbf{u})$ is given by

$$\left|\frac{u_1^{q^{i_1}}\cdots u_r^{q^{i_r}}}{L_{i_1}^{s_1}\cdots L_{i_r}^{s_r}}\right|_\infty=q^{\frac{q}{q-1}(s_1+\cdots+s_r)}\cdot\left|\frac{u_1}{\theta^{s_1q/(q-1)}}\right|_\infty^{q^{i_1}}\cdots\left|\frac{u_r}{\theta^{s_rq/(q-1)}}\right|_\infty^{q^{i_r}}.$$

Therefore the absolute values above have a unique maximal one when $i_1=\cdots=i_r=0$ for any $\mathbf{u}\in\mathbb{D}''_{\mathfrak{s}}\cap(\mathbb{C}_\infty^\times)^r$. Note that $\mathbf{u}_\ell\in\mathbb{D}'_{\mathfrak{s}_\ell}\subset\mathbb{D}''_{\mathfrak{s}_\ell}$ by (5.1.2) and [CM17, Prop. 5.2.2 and Rmk. 5.2.6], whence $\operatorname{Li}^\star_{\mathfrak{s}_\ell}(\mathbf{u}_\ell)\neq0$.

The aim of this section is to deform the identity (5.2.1) to an identity of power series.

**Lemma 5.2.3.** *Fix an index $\mathfrak{s}=(s_1,\ldots,s_r)\in\mathbb{N}^r$. Let $(\mathfrak{s}_\ell,\mathbf{u}_\ell,b_\ell(t))$ and $T_{\mathfrak{s}}$ be given in (5.2.1). Then we have the following identity*

$$\zeta_A^{\mathrm{mot}}(\mathfrak{s})=\sum_{\ell=1}^{T_{\mathfrak{s}}}b_\ell(t)\cdot(-1)^{\mathrm{dep}(\mathfrak{s}_\ell)-1}\mathfrak{Li}^\star_{\mathfrak{s}_\ell,\mathbf{u}_\ell}. \tag{5.2.4}$$

5.2.1. *Formula of $\zeta_A^{\mathrm{mot}}(\mathfrak{s})$ in terms of $\mathfrak{Li}_{\mathfrak{s},\mathbf{u}}$.* We will give the proof of Lemma 5.2.3 shortly, but first we discuss some new ideas in order to connect $\zeta_A^{\mathrm{mot}}(\mathfrak{s})$ to $\mathfrak{Li}_{\mathfrak{s},\mathbf{u}}$. Fix an index $\mathfrak{s}=(s_1,\ldots,s_r)\in\mathbb{N}^r$. For each $1\le i\le r$, we expand the Anderson-Thakur polynomial $H_{s_i-1}(t)$ as

$$H_{s_i-1}(t)=\sum_{j=0}^{n_i}u_{ij}t^j,$$

where $u_{ij} \in A$ with $|u_{ij}|_\infty < |\theta|_\infty^{\frac{s_i q}{q-1}}$ and $u_{in_i} \neq 0$. Following the notation of [CM17] we define

$$J_{\mathfrak{s}} := \{0, 1, \dots, n_1\} \times \cdots \times \{0, 1, \dots, n_r\}.$$

For each $\mathbf{j} = (j_1, \dots, j_r) \in J_{\mathfrak{s}}$, we put

$$\mathbf{u_j} := (u_{1j_1}, \dots, u_{rj_r}) \in A^r \text{ and } a_{\mathbf{j}}(t) := t^{j_1+\cdots+j_r} \in \mathbb{F}_q[t]. \tag{5.2.5}$$

One then observes that

$$H_{s_1-1}^{(i_1)} \cdots H_{s_r-1}^{(i_r)} = \left(\sum_{j=0}^{n_1} u_{1j}^{q^{i_1}} t^j\right) \cdots \left(\sum_{j=0}^{n_r} u_{rj}^{q^{i_r}} t^j\right) = \sum_{\mathbf{j}=(j_1,\dots,j_r)\in J_{\mathfrak{s}}} a_{\mathbf{j}}(t) u_{1j_1}^{q^{i_1}} \cdots u_{rj_r}^{q^{i_r}}.$$

Dividing the above equality by $\mathbb{L}_{i_1}^{s_1} \cdots \mathbb{L}_{i_r}^{s_r}$ and then summing over all $i_1 > \cdots i_r \geq 0$ we find the following identity from the definitions of $\zeta_A^{\mathrm{mot}}(\mathfrak{s})$ and $\mathfrak{Li}_{\mathfrak{s},\mathbf{u}}$.

**Proposition 5.2.6.** *Let $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ and let $J_{\mathfrak{s}}$ be defined as above. Then we have the following identity*

$$\zeta_A^{\mathrm{mot}}(\mathfrak{s}) = \sum_{\mathbf{j}\in J_{\mathfrak{s}}} a_{\mathbf{j}}(t)\, \mathfrak{Li}_{\mathfrak{s},\mathbf{u_j}}. \tag{5.2.7}$$

*Remark* 5.2.8. When we specialize the both sides of (5.2.7) at $t = \theta$, then we obtain the identity

$$\Gamma_{\mathfrak{s}} \zeta_A(\mathfrak{s}) = \sum_{\mathbf{j}\in J_{\mathfrak{s}}} a_{\mathbf{j}}(\theta)\, \mathrm{Li}_{\mathfrak{s}}(\mathbf{u_j}) \tag{5.2.9}$$

given in [C14, Thm. 5.5.2].

5.2.2. *Review of the identity* (5.2.1). We first mention that the arguments of proving the identity (5.2.4) are essentially the same as the arguments of deriving (5.2.1), and so we quickly review the ideas how we derive (5.2.1). As we have the formula (5.2.9), it suffices to express the CMPL $\mathrm{Li}_{\mathfrak{s}}(\mathbf{u_j})$ of the right hand side of (5.2.9) in terms of linear combination of $\mathrm{Li}_{\mathfrak{s}_\ell}^\star$ with coefficients $\pm 1$. It is easy to achieve this goal by using inclusion-exclusion principle on the set

$$\{i_1 > \cdots > i_r \geq 0\}.$$

Note that the coefficients $b_\ell(\theta)$ arise from some $a_{\mathbf{j}}(\theta)$ up to $\pm 1$.

5.2.3. *Proof of Lemma 5.2.3.* Now we prove the identity (5.2.4). First, we start with the identity (5.2.7). We then use inclusion-exclusion principle on the set

$$\{i_1 > \cdots > i_r \geq 0\}$$

to express the $\mathfrak{Li}_{\mathfrak{s},\mathbf{u_j}}$ of the right hand side of (5.2.7) as a linear combinations of $\mathfrak{Li}_{\mathfrak{s}_\ell,\mathbf{u}_\ell}^\star$ with coefficients $\pm 1$. Since such a procedure is completely the same as in Sec. 5.2.2 and since the coefficients of the right hand side of (5.2.7) and (5.2.9) are the same when replacing $t$ by $\theta$, by going through the details we obtain the desired formula (5.2.4).

