ZETA FUNCTIONS FOR TWO-DIMENSIONAL SHIFTS OF FINITE TYPE

JUNG-CHAO BAN*, WEN-GUEI HU, SONG-SUN LIN**, AND YIN-HENG LIN

Abstract. This work is concerned with zeta functions of two-dimensional shifts of finite type. A two-dimensional zeta function $\zeta^0(s)$ which generalizes the Artin-Mazur zeta function was given by Lind for $\mathbb{Z}^2$-action ϕ. The n-th order zeta function ζ_n of ϕ on $\mathbb{Z}_{n\times\infty}$, $n \geq 1$, is studied first. The trace operator $\mathbf{T}_n$ which is the transition matrix for x-periodic patterns of period n with height 2 is rotationally symmetric. The rotational symmetry of $\mathbf{T}_n$ induces the reduced trace operator τ_n and $\zeta_n = (\det(I - s^n\tau_n))^{-1}$. The zeta function $\zeta = \prod_{n=1}^{\infty} (\det(I - s^n\tau_n))^{-1}$ in the x-direction is now a reciprocal of an infinite product of polynomials. The zeta function can be presented in the y-direction and in the coordinates of any unimodular transformation in $GL_2(\mathbb{Z})$. Therefore, there exists a family of zeta functions that are meromorphic extensions of the same analytic function $\zeta^0(s)$. The Taylor series at the origin for these zeta functions are equal with integer coefficients, yielding a family of identities which are of interest in number theory. The method applies to thermodynamic zeta functions for the Ising model with finite range interactions.

1. Introduction

Various zeta functions have been investigated in the fields of number theory, geometry, dynamical systems and statistical physics. This work studies the zeta functions in a manner that follows the work of Artin and Mazur [1], Bowen and Lanford [6], Ruelle [30] and Lind [21]. First, recall the zeta function that was defined by Artin and Mazur.

Let $\phi : \mathbb{X} \longrightarrow \mathbb{X}$ be a homeomorphism of a compact space and $\Gamma_n(\phi)$ denote the number of fixed points of ϕ^n. The zeta function $\zeta_\phi(s)$ for ϕ defined in [1] is

$$\zeta_\phi(s) = \exp\left(\sum_{n=1}^{\infty} \frac{\Gamma_n(\phi)}{n} s^n\right). \tag{1.1}$$

Later, Bowen and Lanford [6] demonstrated that if ϕ is a shift of finite type, then $\zeta_\phi(s)$ is a rational function. In the simplest case, when a shift is generated by a transition matrix A in $\mathbb{Z}$, (1.1) is computed explicitly as

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$$\zeta_A(s) = \exp\left(\sum_{n=1}^{\infty} \frac{trA^n}{n} s^n\right) \tag{1.2}$$

$$= (\det(I - sA))^{-1}, \tag{1.3}$$

and then

$$\zeta_A(s) = \prod_{\lambda\in\Sigma(A)} (1 - \lambda s)^{-\chi(\lambda)}, \tag{1.4}$$

where $\chi(\lambda)$ is a non-negative integer that is the algebraic multiplicity of eigenvalue λ and $\Sigma(A)$ is the spectrum of A. $\zeta_A(s)$ is a rational function which involves only eigenvalues of A.

Lind [21] extended (1.1) to $\mathbb{Z}^d$-action as follows. For $\mathbb{Z}^d$-action, $d \geq 1$, let ϕ be an action of $\mathbb{Z}^d$ on $\mathbb{X}$. Denote the set of finite-index subgroups of $\mathbb{Z}^d$ by $\mathcal{L}_d$. The zeta function ζ_ϕ defined by Lind is

$$\zeta_\phi(s) = \exp\left(\sum_{L\in\mathcal{L}_d} \frac{\Gamma_L(\phi)}{[L]} s^{[L]}\right), \tag{1.5}$$

where $[L] = index[\mathbb{Z}^d/L]$ and $\Gamma_L(\phi)$ is the number of fixed points by $\phi^{\mathbf{n}}$ for all $\mathbf{n} \in L$. Lind [21] obtained some important results for ζ_ϕ, such as conjugacy invariant and product formulae, and computed ζ_ϕ explicitly for some interesting examples. Furthermore, he raised some problems, including the following two.

Problem 7.2. [21] *For "finitely determined" $\mathbb{Z}^d$-actions ϕ such as shifts of finite type, is there a reasonable finite description of $\zeta_\phi(s)$?*

Problem 7.5. [21] *Compute explicitly the thermodynamic zeta function for the 2-dimensional Ising model, where α is the $\mathbb{Z}^2$ shift action on the space of configurations.*

The present authors previously studied pattern generation problems in $\mathbb{Z}^d$, $d \geq 2$, and developed several approaches such as the use of higher order transition matrices and trace operators to compute spatial entropy [2, 3]. The work of Ruelle [30] and Lind [21] indicated that our methods could also be adopted to study zeta functions.

In this investigation, Problems 7.2 and 7.5 are answered when ϕ is a shift of finite type. The following paragraphs briefly introduce relevant results.

Let $\mathbb{Z}_{m\times m}$ be the $m \times m$ square lattice in $\mathbb{Z}^2$ and $\mathcal{S}$ be the finite set of symbols (alphabets or colors). $\mathcal{S}^{\mathbb{Z}_{m\times m}}$ is the set of all local patterns (or configurations) on $\mathbb{Z}_{m\times m}$. A given subset $\mathcal{B} \subset \mathcal{S}^{\mathbb{Z}_{m\times m}}$ is called a basic set of admissible local patterns. $\Sigma(\mathcal{B})$ is the set of all global patterns defined on $\mathbb{Z}^2$ which can be generated by $\mathcal{B}$. For simplicity, this introduction presents only the results of $\mathbb{Z}_{2\times 2}$ with two symbols $\mathcal{S} = \{0, 1\}$. Section 4 considers the general case.

As presented elsewhere [21], $\mathcal{L}_2$ can be parameterized in Hermite normal form [24]:

$$\mathcal{L}_2 = \left\{ \begin{bmatrix} n & l \\ 0 & k \end{bmatrix} \mathbb{Z}^2 : \ n \geq 1, \ k \geq 1 \ and \ 0 \leq l \leq n-1 \right\}.$$

Given a basic set $\mathcal{B}$, denote by $P_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\right)$ the set of all $\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}$-periodic and $\mathcal{B}$-admissible patterns and $\Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\right)$ is the number of $P_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\right)$.

The zeta function, defined by (1.5), is denoted by

$$\zeta^0_{\mathcal{B}} = \exp\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=0}^{n-1}\frac{1}{nk}\Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\right)s^{nk}\right). \tag{1.6}$$

In [21], $\zeta^0_{\mathcal{B}}$ is shown analytically in $|s| < \exp(-g(\mathcal{B}))$, where

$$g(\mathcal{B}) \equiv \limsup_{[L]\to\infty}\frac{1}{[L]}\log\Gamma_{\mathcal{B}}(L). \tag{1.7}$$

In this work, the sum of n and k in (1.6) is treated separately as an iterated sum. Indeed, for any $n \geq 1$, define the n-th order zeta function $\zeta_n(s) \equiv \zeta_{\mathcal{B},n}(s)$ (in x-direction) as

$$\zeta_n(s) = \exp\left(\frac{1}{n}\sum_{k=1}^{\infty}\sum_{l=0}^{n-1}\frac{1}{k}\Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\right)s^{nk}\right); \tag{1.8}$$

the zeta function $\zeta(s) \equiv \zeta_{\mathcal{B}}(s)$ is given by

$$\zeta(s) = \prod_{n=1}^{\infty}\zeta_n(s). \tag{1.9}$$

The first observation of (1.8) is that, for $n \geq 1$ and $l \geq 1$, any $\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}$-periodic pattern is $\begin{bmatrix} n & 0 \\ 0 & \frac{nk}{(n,l)} \end{bmatrix}$-periodic, where (n,l) is the greatest common divisor (GCD) of n and l. Therefore, $\begin{bmatrix} n & 0 \\ 0 & k \end{bmatrix}$-periodicity of patterns must be investigated in details.

The trace operators $\mathbf{T}_n \equiv \mathbf{T}_n(\mathcal{B})$ that were introduced in [3] are useful in studying $\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}$-periodic and the $\mathcal{B}$-admissible pattern, where $\mathbf{T}_n = [t_{n;i,j}]$ is a $2^n \times 2^n$ matrix with $t_{n;i,j} \in \{0,1\}$. $\mathbf{T}_n(\mathcal{B})$ represents the set of patterns that are $\mathcal{B}$-admissible and x-periodic of period n with height 2. The trace operator $\mathbf{T}_n$ can be used to construct (doubly) periodic $\mathcal{B}$-admissible patterns. Indeed, for $k \geq 1$ and $0 \leq l \leq n-1$,

$$\Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\right) = tr(\mathbf{T}_n^k R_n^l), \tag{1.10}$$

where R_n is a $2^n \times 2^n$ rotational matrix defined by

$$\begin{cases} R_{n;i,2i-1} = 1 \text{ and } R_{n;2^{n-1}+i,2i} = 1 & \text{for } 1 \leq i \leq 2^{n-1}, \\ R_{n;i,j} = 0 & \text{otherwise.} \end{cases}$$

Denote by $\mathbf{R}_n = \sum_{l=0}^{n-1} R_n^l$; now based on (1.10), $\zeta_n(s)$ becomes

$$\zeta_n(s) = \exp\left(\frac{1}{n}\sum_{k=1}^{\infty}\frac{1}{k}tr(\mathbf{T}_n^k\mathbf{R}_n)s^{nk}\right), \tag{1.11}$$

which is a generalization of (1.2).

To elucidate the method used to study (1.11), $\mathbf{T}_n$ is firstly assumed to be symmetric. Then $\mathbf{T}_n$ can be expressed in Jordan canonical form as

$$\mathbf{T}_n = \mathbf{U}\mathbf{J}\mathbf{U}^t \tag{1.12}$$

where the eigen-matrix $\mathbf{U} = (U_1, ..., U_N)$ is an $N \times N$ matrix which consists of linearly independent (column) eigenvectors U_j, $1 \leq j \leq N$ and $N \equiv 2^n$. Jordan matrix $\mathbf{J} = diag(\lambda_j)$ is a diagonal $N \times N$ matrix, which comprises eigenvalues λ_j, $1 \leq j \leq N$. Now,

$$\begin{aligned} & \tfrac{1}{n}\sum_{k=1}^{\infty} \tfrac{1}{k} tr(\mathbf{T}_n^k \mathbf{R}_n) s^{nk} \\ = \ & \tfrac{1}{n} tr(\mathbf{U}(\sum_{k=1}^{\infty} \tfrac{1}{k}\mathbf{J}^k s^{nk})\mathbf{U}^t \mathbf{R}_n) \\ = \ & \sum_{j=1}^{N} \tfrac{1}{n} \left|\mathbf{R}_n \circ U_j U_j^t\right| \log(1-\lambda_j s^n)^{-1} \end{aligned} \tag{1.13}$$

can be proven, where $\circ$ is a Hadamard product: if $A = [a_{i,j}]_{M\times M}$ and $B = [b_{i,j}]_{M\times M}$, then $A \circ B = [a_{i,j}b_{i,j}]_{M\times M}$.

Evaluating the coefficients $|\mathbf{R}_n \circ U_j U_j^t|$ of $\log(1-\lambda_j s^n)^{-1}$ is important. Now, the R_n-symmetry of $\mathbf{T}_n$ is crucial. Indeed, let U be an eigenvector of $\mathbf{T}_n$ with eigenvalue λ, such that $\mathbf{T}_n U = \lambda U$; then $R_n^l U$ is also eigenvector of $\mathbf{T}_n$:

$$\mathbf{T}_n(R_n^l U) = \lambda R_n^l U \tag{1.14}$$

for all $0 \leq l \leq n-1$. Notably, $R_n^n = I_{2^n}$, where I_{2^n} is the $2^n \times 2^n$ identity matrix.

U is called R_n-symmetric, if $R_n^l U = U$ for all $0 \leq l \leq n-1$. In this case,

$$\frac{1}{n}|\mathbf{R}_n \circ UU^t| = 1. \tag{1.15}$$

U is called anti-symmetric if $\sum\limits_{l=0}^{n-1} R_n^l U = 0$. In this case,

$$\frac{1}{n}|\mathbf{R}_n \circ UU^t| = 0. \tag{1.16}$$

Additionally, for any given eigenvalue λ, the associated eigenspace E_λ can be proven to be spanned by symmetric eigenvectors $\overline{U}_j$, $1 \leq j \leq p_\lambda$, and anti-symmetric eigenvectors U'_j, $1 \leq j \leq q_\lambda$: $E_\lambda = \{\overline{U}_1, \cdots, \overline{U}_{p_\lambda}, U'_1, \cdots, U'_{q_\lambda}\}$, where $p_\lambda + q_\lambda = dim(E_\lambda)$ and p_λ or q_λ can be zero.

Therefore, for each eigenvalue λ of $\mathbf{T}_n$,

$$\chi(\lambda) \equiv \frac{1}{n}\sum_{\lambda_j=\lambda} |\mathbf{R}_n \circ U_j U_j^t| = p_\lambda \tag{1.17}$$

is the number of linearly independent symmetric eigenvectors of $\mathbf{T}_n$ with respect to λ, a non-negative integer. Moreover, $p_\lambda \geq 1$ can be shown if λ is the largest eigenvalue. Hence, choosing eigen-matrix $\mathbf{U}$ in (1.12), which consists of symmetric and anti-symmetric eigenvectors, yields

$$\zeta_n(s) = \prod_{\lambda\in\Sigma(\mathbf{T}_n)} (1-\lambda s^n)^{-\chi(\lambda)} \tag{1.18}$$

as a rational function, as in (1.4).

From the rotational matrix R_n, for $1 \leq i \leq 2^n$, the equivalent class $C_n(i)$ of i is defined as $C_n(i) = \left\{ j \middle| \left(R_n^l\right)_{i,j} = 1 \text{ for some } 1 \leq l \leq n \right\}$. The index set $\mathcal{I}_n$ of n is defined by $\mathcal{I}_n = \left\{ i \middle| 1 \leq i \leq 2^n, i \leq j \text{ for all } j \in C_n(i) \right\}$ and χ_n is the cardinal number of $\mathcal{I}_n$. Indeed, χ_n is the number of necklaces that can be made from n beads of two colors when the necklaces can be rotated but not turned over. Furthermore,

$$\chi_n = \frac{1}{n}\sum_{d|n}\phi(d)2^{n/d}, \tag{1.19}$$

where $\phi(d)$ is the Euler totient function.

Then, the reduced trace operator $\tau_n = [\tau_{n;i,j}]$ of $\mathbf{T}_n$ is a $\chi_n \times \chi_n$ matrix that is defined by

$$\tau_{n;i,j} = \sum_{k\in C_n(j)} t_{n;i,k} \tag{1.20}$$

for each $i, j \in \mathcal{I}_n$. $\lambda \in \Sigma(\mathbf{T}_n)$ with $\chi(\lambda) \geq 1$ can be verified if and only if $\lambda \in \Sigma(\tau_n)$. Moreover, $\chi(\lambda)$ is the algebraic multiplicity of τ_n with eigenvalue λ. Therefore,

$$\zeta_n(s) = \left(\det\left(I - s^n\tau_n\right)\right)^{-1}, \tag{1.21}$$

a similar formula as in (1.3). Hence, the zeta function $\zeta(s)$ is obtained as

$$\zeta(s) = \prod_{n=1}^{\infty} \left(\det\left(I - s^n\tau_n\right)\right)^{-1}, \tag{1.22}$$

which is an infinite product of rational functions. Equation (1.22) generalizes (1.3) and is a solution to Lind's Problem 7.2. Furthermore, according to (1.22), the coefficients of Taylor series for $\zeta(s)$ at $s = 0$ are integers, as obtained by Lind [21].

As presented elsewhere [3], an another trace operator $\widehat{\mathbf{T}}_n$ is $\mathcal{B}$-admissible and y-periodic of period n with width 2 along the x-axis. $\mathcal{L}_2$ can be parameterized as another Hermite normal form,

$$\mathcal{L}_2 = \left\{ \begin{bmatrix} k & 0 \\ l & n \end{bmatrix} \mathbb{Z}^2 : \ n \geq 1, \ k \geq 1 \ and \ 0 \leq l \leq n-1 \right\}.$$

Again, $P_{\mathcal{B}}\left(\begin{bmatrix} k & 0 \\ l & n \end{bmatrix}\right)$ represents the set of all $\begin{bmatrix} k & 0 \\ l & n \end{bmatrix}$-periodic and $\mathcal{B}$-admissible patterns and $\Gamma_{\mathcal{B}}\left(\begin{bmatrix} k & 0 \\ l & n \end{bmatrix}\right)$ denote the numbers of $P_{\mathcal{B}}\left(\begin{bmatrix} k & 0 \\ l & n \end{bmatrix}\right)$. The n-th order zeta function $\widehat{\zeta}_n(s)$ is defined by

$$\widehat{\zeta}_n(s) = \exp\left(\frac{1}{n}\sum_{k=1}^{\infty}\sum_{l=0}^{n-1}\frac{1}{k}\Gamma_{\mathcal{B}}\left(\begin{bmatrix} k & 0 \\ l & n \end{bmatrix}\right) s^{nk}\right), \tag{1.23}$$

and the zeta function $\widehat{\zeta}(s)$ is defined by

$$\widehat{\zeta}(s) = \prod_{n=1}^{\infty} \widehat{\zeta}_n(s). \tag{1.24}$$

Similar results for $\widehat{\zeta}_n(s)$ and $\widehat{\zeta}(s)$ can be obtained by using $\widehat{\tau}_n$ instead of τ_n. Indeed,

$$\widehat{\zeta}_n(s) = \prod_{\lambda\in\Sigma(\widehat{\mathbf{T}}_n)} (1-\lambda s^n)^{-\widehat{\chi}(\lambda)} \tag{1.25}$$

$$= \left(\det\left(I - s^n \widehat{\tau}_n\right)\right)^{-1}, \tag{1.26}$$

and

$$\widehat{\zeta}(s) = \prod_{n=1}^{\infty} \prod_{\lambda\in\Sigma(\widehat{\mathbf{T}}_n)} (1-\lambda s^n)^{-\widehat{\chi}(\lambda)} \tag{1.27}$$

$$= \prod_{n=1}^{\infty} \left(\det\left(I - s^n \widehat{\tau}_n\right)\right)^{-1}. \tag{1.28}$$

Since ζ and $\widehat{\zeta}$ are rearrangements of $\zeta^0_{\mathcal{B}}$, the uniqueness of the analytic function implies

$$\zeta(s) = \widehat{\zeta}(s) = \zeta^0_{\mathcal{B}} \text{ for } |s| < \exp(-g(\mathcal{B})). \tag{1.29}$$

The construction of the zeta functions ζ and $\widehat{\zeta}$ in rectangular coordinates can be extended to an inclined coordinates system. Indeed, let the unimodular transformation γ be an element of the unimodular group $GL_2(\mathbb{Z})$: $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, a, b, c and d are integers and $ad - bc = \pm 1$. The lattice L_γ is defined by

$$L_\gamma \equiv \begin{pmatrix} n & l \\ 0 & k \end{pmatrix}_\gamma \mathbb{Z}^2 = \begin{pmatrix} na & la+kc \\ nb & lb+kd \end{pmatrix} \mathbb{Z}^2. \tag{1.30}$$

The n-th order zeta function of $\zeta^0_{\mathcal{B}}(s)$ with respect to γ is defined by

$$\zeta_{\mathcal{B};\gamma;n}(s) = \exp\left(\frac{1}{n} \sum_{k=1}^{\infty} \sum_{l=0}^{n-1} \frac{1}{k} \Gamma_{\mathcal{B}} \left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}_\gamma \right) s^{nk} \right), \tag{1.31}$$

and the zeta function $\zeta_{\mathcal{B};\gamma}$ with respect to γ is given by

$$\zeta_{\mathcal{B};\gamma}(s) \equiv \prod_{n=1}^{\infty} \zeta_{\mathcal{B};\gamma;n}(s). \tag{1.32}$$

The n-th order rotational matrix $R_{\gamma;n}$, trace operator $\mathbf{T}_{\gamma;n}(\mathcal{B})$ and reduced trace operator $\tau_{\gamma;n}$ can also be introduced and

$$\begin{aligned} \zeta_{\mathcal{B};\gamma;n}(s) &= \prod_{\lambda\in\Sigma(\mathbf{T}_{\gamma;n}(\mathcal{B}))} (1-\lambda s^n)^{-\chi_{\gamma;n}(\lambda)} \\ &= \left(\det\left(I - s^n \tau_{\gamma;n}\right)\right)^{-1}, \end{aligned} \tag{1.33}$$

where the exponent $\chi_{\gamma;n}(\lambda)$ is the number of linearly independent $R_{\gamma;n}$-symmetric eigenvectors and generalized eigenvectors of $\mathbf{T}_{\gamma;n}(\mathcal{B})$ with respect to eigenvalue λ and the coordinates γ. Therefore, the zeta function $\zeta_{\mathcal{B};\gamma}$ is given by

$$\zeta_{\mathcal{B};\gamma}(s)=\prod_{n=1}^{\infty}(\det(I-s^{n}\tau_{\gamma;n}))^{-1}. \tag{1.34}$$

Since the iterated sum in (1.31) and (1.32) is a rearrangement of $\zeta^{0}_{\mathcal{B}}(s)$,

$$\zeta_{\mathcal{B};\gamma}(s)=\zeta^{0}_{\mathcal{B}}(s) \tag{1.35}$$

for $|s|<\exp(-g(\mathcal{B}))$. The identity(1.35) yields a family of identities when $\zeta_{\mathcal{B};\gamma}$ is expressed as a Taylor series at the origin $s=0$ (Theorem 6.4).

Equations (1.29) or (1.35) give some interesting results even in very simple cases. For instance, let $\mathbf{H}_2=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$ and $\mathbf{V}_2=\begin{bmatrix}1&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&1\end{bmatrix}$ be the given horizontal and vertical transition matrix, respectively; then $\mathbf{T}_2=\mathbf{V}_2$ and $\widehat{\mathbf{T}}_2=\mathbf{H}_2$. Furthermore, as in Example 6.12,

$$\zeta(s)=\prod_{n=1}^{\infty}(1-2s^{n})^{-1}, \tag{1.36}$$

$$\widehat{\zeta}(s)=\prod_{n=1}^{\infty}(1-s^{n})^{-\chi_n}, \tag{1.37}$$

and $\chi_n=\frac{1}{n}\sum\limits_{l=1}^{\infty}2^{(n,l)}$ can be shown; for details, see Example 6.12.

The thermodynamic zeta function [30] with weight function $\theta:\mathbb{X}\to(0,\infty)$ was defined by Lind [21] as

$$\zeta^{0}_{\alpha,\theta}(s)=\exp\left(\sum_{L\in\mathcal{L}_d}\left\{\sum_{x\in fix_L(\alpha)}\prod_{k\in\mathbb{Z}^d/L}\theta\left(\alpha^{k}x\right)\right\}\frac{s^{[L]}}{[L]}\right). \tag{1.38}$$

For the Ising model, where α is a shift of finite type given by $\mathcal{B}$ and the weight function θ is a potential with finite range, the previous arguments apply. Indeed, the zeta function is

$$\zeta_{Ising;\mathcal{B}}(s)=\prod_{n=1}^{\infty}\prod_{\lambda\in\Sigma(\mathbf{T}_{Ising;n}(\mathcal{B}))}(1-\lambda s^{n})^{-\chi(\lambda)} \tag{1.39}$$

$$=\prod_{n=1}^{\infty}\left(\det\left(I-s^{n}\tau_{Ising;n}\right)\right)^{-1}, \tag{1.40}$$

where $\chi(\lambda)$ is the number of linearly independent symmetric eigenvectors and generalized eigenvectors of $\mathbf{T}_{Ising;n}$ with eigenvalue λ and $\tau_{Ising;n}$ is the associated reduced trace operator. (1.40) is a solution of Lind's Problem 7.5.

The methods also apply to higher dimensional shifts of finite type. The results will appear elsewhere.

Some references that are related to our work are listed here. Zeta functions and related topics [1, 6, 8, 10, 11, 14, 15, 21, 22, 23, 25, 26, 27, 29, 30, 32, 33]; patterns generation problems and lattices dynamical systems [2, 3, 7, 13], and phase-transition in statistical physics [4, 5, 16, 17, 18, 19, 20, 28] have all been covered elsewhere.

The rest of this paper is organized as follows. Section 2 reviews the ordering matrices of local patterns and trace operators $\mathbf{T}_n$ for x-periodic patterns. The R_n-symmetry of $\mathbf{T}_n$ is investigated. Then, (1.10) and (1.11) are derived. Section 3 proves the rationality of the n-th order of the zeta function ζ_n, $n \geq 1$. Section 4 describes how to extend the techniques employed in previous sections to study the problems raised by more symbols on larger lattices, which is also useful in the study of zeta functions in inclined coordinates. Section 5 elucidates the zeta function presented in inclined coordinates that is obtained by unimodular transformations. Section 6 discusses the analyticity of zeta functions. The meromorphic extension of zeta function is studied. All meromorphic extensions are equal on $|s| < \exp(-g(\mathcal{B}))$. Section 7 investigates the zeta function of the solution set of equations on $\mathbb{Z}^2$ with numbers from a finite field. Section 8 studies the thermodynamic zeta function for the Ising model with a finite range potential.

2. Periodic patterns

This section first reviews the ordering matrices of local patterns and trace operators [2, 3]. It then derives rotational matrices R_n and $\mathbf{R}_n$, and studies their properties. The R_n-symmetry of the trace operator is also discussed. Finally, some properties of periodic patterns in $\mathbb{Z}^2$ are investigated. In particular, the $\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}$-periodic pattern is proven to be $\begin{bmatrix} n & 0 \\ 0 & \frac{nk}{(n,l)} \end{bmatrix}$-periodic.

For clarity, two symbols on the 2×2 lattice $\mathbb{Z}_{2\times 2}$ are initially examined. Section 4 addresses more general situations.

2.1. Ordering matrices and Trace operators.

For given positive integers N_1 and N_2, the rectangular lattice $\mathbb{Z}_{N_1\times N_2}$ is defined by

$$\mathbb{Z}_{N_1\times N_2} = \{(n_1,n_2) | 1 \leq n_1 \leq N_1 \text{ and } 1 \leq n_2 \leq N_2\}.$$

In particular, $\mathbb{Z}_{2\times 2} = \{(1,1),(2,1),(1,2),(2,2)\}$. Define the set of all global patterns on $\mathbb{Z}^2$ with two symbols $\{0,1\}$ by

$$\Sigma_2^2 = \{0,1\}^{\mathbb{Z}^2} = \left\{U | U : \mathbb{Z}^2 \to \{0,1\}\right\}.$$

Here, $\mathbb{Z}^2 = \{(n_1,n_2) | n_1, n_2 \in \mathbb{Z}\}$, the set of all planar lattice points (vertices).

The set of all local patterns on $\mathbb{Z}_{N_1\times N_2}$ is defined by

$$\Sigma_{N_1\times N_2} = \{U|_{\mathbb{Z}_{N_1\times N_2}} : U \in \Sigma_2^2\}.$$

Now, for any given $\mathcal{B} \subset \Sigma_{2\times 2}$, $\mathcal{B}$ is called a basic set of admissible local patterns. In short, $\mathcal{B}$ is a basic set.

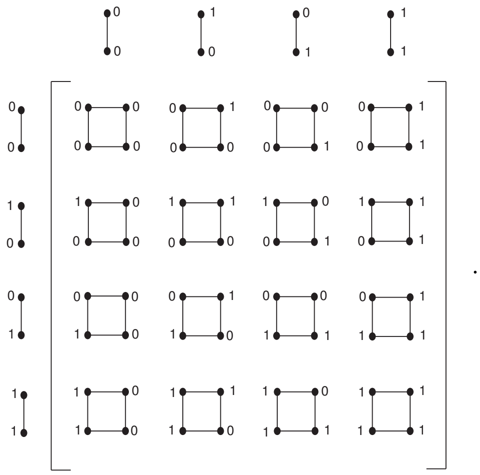
An $N_1 \times N_2$ pattern U is called $\mathcal{B}$-admissible if for any vertex (lattice point) (n_1, n_2) with $0 \le n_1 \le N_1 - 1$ and $0 \le n_2 \le N_2 - 1$, there exists a 2×2 admissible pattern $(\beta_{k_1,k_2})_{1 \le k_1, k_2 \le 2} \in \mathcal{B}$ such that

$$U_{n_1+k_1, n_2+k_2} = \beta_{k_1,k_2},$$

for $1 \le k_1, k_2 \le 2$. Denote by $\Sigma_{N_1 \times N_2}(\mathcal{B})$ the set of all $\mathcal{B}$-admissible patterns on $\mathbb{Z}_{N_1 \times N_2}$. As presented elsewhere [2], the ordering matrices $\mathbf{X}_{2\times 2}$ and $\mathbf{Y}_{2\times 2}$ are introduced to arrange systematically all local patterns in $\Sigma_{2\times 2}$.

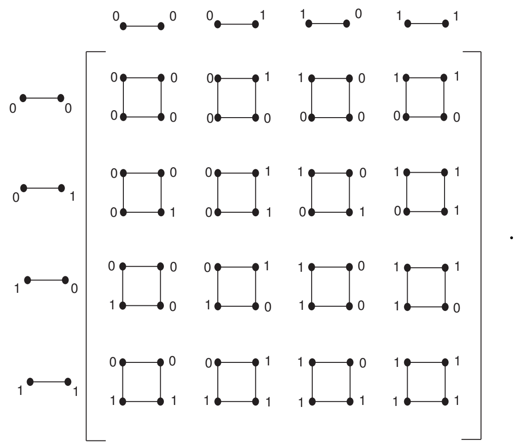
Indeed, the horizontal ordering matrix $\mathbf{X}_{2\times 2} = [x_{p,q}]_{4\times 4}$ is defined by

(2.1)

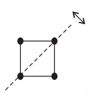


The vertical ordering matrix $\mathbf{Y}_{2\times 2} = [y_{i,j}]_{4\times 4}$ is defined by

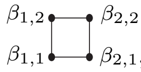
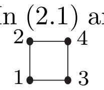
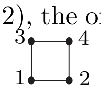
(2.2)



It is clear that the local pattern $y_{i,j}$ in $\mathbf{Y}_{2\times 2}$ is the reflection $\frac{\pi}{4}$ of $x_{i,j}$ in $\mathbf{X}_{2\times 2}$, i.e,



. The reflection can be represented by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ in $GL_2(\mathbb{Z})$ with determinant -1.

In (2.1) and (2.2), the orders of the pattern  $\beta_{2,1}$, $\beta_{i,j} \in \{0,1\}$, are given by  and  respectively. More precisely, in (2.1), $x_{p,q}$ is ordered by

$$p = 2\beta_{1,1} + \beta_{1,2} + 1$$

and

$$q = 2\beta_{2,1} + \beta_{2,2} + 1,$$

and in (2.2), $y_{i,j}$ is ordered by

$$i = 2\beta_{1,1} + \beta_{2,1} + 1$$

and

$$j = 2\beta_{1,2} + \beta_{2,2} + 1.$$

$\mathbf{X}_{2\times 2}$ and $\mathbf{Y}_{2\times 2}$ are clearly related as follows.

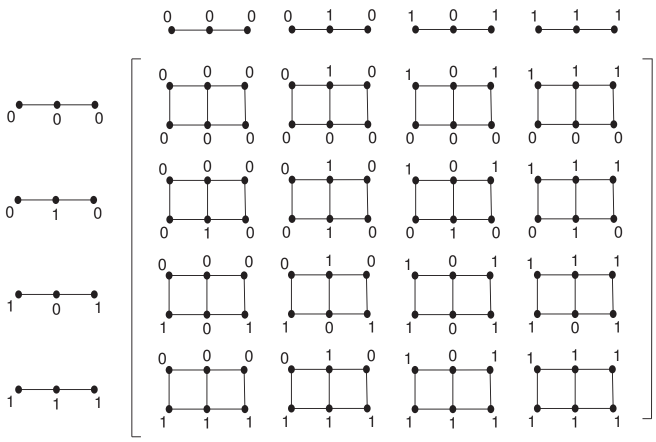
$$\mathbf{X}_{2\times 2} = \begin{bmatrix} y_{1,1} & y_{1,2} & y_{2,1} & y_{2,2} \\ y_{1,3} & y_{1,4} & y_{2,3} & y_{2,4} \\ y_{3,1} & y_{3,2} & y_{4,1} & y_{4,2} \\ y_{3,3} & y_{3,4} & y_{4,3} & y_{4,4} \end{bmatrix} \tag{2.3}$$

and

$$\mathbf{Y}_{2\times 2} = \begin{bmatrix} x_{1,1} & x_{1,2} & x_{2,1} & x_{2,2} \\ x_{1,3} & x_{1,4} & x_{2,3} & x_{2,4} \\ x_{3,1} & x_{3,2} & x_{4,1} & x_{4,2} \\ x_{3,3} & x_{3,4} & x_{4,3} & x_{4,4} \end{bmatrix}. \tag{2.4}$$

The set $\mathbf{C}_{2\times 2} = [c_{i,j}]$, which consists of all x-periodic patterns of period 2 with height 2 can be constructed from $\mathbf{Y}_{2\times 2}$ as follows.

(2.5)



$$= \begin{bmatrix} c_{1,1} & c_{1,2} & c_{1,3} & c_{1,4} \\ c_{2,1} & c_{2,2} & c_{2,3} & c_{2,4} \\ c_{3,1} & c_{3,2} & c_{3,3} & c_{3,4} \\ c_{4,1} & c_{4,2} & c_{4,3} & c_{4,4} \end{bmatrix}.$$

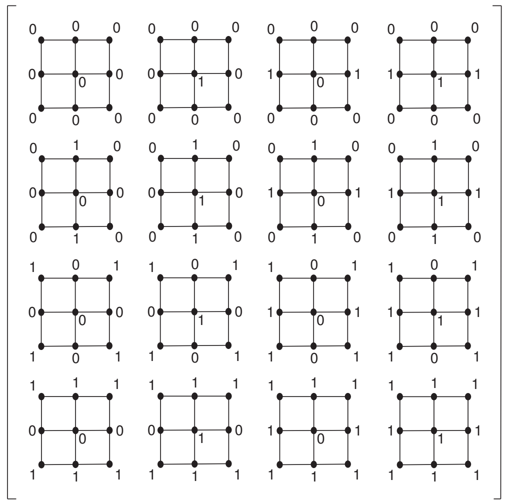
The patterns in $\mathbf{C}_{2\times 2}$ are expressed as elements in $\Sigma_{3\times 2}$ and are understood to be extendable periodically in the x-direction to all of $\mathbb{Z}_{\infty\times 2}$. Notably,

$$\left\{ \begin{array}{lll} c_{1,2} \cong c_{1,3}, & c_{2,1} \cong c_{3,1}, & c_{2,2} \cong c_{3,3}, \\ c_{2,3} \cong c_{3,2}, & c_{2,4} \cong c_{3,4}, & c_{4,2} \cong c_{4,3}, \end{array} \right. \tag{2.6}$$

where $c_{i,j} \cong c_{i',j'}$ means that $c_{i',j'}$ is an x-translation by one step from $c_{i,j}$. Later, the translation invariance property (2.6) will be shown to imply R_2-symmetry of the trace operator $\mathbf{T}_2$.