*Remark* 5.2.10. The formula (5.2.4) will be used in the proof of Theorem 6.2.1. The key idea of the proof is that formula (5.2.4) is the deformation of (5.2.1). In the proof of Theorem 6.2.1 we do not need to know the precise coefficients $b_\ell$, so we avoid presenting the repetitive details given in [CM17, p. 23]. One certainly could write down the precise coefficients $b_\ell$ by going through the procedure mentioned above.

## 6. Explicit formulae for $Z_{\mathfrak{s}}$

For a given index $\mathfrak{s}=(s_1,\ldots,s_r)\in\mathbb{N}^r$, in [CM17] one explicitly constructs a uniformizable $t$-module $G_{\mathfrak{s}}$ defined over $A$, a special point $\mathbf{v}_{\mathfrak{s}}\in G_{\mathfrak{s}}(A)$ and a vector $Z_{\mathfrak{s}}\in\operatorname{Lie}G_{\mathfrak{s}}(\mathbb{C}_\infty)$ so that

- $\operatorname{Exp}_{G_{\mathfrak{s}}}(Z_{\mathfrak{s}})=\mathbf{v}_{\mathfrak{s}}$, and
- the $d_1$th coordinate of $Z_{\mathfrak{s}}$ gives $\Gamma_{\mathfrak{s}}\zeta_A(\mathfrak{s})$,

where we recall that $d_i:=s_i+\cdots+s_r$ for $i=1,\ldots,r$. The purpose of this section is to give explicit formulae for all coordinates of $Z_{\mathfrak{s}}$ in terms of hyperderivatives of $t$-motivic MZV's and $t$-motivic CMSPL's.

### 6.1. Review of the constructions of $G_{\mathfrak{s}},\mathbf{v}_{\mathfrak{s}},Z_{\mathfrak{s}}$.

In this subsection, for a fixed index $\mathfrak{s}=(s_1,\ldots,s_r)\in\mathbb{N}^r$ we recall the constructions of the $t$-module $G_{\mathfrak{s}}$ defined over $A$, the special point $\mathbf{v}_{\mathfrak{s}}\in G_{\mathfrak{s}}(A)$ and the vector $Z_{\mathfrak{s}}\in\operatorname{Lie}G_{\mathfrak{s}}(\mathbb{C}_\infty)$ given in [CM17].

6.1.1. *Review of fiber coproducts of dual $t$-motives.* Let $\mathbb{K}\subset\mathbb{C}_\infty$ be an algebraically closed subfield containing $K$. Fix a dual $t$-motive $\mathcal{N}$, and let $\{\mathcal{M}_1,\ldots,\mathcal{M}_T\}$ be dual $t$-motives so that we have an embedding $\mathcal{N}\hookrightarrow\mathcal{M}_i$ as left $\mathbb{K}[t,\sigma]$-modules and the quotient $\mathcal{M}_i/\mathcal{N}$ is either zero or a dual $t$-motive for each $i$. Let $\mathbf{n}$ be a $\mathbb{K}[t]$-basis of $\mathcal{N}$ and denote by $\mathbf{n}_i$ the image of $\mathbf{n}$ under the embedding $\mathcal{N}\hookrightarrow\mathcal{M}_i$. Under the assumptions on $\mathcal{M}_i$, we note that for each $i$ the set $\mathbf{n}_i$ is either a $\mathbb{K}[t]$-basis of $\mathcal{M}_i$ or can be extended to a $\mathbb{K}[t]$-basis of $\mathcal{M}_i$.

We define $\mathcal{M}$ to be the fiber coproduct of $\{\mathcal{M}_i\}_{i=1}^T$ over $\mathcal{N}$ denoted by $\mathcal{M}_1\sqcup_{\mathcal{N}}\cdots\sqcup_{\mathcal{N}}\mathcal{M}_T$. As a left $\mathbb{K}[t]$-module, $\mathcal{M}$ is defined by the following quotient module

$$\mathcal{M}:=\bigoplus_{i=1}^T\mathcal{M}_i/\left(\operatorname{Span}_{\mathbb{K}[t]}\left\{x_i'-x_j'|\ \forall\ x\in\mathbf{n}\ \forall 1\leq i,j\leq T\right\}\right),$$

where $x_i'$ denotes the image of $x$ under the embedding $\mathcal{N}\hookrightarrow\mathcal{M}_i$ for $i=1,\ldots,T$. It is shown in [CM17, Sec. 2.4.2] that the $\mathbb{K}[t]$-module $\left(\operatorname{Span}_{\mathbb{K}[t]}\left\{x_i'-x_j'|\ \forall\ x\in\mathbf{n}\ \forall 1\leq i,j\leq T\right\}\right)$ is stable under the $\sigma$-action, and hence $\mathcal{M}$ is a left $\mathbb{K}[t,\sigma]$-module. In fact, $\mathcal{M}$ is shown to be a dual $t$-motive.

6.1.2. *The set up.* Recall the formula $\Gamma_{\mathfrak{s}}\zeta_A(\mathfrak{s})=\sum_{\ell=1}^{T_{\mathfrak{s}}}b_\ell(\theta)\cdot(-1)^{\operatorname{dep}(\mathfrak{s}_\ell)-1}\operatorname{Li}^\star_{\mathfrak{s}_\ell}(\mathbf{u}_\ell)$ given in (5.2.1). To simplify notation we fix $T=T_{\mathfrak{s}}$ and let $s$ be the cardinality of those triples $(\mathfrak{s}_\ell,\mathbf{u}_\ell,b_\ell(\theta))$ with $\operatorname{dep}(\mathfrak{s}_\ell)=1$. We then renumber the indexes $\ell$ of $\{1,\ldots,T\}$ such that

- $1\leq\ell\leq s$ if $\operatorname{dep}(\mathfrak{s}_\ell)=1$, and
- $s+1\leq\ell\leq T$ for $\operatorname{dep}(\mathfrak{s}_\ell)\geq 2$,

where $\ell$ corresponds to the triple $(\mathfrak{s}_\ell,\mathbf{u}_\ell,b_\ell(\theta))$.

Now for each $1\leq\ell\leq T$ we define matrices $\Phi_\ell\in\operatorname{Mat}_{\operatorname{dep}(\mathfrak{s}_\ell)+1}(\mathbb{K}[t])$ and $\Phi'_\ell\in\operatorname{Mat}_{\operatorname{dep}(\mathfrak{s}_\ell)}(\mathbb{K}[t])$ using the $\operatorname{dep}(\mathfrak{s}_\ell)$-tuple $\widetilde{\mathfrak{s}}_\ell$ (recall that $\widetilde{\cdot}$ reverses the order of the tuple) as in (2.3.2) and (2.3.4), respectively, with $\mathfrak{Q}=\widetilde{\mathbf{u}}_\ell$. Further, define the Frobenius module $M_\ell$ and the dual $t$-motive $M'_\ell$ as in Sec. 2.3 with sigma actions given by $\Phi_\ell$ and $\Phi'_\ell$, respectively. For each $(\widetilde{\mathfrak{s}}_\ell,\widetilde{\mathbf{u}}_\ell)$, let $(G_\ell,\rho_\ell)$ be the $t$-module associated to $M'_\ell$, i.e., $\rho_\ell$ is given in (4.1.3), and let $\mathbf{v}_\ell$ be the special point of $G_\ell$ given in (4.1.4). Note that $\mathbf{v}_\ell$ arises from $M_\ell$ by Remark 4.1.5.

Since each $\widetilde{\mathbf{u}}_\ell$ is an integral point, by Sec. 4.1 we see that $G_\ell$ is defined over $A$ and $\mathbf{v}_\ell\in G_\ell(A)$. Note further that by [CM17, Thm. 4.2.3] the logarithm $\operatorname{Log}_{G_\ell}$ converges at the special point $\mathbf{v}_\ell$, and the $d_1$th coordinate of $\operatorname{Log}_{G_\ell}(\mathbf{v}_\ell)$ is equal to

$$(-1)^{\operatorname{dep}(\mathfrak{s}_\ell)-1}\operatorname{Li}^\star_{\mathfrak{s}_\ell}(\mathbf{u}_\ell),$$

where $d_1 := \mathrm{wt}(\mathfrak{s}) = \mathrm{wt}(\mathfrak{s}_\ell)$ for all $\ell$. Finally, we put $Z_\ell := \mathrm{Log}_{G_\ell}(\mathbf{v}_\ell)$ for each $\ell$. We note that the above setting is the same as [CM17, p. 24-25].

6.1.3. *The $t$-module $G_{\mathfrak{s}}$.* For each $\ell$, we let $\rho_\ell$ be the map defining the $t$-module structure on $G_\ell$. By (4.1.3) we see that if $\ell \geq s+1$, $\rho_\ell(t)$ is a right upper triangular block matrix with $[t]_{d_1}$ located as upper left square. So $\rho_\ell(t)$ has the shape

$$\rho_\ell(t) = \begin{pmatrix} [t]_{d_1} & F_\ell \\ & \rho_\ell(t)' \end{pmatrix},$$

where $F_\ell$ and $\rho_\ell'$ are matrices over $A[\tau]$ which one could calculate explicitly, but going forward we only need to know that $[t]_{d_1}$ is the top left block without knowing the precise sizes of $F_\ell$ and $\rho_\ell'$. Note that if $\ell \leq s$ then $\rho_\ell(t) = [t]_{d_1}$. We define the $t$-module $(G_{\mathfrak{s}}, \rho)$ defined over $A$ to be the $t$-module associated to the dual $t$-motive

$$\mathcal{M}_{\mathfrak{s}} = M_1' \sqcup_{\mathbf{C}^{\otimes d_1}} \cdots \sqcup_{\mathbf{C}^{\otimes d_1}} M_T',$$

which is the fiber coproduct of the dual $t$-motives $\{M_\ell'\}_{\ell=1}^T$ over $C^{\otimes d_1}$. We claim that $\rho(t)$ is a block upper triangular matrix given by

$$\begin{pmatrix} [t]_{d_1} & F_{s+1} & \cdots & F_T \\ & \rho_{s+1}(t)' & & \\ & & \ddots & \\ & & & \rho_T(t)' \end{pmatrix}, \tag{6.1.1}$$

where again $F_i$ and $\rho_i'$ are certain matrices over $A[\tau]$ whose exact description we do not require going forward. We will prove the above claim after giving a brief definition.

**Definition 6.1.2.** *For any vector $\mathbf{z} \in \mathrm{Mat}_{n\times 1}(\mathbb{C}_\infty)$ with $n \geq d_1$, we define $\hat{\mathbf{z}}$ as the vector of the first $d_1$ coordinates of $\mathbf{z}$, and $\mathbf{z}_-$ as the vector of the remaining $n-d_1$ coordinates, i.e.,*

$$\mathbf{z} = \begin{pmatrix} \hat{\mathbf{z}} \\ \mathbf{z}_- \end{pmatrix}$$

*for which $\hat{\mathbf{z}}$ is of length $d_1$ and $\mathbf{z}_-$ is of length $n-d_1$.*

To prove the claim, we note that from the construction of fiber coproducts of dual $t$-motives there is a natural morphism of $t$-modules

$$\pi : \bigoplus_{\ell=1}^T G_\ell \to G_{\mathfrak{s}}$$

given by

$$(\mathbf{z}_1, \ldots, \mathbf{z}_T) \mapsto \left( \sum_{\ell=1}^T \hat{\mathbf{z}}_\ell^{\mathrm{tr}}, \mathbf{z}_{s+1-}^{\mathrm{tr}}, \ldots, \mathbf{z}_{T-}^{\mathrm{tr}} \right)^{\mathrm{tr}}.$$

Note further that given a point $\mathbf{z} \in G_{\mathfrak{s}}$, we can pick a suitable point $\mathbf{z}_\ell \in G_\ell$ for each $1 \leq \ell \leq T$ so that $\hat{\mathbf{z}}_\ell = \mathbf{0}$ for all $\ell \neq s+1$ and $\pi(\mathbf{z}_1, \ldots, \mathbf{z}_T) = \mathbf{z}$. Since $\pi$ is a morphism of

$t$-modules, we have

$$\begin{aligned}
\rho(t)(\mathbf{z}) &= \rho(t)\left(\pi\left(\mathbf{z}_1,\dots,\mathbf{z}_T\right)\right) = \pi\left(\rho_1(t)(\mathbf{z}_1),\dots,\rho_T(t)(\mathbf{z}_T)\right)\\
&= \pi\left(\mathbf{0},\dots,\mathbf{0},\begin{pmatrix}[t]_{d_1}\hat{\mathbf{z}}_{s+1}+F_{s+1}\mathbf{z}_{s+1-}\\ \rho_{s+1}(t)'\mathbf{z}_{s+1-}\end{pmatrix},\begin{pmatrix}F_{s+2}\mathbf{z}_{s+2-}\\ \rho_{s+2}(t)'\mathbf{z}_{s+2-}\end{pmatrix},\dots,\begin{pmatrix}F_T\mathbf{z}_{T-}\\ \rho_T(t)'\mathbf{z}_{T-}\end{pmatrix}\right)\\
&= \begin{pmatrix}[t]_{d_1}\hat{\mathbf{z}}_{s+1}+\displaystyle\sum_{\ell=s+1}^{T}F_\ell\mathbf{z}_{\ell-}\\ \rho_{s+1}(t)'\mathbf{z}_{s+1-}\\ \vdots\\ \rho_T(t)'\mathbf{z}_{T-}\end{pmatrix} = \begin{pmatrix}[t]_{d_1} & F_{s+1} & \cdots & F_T\\ & \rho_{s+1}(t)' & & \\ & & \ddots & \\ & & & \rho_T(t)'\end{pmatrix}\begin{pmatrix}\hat{\mathbf{z}}_{s+1}\\ \mathbf{z}_{s+1-}\\ \vdots\\ \mathbf{z}_{T-}\end{pmatrix}\\
&= \begin{pmatrix}[t]_{d_1} & F_{s+1} & \cdots & F_T\\ & \rho_{s+1}(t)' & & \\ & & \ddots & \\ & & & \rho_T(t)'\end{pmatrix}\mathbf{z}.
\end{aligned}$$