Finally, $\mathbf{P}_{2\times 2}$ denotes the set of $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$-periodic patterns, which can be recorded from $\mathbf{C}_{2\times 2}$ or $\mathbf{Y}_{2\times 2}$ as an element in $\Sigma_{3\times 3}$ as follows.

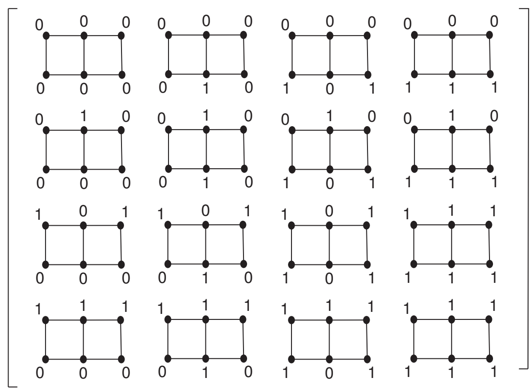
(2.7) $\mathbf{P}_{2\times 2} =$



.

Notably, the upper two rows from the top of each pattern in $\mathbf{P}_{2\times 2}$ is $\mathbf{C}^t_{2\times 2}$, where $\mathbf{C}^t_{2\times 2}$ is the transpose of $\mathbf{C}_{2\times 2}$;

(2.8) $\mathbf{C}^t_{2\times 2} =$



.

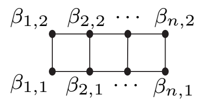
Therefore, $\mathbf{P}_{2\times 2}$ can be regarded as a "Hadamard type product $\bullet$" of $\mathbf{C}_{2\times 2}$ with $\mathbf{C}^t_{2\times 2}$, given by the following construction.

$$\mathbf{P}_{2\times 2} = \mathbf{C}_{2\times 2} \bullet \mathbf{C}^t_{2\times 2}; \tag{2.9}$$

the lower two rows of each pattern in $\mathbf{P}_{2\times 2}$ come from $\mathbf{C}_{2\times 2}$, and the upper two rows come from $\mathbf{C}^t_{2\times 2}$; they are glued together by the middle row. Equation (2.9) is the prototype for constructing doubly periodic patterns of $\mathbb{Z}^2$ from x-periodic patterns. Later, this idea will be generalized to all doubly periodic patterns.

The y-ordering matrices of patterns in $\Sigma_{n\times 2}$, $n \geq 2$, can be ordered analogously by

(2.10) $$\mathbf{Y}_{n\times 2} = [y_{n;i,j}] = \left[\quad \right]_{2^n \times 2^n},$$



where

(2.11) $$\begin{cases} i = \psi(\beta_{1,1}\beta_{2,1}\cdots\beta_{n,1}), \\ j = \psi(\beta_{1,2}\beta_{2,2}\cdots\beta_{n,2}), \end{cases}$$

and the n-th order counting function $\psi \equiv \psi_n : \{0,1\}^{\mathbb{Z}_n} \to \{j | 1 \leq j \leq 2^n\}$ is defined by

(2.12) $$\psi(\beta_1\beta_2\cdots\beta_n) = 1 + \sum_{j=1}^{n} \beta_j 2^{(n-j)}.$$

The recursive formulas for generating $\mathbf{Y}_{n\times 2}$ from $\mathbf{Y}_{2\times 2}$, taken from another investigation [2], is as follows.

Let

(2.13) $$\mathbf{Y}_{n\times 2} = \begin{bmatrix} Y_{n\times 2;1} & Y_{n\times 2;2} \\ Y_{n\times 2;3} & Y_{n\times 2;4} \end{bmatrix},$$

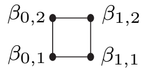
$Y_{n\times 2;i}$ be a $2^{n-1} \times 2^{n-1}$ matrix of patterns. Then,

(2.14) $$\mathbf{Y}_{(n+1)\times 2} = \begin{bmatrix} x_{1,1}Y_{n\times 2;1} & x_{1,2}Y_{n\times 2;2} & x_{2,1}Y_{n\times 2;1} & x_{2,2}Y_{n\times 2;2} \\ x_{1,3}Y_{n\times 2;3} & x_{1,4}Y_{n\times 2;4} & x_{2,3}Y_{n\times 2;3} & x_{2,4}Y_{n\times 2;4} \\ x_{3,1}Y_{n\times 2;1} & x_{3,2}Y_{n\times 2;2} & x_{4,1}Y_{n\times 2;1} & x_{4,2}Y_{n\times 2;2} \\ x_{3,3}Y_{n\times 2;3} & x_{3,4}Y_{n\times 2;4} & x_{4,3}Y_{n\times 2;3} & x_{4,4}Y_{n\times 2;4} \end{bmatrix}$$

is a $2^{n+1} \times 2^{n+1}$ matrix.

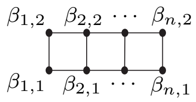
The entries in $\mathbf{Y}_{(n+1)\times 2}$ are explained as follows; if

$$x_{p,q} =$$



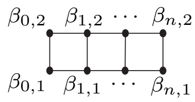
and

$$Y_{n\times 2;q} = \quad ,$$



then

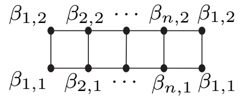
(2.15) $$x_{p,q}Y_{n\times 2;q} =$$



such that the second column of $x_{p,q}$ and the first column of $Y_{n\times 2;q}$ are mutually overlapping.

Hence, x-periodic patterns of period n with height 2 can be expressed in $\Sigma_{(n+1)\times 2}$, and recorded as an element in $\mathbf{C}_{n\times 2}$ by

(2.16) $$\mathbf{C}_{n\times 2} = \left[\quad \right]_{2^n \times 2^n}.$$



Now, given any basic set $\mathcal{B}$, define the associated horizontal and vertical transition matrices $\mathbf{H}_2 = \mathbf{H}_2(\mathcal{B}) = [a_{p,q}]$ and $\mathbf{V}_2 = \mathbf{V}_2(\mathcal{B}) = [x_{i,j}]$ by

(2.17) $$a_{p,q} = \begin{cases} 1 & \text{if} \quad x_{p,q} \in \mathcal{B}, \\ 0 & \text{if} \quad x_{p,q} \notin \mathcal{B}, \end{cases}$$

and

(2.18) $$b_{i,j} = \begin{cases} 1 & \text{if} \quad y_{i,j} \in \mathcal{B}, \\ 0 & \text{if} \quad y_{i,j} \notin \mathcal{B}, \end{cases}$$

respectively.

(2.19) $$\mathbf{H}_2 = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{bmatrix} = \begin{bmatrix} b_{1,1} & b_{1,2} & b_{2,1} & b_{2,2} \\ b_{1,3} & b_{1,4} & b_{2,3} & b_{2,4} \\ b_{3,1} & b_{3,2} & b_{4,1} & b_{4,2} \\ b_{3,3} & b_{3,4} & b_{4,3} & b_{4,4} \end{bmatrix},$$

and

(2.20) $$\mathbf{V}_2 = \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} \\ b_{2,1} & b_{2,2} & b_{2,3} & b_{2,4} \\ b_{3,1} & b_{3,2} & b_{3,3} & b_{3,4} \\ b_{4,1} & b_{4,2} & b_{4,3} & b_{4,4} \end{bmatrix} = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{2,1} & a_{2,2} \\ a_{1,3} & a_{1,4} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{4,1} & a_{4,2} \\ a_{3,3} & a_{3,4} & a_{4,3} & a_{4,4} \end{bmatrix}.$$

The associated column matrices $\widetilde{\mathbf{H}}_2$ of $\mathbf{H}_2$ and $\widetilde{\mathbf{V}}_2$ of $\mathbf{V}_2$ are defined as

(2.21) $$\widetilde{\mathbf{H}}_2 = \begin{bmatrix} a_{1,1} & a_{2,1} & a_{2,1} & a_{2,2} \\ a_{3,1} & a_{4,1} & a_{3,2} & a_{4,2} \\ a_{1,3} & a_{2,3} & a_{1,4} & a_{2,4} \\ a_{3,3} & a_{4,3} & a_{3,4} & a_{4,4} \end{bmatrix}$$

and

(2.22) $$\widetilde{\mathbf{V}}_2 = \begin{bmatrix} b_{1,1} & b_{2,1} & b_{2,1} & b_{2,2} \\ b_{3,1} & b_{4,1} & b_{3,2} & b_{4,2} \\ b_{1,3} & b_{2,3} & b_{1,4} & b_{2,4} \\ b_{3,3} & b_{4,3} & b_{3,4} & b_{4,4} \end{bmatrix},$$

respectively.

The trace operators $\mathbf{T}_2 = \mathbf{T}_2(\mathcal{B})$ and $\widehat{\mathbf{T}}_2 = \widehat{\mathbf{T}}_2(\mathcal{B})$ which were introduced in [3] are defined as

(2.23) $$\mathbf{T}_2 = \mathbf{V}_2 \circ \widetilde{\mathbf{H}}_2 \quad \textit{and} \quad \widehat{\mathbf{T}}_2 = \mathbf{H}_2 \circ \widetilde{\mathbf{V}}_2,$$

where $\circ$ is the Hadamard product: if $A = [\alpha_{i,j}]_{p\times p}$ and $B = [\beta_{i,j}]_{p\times p}$, then $A \circ B = [\alpha_{i,j}\beta_{i,j}]_{p\times p}$. More precisely,

$$\mathbf{T}_2 = [t_{i,j}]_{2^2\times 2^2} = \begin{bmatrix} a_{1,1}a_{1,1} & a_{1,2}a_{2,1} & a_{2,1}a_{1,2} & a_{2,2}a_{2,2} \\ a_{1,3}a_{3,1} & a_{1,4}a_{4,1} & a_{2,3}a_{3,2} & a_{2,4}a_{4,2} \\ a_{3,1}a_{1,3} & a_{3,2}a_{2,3} & a_{4,1}a_{1,4} & a_{4,2}a_{2,4} \\ a_{3,3}a_{3,3} & a_{3,4}a_{4,3} & a_{4,3}a_{3,4} & a_{4,4}a_{4,4} \end{bmatrix} \tag{2.24}$$

and

$$\widehat{\mathbf{T}}_2 = \left[\widehat{t}_{i,j}\right]_{2^2\times 2^2} = \begin{bmatrix} b_{1,1}b_{1,1} & b_{1,2}b_{2,1} & b_{2,1}b_{1,2} & b_{2,2}b_{2,2} \\ b_{1,3}b_{3,1} & b_{1,4}b_{4,1} & b_{2,3}b_{3,2} & b_{2,4}b_{4,2} \\ b_{3,1}b_{1,3} & b_{3,2}b_{2,3} & b_{4,1}b_{1,4} & b_{4,2}b_{,24} \\ b_{3,3}b_{3,3} & b_{3,4}b_{4,3} & b_{4,3}b_{3,4} & b_{4,4}b_{4,4} \end{bmatrix}. \tag{2.25}$$

From (2.5), (2.20) and (2.24), clearly

$$t_{i,j} = \begin{cases} 1 & if \quad c_{i,j} \text{ is } \mathcal{B}\text{-admissible}, \\ 0 & if \quad c_{i,j} \text{ is not } \mathcal{B}\text{-admissible}, \end{cases} \tag{2.26}$$

where $c_{i,j} \in \mathbf{C}_{2\times 2}$.

Therefore, $\mathbf{T}_2$ is the transition matrix of the $\mathcal{B}$-admissible and x-periodic patterns of period 2 with height 2. Similarly, $\widehat{\mathbf{T}}_2$ is the transition matrix of $\mathcal{B}$-admissible and y-periodic patterns of period 2 with width 2.

The translation invariance property (2.6) of $\mathbf{C}_{2\times 2}$ implies the following symmetry of $\mathbf{T}_2$;

$$\begin{cases} t_{1,2} = t_{1,3}, & t_{2,1} = t_{3,1}, & t_{2,2} = t_{3,3}, \\ t_{2,3} = t_{3,2}, & t_{2,4} = t_{3,4}, & t_{4,2} = t_{4,3}. \end{cases} \tag{2.27}$$

The symmetry of (2.6) or (2.27) can also be identified as the rotational symmetry of a cylinder since elements in $\mathbf{C}_{2\times 2}$ can be regarded as cylindrical patterns.

The recursive formulas of $\mathbf{Y}_{n\times 2}$ can also be applied to $\mathbf{V}_n$. Indeed, if

$$\mathbf{V}_n = \begin{bmatrix} V_{n;1} & V_{n;2} \\ V_{n;3} & V_{n;4} \end{bmatrix}_{2^n\times 2^n},$$

where $V_{n;j}$ is a $2^{n-1} \times 2^{n-1}$ matrix, then

$$\mathbf{V}_{n+1} = \begin{bmatrix} a_{1,1}V_{n;1} & a_{1,2}V_{n;2} & a_{2,1}V_{n;1} & a_{2,2}V_{n;2} \\ a_{1,3}V_{n;3} & a_{1,4}V_{n;4} & a_{2,3}V_{n;3} & a_{2,4}V_{n;4} \\ a_{3,1}V_{n;1} & a_{3,2}V_{n;2} & a_{4,1}V_{n;1} & a_{4,2}V_{n;2} \\ a_{3,3}V_{n;3} & a_{3,4}V_{n;4} & a_{4,3}V_{n;3} & a_{4,4}V_{n;4} \end{bmatrix} \tag{2.28}$$

with

$$V_{n+1;i} = \begin{bmatrix} a_{i,1}V_{n;1} & a_{i,2}V_{n;2} \\ a_{i,3}V_{n;3} & a_{i,4}V_{n;4} \end{bmatrix}. \tag{2.29}$$

The n-th order trace operator $\mathbf{T}_n$ is defined as

$$\mathbf{T}_n = \mathbf{V}_n \circ \begin{bmatrix} E_{2^{n-2}} \otimes \begin{bmatrix} a_{1,1} & a_{2,1} \\ a_{3,1} & a_{4,1} \end{bmatrix} & E_{2^{n-2}} \otimes \begin{bmatrix} a_{1,2} & a_{2,2} \\ a_{3,2} & a_{4,2} \end{bmatrix} \\ E_{2^{n-2}} \otimes \begin{bmatrix} a_{1,3} & a_{2,3} \\ a_{3,3} & a_{4,3} \end{bmatrix} & E_{2^{n-2}} \otimes \begin{bmatrix} a_{1,4} & a_{2,4} \\ a_{3,4} & a_{4,4} \end{bmatrix} \end{bmatrix}, \tag{2.30}$$

where $\otimes$ is the Kroncker (tensor) product and E_j is the $j \times j$ full matrix.

Now, $\mathbf{T}_n$ represents the transition matrix of $\mathcal{B}$-admissible x-periodic patterns of period n with height 2. Similarly, $\widehat{\mathbf{T}}_n$ represents the transition matrix of $\mathcal{B}$-admissible y-periodic patterns of period n with width 2.

2.2. **Rotational matrices.** In this subsection, the rotational matrices R_n and the invariance property of $\mathbf{C}_{n\times 2}$ under R_n are investigated and the R_n-symmetry of $\mathbf{T}_n$ is then proven.

The shift of any n-sequence $\overline{\beta} = (\beta_1\beta_2\cdots\beta_{n-1}\beta_n)$, $n \geq 2$, $\beta_j \in \{0,1\}$, is defined by

$$\sigma((\beta_1\beta_2\cdots\beta_{n-1}\beta_n)) \equiv \sigma_n((\beta_1\beta_2\cdots\beta_{n-1}\beta_n)) = (\beta_2\beta_3\cdots\beta_n\beta_1). \tag{2.31}$$

The subscript of σ_n is omitted for brevity. Notably, the shift (to the left) of any one-dimensional periodic sequence $(\beta_1\beta_2\cdots\beta_n\beta_1\cdots)$ of period n becomes $(\beta_2\beta_3\cdots\beta_n\beta_1\beta_2\cdots)$.

The $2^n \times 2^n$ rotational matrix $R_n = [R_{n;i,j}], R_{n;i,j} \in \{0,1\}$, is defined by

$$R_{n;i,j} = 1 \quad \text{if and only if}$$

$$i = \psi(\beta_1\beta_2\cdots\beta_n) \quad \text{and} \quad j = \psi(\sigma(\beta_1\beta_2\cdots\beta_n)) = \psi(\beta_2\beta_3\cdots\beta_n\beta_1). \tag{2.32}$$

From (2.32), for convenience, denote by

$$j = \sigma(i). \tag{2.33}$$

Clearly, R_n is a permutation matrix: each row and column of R_n has one and only one element with a value of 1. Indeed, R_n can be written explicitly as follows.