Finally, we mention that the special point $\mathbf{v}_{\mathfrak{s}}\in G_{\mathfrak{s}}(A)$ and the vector $Z_{\mathfrak{s}}\in\operatorname{Lie}G_{\mathfrak{s}}(\mathbb{C}_\infty)$ in [CM17] are defined by

$$\mathbf{v}_{\mathfrak{s}} := \pi\left((\rho_1(b_1(t))(\mathbf{v}_1),\dots,\rho_T(b_T(t))(\mathbf{v}_T))\right)$$

and

$$Z_{\mathfrak{s}} := \partial\pi\left((\partial\rho_1(b_1(t))Z_1,\dots,\partial\rho_T(b_T(t))Z_T)\right),$$

and one has the fact [CM17, Thm. 1.2.2] that

$$\operatorname{Exp}_{G_{\mathfrak{s}}}(Z_{\mathfrak{s}}) = \mathbf{v}_{\mathfrak{s}}.$$

6.2. **The main result.** The primary result of this paper is stated as follows.

**Theorem 6.2.1.** *For any index $\mathfrak{s}=(s_1,\dots,s_r)\in\mathbb{N}^r$, we let $d_1:=s_1+\cdots+s_r$ and let $Z_{\mathfrak{s}}\in\operatorname{Lie}G_{\mathfrak{s}}(\mathbb{C}_\infty)$ be the vector given as above. For each $s+1\le\ell\le T$, we set*

$$\mathbf{Y}_\ell := \mathbf{Y}_{\tilde{\mathfrak{s}}_\ell,\tilde{\mathbf{u}}_\ell},$$

*which is defined in Theorem 4.2.2. Then $Z_{\mathfrak{s}}$ has the following explicit formula*

$$Z_{\mathfrak{s}} = \begin{pmatrix}\left(\partial_t^{d_1-1}\zeta_A^{\mathrm{mot}}(\mathfrak{s})\right)|_{t=\theta}\\ \vdots\\ \left(\partial_t^{1}\zeta_A^{\mathrm{mot}}(\mathfrak{s})\right)|_{t=\theta}\\ \zeta_A^{\mathrm{mot}}(\mathfrak{s})|_{t=\theta}\\ (\partial\rho_{s+1}(b_{s+1})(\mathbf{Y}_{s+1}))_-\\ \vdots\\ (\partial\rho_T(b_T)(\mathbf{Y}_T))_-\end{pmatrix} = \begin{pmatrix}\left(\partial_t^{d_1-1}\zeta_A^{\mathrm{mot}}(\mathfrak{s})\right)|_{t=\theta}\\ \vdots\\ \left(\partial_t^{1}\zeta_A^{\mathrm{mot}}(\mathfrak{s})\right)|_{t=\theta}\\ \Gamma_{\mathfrak{s}}\zeta_A(\mathfrak{s})\\ (\partial\rho_{s+1}(b_{s+1})(\mathbf{Y}_{s+1}))_-\\ \vdots\\ (\partial\rho_T(b_T)(\mathbf{Y}_T))_-\end{pmatrix},$$

*where $b_\ell\in\mathbb{F}_q[t]$ is given in (5.2.4) and $\partial\rho_\ell(b_\ell)(\mathbf{Y}_\ell)$ is explicitly given in Corollary 4.2.4 for each $s+1\le\ell\le T$, and the notation $(\cdot)_-$ is defined in Definition 6.1.2.*

*Proof.* Note that by Proposition 4.1.6, $\partial\rho_\ell(b_\ell)$ is given explicitly as diagonal block matrices, and the first block matrix is given by

$$\begin{pmatrix} b_\ell(\theta) & (\partial_t^1 b_\ell)\,|_{t=\theta} & \cdots & \left(\partial_t^{d_1-1} b_\ell\right)|_{t=\theta} \\ & \ddots & \ddots & \vdots \\ & & \ddots & (\partial_t^1 b_\ell)\,|_{t=\theta} \\ & & & b_\ell(\theta) \end{pmatrix}. \tag{6.2.2}$$

Recall by Theorem 4.2.2 that $Z_\ell = \begin{pmatrix} \hat{Z}_\ell \\ Z_{\ell-} \end{pmatrix}$, where

$$\hat{Z}_\ell = \begin{pmatrix} (-1)^{\mathrm{dep}(\mathfrak{s}_\ell)-1}\left(\partial_t^{d_1-1}\mathfrak{Li}^\star_{\mathfrak{s}_\ell,\mathbf{u}_\ell}\right)|_{t=\theta} \\ \vdots \\ (-1)^{\mathrm{dep}(\mathfrak{s}_\ell)-1}\left(\partial_t^{1}\mathfrak{Li}^\star_{\mathfrak{s}_\ell,\mathbf{u}_\ell}\right)|_{t=\theta} \\ (-1)^{\mathrm{dep}(\mathfrak{s}_\ell)-1}\left(\mathfrak{Li}^\star_{\mathfrak{s}_\ell,\mathbf{u}_\ell}\right)|_{t=\theta} \end{pmatrix} = \begin{pmatrix} (-1)^{\mathrm{dep}(\mathfrak{s}_\ell)-1}\left(\partial_t^{d_1-1}\mathfrak{Li}^\star_{\mathfrak{s}_\ell,\mathbf{u}_\ell}\right)|_{t=\theta} \\ \vdots \\ (-1)^{\mathrm{dep}(\mathfrak{s}_\ell)-1}\left(\partial_t^{1}\mathfrak{Li}^\star_{\mathfrak{s}_\ell,\mathbf{u}_\ell}\right)|_{t=\theta} \\ (-1)^{\mathrm{dep}(\mathfrak{s}_\ell)-1}\mathfrak{Li}^\star_{\mathfrak{s}_\ell}(\mathbf{u}_\ell) \end{pmatrix}.$$

Since $Z_{\mathfrak{s}} := \partial\pi\left(\partial\rho_1(b_1(t))Z_1, \ldots, \partial\rho_T(b_T(t))Z_T\right)$, by the definition of $\pi$ we have

$$Z_{\mathfrak{s}} = \left(\sum_{\ell=1}^{T} \widehat{\partial\rho_\ell(b_\ell)(Z_\ell)}^{\mathrm{tr}}, \left(\partial\rho_{s+1}(b_{s+1})(Z_{s+1})^{\mathrm{tr}}\right)_-, \ldots, \left(\partial\rho_T(b_T)(Z_T)^{\mathrm{tr}}\right)_-\right)^{\mathrm{tr}}. \tag{6.2.3}$$