Lemma 2.1.

$$\begin{cases} R_{n;i,2i-1} = 1 \textit{ and } R_{n;2^{n-1}+i,2i} = 1 & \textit{for } 1 \leq i \leq 2^{n-1}, \\ R_{n;i,j} = 0 & \textit{otherwise}, \end{cases} \tag{2.34}$$

or equivalently,

$$\sigma(i) \equiv \sigma_n(i) = \begin{cases} 2i-1 & \textit{for} \quad 1 \leq i \leq 2^{n-1}, \\ 2(i-2^{n-1}) & \textit{for} \quad 1+2^{n-1} \leq i \leq 2^n. \end{cases} \tag{2.35}$$

Furthermore, $R_n^n = I_{2^n}$ *and for any* $1 \leq j \leq n-1$,

$$(R_n^j)_{i,\sigma^j(i)} = 1. \tag{2.36}$$

Proof. Clearly,

$$\psi(\beta_1\beta_2\cdots\beta_n) \leq 2^{n-1} \quad \text{if} \quad \beta_1 = 0,$$

and

$$\psi(\beta_1\beta_2\cdots\beta_n) \geq 1 + 2^{n-1} \quad \text{if} \quad \beta_1 = 1.$$

From (2.12),

$$\psi(\beta_2\beta_3\cdots\beta_n\beta_1) = \begin{cases} 2i-1 & \text{if} \quad \beta_1 = 0, \\ 2(i-2^{n-1}) & \text{if} \quad \beta_1 = 1, \end{cases}$$

can be verified. Equations (2.34) and (2.35) follow.

Finally, (2.36) follows easily from (2.32) and (2.33). □

The equivalent class $C_n(i)$ of i is defined by

$$\begin{aligned} C_n(i) = & \quad \{\sigma^j(i) | 0 \leq j \leq n-1\} \\ = & \quad \left\{ j \middle| \left(R_n^l\right)_{i,j} = 1 \text{ for some } 1 \leq l \leq n \right\}. \end{aligned} \tag{2.37}$$

Clearly, either $C_n(i) = C_n(j)$ or $C_n(i) \cap C_n(j) = \emptyset$. Let i be the smallest element in its equivalent class, and the index set $\mathcal{I}_n$ of n is defined by

$$\begin{aligned} \mathcal{I}_n = & \quad \{i | 1 \leq i \leq 2^n, i \leq \sigma^q(i), 1 \leq q \leq n-1\} \\ = & \quad \left\{ i \middle| 1 \leq i \leq 2^n, i \leq j \text{ for all } j \in C_n(i) \right\}. \end{aligned} \tag{2.38}$$

Therefore, for each $n \geq 1$, $\{j | 1 \leq j \leq 2^n\} = \underset{i \in \mathcal{I}_n}{\cup} C_n(i)$. The cardinal number of $\mathcal{I}_n$ is denoted by χ_n. Notably, χ_n can be identified as the number of necklaces that can be made from n beads of two colors, when the necklaces can be rotated but not turned over [34]. χ_n is expressed as

$$\chi_n = \frac{1}{n} \sum_{d|n} \phi(d) 2^{n/d} \tag{2.39}$$

where $\phi(n)$ is the Euler totient function, which counts the numbers smaller or equal to n and prime relative to n,

$$\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right). \tag{2.40}$$

For $n = 2, 3$ and 4, R_n and $C_n(i)$ are as follows.

Example 2.2. *R_n, $\mathcal{I}_n$ and $C_n(i)$ for $n = 2, 3$ and 4*

(i) $R_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$,

and $\begin{cases} C_2(1) = \{1\}, & 1 \to 1, \\ C_2(2) = C_2(3) = \{2, 3\}, & 2 \to 3 \to 2, \\ C_2(4) = \{4\}, & 4 \to 4, \\ \mathcal{I}_2 = \{1, 2, 4\}. & \end{cases}$

(ii) $R_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$,

and $\begin{cases} C_3(1) = \{1\}, & 1 \to 1, \\ C_3(2) = \{2,3,5\}, & 2 \to 3 \to 5 \to 2, \\ C_3(4) = \{4,7,6\}, & 4 \to 7 \to 6 \to 4, \\ C_3(8) = \{8\}, & 8 \to 8, \\ \mathcal{I}_3 = \{1,2,4,8\}. & \end{cases}$

(iii) *For* R_4, $\begin{cases} 1 \to 1, \\ 2 \to 3 \to 5 \to 9 \to 2, \\ 4 \to 7 \to 13 \to 10 \to 4, \\ 6 \to 11 \to 6, \\ 8 \to 15 \to 14 \to 12 \to 8, \\ 16 \to 16, \\ \mathcal{I}_4 = \{1,2,4,6,8,16\}. \end{cases}$

The following proposition shows the permutation character of R_n.

Proposition 2.3. *Let* $\mathbf{M} = [M_{i,j}]_{2^n \times 2^n}$ *be a matrix where* $M_{i,j}$ *denotes a number or a pattern or a set of patterns. Then,*

$$(R_n\mathbf{M})_{i,j} = M_{\sigma(i),j} \text{ and } (\mathbf{M}R_n)_{i,j} = M_{i,\sigma^{-1}(j)}. \tag{2.41}$$

Furthermore, for any $l \geq 1$,

$$(R_n^l\mathbf{M})_{i,j} = M_{\sigma^l(i),j} \text{ and } (\mathbf{M}R_n^l)_{i,j} = M_{i,\sigma^{-l}(j)}. \tag{2.42}$$

Proof. For any $1 \leq i,j \leq 2^n$, by (2.36),

$$\begin{aligned} (R_n\mathbf{M})_{i,j} &= \sum_q R_{n;i,q}M_{q,j} \\ &= R_{n;i,\sigma(i)}M_{\sigma(i),j} \\ &= M_{\sigma(i),j}. \end{aligned}$$

Similarly,

$$\begin{aligned} (\mathbf{M}R_n)_{i,j} &= \sum_q M_{i,q}R_{n;q,j} \\ &= M_{i,\sigma^{-1}(j)}R_{n;\sigma^{-1}(j),j} \\ &= M_{i,\sigma^{-1}(j)}. \end{aligned}$$

Applying (2.41) l times yields (2.42).

The proof is complete. □

In the following, x-periodic patterns of period n with height $k \geq 1$ are studied. More notation is required.

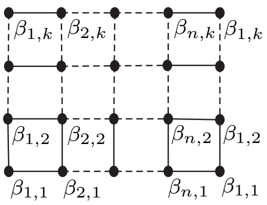
Definition 2.4.

(i) *For any* $n \geq 1$, *let* $(\beta_1\beta_2\cdots\beta_n)^\infty$ *be a periodic sequence of period n, denoted by* $\overline{\beta} = (\beta_1\cdots\beta_n)$. $\sigma(\overline{\beta}) = \sigma((\beta_1\beta_2\cdots\beta_n)) = (\beta_2\beta_3\cdots\beta_n\beta_1)$. *For any fixed* $n \geq 1$ *and any* $j \geq 1$, *denote by* $\overline{\beta}_j = (\beta_{1,j}\beta_{2,j}\cdots\beta_{n,j})$ *a periodic sequence of period n.*

(ii) *For fixed* $n \geq 1$ *and any* $k \geq 1$, *denote by*

$$[\overline{\beta}_1\overline{\beta}_2\cdots\overline{\beta}_k] = (\beta_{1,1}\beta_{2,1}\cdots\beta_{n,1})^{\infty} \oplus (\beta_{1,2}\beta_{2,2}\cdots\beta_{n,2})^{\infty} \oplus \cdots (\beta_{1,k}\beta_{2,k}\cdots\beta_{n,k})^{\infty}$$

$=$



a x-periodic pattern of period n with height k.

(iii) *A Hadamard type product* $\bullet$ *of patterns is defined as follows.*

$$[\overline{\beta}_1\overline{\beta}_2] \bullet [\overline{\beta}_2\overline{\beta}_3] = [\overline{\beta}_1\overline{\beta}_2\overline{\beta}_3]$$

and

$$[\overline{\beta}_1\overline{\beta}_2\cdots\overline{\beta}_k] = [\overline{\beta}_1\overline{\beta}_2] \bullet [\overline{\beta}_2\overline{\beta}_3] \bullet \cdots \bullet [\overline{\beta}_{k-1}\overline{\beta}_k].$$

(iv) *A* $2^n \times 2^n$ *ordering matrix* $\mathbf{C}_{n\times k} = [C_{n\times k;i,j}]$ *of x-periodic patterns of period n with height* $k \geq 2$ *is defined by*

$$C_{n\times k;i,j} = \{[\overline{\beta}_1\overline{\beta}_2\cdots\overline{\beta}_k] | \psi(\overline{\beta}_1) = i \text{ and } \psi(\overline{\beta}_k) = j\}.$$

(v) *For* $n \geq 1$ *and* $k \geq 2$, *denote by* $\mathbf{D}_{n,k}$ *the ordering matrix of patterns, which consists of a first row* $\overline{\beta}_1$ *and the k-th row* $\overline{\beta}_k$ *of* $\mathbf{C}_{n\times k}$:

$$D_{n,k;i,j} = \{[\overline{\beta}_1\overline{\beta}_k] | [\overline{\beta}_1\overline{\beta}_2\cdots\overline{\beta}_k] \in C_{n\times k},\ \psi(\overline{\beta}_1) = i \text{ and } \psi(\overline{\beta}_k) = j\}.$$

Some remarks should be made.

Remark 2.5.

(1) *For any* $n \geq 1$, *the length of* $\overline{\beta}$ *in (i) and* $\overline{\beta}_j$ *in (ii) depends on* n. *For simplicity, these dependencies are omitted.*

(2) *The product* $\bullet$ *defined in (iii) applies only when the top row of the first pattern is identical to the first row of the second pattern.*

(3) *In (iv), when* $k = 2$, *(2.16) applies.*

(4) $C_{n\times k;i,j}$ *is a set of patterns with the same first and k-th rows.* $\mathbf{D}_{n,k}$ *is exactly* $\mathbf{C}_{n\times 2}$, *but, importantly, in* $\mathbf{C}_{n\times k}$, *all patterns in the entry* $C_{n\times k;i,j}$ *have the same top and first rows, which can be used to construct y-periodic patterns with a shift in the (k+1)-th row.*

In the following lemma, R_n is used to shift the first row in $\mathbf{D}^t_{n,k}$.

Lemma 2.6. *Let* $i = \psi(\overline{\beta}_1)$ *and* $j = \psi(\overline{\beta}_k)$. *Then*

(i) $(R_n\mathbf{D}^t_{n,k})_{i,j} = [\overline{\beta}_k\sigma(\overline{\beta}_1)]$,

(ii) $(\mathbf{C}_{n\times k} \bullet R_n\mathbf{D}^t_{n,k})_{i,j} = [\overline{\beta}_1\overline{\beta}_2\cdots\overline{\beta}_k\sigma(\overline{\beta}_1)]$.

Proof. (i) follows easily from Proposition 2.3 and part (v) of Definition 2.4. From parts (i) and (iii) of Definition 2.4, a product in (ii) is legitimate since the top row of $\mathbf{C}_{n\times k}$ and the first row of $R_nD^t_{n,k}$ are $\overline{\beta}_k$, and (ii) follows from (i). □

Furthermore, the following result shows that the patterns in $\mathbf{C}_{n\times k} \bullet R^l_n\mathbf{D}^t_{n,k}$ are the same as the patterns in $diag(\mathbf{C}_{n\times(k+1)}R^{n-l}_n)$ where $diag(\mathbf{M})$ is the diagonal part of $\mathbf{M}$, such that $diag(\mathbf{M}) = I \circ \mathbf{M}$. They are important in constructing y-periodic patterns.

Proposition 2.7. *For any $n \geq 2$, $k \geq 1$ and $0 \leq l \leq n$,*

$$\begin{aligned} \textit{patterns in } \mathbf{C}_{n\times k} \bullet R_n^l \mathbf{D}_{n,k}^t = & \ \textit{patterns in } diag(\mathbf{C}_{n\times(k+1)} R_n^{n-l}) \\ = & \ \{[\overline{\beta}_1 \cdots \overline{\beta}_k \sigma^l(\overline{\beta}_1)] | [\overline{\beta}_1 \cdots \overline{\beta}_k] \in \mathbf{C}_{n\times k}\}. \end{aligned}$$

Proof. By (2.42), for any $0 \leq l \leq n-1$, $1 \leq i,j \leq 2^{n+1}$,

$$(\mathbf{C}_{n\times(k+1)} R_n^{n-l})_{i,j} = \{[\overline{\beta}_1 \cdots \overline{\beta}_k \sigma^{l-n}(\overline{\beta}_{k+1})] : \psi(\overline{\beta}_1) = i \text{ and } \psi(\overline{\beta}_{k+1}) = j\}.$$

Since $\psi(\overline{\beta}_{k+1}) = \psi(\overline{\beta}_1) = i$ implies $\overline{\beta}_{k+1} = \overline{\beta}_1$,

$$(\mathbf{C}_{n\times(k+1)} R_n^{n-l})_{i,i} = \{[\overline{\beta}_1 \cdots \overline{\beta}_k \sigma^{l-n}(\overline{\beta}_1)] : \psi(\overline{\beta}_1) = i\}.$$

However, for any $1 \leq i,j \leq 2^n$, part (ii) of Lemma 2.6 implies

$$(\mathbf{C}_{n\times k} \bullet R_n^l D_{n,k}^t)_{i,j} = [\overline{\beta}_1 \overline{\beta}_2 \cdots \overline{\beta}_k \sigma^l(\overline{\beta}_1)].$$

Now, for any $0 \leq l \leq n-1$ and $\overline{\beta} = (\beta_1 \cdots \beta_n)$,

$$\sigma^l(\overline{\beta}) = \sigma^{l-n}(\overline{\beta}).$$

The proof is complete. □

The rotational symmetry of $\mathbf{T}_n$ is determined by studying $\mathbf{C}_{n\times 2}$ in more detail. Given a basic admissible set $\mathcal{B} \subset \Sigma_{2\times 2}$, $\mathbf{T}_n$ is defined by (2.30). Let $[\overline{\beta}_1 \overline{\beta}_2] \in \mathbf{C}_{n\times 2}$, for $1 \leq j \leq n$, denote

$$p_j = 2\beta_{j,1} + \beta_{j,2} + 1,$$

then the associated entry in $\mathbf{T}_n$ is

$$\mathbf{T}_n([\overline{\beta}_1 \overline{\beta}_2]) \equiv a_{p_1,p_2} a_{p_2,p_3} \cdots a_{p_n,p_1}. \tag{2.43}$$

$[\overline{\beta}_1 \overline{\beta}_2]$ is $\mathcal{B}$-admissible if and only if $a_{p_j,p_{j+1}} = 1$ for all $1 \leq j \leq n$, where $p_{n+1} = p_1$.

Theorem 2.8. *For any $n \geq 2$, the trace operator $\mathbf{T}_n = [t_{n;i,j}]_{2^n\times 2^n}$ has the following R_n-symmetry:*

$$t_{n;\sigma^l(i),\sigma^l(j)} = t_{n;i,j} \tag{2.44}$$

for all $1 \leq i,j \leq 2^n$ and $0 \leq l \leq n-1$.

Proof. Given $[\overline{\beta}_1 \overline{\beta}_2] \in \mathbf{C}_{n\times 2}$, all $[\sigma^l(\overline{\beta}_1)\sigma^l(\overline{\beta}_2)]$, $0 \leq l \leq n-1$, represent similar x-periodic patterns. The entry of $[\sigma^l(\overline{\beta}_1)\sigma^l(\overline{\beta}_2)]$ in $\mathbf{T}_n$ is

$$\mathbf{T}_n([\sigma^l(\overline{\beta}_1)\sigma^l(\overline{\beta}_2)]) = a_{p_{l+1}p_{l+2}} a_{p_{l+2}p_{l+3}} \cdots a_{p_n p_1} a_{p_1 p_2} \cdots a_{p_l p_{l+1}}. \tag{2.45}$$

Comparing (2.43) with (2.45) clearly reveals that

$$\mathbf{T}_n([\overline{\beta}_1 \overline{\beta}_2]) = \mathbf{T}_n([\sigma^l(\overline{\beta}_1)\sigma^l(\overline{\beta}_2)]) \tag{2.46}$$

for all $0 \leq l \leq n-1$. Additionally, if $\mathbf{T}_n = [t_{n;i,j}]$ with $i = \psi(\overline{\beta}_1)$ and $j = \psi(\overline{\beta}_2)$, then (2.46) implies

$$t_{n;\sigma^l(i),\sigma^l(j)} = t_{n;i,j} \text{ for all } 0 \leq l \leq n-1.$$

The proof is complete. □

Proposition 2.7 and Theorem 2.8 yield the following theorem.

Theorem 2.9. *For any $n \geq 2$ and $k \geq 2$, $0 \leq l \leq n-1$,*

$$|\mathbf{T}_n^{k-1} \circ R_n^l \mathbf{T}_n^t| = tr(\mathbf{T}_n^k R_n^{n-l}) \tag{2.47}$$

and

$$|\mathbf{T}_n^{k-1} \circ \mathbf{R}_n \mathbf{T}_n^t| = tr(\mathbf{T}_n^k \mathbf{R}_n), \tag{2.48}$$

where

$$\mathbf{R}_n = \sum_{l=0}^{n-1} R_n^l. \tag{2.49}$$

Proof. From Proposition 2.7, (2.43) and the properties of $\mathbf{T}_n$, (2.47) follows. Equations (2.47) and (2.49) yield (2.48). The proof is complete. □

2.3. **Periodic patterns.** This subsection studies in detail (double) periodic patterns in $\mathbb{Z}^2$. Indeed, consider a lattice L with Hermite normal form,

$$L = \begin{bmatrix} n & l \\ 0 & k \end{bmatrix} \mathbb{Z}^2, \tag{2.50}$$

where $n \geq 1$, $k \geq 1$ and $0 \leq l \leq n-1$.