Recall that $\partial\rho_\ell(b_\ell)$ is a diagonal block matrix with the first block given as (6.2.2), whence we have that

$$\begin{aligned} &\textstyle\sum_{\ell=1}^T \widehat{\partial\rho_\ell(b_\ell)(Z_\ell)} \\ = \;& \sum_{\ell=1}^T \begin{pmatrix} b_\ell(\theta) & (\partial_t^1 b_\ell)\,|_{t=\theta} & \cdots & \left(\partial_t^{d_1-1} b_\ell\right)|_{t=\theta} \\ & \ddots & \ddots & \vdots \\ & & \ddots & (\partial_t^1 b_\ell)\,|_{t=\theta} \\ & & & b_\ell(\theta) \end{pmatrix} \begin{pmatrix} (-1)^{\mathrm{dep}(\mathfrak{s}_\ell)-1}\cdot\left(\partial_t^{d_1-1}\mathfrak{Li}^\star_{\mathfrak{s}_\ell,\mathbf{u}_\ell}(t)\right)|_{t=\theta} \\ (-1)^{\mathrm{dep}(\mathfrak{s}_\ell)-1}\cdot\left(\partial_t^{d_1-2}\mathfrak{Li}^\star_{\mathfrak{s}_\ell,\mathbf{u}_\ell}(t)\right)|_{t=\theta} \\ \vdots \\ (-1)^{\mathrm{dep}(\mathfrak{s}_\ell)-1}\cdot\mathfrak{Li}^\star_{\mathfrak{s}_\ell,\mathbf{u}_\ell}(t)|_{t=\theta} \end{pmatrix} \\ = \;& \begin{pmatrix} \sum_{\ell=1}^T \partial_t^{d_1-1}\left((-1)^{\mathrm{dep}(\mathfrak{s}_\ell)-1}\cdot b_\ell(t)\mathfrak{Li}^\star_{\mathfrak{s}_\ell,\mathbf{u}_\ell}(t)\right)|_{t=\theta} \\ \sum_{\ell=1}^T \partial_t^{d_1-2}\left((-1)^{\mathrm{dep}(\mathfrak{s}_\ell)-1}\cdot b_\ell(t)\mathfrak{Li}^\star_{\mathfrak{s}_\ell,\mathbf{u}_\ell}(t)\right)|_{t=\theta} \\ \vdots \\ \sum_{\ell=1}^T (-1)^{\mathrm{dep}(\mathfrak{s}_\ell)-1}\cdot b_\ell(t)\mathfrak{Li}^\star_{\mathfrak{s}_\ell,\mathbf{u}_\ell}(t)|_{t=\theta} \end{pmatrix} \\ = \;& \begin{pmatrix} \left(\partial_t^{d_1-1}\zeta_A^{\mathrm{mot}}(\mathfrak{s})\right)|_{t=\theta} \\ \vdots \\ (\partial_t^1\zeta_A^{\mathrm{mot}}(\mathfrak{s}))\,|_{t=\theta} \\ \zeta_A^{\mathrm{mot}}(\mathfrak{s})|_{t=\theta} \end{pmatrix} = \begin{pmatrix} \left(\partial_t^{d_1-1}\zeta_A^{\mathrm{mot}}(\mathfrak{s})\right)|_{t=\theta} \\ \vdots \\ (\partial_t^1\zeta_A^{\mathrm{mot}}(\mathfrak{s}))\,|_{t=\theta} \\ \Gamma_{\mathfrak{s}}\zeta_A(\mathfrak{s}) \end{pmatrix}, \end{aligned}$$

where the second equality comes from Proposition 3.3.8(2) and the third equality arises from linearity of hyperderivatives and (5.2.4).

Since by definition $G_\ell = G_{\tilde{\mathfrak{s}}_\ell,\tilde{\mathbf{u}}_\ell}$, $\mathbf{v}_\ell = \mathbf{v}_{\tilde{\mathfrak{s}}_\ell,\tilde{\mathbf{u}}_\ell}$ and $Z_\ell = \mathrm{Log}_{G_\ell}(\mathbf{v}_\ell)$, we have $Z_\ell = \mathbf{Y}_\ell := \mathbf{Y}_{\tilde{\mathfrak{s}}_\ell,\tilde{\mathbf{u}}_\ell}$ defined in Theorem 4.2.2 and so the explicit formulae of the remaining coordinates of $Z_{\mathfrak{s}}$ follow from (6.2.3) and Corollary 4.2.4. □

**Example 6.2.4.** *Take $q = 2$ and $\mathfrak{s} = (1, 3)$. The 4th coordinate of $Z_{(1,3)}$ is given in* [CM19, Example 5.4.2]*; however we can give other coordinates explicitily here. In this case, we have*

$\Gamma_1 = 1$, $\Gamma_3 = \theta^2+\theta$, $H_{1-1}=1$, $H_{3-1}=t+\theta^2$, $J_{(1,3)}=\{(0,0),(0,1)\}$, $\mathbf{u}_{(0,0)}=(1,\theta^2)$, $\mathbf{u}_{(0,1)}=(1,1)$, $a_{(0,0)}=1$, $a_{(0,1)}=t$. *Thus we have*

$$\begin{aligned}(\theta^2+\theta)\zeta_A(1,3) &= \mathrm{Li}_{(1,3)}(1,\theta^2)+\theta\,\mathrm{Li}_{(1,3)}(1,1)\\ &= \mathrm{Li}^\star_{(1,3)}(1,\theta^2)-\mathrm{Li}^\star_4(\theta^2)+\theta\,\mathrm{Li}^\star_{(1,3)}(1,1)-\theta\,\mathrm{Li}^\star_4(1)\\ &= (-1)^{1-1}\,\mathrm{Li}^\star_4(\theta^2)+\theta\cdot(-1)^{1-1}\,\mathrm{Li}^\star_4(1)\\ &\qquad+(-1)^{2-1}\,\mathrm{Li}^\star_{(1,3)}(1,\theta^2)+\theta\cdot(-1)^{2-1}\,\mathrm{Li}^\star_{(1,3)}(1,1),\\ \zeta_A^{\mathrm{mot}}(1,3) &= (-1)^{1-1}\mathfrak{Li}^\star_{4,\theta^2}(t)+t\cdot(-1)^{1-1}\mathfrak{Li}^\star_{4,1}(t)\\ &\qquad+(-1)^{2-1}\mathfrak{Li}^\star_{(1,3),(1,\theta^2)}(t)+t\cdot(-1)^{2-1}\mathfrak{Li}^\star_{(1,3),(1,1)}(t),\end{aligned}$$

*and* $(b_1(t),\mathfrak{s}_1,\mathbf{u}_1)=(1,4,\theta^2)$, $(b_2(t),\mathfrak{s}_2,\mathbf{u}_2)=(t,4,1)$, $(b_3(t),\mathfrak{s}_3,\mathbf{u}_3)=(1,(1,3),(1,\theta^2))$, $(b_4(t),\mathfrak{s}_4,\mathbf{u}_4)=(t,(1,3),(1,1))$.

*For* $\ell=1$, *we have* $G_1=\mathbf{C}^{\otimes 4}$, *and points*

$$\mathbf{v}_1=(0,0,0,\theta^2)^{\mathrm{tr}}\in\mathbf{C}^{\otimes 4}(A),$$

$$Z_1=\left(\left(\partial_t^3\mathfrak{Li}^\star_{4,\theta^2}\right)|_{t=\theta},\left(\partial_t^2\mathfrak{Li}^\star_{4,\theta^2}\right)|_{t=\theta},\left(\partial_t^1\mathfrak{Li}^\star_{4,\theta^2}\right)|_{t=\theta},\mathrm{Li}^\star_4(\theta^2)\right)^{\mathrm{tr}}\in\mathrm{Lie}\,\mathbf{C}^{\otimes 4}(\mathbb{C}_\infty).$$

*For* $\ell=2$, *we have* $G_2=\mathbf{C}^{\otimes 4}$, *and points*

$$\mathbf{v}_2=(0,0,0,1)^{\mathrm{tr}}\in\mathbf{C}^{\otimes 4}(A),$$

$$Z_2=\left(\left(\partial_t^3\mathfrak{Li}^\star_{4,1}\right)|_{t=\theta},\left(\partial_t^2\mathfrak{Li}^\star_{4,1}\right)|_{t=\theta},\left(\partial_t^1\mathfrak{Li}^\star_{4,1}\right)|_{t=\theta},\mathrm{Li}^\star_4(1)\right)^{\mathrm{tr}}\in\mathrm{Lie}\,\mathbf{C}^{\otimes 4}(\mathbb{C}_\infty).$$

*We also have*

$$\rho_2(t)(\mathbf{v}_2)=[t]_4\mathbf{v}_2=(0,0,1,\theta)^{\mathrm{tr}}\in\mathbf{C}^{\otimes 4}(A),$$

$$\partial\rho_2(t)Z_2=\left(\left(\partial_t^3 t\mathfrak{Li}^\star_{4,1}\right)|_{t=\theta},\left(\partial_t^2 t\mathfrak{Li}^\star_{4,1}\right)|_{t=\theta},\left(\partial_t^1 t\mathfrak{Li}^\star_{4,1}\right)|_{t=\theta},\theta\,\mathrm{Li}^\star_4(1)\right)^{\mathrm{tr}}\in\mathrm{Lie}\,\mathbf{C}^{\otimes 4}(\mathbb{C}_\infty).$$

*For* $\ell=3$, *we have* $G_3=\mathbb{G}_a^5$ *with the* $t$-*action*

$$\rho_3(t)=\left(\begin{array}{cccc|c}\theta & 1 & & & \\ & \theta & 1 & & \\ & & \theta & 1 & \\ \tau & & & \theta & -\theta^2\tau \\ \hline & & & & \theta+\tau\end{array}\right),$$

*and points*

$$\mathbf{v}_3=(0,0,0,-\theta^2,1)^{\mathrm{tr}}\in G_3(A),$$

$$Z_3=\begin{pmatrix}-\left(\partial_t^3\mathfrak{Li}^\star_{(1,3),(1,\theta^2)}\right)|_{t=\theta}\\ -\left(\partial_t^2\mathfrak{Li}^\star_{(1,3),(1,\theta^2)}\right)|_{t=\theta}\\ -\left(\partial_t^1\mathfrak{Li}^\star_{(1,3),(1,\theta^2)}\right)|_{t=\theta}\\ -\mathrm{Li}^\star_{(1,3)}(1,\theta^2)\\ \mathrm{Li}^\star_1(1)\end{pmatrix}\in\mathrm{Lie}\,G_3(\mathbb{C}_\infty).$$

*For* $\ell=4$, *we have* $G_4=\mathbb{G}_a^5$ *with the* $t$-*action*

$$\rho_4(t)=\left(\begin{array}{cccc|c}\theta & 1 & & & \\ & \theta & 1 & & \\ & & \theta & 1 & \\ \tau & & & \theta & -\tau \\ \hline & & & & \theta+\tau\end{array}\right),$$

*and*

$$\mathbf{v}_4 = (0,0,0,-1,1)^{\mathrm{tr}} \in G_4(A),$$

$$Z_4 = \begin{pmatrix} -\left(\partial_t^3 \mathfrak{Li}^\star_{(1,3),(1,1)}\right)|_{t=\theta} \\ -\left(\partial_t^2 \mathfrak{Li}^\star_{(1,3),(1,1)}\right)|_{t=\theta} \\ -\left(\partial_t^1 \mathfrak{Li}^\star_{(1,3),(1,1)}\right)|_{t=\theta} \\ -\operatorname{Li}^\star_{(1,3)}(1,1) \\ \operatorname{Li}^\star_1(1) \end{pmatrix} \in \operatorname{Lie} G_4(\mathbb{C}_\infty).$$

*We also have*

$$\rho_4(t)(\mathbf{v}_4) = (0,0,1,\theta+1,\theta+1)^{\mathrm{tr}} \in G_4(A),$$

$$\partial\rho_4(t)Z_4 = \begin{pmatrix} -\left(\partial_t^3 t\mathfrak{Li}^\star_{(1,3),(1,1)}\right)|_{t=\theta} \\ -\left(\partial_t^2 t\mathfrak{Li}^\star_{(1,3),(1,1)}\right)|_{t=\theta} \\ -\left(\partial_t^1 t\mathfrak{Li}^\star_{(1,3),(1,1)}\right)|_{t=\theta} \\ -\theta\operatorname{Li}^\star_{(1,3)}(1,1) \\ \theta\operatorname{Li}^\star_1(1) \end{pmatrix} \in \operatorname{Lie} G_4(\mathbb{C}_\infty).$$

*Therefore we have* $G_{(1,3)} = \mathbb{G}_a^6$ *with the* $t$*-action*

$$\rho(t) = \left(\begin{array}{cccc|c|c} \theta & 1 & & & & \\ & \theta & 1 & & & \\ & & \theta & 1 & & \\ \tau & & & \theta & -\theta^2\tau & -\tau \\ \hline & & & & \theta+\tau & \\ \hline & & & & & \theta+\tau \end{array}\right),$$