A pattern $U = (\alpha_{i,j})_{i,j \in \mathbb{Z}}$ is called L-periodic if every $i, j \in \mathbb{Z}$

$$\alpha_{i+np+lq, j+kq} = \alpha_{i,j} \tag{2.51}$$

for all $p, q \in \mathbb{Z}$.

The periodicity of $\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}$ and $\begin{bmatrix} n & 0 \\ 0 & k' \end{bmatrix}$ are closely related as follows.

Proposition 2.10. *For any $n \geq 2$, $k \geq 1$ and $0 \leq l \leq n-1$, $\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}$-periodic patterns are $\begin{bmatrix} n & 0 \\ 0 & \frac{nk}{(n,l)} \end{bmatrix}$-periodic where (n, l) is the greatest common divisor (GCD) of n and l.*

Proof. By (2.51), the $\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}$-periodic pattern is easily identified as $\begin{bmatrix} n & l \cdot m \\ 0 & k \cdot m \end{bmatrix}$-periodic for all $m \in \mathbb{N}$.

By taking $m = \frac{n}{(n,l)}$, the result holds. □

Given an admissible set $\mathcal{B} \subset \Sigma_{2\times 2}$, defined on square lattice $\mathbb{Z}_{2\times 2}$, the periodic patterns that are $\mathcal{B}$-admissible must be verified on $\mathbb{Z}_{2\times 2}$.

Let $\mathbb{Z}_{2\times 2}((i,j))$ be the square lattice with the left-bottom vertex (i, j):

$$\mathbb{Z}_{2\times 2}((i,j)) = \{(i,j), (i+1,j), (i,j+1), (i+1,j+1)\}.$$

Now, the admissibility is demonstrated to have to be verified on finite square lattices.

Proposition 2.11. *An L-periodic pattern U is $\mathcal{B}$-admissible if and only if*

$$U\big|_{\mathbb{Z}_{2\times 2}((i,j))} \in \mathcal{B} \tag{2.52}$$

for any $0 \leq i \leq n-1$ and $0 \leq j \leq k-1$.

Proof. The proof follows easily from (2.51). The details are left to the reader. □

Suppose $U = (\alpha_{i,j})_{i,j\in\mathbb{Z}}$ is an L-periodic pattern. For convenience, let

$$\beta_{i+1,j+1} = \alpha_{i,j}$$

for all $i, j \in \mathbb{Z}$.

According to proposition 2.11, the admissibility of U is determined by

$$(\alpha_{i,j})_{0\leq i\leq n, 0\leq j\leq k} = (\beta_{i,j})_{1\leq i\leq n+1, 1\leq j\leq k+1},$$

and $(\beta_{i,j})_{1\leq i\leq n+1,1\leq j\leq k+1}$ with the periodic property (2.51). Therefore, the following theorem can be obtained.

Theorem 2.12. *Given a basic admissible set $\mathcal{B} \subset \Sigma_{2\times 2}$, an L-periodic pattern U is $\mathcal{B}$-admissible if and only if*

$$[\overline{\beta}_1\overline{\beta}_2\cdots\overline{\beta}_k] \text{ and } [\overline{\beta}_k\sigma^{n-l}(\overline{\beta}_1)] \text{ are } \mathcal{B}\text{-admissible.} \tag{2.53}$$

Theorems 2.7 and 2.12 yield the following main results.

Theorem 2.13. *For $n \geq 1$, $0 \leq l \leq n-1$ and $k \geq 1$, denote by $\Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\right)$ the cardinal number of the set of $\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}$-periodic and $\mathcal{B}$-admissible patterns. For $n \geq 2$, $0 \leq l \leq n-1$ and $k \geq 2$,*

$$\Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\right) = tr\left(\mathbf{T}_n^k R_n^l\right) = |\mathbf{T}_n^{k-1} \circ R_n^{n-l}\mathbf{T}_n^t| \tag{2.54}$$

and

$$\sum_{l=0}^{n-1}\Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\right) = tr\left(\mathbf{T}_n^k \mathbf{R}_n\right) = |\mathbf{T}_n^{k-1} \circ \mathbf{R}_n\mathbf{T}_n^t|. \tag{2.55}$$

For $n \geq 2$ and $0 \leq l \leq n-1$,

$$\Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & 1 \end{bmatrix}\right) = tr(\mathbf{T}_n R_n^l) = |diag(\mathbf{T}_n) \circ R_n^{n-l}\mathbf{T}_n^t| \tag{2.56}$$

and

$$\sum_{l=0}^{n-1}\Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & 1 \end{bmatrix}\right) = tr(\mathbf{T}_n \mathbf{R}_n) = |diag(\mathbf{T}_n) \circ \mathbf{R}_n\mathbf{T}_n^t|. \tag{2.57}$$

Furthermore, let

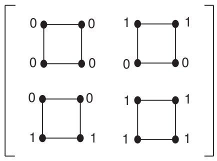
$$\mathbf{T}_1 = \begin{bmatrix} a_{1,1}a_{1,1} & a_{2,2}a_{2,2} \\ a_{3,3}a_{3,3} & a_{4,4}a_{4,4} \end{bmatrix} \text{ and } R_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \tag{2.58}$$

then

$$\Gamma_{\mathcal{B}}\left(\left[\begin{array}{cc}1 & 0\\ 0 & k\end{array}\right]\right)=tr(\mathbf{T}_1^k). \tag{2.59}$$

Proof. By Proposition 2.7, Theorem 2.12 and the construction of $\mathbf{T}_n$, the results (2.54) to (2.57) hold for $n\geq 2$, $0\leq l\leq n-1$ and $k\geq 1$.

For $n=1$, define

(2.60) $\mathbf{C}_{1\times 2}=$  ,

which is the collection of x-periodic patterns of period 1 with height 2. Then $\mathcal{B}$-admissible patterns of $\mathbf{C}_{1\times 2}$ are represented by $\mathbf{T}_1$ as defined in (2.58).

Theorem 2.12 and the construction of $\mathbf{T}_1$ easily yields (2.59).

The proof is complete. □

The n-th order zeta function $\zeta_n(s)$ can now be obtained as follows.

Theorem 2.14. *For $n\geq 1$,*

$$\zeta_n(s)=\exp\left(\frac{1}{n}\sum_{k=1}^{\infty}\frac{1}{k}tr(\mathbf{T}_n^k\mathbf{R}_n)s^{kn}\right). \tag{2.61}$$

Proof. The results follow from Theorem 2.13.

The proof is complete. □

3. Rationality of ζ_n

This section proves that ζ_n is a rational function, as specified by (1.21). To elucidate the method, the symmetric $\mathbf{T}_n$ is considered initially. For any $n\geq 1$, let

$$N=2^n. \tag{3.1}$$

Let λ_j be an eigenvalue of $\mathbf{T}_n$:

$$\mathbf{T}_nU_j=\lambda_jU_j, \tag{3.2}$$

$1\leq j\leq N$. If $\mathbf{T}_n$ is symmetric, then the Jordan form of $\mathbf{T}_n$ [12] is

$$\mathbf{T}_n=\mathbf{U}\mathbf{J}\mathbf{U}^t, \tag{3.3}$$

where

$$\mathbf{U}^t=\mathbf{U}^{-1}. \tag{3.4}$$

The eigen-matrix $\mathbf{U}$ in (3.3) is defined by

$$\mathbf{U}=[U_1,U_2,\cdots,U_N]_{N\times N}=[u_{i,j}]_{N\times N}, \tag{3.5}$$

where $U_j=(u_{1,j},u_{2,j},\cdots,u_{N,j})^t$ is the j-th (column) eigenvector, and

$$\mathbf{J} = diag(\lambda_1, \lambda_2, \cdots, \lambda_N). \tag{3.6}$$

λ_j can be arranged such that $\lambda_1 \geq |\lambda_2| \geq \cdots \geq |\lambda_N|$. Equation (3.4) implies

$$\sum_{p=1}^{N} u_{i,p}u_{j,p} = \delta_{i,j} \text{ and } \sum_{q=1}^{N} u_{q,i}u_{q,j} = \delta_{i,j}. \tag{3.7}$$

Now, Theorem 3.1 will be proven.

Theorem 3.1. *Assume* $\mathbf{T}_n$ *is symmetric; then*

$$\frac{1}{n}\sum_{l=0}^{n-1}\Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\right) = \frac{1}{n} tr\left(\mathbf{T}_n^k \mathbf{R}_n\right) = \sum_{\lambda\in\Sigma(\mathbf{T}_n)} \chi(\lambda)\lambda^k, \tag{3.8}$$

where $\Sigma(\mathbf{T}_n)$ *is the spectrum of* $\mathbf{T}_n$*,*

$$\chi(\lambda) = \sum_{\lambda_j=\lambda} \chi(\lambda_j) \tag{3.9}$$

and

$$\begin{aligned} \chi(\lambda_j) &= \tfrac{1}{n}|\mathbf{R}_n \circ U_j U_j^t| \\ &= \tfrac{1}{n}\sum_{i\in\mathcal{I}_n} \tfrac{\omega_{n,i}}{n}\left(\sum_{l=0}^{n-1} u_{\sigma^l(i),j}\right)^2, \end{aligned} \tag{3.10}$$

where $\omega_{n,i}$ *is the cardinal number of* $C_n(i)$*. Moreover,*

$$\zeta_n(s) = \prod_{\lambda\in\Sigma(\mathbf{T}_n)} (1-\lambda s^n)^{-\chi(\lambda)}. \tag{3.11}$$

Proof. Clearly,

$$\begin{aligned} & tr\left(\mathbf{T}_n^k \mathbf{R}_n\right) \\ = \; & tr\left(\mathbf{U} diag(\lambda_j)\mathbf{U}^t\mathbf{R}_n\right) \\ = \; & \sum_{j=1}^{N}\left\{\sum_{i=1}^{N} u_{i,j}\sum_{p=1}^{N} u_{p,j}\left(\sum_{l=1}^{n-1} R_{n;p,i}^l\right)\right\}\lambda_j. \end{aligned}$$

For each j, $1 \leq j \leq N$,

$$\begin{aligned} & \sum_{i=1}^{N} u_{i,j}\left(\sum_{p=1}^{N} u_{p,j}\sum_{l=0}^{n-1} R_{n;p,i}^l\right) \\ = \; & \sum_{i=1}^{N} u_{i,j}\left(\sum_{l=0}^{n-1} u_{\sigma^{-l}(i),j}\right) \\ = \; & \sum_{i\in\mathcal{I}_n}\tfrac{\omega_{n,i}}{n}\left(\sum_{l=0}^{n-1} u_{\sigma^l(i),j}\right)\left(\sum_{l=0}^{n-1} u_{\sigma^{-l}(i),j}\right) \\ = \; & \sum_{i\in\mathcal{I}_n}\tfrac{\omega_{n,i}}{n}\left(\sum_{l=0}^{n-1} u_{\sigma^l(i),j}\right)^2. \end{aligned}$$

The following is easily verified;

$$|\mathbf{R}_n \circ U_j U_j^t| = \sum_{i\in\mathcal{I}_n} \frac{\omega_{n,i}}{n} \left(\sum_{l=0}^{n-1} u_{\sigma^l(i),j} \right)^2 . \tag{3.12}$$

Then, (3.8)∼(3.10) follow.
From [9],

$$\sum_{k=1}^{\infty} \frac{1}{k} \mathbf{J}^k s^{kn} = diag\left(\log(1-\lambda_j s^n)^{-1}\right). \tag{3.13}$$

Therefore, (3.11) holds.
The proof is complete. □

We now extend Theorem 3.1 to general $\mathbf{T}_n$. In this case, the Jordan form for $\mathbf{T}_n$ is

$$\mathbf{T}_n = \mathbf{U}\mathbf{J}\mathbf{U}^{-1}, \tag{3.14}$$

where $\mathbf{U}$ is given as(3.5) and U_j, $1 \le j \le N$, is an eigenvector or generalized eigenvector [9, 12]. Denote by

$$\mathbf{U}^{-1} = [w_{i,j}] = [W_1; W_2; \cdots ; W_N]_{N\times N} \tag{3.15}$$

with $W_i = (w_{i1}, w_{i2}, \cdots , w_{iN})$, the i-th row vector.

$$\mathbf{J} = diag(J_1, J_2, \cdots , J_Q), \tag{3.16}$$

where J_q is the Jordan block, $1 \le q \le Q$:

$$J_q = \begin{bmatrix} \lambda_q & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_q & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_q & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_q \end{bmatrix}_{M_q\times M_q}, \tag{3.17}$$

$M_q \ge 1$.
As is well-known [9], for any Jordan block

$$J = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}_{M\times M} \tag{3.18}$$

and

$$\log(I-tJ)=\begin{bmatrix}\mu_{1,1} & \mu_{1,2} & \mu_{1,3} & \cdots & \mu_{1,M}\\ 0 & \mu_{2,2} & \mu_{2,3} & \cdots & \mu_{2,M}\\ & & \ddots & & \vdots\\ & & & & \\ 0 & 0 & \cdots & 0 & \mu_{M,M}\end{bmatrix}, \tag{3.19}$$

where

$$\mu_{i,i+j-1}=\mu_{1,j} \quad \text{for } 1\le j\le M \text{ and } 1\le i\le M+1-j, \tag{3.20}$$

and

$$\mu_{i,j}=0 \quad \text{if } i>j. \tag{3.21}$$

In particular, $1\le i\le M$,

$$\mu_{i,i}=\log(1-\lambda t). \tag{3.22}$$

Therefore,

$$\begin{aligned}&\sum_{k=1}^{\infty}\tfrac{1}{k}\mathbf{J}^k s^{kn}\\ =\ & -\log(I-s^n\mathbf{J})\\ =\ & -diag\,(\log(I-s^nJ_1),\cdots,\log(I-s^nJ_Q))\\ =\ & -[\mu_{i,j}]_{N\times N},\end{aligned} \tag{3.23}$$

where

$$\log(I-s^nJ_q)=\begin{bmatrix}\mu_{q;1,1} & \mu_{q;1,2} & \mu_{q;1,3} & \cdots & \mu_{q;1,M_q}\\ 0 & \mu_{q;2,2} & \mu_{q;2,3} & \cdots & \mu_{q;2,M_q}\\ & & \ddots & & \vdots\\ & & & & \\ 0 & 0 & \cdots & 0 & \mu_{q;M_q,M_q}\end{bmatrix} \tag{3.24}$$

and

$$\mu_{q;i,i}=\log(1-\lambda_q s^n),\ 1\le q\le Q. \tag{3.25}$$

Now, Theorem 3.1 is generalized for general $\mathbf{T}_n$.

Theorem 3.2. *For $n\ge 1$, in (3.14) and (3.15) the generalized eigen-matrix is denoted by*

$$\mathbf{U}=[U_{1,1}\cdots U_{1,M_1};\cdots;U_{q,1}\cdots U_{q,M_q};\cdots;U_{Q,1}\cdots U_{Q,M_Q}]_{N\times N},$$

and its inverse is denoted by

$$\mathbf{U}^{-1}=[W_{1,1};\cdots;W_{1,M_1};\cdots;W_{q,1};\cdots;W_{q,M_q};\cdots;W_{Q,1};\cdots;W_{Q,M_Q}]_{N\times N}.$$

Then,

$$\zeta_n(s) = \prod_{q=1}^{Q} \prod_{1\le i\le j\le M_q} \exp\left(-\chi_{q;i,j}\mu_{q;i,j}\right), \tag{3.26}$$

where

$$\begin{aligned} \chi_{q;i,j} &= \tfrac{1}{n}|\mathbf{R}_n \circ U_{q,i}W_{q,j}| \\ &= \tfrac{1}{n}\sum_{p\in\mathcal{I}_n} \tfrac{\omega_{n,p}}{n}\left(\sum_{l=0}^{n-1} u_{q;\sigma^l(p),i}\right)\left(\sum_{l=0}^{n-1} w_{q;j,\sigma^l(p)}\right). \end{aligned} \tag{3.27}$$

In particular, if

$$\mu_{q;i,j} = 0 \text{ for all } i \neq j, \tag{3.28}$$

then

$$\begin{aligned} \zeta_n(s) &= \prod_{q=1}^{Q}(1-\lambda_q s^n)^{-\chi_q} \\ &= \prod_{\lambda\in\Sigma(\mathbf{T}_n)}(1-\lambda s^n)^{-\chi(\lambda)}, \end{aligned} \tag{3.29}$$

where

$$\chi_q = \frac{1}{n}\sum_{i=1}^{M_q}|\mathbf{R}_n \circ U_{q;i}W_{q;i}| \tag{3.30}$$

and

$$\chi(\lambda) = \sum_{\lambda_q=\lambda}\chi_q. \tag{3.31}$$

Proof. From (3.14) and (3.21),

$$\zeta_n(s) = \exp\left(\frac{1}{n}tr\left(\mathbf{U}diag(\log(I-s^nJ_1),\cdots,\log(I-s^nJ_Q))\mathbf{U}^{-1}\mathbf{R}_n\right)\right).$$

Now,

$$\begin{aligned} & tr\left(\mathbf{U}diag(\log(I-s^nJ_1),\cdots,\log(I-s^nJ_Q))\mathbf{U}^{-1}\mathbf{R}_n\right) \\ =\ & \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{r=1}^{N}\sum_{p=1}^{N} u_{p,i}\mu_{i,j}w_{j,r}\left(\sum_{l=0}^{n-1}R^l_{n;r,p}\right) \\ =\ & \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{p\in\mathcal{I}_n}\frac{\omega_{n,p}}{n}\left(\sum_{l=0}^{n-1}u_{\sigma^l(p),i}\right)\left(\sum_{l=0}^{n-1}w_{j,\sigma^{-l}(p)}\right)\mu_{i,j}. \end{aligned}$$

Therefore, (3.26) follows. Clearly, if (3.28) holds, then (3.29) holds. The proof is complete. □

In the rest of the section, (3.28) is proven and $\chi(\lambda)$ is shown to be a nonnegative integer. Therefore, ζ_n is a rational function. Some of the symmetry properties of the eigenvectors associated with the R_n-symmetry of $\mathbf{T}_n$ are investigated first.

Lemma 3.3. *For $n \geq 1$, if*

$$\mathbf{T}_n U = \lambda U, \tag{3.32}$$

then for any $0 \leq l \leq n-1$,

$$\mathbf{T}_n(R_n^l U) = \lambda R_n^l U. \tag{3.33}$$

Therefore, if U is an eigenvector, then $R_n^l U$ is also an eigenvector.

Furthermore, if

$$(\mathbf{T}_n - \lambda)^q U = 0 \tag{3.34}$$

for some $q \geq 2$, then for any $0 \leq l \leq n-1$,

$$(\mathbf{T}_n - \lambda)^q (R_n^l U) = 0. \tag{3.35}$$

Therefore, if U is a generalized eigenvector, then $R_n^l U$ is also a generalized eigenvector.

Proof. Assume that (3.32) holds and $U = (u_1, u_2, \cdots, u_N)^t$. Then,

$$R_n U = (u_{\sigma(1)}, u_{\sigma(2)}, \cdots, u_{\sigma(N)})^t. \tag{3.36}$$

According to (2.41), $\mathbf{T}_n = [t_{n;i,j}]_{N\times N}$ exhibits R_n-symmetry

$$t_{n;\sigma(i),\sigma(j)} = t_{n;i,j}$$

for all $1 \leq i, j \leq N$. Therefore, for any $1 \leq i \leq N$,

$$\begin{aligned}(\mathbf{T}_n(R_n U))_i &= \sum_{j=1}^{N} t_{n;i,j} u_{\sigma(j)} \\ &= \sum_{j=1}^{N} t_{n;\sigma(i),\sigma(j)} u_{\sigma(j)} \\ &= \lambda u_{\sigma(i)} = \lambda (R_n U)_i.\end{aligned}$$

Hence, $R_n U$ is an eigenvector of $\mathbf{T}_n$ with eigenvalue λ. Similarly, $R_n^l U$ is an eigenvector for any $0 \leq l \leq n-1$. Equation (2.41) can also be applied easily to verify (3.35) when (3.34) holds. Details are omitted.

The proof is complete. □

Based on Lemma 3.3, the equivalent class $\mathcal{R}(U)$ of eigenvector U is introduced by R_n.

Definition 3.4. *For any $N \times 1$ column vector U,*

$$\mathcal{R}(U) = \left\{ R_n^l U | 0 \leq l \leq n-1 \right\}. \tag{3.37}$$

U is called (R_n-) symmetric if $\mathcal{R}(U) = \{U\}$, such meaning that $u_j = u_i$ for all $j \in C_n(i)$ or

$$R_n^l U = U \tag{3.38}$$

for all $0 \leq l \leq n-1$. U *is called* (R_n-) *anti-symmetric if* $\sum_{l=0}^{n-1} R_n^l U = 0$, *such meaning*

$$\sum_{l=0}^{n-1} U_{\sigma^l(i)} = 0 \tag{3.39}$$

for all $i \in \mathcal{I}_n$.

For a symmetric eigenvector U, the following property is observed.

Lemma 3.5. *Let* $U = (u_1, u_2, \cdots, u_N)^t$ *and* $W = (w_1, w_2, \cdots, w_N)$,

$$\frac{1}{n}|\mathbf{R}_n \circ UW| = \sum_{i \in \mathcal{I}_n} \frac{1}{\omega_{n,i}} \left(\sum_{j \in C_n(i)} u_j \right) \left(\sum_{j \in C_n(i)} w_j \right). \tag{3.40}$$

Furthermore, if U *is symmetric, then*

$$\frac{1}{n}|\mathbf{R}_n \circ UW| = WU = \sum_{j=1}^{N} u_j w_j. \tag{3.41}$$

In particular, if $|U| = 1$, *then*

$$\frac{1}{n}|\mathbf{R}_n \circ UU^t| = 1. \tag{3.42}$$

Proof. Clearly,

$$\sum_{l=0}^{n-1} u_{\sigma^l(i)} = \frac{n}{\omega_{n,i}} \sum_{j \in C_n(i)} u_j, \tag{3.43}$$

and

$$\sum_{l=0}^{n-1} w_{\sigma^l(i)} = \frac{n}{\omega_{n,i}} \sum_{j \in C_n(i)} w_j. \tag{3.44}$$

Therefore, substituting (3.43) and (3.44) into (3.27) yields

$$\frac{1}{n}|\mathbf{R}_n \circ UW| = \sum_{i \in \mathcal{I}_n} \frac{1}{\omega_{n,i}} \left(\sum_{j \in C_n(i)} u_j \right) \left(\sum_{j \in C_n(i)} w_j \right).$$

If U is symmetric, then

$$\sum_{j \in C_n(i)} u_j = \omega_{n,i} u_i.$$

Hence,

$$\frac{1}{n}|\mathbf{R}_n \circ UW| = \sum_{i\in\mathcal{I}_n}\left(\sum_{j\in C_n(i)} u_i w_j\right) = \sum_{j=1}^{N} u_j w_j = WU.$$

The proof is complete. □

The following non-singular matrix Q_n is very useful in finding symmetric and anti-symmetric eigenvectors of $\mathbf{T}_n$.

Lemma 3.6. *For $n \geq 2$, the $n \times n$ matrix*

$$Q_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\ 1 & -\frac{1}{n-1} & -\frac{1}{n-1} & \cdots & -\frac{1}{n-1} & -\frac{1}{n-1} & -\frac{1}{n-1} & -\frac{1}{n-1} \\ 0 & 1 & -\frac{1}{n-2} & \cdots & -\frac{1}{n-2} & -\frac{1}{n-2} & -\frac{1}{n-2} & -\frac{1}{n-2} \\ & & & \vdots & & & & \\ 0 & 0 & 0 & \cdots & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & -1 \end{bmatrix} \tag{3.45}$$

is non-singular and

(3.46)

$$\overline{Q}_n = \begin{bmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} \\ \sqrt{\frac{n-1}{n}} & -\frac{1}{\sqrt{n(n-1)}} & -\frac{1}{\sqrt{n(n-1)}} & \cdots & -\frac{1}{\sqrt{n(n-1)}} & -\frac{1}{\sqrt{n(n-1)}} \\ 0 & \sqrt{\frac{n-2}{n-1}} & -\frac{1}{\sqrt{(n-1)(n-2)}} & \cdots & -\frac{1}{\sqrt{(n-1)(n-2)}} & -\frac{1}{\sqrt{(n-1)(n-2)}} \\ & & & \vdots & & \\ 0 & 0 & 0 & \cdots & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

is orthogonal.

Proof. The non-singularity of Q_n and orthogonality of $\overline{Q}_n$ can be verified directly; the details are omitted. □

In the following lemma, when $\overline{Q}_n$ is used, $\mathcal{R}(U)$ can be replaced by symmetric and anti-symmetric eigenvectors.

Lemma 3.7. *For $n \geq 2$, given eigenvector U, define*

$$\overline{U}_1 = \frac{1}{\sqrt{n}}\sum_{l=0}^{n-1} R_n^l U \tag{3.47}$$

and, $2 \leq j \leq n$,

$$\overline{U}_j = \sqrt{\frac{n-j+1}{n-j+2}} R_n^{j-2} U - \frac{1}{\sqrt{n-j+1}\sqrt{n-j+2}} \sum_{k=j-1}^{n-1} R_n^k U. \tag{3.48}$$

If $\mathcal{R}(U)$ has rank κ, for some κ, $1 \leq \kappa \leq n$,

(i) *then $\left\{\overline{U}_j\right\}_{j=1}^n$ also has rank κ;*

(ii) *if $\overline{U}_1 \neq 0$, then $\overline{U}_1$ is symmetric, and for each j, $2 \leq j \leq n$, $\overline{U}_j$ is anti-symmetric.*

Proof. Clearly,

$$\left(\overline{U}_1, \overline{U}_2, \cdots, \overline{U}_n\right)^t = \overline{Q}_n \left(U, R_n U, \cdots, R_n^j U, \cdots, R_n^{n-1} U\right)^t.$$

Since $\overline{Q}_n$ is orthogonal, (i) holds.

Since $R_n(\overline{U}_1) = \overline{U}_1$, $\overline{U}_1$ is symmetric. For $2 \leq j \leq n$ and $i \in \mathcal{I}_n$,

$$\begin{aligned}
\sum_{l=0}^{n-1} (\overline{U}_j)_{\sigma^l(i)} &= \sqrt{\tfrac{n-j+1}{n-j+2}} \left(\sum_{l=0}^{n-1} (R_n^{j-2} U)_{\sigma^l(i)} - \tfrac{1}{n-j+1} \sum_{k=j-1}^{n-1} \sum_{l=0}^{n-1} (R_n^k U)_{\sigma^l(i)} \right) \\
&= \sqrt{\tfrac{n-j+1}{n-j+2}} \left(\sum_{l=0}^{n-1} u_{\sigma^l(i)} - \tfrac{1}{n-j+1} \sum_{k=j-1}^{n-1} \sum_{l=0}^{n-1} u_{\sigma^l(i)} \right) \\
&= 0.
\end{aligned}$$

Therefore, $\overline{U}_j$ is anti-symmetric for any $2 \leq j \leq n$.

The proof is complete. □

The main result can now be proven.

Theorem 3.8. *For $n \geq 1$,*

$$\frac{1}{n} tr\left(\mathbf{T}_n^k \mathbf{R}_n\right) = \sum_{\lambda \in \Sigma(\mathbf{T}_n)} \chi(\lambda) \lambda^k \tag{3.49}$$

and

$$\zeta_n(s) = \prod_{\lambda \in \Sigma(\mathbf{T}_n)} (1 - \lambda s^n)^{-\chi(\lambda)}, \tag{3.50}$$

where $\chi(\lambda)$ is the number of linearly independent symmetric eigenvectors and generalized eigenvectors of $\mathbf{T}_n$ with eigenvalue λ.

Proof. The case of symmetric $\mathbf{T}_n$ is considered first. Let E_λ be the eigenspace of $\mathbf{T}_n$ with eigenvalue λ. By Lemma 3.7, E_λ is spanned by linearly independent symmetric unit eigenvectors $\overline{U}_1, \overline{U}_2, \cdots, \overline{U}_p$ and anti-symmetric unit eigenvectors $U_1', U_2', \cdots, U_{p'}'$, where $p + p' = dim(E_\lambda)$ and p or p' may be zero.

Now,

$$\begin{aligned}
\chi(\lambda) &= \tfrac{1}{n} \left(\sum_{j=1}^{p} |\mathbf{R}_n \circ \overline{U}_j \overline{U}_j^t| + \sum_{j=1}^{p'} |\mathbf{R}_n \circ U_j' (U_j')^t| \right) \\
&= p,
\end{aligned} \tag{3.51}$$

which is the number of linearly independent symmetric eigenvectors of $\mathbf{T}_n$ with eigenvalue λ.

For general $\mathbf{T}_n$, in Jordan canonical form (3.14) and (3.16), $\mathbf{U}$ can be decomposed into

$$\mathbf{U} = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_Q}.$$

Each E_{λ_j} is spanned by unit symmetric eigenvectors and generalized eigenvectors $\overline{U}_{j,1}, \overline{U}_{j,2}, \cdots, \overline{U}_{j,p_j}$ and anti-symmetric eigenvectors and generalized eigenvectors $U'_{j,1}, U'_{j,2}, \cdots, U'_{j,p'_j}$, and $p_j + p'_j = dim(E_{\lambda_j})$.

The inverse matrix is

$$\mathbf{U}^{-1} = \left[\overline{W}_{1,1}; \cdots; \overline{W}_{1,p_1}; W'_{1,1}; \cdots; W'_{1,p'_1}; \cdots; \overline{W}_{Q,1}; \cdots; \overline{W}_{Q,p_Q}; W'_{Q,1}; \cdots; W'_{Q,p'_Q}\right].$$

Lemma 3.5 implies

$$\frac{1}{n}|\mathbf{R}_n \circ \overline{U}_{j,i}\overline{W}_{j',k}| = \delta_{jj'}\delta_{ik}$$

and

$$\frac{1}{n}|\mathbf{R}_n \circ U'_{j,i}W'_{j',k}| = 0.$$

Therefore,

$$\begin{aligned}\chi(\lambda_j) &= p_j \\ &= \text{the number of linearly independent symmetric eigenvectors and} \\ &\quad\ \text{generalized eigenvectors of } \mathbf{T}_n \text{ with eigenvalue } \lambda_j.\end{aligned}$$

The result follows.

The proof is complete. □

To further study the eigenvalue λ with symmetric eigenvectors and generalized eigenvectors of $\mathbf{T}_n$, the following reduced trace operator τ_n of $\mathbf{T}_n$ is introduced.

Definition 3.9. *For $n \geq 1$,*

$$\mathbf{T}_n = [t_{n;i,j}].$$

For each $i, j \in \mathcal{I}_n$, define

$$\tau_{n;i,j} = \sum_{k \in C_n(j)} t_{n;i,k} \tag{3.52}$$

and denote the reduced trace operator of $\mathbf{T}_n$ by

$$\tau_n = [\tau_{n;i,j}], \tag{3.53}$$

which is a $\chi_n \times \chi_n$ matrix.

The following theorem indicates that τ_n is more effective in computing the eigenvalues with rotationally symmetric eigenvectors and generalized eigenvectors of $\mathbf{T}_n$. See also examples 7.2 and 7.3.

Theorem 3.10. *$\lambda \in \Sigma(\mathbf{T}_n)$ with $\chi(\lambda) \geq 1$ if and only if $\lambda \in \Sigma(\tau_n)$. Moreover, $\chi(\lambda)$ is the algebraic multiplicity of τ_n with eigenvalue λ. Furthermore,*

$$\frac{1}{n}\sum_{l=0}^{n-1}\Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\right) = \sum_{\lambda\in\Sigma(\tau_n)}\chi(\lambda)\lambda^k = tr(\tau_n^k), \tag{3.54}$$

and

$$\zeta_n(s) = \exp\left(\sum_{k=1}^{\infty}\frac{tr(\tau_n^k)}{k}s^{nk}\right). \tag{3.55}$$

Proof. Let $\lambda \in \Sigma(\mathbf{T}_n)$ be an eigenvalue with rotationally symmetric eigenvector $U = (u_1, u_2, \cdots, u_{2^n})^t$, where $u_i = u_j$ for any $i \in \mathcal{I}_n$ and $j \in C_n(i)$.

Define $V = (u_1, \cdots, u_i, \cdots, u_{2^n})^t$ for $i \in \mathcal{I}_n$. Then, clearly, $\mathbf{T}_n U = \lambda U$ implies $\tau_n V = \lambda V$.

On the other hand, if $\tau_n V = \lambda V$ and $V = (v_1, \cdots, v_i, \cdots, v_{2^n})^t$, then V can be extended to U, a 2^n-vector, by $u_j = v_i$ for $i \in \mathcal{I}_n$ and $j \in C_n(i)$. Then, $\mathbf{T}_n U = \lambda U$ and U is rotationally symmetric. The arguments also hold for a generalized eigenvector.

Finally, (3.54) follows from (2.55) and (3.49), and (3.55) follows from (1.8) and (3.54).

The proof is complete. □

Remark 3.11. *According to Theorem 3.10, the following is easily verified;*

$$\sum_{\lambda\in\Sigma(\mathbf{T}_n)}\chi(\lambda) = \sum_{\lambda\in\Sigma(\tau_n)}\chi(\lambda) = \chi_n. \tag{3.56}$$

Theorem 3.10 yields the following result.

Theorem 3.12. *For $n \geq 1$,*

$$\zeta_n(s) = \left(\det\left(I - s^n\tau_n\right)\right)^{-1} \tag{3.57}$$

$$= \prod_{\lambda\in\Sigma(\tau_n)}\left(1 - \lambda s^n\right)^{-\chi_n(\lambda)}, \tag{3.58}$$

where $\chi_n(\lambda)$ is the algebraic multiplicity of $\lambda \in \Sigma(\tau_n)$ and

$$\zeta(s) = \prod_{n=1}^{\infty}\left(\det\left(I - s^n\tau_n\right)\right)^{-1} \tag{3.59}$$

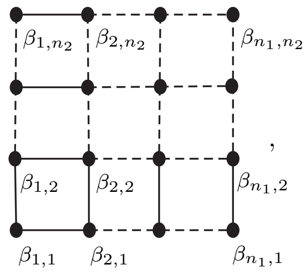
$$= \prod_{n=1}^{\infty}\prod_{\lambda\in\Sigma(\tau_n)}\left(1 - \lambda s^n\right)^{-\chi_n(\lambda)}. \tag{3.60}$$

4. More symbols on larger lattice

This section extends the results found in previous sections to any finite number of symbols $p \geq 2$ on any finite square lattice $\mathbb{Z}_{m\times m}$, $m \geq 2$. The results are outlined here and the details are left to the reader. The proofs of the theorems are sketched only or omitted for brevity.

For fixed positive integers $p \geq 2$ and $m \geq 2$, the set of symbols is denoted by $\mathcal{S}_p = \{0, 1, 2, \cdots, p-1\}$ and the basic square lattice is $\mathbb{Z}_{m\times m}$.