*and*

$$\mathbf{v}_{(1,3)} = \pi(\mathbf{v}_1, \rho_2(t)(\mathbf{v}_2), \mathbf{v}_3, \rho_4(t)(\mathbf{v}_4)) = (0,0,0,1,1,\theta+1)^{\mathrm{tr}} \in G_{(1,3)}(A),$$

$$Z_{(1,3)} = \begin{pmatrix} (\partial_t^3 \zeta_A^{\mathrm{mot}}(1,3))|_{t=\theta} \\ (\partial_t^2 \zeta_A^{\mathrm{mot}}(1,3))|_{t=\theta} \\ (\partial_t^1 \zeta_A^{\mathrm{mot}}(1,3))|_{t=\theta} \\ (\theta^2+\theta)\zeta_A(1,3) \\ \operatorname{Li}^\star_1(1) \\ \theta\operatorname{Li}^\star_1(1) \end{pmatrix} \in \operatorname{Lie} G_{(1,3)}(\mathbb{C}_\infty).$$

**Corollary 6.2.5.** *Let notation and hypotheses be given in Theorem 6.2.1. Then for any polynomial* $c(t) \in \mathbb{F}_q[t]$*, we have*

$$\partial\rho(c(t))Z_{\mathfrak{s}} = \begin{pmatrix} \left(\partial_t^{d_1-1} c(t)\zeta_A^{\mathrm{mot}}(\mathfrak{s})\right)|_{t=\theta} \\ \vdots \\ (\partial_t^1 c(t)\zeta_A^{\mathrm{mot}}(\mathfrak{s}))|_{t=\theta} \\ c(t)\zeta_A^{\mathrm{mot}}(\mathfrak{s})|_{t=\theta} \\ (\partial\rho_{s+1}(cb_{s+1})(\mathbf{Y}_{s+1}))_- \\ \vdots \\ (\partial\rho_T(cb_T)(\mathbf{Y}_T))_- \end{pmatrix} = \begin{pmatrix} \left(\partial_t^{d_1-1} c(t)\zeta_A^{\mathrm{mot}}(\mathfrak{s})\right)|_{t=\theta} \\ \vdots \\ (\partial_t^1 c(t)\zeta_A^{\mathrm{mot}}(\mathfrak{s}))|_{t=\theta} \\ c(\theta)\Gamma_{\mathfrak{s}}\zeta_A(\mathfrak{s}) \\ (\partial\rho_{s+1}(cb_{s+1})(\mathbf{Y}_{s+1}))_- \\ \vdots \\ (\partial\rho_T(cb_T)(\mathbf{Y}_T))_- \end{pmatrix},$$

*where* $\partial\rho_\ell(cb_\ell)(\mathbf{Y}_\ell)$ *is explicitly given in Corollary 4.2.4 for* $s+1 \le \ell \le T$*.*

*Proof.* The arguments are entirely the same as above and so we omit them. ∎

6.3. **Monomials of MZV's.** Given any two MZV's with weight $n_1$ and $n_2$, Thakur showed in [T10] that the product of these two MZV's is an $\mathbb{F}_p$-linear combination of certain MZV's of weight $n_1+n_2$, where $\mathbb{F}_p$ is the prime field of $K$. It follows that for any indexes $\mathbf{k}_1,\dots,\mathbf{k}_m$, there exist some indexes $\mathfrak{s}_1,\dots,\mathfrak{s}_n$ of wight $w:=\mathrm{wt}(\mathbf{k}_1)+\cdots+\mathrm{wt}(\mathbf{k}_m)$ and coefficients $a_1,\dots,a_n\in\mathbb{F}_p$ so that

$$\zeta_A(\mathbf{k}_1)\cdots\zeta_A(\mathbf{k}_m)=a_1\zeta_A(\mathfrak{s}_1)+\cdots+a_n\zeta_A(\mathfrak{s}_n). \tag{6.3.1}$$

For each $\mathfrak{s}_i$ above, we let $(G_{\mathfrak{s}_i},\rho_{\mathfrak{s}_i})$ be the $t$-module over $A$ and $\mathbf{v}_{\mathfrak{s}_i}\in G_{\mathfrak{s}_i}(A)$ be the special point defined in Sec. 6.1.3. Let $Z_{\mathfrak{s}_i}$ be given as in Theorem 6.2.1, and note that $\mathrm{Exp}_{G_{\mathfrak{s}_i}}(Z_{\mathfrak{s}_i})=\mathbf{v}_{\mathfrak{s}_i}$. Recall that $G_{\mathfrak{s}_i}$ is the $t$-module associated to the dual $t$-motive $\mathcal{M}_{\mathfrak{s}_i}$ containing $C^{\otimes w}$ as a sub-dual-$t$-motive. We put

$$\mathcal{M}:=\mathcal{M}_{\mathfrak{s}_1}\sqcup_{C^{\otimes w}}\cdots\sqcup_{C^{\otimes w}}\mathcal{M}_{\mathfrak{s}_n},$$

which is the fiber coprodcut of $\{\mathcal{M}_{\mathfrak{s}_i}\}_{i=1}^n$ over $C^{\otimes w}$, and let $(G,\rho)$ be the $t$-module over $A$ associated to the dual $t$-motive $\mathcal{M}$ in Sec. 2.3.2. Using the arguments above, we can write $(G,\rho)$ explicitly as follows.

We first note that for each $i$, $\rho_{\mathfrak{s}_i}(t)$ is a right upper triangular block matrix with $[t]_w$ located at upper left square. That is, $\rho_{\mathfrak{s}_i}(t)$ has the shape of the form

$$\begin{pmatrix}[t]_w & B_{\mathfrak{s}_i}\\ & \rho_{\mathfrak{s}_i}(t)'\end{pmatrix}.$$

Then $\rho(t)$ is given by

$$\begin{pmatrix}[t]_w & B_{\mathfrak{s}_1} & \cdots & B_{\mathfrak{s}_n}\\ & \rho_{\mathfrak{s}_1}(t)' & & \\ & & \ddots & \\ & & & \rho_{\mathfrak{s}_n}(t)'\end{pmatrix}.$$

We further note that there is a natural morphism of $t$-modules

$$\pi:\bigoplus_{i=1}^n G_{\mathfrak{s}_i}\to G$$

given by

$$(\mathbf{z}_{\mathfrak{s}_1},\dots,\mathbf{z}_{\mathfrak{s}_n})\mapsto\left(\sum_{i=1}^n\hat{\mathbf{z}}_{\mathfrak{s}_i}^{\mathrm{tr}},\mathbf{z}_{\mathfrak{s}_1-}^{\mathrm{tr}},\dots,\mathbf{z}_{\mathfrak{s}_n-}^{\mathrm{tr}}\right)^{\mathrm{tr}},$$

where $\hat{\mathbf{z}}$ and $\mathbf{z}_-$ are defined in Definition 6.1.2 by putting $d_1:=w$ there.