For any $n_1, n_2 \geq 1$, local patterns on $\mathbb{Z}_{n_1\times n_2}$ are denoted by $[\beta_{\alpha_1,\alpha_2}]_{1:n_1,1:n_2}$:



where $\beta_{\alpha_1,\alpha_2} \in \mathcal{S}_p$. Here, $[\beta_{\alpha_1,\alpha_2}]_{n_1:n_1',n_2:n_2'}$ means $n_1 \leq \alpha_1 \leq n_1'$ and $n_2 \leq \alpha_2 \leq n_2'$.

Define the counting function on $\mathcal{S}_p^{\mathbb{Z}_{n_1\times n_2}}$ by

$$\psi\left([\beta_{\alpha_1,\alpha_2}]_{1:n_1,1:n_2}\right) = 1 + \sum_{\alpha_1=1}^{n_1}\sum_{\alpha_2=1}^{n_2}\beta_{\alpha_1,\alpha_2}p^{n_2(n_1-\alpha_1)+(n_2-\alpha_2)}. \tag{4.1}$$

For any fixed $n \geq 1$, $\mathbf{Y}_{n\times m} = [y_{n\times m;i,j}]$ denote the ordering matrix of local patterns $[\beta_{\alpha_1,\alpha_2}]_{1:n,1:m}$on $\mathbb{Z}_{n\times m}$.

Pattern $[\beta_{\alpha_1,\alpha_2}]_{1:n,1:m}$ can be rewritten as

$$[\beta_{\alpha_1,\alpha_2}]_{1:n,1:m} = [\beta_{\alpha_1,\alpha_2}]_{1:n,1:m-1} \bullet [\beta_{\alpha_1,\alpha_2}]_{1:n,2:m},$$

and thus recorded as an element $y_{n\times m;i,j}$ in $\mathbf{Y}_{n\times m}$ by

$$[\beta_{\alpha_1,\alpha_2}]_{1:n,1:m} = y_{n\times m;i,j} \tag{4.2}$$

with

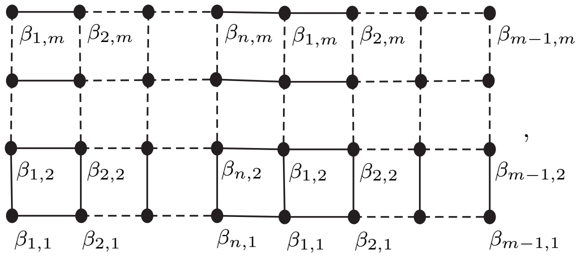
$$i \equiv \psi\left([\beta_{\alpha_1,\alpha_2}]_{1:n,1:m-1}\right) = 1 + \sum_{\alpha_1=1}^{n}\sum_{\alpha_2=1}^{m-1}\beta_{\alpha_1,\alpha_2}p^{(m-1)(n-\alpha_1)+(m-1-\alpha_2)} \tag{4.3}$$

and

$$j \equiv \psi\left([\beta_{\alpha_1,\alpha_2}]_{1:n,2:m}\right) = 1 + \sum_{\alpha_1=1}^{n}\sum_{\alpha_2=1}^{m-1}\beta_{\alpha_1,\alpha_2+1}p^{(m-1)(n-\alpha_1)+(m-1-\alpha_2)}. \tag{4.4}$$

Notably, $\mathbf{Y}_{n\times m}$ is a $p^{n(m-1)}\times p^{n(m-1)}$ matrix that has $p^{n\times m}$ non-trivial elements only and leaves $p^{2n(m-1)} - p^{mn}$ empty sites when $m \geq 3$.

For any fixed $n \geq m$, such as in (2.16), the x-periodic patterns of period n with height m can be recorded as $C_{n\times m;i,j}$ in $\mathbf{C}_{n\times m}$ by

$C_{n\times m;i,j} =$ 

,

where i and j are given as in (4.3) and (4.4), respectively.

Now, given a basic admissible set of patterns $\mathcal{B} \subset \mathcal{S}_p^{\mathbb{Z}_{m\times m}}$, the associated vertical transition matrix $\mathbf{V}_{n\times m} = \mathbf{V}_{n\times m}(\mathcal{B}) = [b_{n\times m;i,j}]$ for $n \geq m$ is defined by

$$(4.5) \qquad b_{n\times m;i,j} = 1 \text{ if and only if } y_{n\times m;i,j} = [\beta_{\alpha_1,\alpha_2}]_{1:n,1:m} \text{ is } \mathcal{B}\text{-admissible.}$$

Similarly, for any $n \geq m$, the associated trace operator $\mathbf{T}_{n\times m} = [t_{n\times m;i,j}]$ can be defined by

$$(4.6) \qquad t_{n\times m;i,j} = 1 \text{ if and only if } C_{n\times m;i,j} = [\overline{\beta}_1\overline{\beta}_2\cdots\overline{\beta}_m] \text{ is } \mathcal{B}\text{-admissible.}$$

Notably, both $\mathbf{V}_{n\times m}$ and $\mathbf{T}_{n\times m}$ are $p^{n(m-1)\times p^{n(m-1)}}$ matrices with entries in $\{0,1\}$. To verify (4.5),

$$(4.7) \qquad [\beta_{\alpha_1,\alpha_2}]_{k:k+m-1,1:m} \in \mathcal{B}$$

for $1 \leq k \leq n-m+1$, must be checked. Similarly, to verify (4.6), in addition to (4.7), the following must be established;

$$(4.8) \qquad [\beta_{\alpha_1,\alpha_2}] \in \mathcal{B},$$

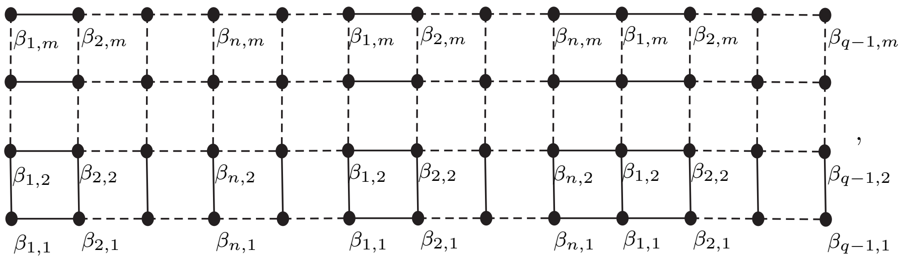
where

$$\alpha_1 = n-m+k+1, \cdots, n, 1, \cdots, k \quad \text{and} \quad \alpha_2 = 1, \cdots, m$$

for $1 \leq k \leq m-1$.

Clearly, when $\mathcal{B} = \mathcal{S}_p^{\mathbb{Z}_{m\times m}}$, then both $\mathbf{V}_{n\times m}^{m-1}$ and $\mathbf{T}_{n\times m}^{m-1}$ are full matrices, such meaning that all of their entries are 1.

For $1 \leq n \leq m-1$, $\left[\frac{m}{n}\right] = p$ and $m = pn + q$, $0 \leq q \leq n-1$. Then, $\mathbf{C}_{n\times m} = [C_{n\times m;i,j}]$ also records all of the x-periodic patterns of period n with height m by expressing $C_{n\times m;i,j}$ as an $(n+m-1)\times m$ pattern as follows:



,

where $[\beta_{\alpha_1,\alpha_2}]_{1:n,1:m}$ repeats p times and i and j are given as in (4.3) and (4.4) , respectively.

Hence, for $1 \leq n \leq m-1$, the associated trace operator $\mathbf{T}_{n\times m} = [t_{n\times m;i,j}]$ can be defined similarly to (4.6).

Notably, for any $k \geq 1$, $(\mathbf{T}^k_{n\times m})_{i,j}$ is the number of $\mathcal{B}$-admissible patterns of the form

$$(4.9)\qquad \begin{aligned} & [\overline{\beta}_1 \cdots \overline{\beta}_m \overline{\beta}_{m+1} \cdots \overline{\beta}_{m+k-1}] \\ = \; & [\overline{\beta}_1 \cdots \overline{\beta}_{m-1}] \bullet [\overline{\beta}_2 \cdots \overline{\beta}_m] \bullet \cdots \bullet [\overline{\beta}_{k+1} \cdots \overline{\beta}_{m+k-1}], \end{aligned}$$

where

$$(4.10)\qquad i = \psi([\overline{\beta}_1 \cdots \overline{\beta}_{m-1}])$$

and

$$(4.11)\qquad j = \psi([\overline{\beta}_{k+1} \cdots \overline{\beta}_{m+k-1}]).$$

Now, for any $n \geq 1$, the corresponding rotational matrix $R_{n\times(m-1)}$ which is a zero-one $p^{n(m-1)} \times p^{n(m-1)}$ matrix is defined by

$$R_{n\times(m-1);i,j} = 1 \text{ if and only if}$$

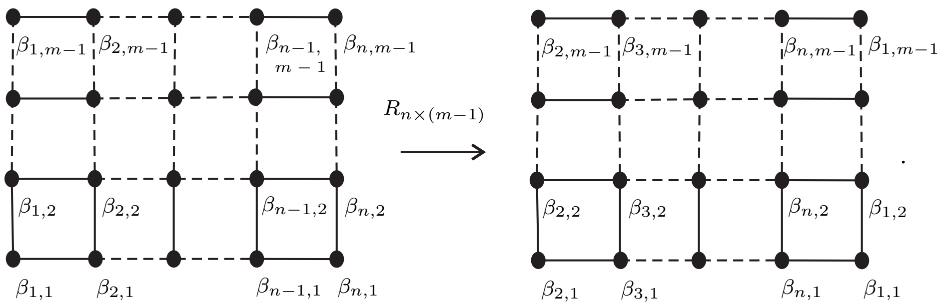
$$(4.12)\qquad j = \sigma(i),$$

where i is given by $1 \leq i \leq p^{n(m-1)}$ which is represented by (4.9) and $1 \leq \sigma(i) \leq p^{n(m-1)}$ is represented by

$$(4.13)\qquad \sigma(i) = \psi\left([\sigma(\overline{\beta}_1)\sigma(\overline{\beta}_2)\cdots\sigma(\overline{\beta}_{m-1})]\right).$$

The explicit expression for $R_{n\times(m-1)}$, like (2.35), can also be obtained and the result is omitted here. When $m = 3$, see (7.27).

Furthermore, $R_{n\times(m-1)}$ clearly sends patterns in $\mathbf{Y_{n\times m-1}}$ into itself as follows.



As (2.37) and (2.38), the equivalent class $C_{n\times(m-1)}(i)$ of i is defined by

$$\begin{aligned} C_{n\times(m-1)}(i) = & \ \{\sigma^j(i) | 0\leq j\leq n-1\} \\ = & \left\{ j \middle| \left(R^l_{n\times(m-1)}\right)_{i,j} = 1 \text{ for some } 1\leq l\leq n \right\}, \end{aligned} \tag{4.14}$$

and the index set $\mathcal{I}_{n\times(m-1)}$ of n is defined by

$$\begin{aligned} \mathcal{I}_{n\times(m-1)} = & \ \{i | 1\leq i\leq p^{n(m-1)}, i\leq \sigma^q(i), 1\leq q\leq n-1\} \\ = & \left\{ i \middle| 1\leq i\leq p^{n(m-1)}, i\leq j \text{ for all } j\in C_{n\times(m-1)}(i) \right\}. \end{aligned} \tag{4.15}$$

The cardinal number of $\mathcal{I}_{n\times(m-1)}$ is denoted by $\chi_{n\times(m-1)}$ and $\chi_{n\times(m-1)}$ is equal to the number of necklaces that can be made from 2^{m-1} colors, when the necklaces can be rotated but not turned over [34]. $\chi_{n\times(m-1)}$ is expressed as

$$\chi_{n\times(m-1)} = \frac{1}{n}\sum_{d|n}\phi(d)\left(2^{m-1}\right)^{n/d}, \tag{4.16}$$

where $\phi(n)$ is the Euler totient function.

Like Proposition 2.3, $R_{n\times(m-1)}$ has the following permutation properties.

Proposition 4.1. *Let $\mathbf{M} = [M_{i,j}]$ be a $p^{n(m-1)}\times p^{n(m-1)}$ matrix, where $M_{i,j}$ is a number or pattern or set of patterns. Then,*

$$(R_{n\times(m-1)}\mathbf{M})_{i,j} = M_{\sigma(i),j} \quad \text{and} \quad (\mathbf{M}R_{n\times(m-1)})_{i,j} = M_{i,\sigma^{-1}(j)}. \tag{4.17}$$

Furthermore, for any $l\geq 1$,

$$(R^l_{n\times(m-1)}\mathbf{M})_{i,j} = M_{\sigma^l(i),j} \quad \text{and} \quad (\mathbf{M}R^l_{n\times(m-1)})_{i,j} = M_{i,\sigma^{-l}(j)}. \tag{4.18}$$

The proof is similar to that of Proposition 2.3 and is omitted here.

Now, define

$$\mathbf{R}_{n\times(m-1)} = \sum_{l=0}^{n-1} R^l_{n\times(m-1)}. \tag{4.19}$$

A similar result to Theorem 2.13 can now be obtained for $\Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\right)$.

Theorem 4.2. *For $n\geq 1$, $k\geq 1$ and $0\leq l\leq n-1$,*

$$\Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\right) = tr\left(\mathbf{T}^k_{n\times m}R^l_{n\times(m-1)}\right) \tag{4.20}$$

and

$$\sum_{l=0}^{n-1}\Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\right) = tr\left(\mathbf{T}^k_{n\times m}\mathbf{R}_{n\times(m-1)}\right). \tag{4.21}$$

Proof. For $k \geq m$ and $l = 0$, $\begin{bmatrix} n & 0 \\ 0 & k \end{bmatrix}$-periodic patterns have the form

$$[\overline{\beta}_1 \cdots \overline{\beta}_k \overline{\beta}_1 \cdots \overline{\beta}_{m-1}]. \tag{4.22}$$

Now, from (4.9)∼(4.11), $\left(\mathbf{T}^k_{n\times m}\right)_{ii}$ have the form (4.22). Hence,

$$\Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & 0 \\ 0 & k \end{bmatrix}\right) = tr\left(\mathbf{T}^k_{n\times m}\right).$$

For $1 \leq l \leq n-1$, $\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}$-periodic patterns has the form

$$[\overline{\beta}_1 \cdots \overline{\beta}_k \sigma^{n-l}(\overline{\beta}_1)\sigma^{n-l}(\overline{\beta}_2)\cdots\sigma^{n-l}(\overline{\beta}_{m-1})]. \tag{4.23}$$

Proposition 4.1 implies that $\left(\mathbf{T}^k_{n\times m} R^l_{n\times(m-1)}\right)_{i,i}$ has the form (4.23). Therefore,

$$\Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\right) = tr\left(\mathbf{T}^k_{n\times m} R^l_{n\times(m-1)}\right).$$

Equations (4.20) and (4.21) follow for $k \geq m$.

Now, (4.20) and (4.21) for $1 \leq k \leq m-1$ must be shown.

When $k = 1$, from (4.9)∼(4.11) $\mathbf{T}_{n\times m;i,i}$ has the form $[\overline{\beta}_1\overline{\beta}_2\cdots\overline{\beta}_m]$ which satisfies

$$[\overline{\beta}_1\overline{\beta}_2\cdots\overline{\beta}_{m-1}] = [\overline{\beta}_2\overline{\beta}_3\cdots\overline{\beta}_m], \tag{4.24}$$

which implies

$$\overline{\beta}_1 = \overline{\beta}_2 = \cdots = \overline{\beta}_m. \tag{4.25}$$

Accordingly,

$$\Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & 0 \\ 0 & 1 \end{bmatrix}\right) = tr\left(\mathbf{T}_{n\times m}\right).$$

Proposition 4.1 can again be applied to verify

$$\Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & 1 \end{bmatrix}\right) = tr\left(\mathbf{T}_{n\times m} R^l_{n\times(m-1)}\right)$$

for any $1 \leq l \leq n-1$.

For any $2 \leq k \leq m-1$, $\left[\frac{m}{k}\right] = p$ and $m = pk + q$, $0 \leq q \leq k-1$. $\begin{bmatrix} n & 0 \\ 0 & k \end{bmatrix}$-periodic patterns have the form

$$[\underbrace{\overline{\beta}_1 \cdots \overline{\beta}_k \cdots \overline{\beta}_1 \cdots \overline{\beta}_k}_{\text{p+1 times}} \overline{\beta}_1 \cdots \overline{\beta}_{q-1}] \tag{4.26}$$

Pattern (4.9) in $\left(\mathbf{T}^k_{n\times m}\right)_{i,i}$ implies

$$[\overline{\beta}_1\overline{\beta}_2\cdots\overline{\beta}_{m-1}] = [\overline{\beta}_{k+1}\overline{\beta}_{k+2}\cdots\overline{\beta}_{m+k-1}],$$

i.e.,

$$\overline{\beta}_j = \overline{\beta}_{j+k} \tag{4.27}$$

for any $1 \leq j \leq m-1$. The relation (4.27) implies that (4.9) has exactly the form of (4.26). Hence, $\Gamma_{\mathcal{B}}\left(\left[\begin{array}{cc} n & 0 \\ 0 & k \end{array}\right]\right) = tr\left(\mathbf{T}^k_{n\times m}\right)$ holds. A similar argument also establishes that (4.20) holds for any $2 \leq k \leq m-1$.