Using the methods of fiber coproduct [CM17, Lem. 3.3.2] and Theorem 6.2.1, we can relate the monomial $\zeta_A(\mathbf{k}_1)\cdots\zeta_A(\mathbf{k}_m)$ to the $w$-th coordinate of a certain logarithmic vector $Z$ and give explicit formulae for all the other coordinates. Before stating the result, we multiply both sides of (6.3.1) by $\Gamma_{\mathfrak{s}_1}\cdots\Gamma_{\mathfrak{s}_n}$ which gives

$$\Gamma_{\mathfrak{s}_1}\cdots\Gamma_{\mathfrak{s}_n}\zeta_A(\mathbf{k}_1)\cdots\zeta_A(\mathbf{k}_m)=c_1\Gamma_{\mathfrak{s}_1}\zeta_A(\mathfrak{s}_1)+\cdots+c_n\Gamma_{\mathfrak{s}_n}\zeta_A(\mathfrak{s}_n), \tag{6.3.2}$$

where for each $i$,

$$c_i:=a_i\prod_{\substack{1\leq j\leq n\\ j\neq i}}\Gamma_{\mathfrak{s}_j}\in A \tag{6.3.3}$$

and denote by $c_i(t):=c_i|_{\theta=t}\in\mathbb{F}_q[t]$. Finally, we denote

$$Z:=\partial\pi\left(\partial\rho_{\mathfrak{s}_1}(c_1(t))Z_{\mathfrak{s}_1},\dots,\partial\rho_{\mathfrak{s}_n}(c_n(t))Z_{\mathfrak{s}_n}\right)\in\mathrm{Lie}\,G(\mathbb{C}_\infty),$$

$$\mathbf{v} := \pi\left(\rho_{\mathfrak{s}_1}(c_1(t))(\mathbf{v}_{\mathfrak{s}_1}), \ldots, \rho_{\mathfrak{s}_n}(c_n(t))(\mathbf{v}_{\mathfrak{s}_n})\right) \in G(A),$$

so that $\operatorname{Exp}_G(Z) = \mathbf{v}$. Then, we have the following commutative diagram,

$$\begin{tikzcd} \bigoplus_{i=1}^{n} G_{\mathfrak{s}_i} \arrow[r, "\pi"] & G \\ \bigoplus_{i=1}^{n} \operatorname{Lie} G_{\mathfrak{s}_i} \arrow[u, "\oplus_{i=1}^{n} \operatorname{Exp}_{G_{\mathfrak{s}_i}}"] \arrow[r, "\partial\pi"] & \operatorname{Lie} G. \arrow[u, "\operatorname{Exp}_G"'] \end{tikzcd}$$

**Theorem 6.3.4.** *Let $\mathbf{k}_1, \ldots, \mathbf{k}_m$ be $m$ indexes and put $w := \operatorname{wt}(\mathbf{k}_1) + \cdots + \operatorname{wt}(\mathbf{k}_m)$. Let $\{\mathfrak{s}_i, c_i\}_{i=1}^{n}$ be given in (6.3.2) and (6.3.3), and $Z_{\mathfrak{s}_i}$ be given in Theorem 6.2.1 for each $i$. Let $G$ be the $t$-module, $Z \in \operatorname{Lie} G(\mathbb{C}_\infty)$ be the logarithmic vector and $\mathbf{v} \in G(A)$ be the special point defined above. Then we have*

(1) *$Z$ is given by*

$$\begin{pmatrix} \sum_{i=1}^{n} \partial\rho_{\mathfrak{s}_i}\widehat{(c_i(t))} Z_{\mathfrak{s}_i} \\ (\partial\rho_{\mathfrak{s}_1}(c_1(t))Z_{\mathfrak{s}_1})_- \\ \vdots \\ (\partial\rho_{\mathfrak{s}_n}(c_n(t))Z_{\mathfrak{s}_n})_- \end{pmatrix},$$

*where $\partial\rho_{\mathfrak{s}_i}(c_i(t))Z_{\mathfrak{s}_i}$ is explicitly given in Corollary 6.2.5 for each $1 \le i \le n$ and where we recall the notation from Definition 6.1.2.*

(2) *The $w$-th coordinate of $Z$ is given by $\Gamma_{\mathfrak{s}_1} \cdots \Gamma_{\mathfrak{s}_n} \zeta_A(\mathbf{k}_1) \cdots \zeta_A(\mathbf{k}_m)$.*

*Proof.* The first assertion follows from the definition of $\partial\pi$. To prove the second one, we first note that for each $i$, we have the following.

- $G_{\mathfrak{s}_i}$ comes from the dual $t$-motive $\mathcal{M}_{\mathfrak{s}_i}$, and contains $\mathbf{C}^{\otimes w}$ as sub-$t$-module.
- $\mathcal{M}$ is the fiber coproduct of $\{\mathcal{M}_{\mathfrak{s}_i}\}_{i=1}^{n}$ over $C^{\otimes w}$ and $G$ is its corresponding $t$-module.
- $\operatorname{Exp}_{G_{\mathfrak{s}_i}}(\partial\rho_{\mathfrak{s}_i}(c_i(t))Z_{\mathfrak{s}_i}) = \rho_{\mathfrak{s}_i}(c_i(t))\,(\mathbf{v}_{\mathfrak{s}_i})$.
- The $w$-th coordinate of $\partial\rho_{\mathfrak{s}_i}(c_i(t))Z_{\mathfrak{s}_i} = c_i(\theta)\Gamma_{\mathfrak{s}_i}\zeta_A(\mathfrak{s}_i)$.

So by [CM17, Lem. 3.3.2] the $w$-th coordinate of $Z$ is given by $c_1\Gamma_{\mathfrak{s}_1}\zeta_A(\mathfrak{s}_1)+\cdots+c_n\Gamma_{\mathfrak{s}_n}\zeta_A(\mathfrak{s}_n)$, which is equal to $\Gamma_{\mathfrak{s}_1} \cdots \Gamma_{\mathfrak{s}_n}\zeta_A(\mathbf{k}_1)\cdots\zeta_A(\mathbf{k}_m)$ by (6.3.2). □

*Remark* 6.3.5. We mention that the coefficients $\{a_i\}$ in (6.3.1) are shown to exist by Thakur [T10], and in general we do not know how to write them down explicitly. In the simplest case that $m = 2$ and $\operatorname{dep}(\mathfrak{s}_1) = \operatorname{dep}(\mathfrak{s}_2) = 1$, Chen has explicit formulae for the coefficients $\{a_i\}$ in [Ch15]. Precisely, we have that for any positive integers $s_1$ and $s_2$ with $n := s_1 + s_2$, then

$$\begin{aligned} \zeta_A(s_1)\zeta_A(s_2) &= \zeta_A(s_1, s_2) + \zeta_A(s_2, s_1) + \zeta_A(s_1 + s_2) \\ &\quad + \sum_{\substack{i+j=n \\ (q-1)\mid j}} \left[(-1)^{s_1-1}\binom{j-1}{s_1-1} + (-1)^{s_2-1}\binom{j-1}{s_2-1}\right]\zeta_A(i,j). \end{aligned} \tag{6.3.6}$$

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