The proof is complete. □

As in (1.6), the n-th order zeta function is given by

$$\zeta_n(s) = \exp\left(\frac{1}{n}\sum_{k=1}^{\infty}\sum_{l=0}^{n-1}\frac{1}{k}\Gamma_{\mathcal{B}}\left(\left[\begin{array}{cc} n & l \\ 0 & k \end{array}\right]\right)s^{kn}\right). \tag{4.28}$$

From Theorem 4.2, the following theorem is obtained.

Theorem 4.3. *For any $n \geq 1$,*

$$\zeta_n(s) = \exp\left(\frac{1}{n}\sum_{k=1}^{\infty}\frac{1}{k}tr\left(\mathbf{T}^k_{n\times m}\mathbf{R}_{n\times(m-1)}\right)s^{nk}\right). \tag{4.29}$$

The proof that $\zeta_n(s)$ is a rational function depends on the fact that $\mathbf{T}_{n\times m}$ is also $R_{n\times(m-1)}$-symmetric.

Proposition 4.4. *For any $n \geq 1$,*

$$t_{n\times m;\sigma(i),\sigma(j)} = t_{n\times m;i,j} \tag{4.30}$$

for any $1 \leq i,j \leq p^{n(m-1)}$.

Then the reduced trace operator of $\mathbf{T}_{n\times m}$ is defined as follows.

Definition 4.5. *For $n \geq 1$, the reduced trace operator $\tau_{n\times m} = [\tau_{n\times m;i,j}]$ of $\mathbf{T}_{n\times m}$ is a $\chi_{n\times(m-1)} \times \chi_{n\times(m-1)}$ matrix defined by*

$$\tau_{n\times m;i,j} = \sum_{k\in C_{n\times(m-1)}(j)} t_{n\times m;i,k} \tag{4.31}$$

for each $i,j \in \mathcal{I}_{n\times(m-1)}$.

The notion of symmetric and anti-symmetric eigenvectors of $\mathbf{T}_{n\times m}$, as in Definition 3.4, must also be introduced.

Definition 4.6. *Let U be an eigenvector of $\mathbf{T}_{n\times m}$. U is called symmetric if*

$$R^l_{n\times(m-1)}U = U \tag{4.32}$$

for all $0 \leq l \leq n-1$, and is called anti-symmetric if

$$\sum_{l=0}^{n-1} R^l_{n\times(m-1)}U = 0. \tag{4.33}$$

The equivalent class $\mathcal{R}(U)$ of U is defined by

$$\mathcal{R}(U)=\left\{R^{l}_{n\times(m-1)}U\middle|0\leq l\leq n-1\right\}. \tag{4.34}$$

Now, the main result can be obtained.

Theorem 4.7. *For any $n\geq 1$,*

$$\zeta_n(s)=\prod_{\lambda\in\Sigma(\mathbf{T}_{n\times m})}(1-\lambda s^n)^{-\chi(\lambda)} \tag{4.35}$$

$$=\left(\det\left(I-s^n\tau_{n\times m}\right)\right)^{-1}, \tag{4.36}$$

where $\chi(\lambda)$ is the number of linearly independent symmetric eigenvectors and generalized eigenvectors of $\mathbf{T}_{n\times m}$ with eigenvalue λ.

The zeta function is

$$\zeta(s)=\prod_{n=1}^{\infty}\left(\det\left(I-s^n\tau_{n\times m}\right)\right)^{-1}. \tag{4.37}$$

Proof. Lemma 3.7: the symmetrization of eigenvectors also holds for the present cases.

The arguments in Section 3 apply here and the results follow. □

5. Zeta functions presented in inclined coordinates

This section will present the zeta function with respect to the inclined coordinates, as determined by applying unimodular transformations. $\mathbb{Z}^2$ is known to be invariant with respect to unimodular transformation. Indeed, Lind [21] proved that $\zeta^0_{\mathcal{B};\gamma}=\zeta^0_{\mathcal{B}}$ for any $\gamma\in GL_2(\mathbb{Z})$: the zeta function is independent of a choice of basis for $\mathbb{Z}^2$. This section presents the constructions of the trace operator $\mathbf{T}_{\gamma;n}(\mathcal{B})$ and the reduced trace operator $\tau_{\gamma;n}(\mathcal{B})$, then determines $\zeta_{\mathcal{B};\gamma;n}$ and $\zeta_{\mathcal{B};\gamma}$. Finally, $\zeta_{\mathcal{B};\gamma}$ is obtained as

$$\zeta_{\mathcal{B};\gamma}(s)=\prod_{n=1}^{\infty}(\det(I-s^n\tau_{\gamma;n}(\mathcal{B})))^{-1}. \tag{5.1}$$

As mentioned in (1.35), $\zeta_{\mathcal{B};\gamma}(s)=\zeta^0_{\mathcal{B}}(s)$ in $|s|<\exp(-g(\mathcal{B}))$, for any $\gamma\in GL_2(\mathbb{Z})$, which yields a family of identities when $\zeta_{\mathcal{B};\gamma}$ is expressed as Taylor series at the origin $s=0$ (Theorem 6.4). Furthermore, for some $\mathcal{B}\subset\Sigma_{2\times 2}$, we may find a $\gamma\in GL_2(\mathbb{Z})$ such that $\zeta_{\mathcal{B};\gamma}$ offers a better description of poles and natural boundary of $\zeta^0_{\mathcal{B}}$ when $\zeta_{\mathcal{B}}$ and $\widehat{\zeta}_{\mathcal{B}}$ fail to do so, see Example 7.4.

For simplicity, only $\mathcal{B}\subset\Sigma_{2\times 2}$ with two symbols are considered. The general cases can be treated analogously.

We begin with the study in the modular group $SL_2(\mathbb{Z})$. The results also hold for any $\gamma\in GL_2(\mathbb{Z})$ with $\det\gamma=-1$.

Recall the modular group

$$SL_2(\mathbb{Z})=\left\{\left[\begin{array}{cc}a&b\\c&d\end{array}\right]\middle|\ a,b,c,d\in\mathbb{Z}\text{ and }ad-bc=1\right\}.$$

$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$ is called a unimodular transformation. Then,

$$\mathbb{Z}^2 = \{p(a,c) + q(b,d) | p, q \in \mathbb{Z}\} \tag{5.2}$$

holds, here $\mathbb{Z}^2$ is the set of lattice points (vertices).

Consider the set of all finite-index subgroups $\mathcal{L}_2$ of $\mathbb{Z}^2$ by

$$\mathcal{L}_2 = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \mathbb{Z}^2 \middle| \; a_{11}a_{22} - a_{12}a_{21} \geq 1, a_{ij} \in \mathbb{Z}, 1 \leq i,j \leq 2 \right\},$$

here $\mathbb{Z}^2 = \left\{ \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} | n_1, n_2 \in \mathbb{Z} \right\}$. An equivalent relation $\sim$ exists in $\mathcal{L}_2$. Two sublattices $L = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \mathbb{Z}^2$ and $L' = \begin{bmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22} \end{bmatrix} \mathbb{Z}^2$ are equivalent if L and L' determine the same sublattice of $\mathbb{Z}^2$: $L' = L$.

The following result states the existence of unique Hermite normal upper (or lower) triangular forms within each equivalent class in $\mathcal{L}_2$.

Proposition 5.1. *For each* $L = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \mathbb{Z}^2 \in \mathcal{L}_2$*, there is a unique* $\begin{bmatrix} n & l \\ 0 & k \end{bmatrix} \mathbb{Z}^2 \in \mathcal{L}_2$*,* $n, k \geq 1$ *and* $0 \leq l \leq n-1$*, and* $\begin{bmatrix} k_1 & 0 \\ l_1 & n_1 \end{bmatrix} \mathbb{Z}^2 \in \mathcal{L}_2$*,* $n_1, k_1 \geq 1$ *and* $0 \leq l_1 \leq n_1 - 1$*, such that they are equivalent, where*

$$nk = n_1 k_1 = a_{11}a_{22} - a_{12}a_{21}. \tag{5.3}$$

The proof can be found elsewhere [24].

For a given $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$, the lattice points in γ-coordinates are

$$(1,0)_\gamma = (a,b) \quad \text{and} \quad (0,1)_\gamma = (c,d),$$

and the unit vectors are

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}_\gamma = \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}_\gamma = \begin{pmatrix} c \\ d \end{pmatrix}.$$

Notably, when $\gamma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, standard rectangular coordinates are used and the subscript γ is omitted.

The parallelogram

$$M_\gamma = \begin{bmatrix} n & l \\ 0 & k \end{bmatrix}_\gamma$$

with respect to γ is defined by

$$M_\gamma \begin{pmatrix} 1 \\ 0 \end{pmatrix}_\gamma = \begin{pmatrix} n \\ 0 \end{pmatrix}_\gamma = \begin{pmatrix} na \\ nb \end{pmatrix} \tag{5.4}$$

and

$$M_\gamma \begin{pmatrix} 0 \\ 1 \end{pmatrix}_\gamma = \begin{pmatrix} l \\ k \end{pmatrix}_\gamma = \begin{pmatrix} la + kc \\ lb + kd \end{pmatrix}. \tag{5.5}$$

Hence,

$$M_\gamma = \begin{bmatrix} na & la+kc \\ nb & lb+kd \end{bmatrix}. \tag{5.6}$$

Let $L_\gamma = M_\gamma \mathbb{Z}^2$. Then,

$$L_\gamma = \gamma^t L \tag{5.7}$$

is easily verified:

$$\gamma^t L = \left[\begin{array}{cc} a & c \\ b & d \end{array}\right]\left[\begin{array}{cc} n & l \\ 0 & k \end{array}\right]\mathbb{Z}^2 = \left[\begin{array}{cc} na & la+kc \\ nb & lb+kd \end{array}\right]\mathbb{Z}^2 = M_\gamma \mathbb{Z}^2 = L_\gamma.$$

The Hermite normal form in Proposition 5.1 indicates the existence and uniqueness of $0 \le l_j \le n_j - 1$, $1 \le k_j$ for $j = 1, 2$, such that

$$L_\gamma = \left[\begin{array}{cc} a & c \\ b & d \end{array}\right]\left[\begin{array}{cc} n & l \\ 0 & k \end{array}\right]\mathbb{Z}^2 = \left[\begin{array}{cc} n_1 & l_1 \\ 0 & k_1 \end{array}\right]\mathbb{Z}^2 = \left[\begin{array}{cc} k_2 & 0 \\ l_2 & n_2 \end{array}\right]\mathbb{Z}^2 \tag{5.8}$$

with $n_1 k_1 = n_2 k_2 = nk$.

Therefore, the n-th order zeta function of $\zeta^0_{\mathcal{B}}(s)$ with respect to γ is defined by

$$\zeta_{\mathcal{B};\gamma;n}(s) = \exp\left(\frac{1}{n}\sum_{k=1}^{\infty}\sum_{l=0}^{n-1}\frac{1}{k}\Gamma_{\mathcal{B}}\left(\left[\begin{array}{cc} n & l \\ 0 & k \end{array}\right]_\gamma\right) s^{nk}\right) \tag{5.9}$$

and the zeta function $\zeta_{\mathcal{B};\gamma}$ with respect to γ is defined by

$$\zeta_{\mathcal{B};\gamma}(s) \equiv \prod_{n=1}^{\infty}\zeta_{\mathcal{B};\gamma;n}(s). \tag{5.10}$$

Since (5.8) holds, the iterated sum in (5.9) and (5.10) is a rearrangement of $\zeta^0_{\mathcal{B}}(s)$. Therefore,

$$\zeta_{\mathcal{B};\gamma}(s) = \zeta^0_{\mathcal{B}}(s) \tag{5.11}$$

for $|s| < \exp(-g(\mathcal{B}))$. See Proposition 6.2 ($i$) and another work [21].

The main purpose of this section is to establish results that are similar to Theorems 3.8, 3.12 and 4.7:

$$\zeta_{\mathcal{B};\gamma;n}(s) = \prod_{\lambda \in \Sigma(\mathbf{T}_{\gamma;n})} (1 - \lambda s^n)^{-\chi_{\gamma;n}} \tag{5.12}$$

$$= \left(\det\left(I - s^n \tau_{\gamma;n}\right)\right)^{-1}, \tag{5.13}$$

where $\mathbf{T}_{\gamma,n}$ is the trace operator with respect to γ and $\tau_{\gamma;n}$ is the associated reduced trace operator of $\mathbf{T}_{\gamma;n}$. The following introduces cylindrical matrix $\mathbf{C}_\gamma$ and rotational symmetrical operator $R_{\gamma,n}$. Only the essential parts of the proofs of the results are presented and the details are left to the reader.

In the following, a unimodular transformation γ is given and fixed. Let $\mathbb{Z}_{\gamma;n\times m}$ be the $n \times m$ lattice with one side in the $\gamma_1 = \left(\begin{array}{c} 1 \\ 0 \end{array}\right)_\gamma = \left(\begin{array}{c} a \\ b \end{array}\right)$ direction and the other side in the $\gamma_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}_\gamma = \begin{pmatrix} c \\ d \end{pmatrix}$ direction. The total number of lattice points on $\mathbb{Z}_{\gamma;n\times m}$ is $n \cdot m$. The ordering matrix $\mathbf{Y}_{\gamma;n\times m} = [y_{\gamma;n\times m;i,j}]$ of local patterns $[\beta_{\gamma;\alpha_1,\alpha_2}]_{1:n,1:m}$ is defined on $\mathbb{Z}_{\gamma;n\times m}$. On $\mathbb{Z}_{\gamma;2\times 2}$ and $\mathbb{Z}_{\gamma;n\times 2}$, $\mathbf{Y}_{\gamma;2\times 2}$ is arranged as in (2.2) and $\mathbf{Y}_{\gamma;n\times 2}$ is defined recursively as in (2.13) and (2.14), except that the horizontal is now in the γ_1 direction and the vertical is in the γ_2 direction. $\mathbf{Y}_{\gamma;n\times m} = [y_{\gamma;n\times m;i,j}]$ is given in (4.2).

The γ_1-periodic patterns of period n with height m on $\mathbb{Z}_{\gamma;(n+1)\times m}$ can be recorded as $C_{\gamma;n\times m;i,j}$ in a cylindrical matrix $\mathbf{C}_{\gamma;n\times m}$. The indices i,j are given by (4.3) and (4.4) with $p=2$.

To illustrate the cylindrical matrix $\mathbf{C}_{\gamma;n\times m}$, consider the following example: given $\gamma = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\mathbf{C}_{\gamma;1\times 3}$ is defined by

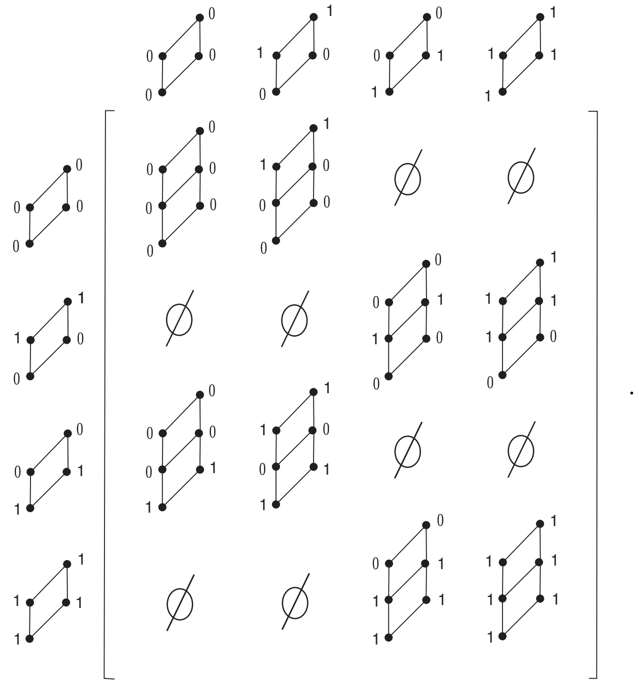


Fig 5.1.

The shift operator σ_γ is defined to shift one step to the left in the γ_1 direction.

Since the admissible local pattern $\mathcal{B}$ is given on square lattice $\mathbb{Z}_{2\times 2}$, the periodic patterns in γ-coordinates that are $\mathcal{B}$-admissible must be verified on $\mathbb{Z}_{2\times 2}$. Let $\mathbb{Z}_{2\times 2}((i,j))$ be the square lattice with the left-bottom vertex (i,j):

$$\mathbb{Z}_{2\times 2}((i,j)) = \{(i,j),(i+1,j),(i,j+1),(i+1,j+1)\}.$$

Now, the admissibility is demonstrated to have to be verified on finite square lattices.

Proposition 5.2. *Given* $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$ *and* $n \geq 1$, $k \geq 1$ *and* $0 \leq l \leq n-1$. *An* $\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}_\gamma$*-periodic pattern* U *is* $\mathcal{B}$*-admissible if and only if*

$$U\Big|_{\mathbb{Z}_{2\times 2}((\xi,\eta)_\gamma)} \in \mathcal{B} \tag{5.14}$$

for any $0 \le \xi \le n-1$ *and* $0 \le \eta \le k-1$.

Proof. Clearly, (5.14) is a necessary condition. Now, only (5.14) must be shown to be sufficient. Since $ad - bc = 1$, if $(\xi,\eta)_\gamma = (i,j)$, then

$$\begin{cases} (i+1,j) = (\xi+d, \eta-b)_\gamma, \\ (i,j+1) = (\xi-c, \eta+a)_\gamma, \\ (i+1,j+1) = (\xi+d-c, \eta+a-b)_\gamma, \end{cases} \tag{5.15}$$

are easily verified.

Now, suppose that (5.14) holds; then, the periodicity and (5.15) imply that (5.14) holds for all $(\xi,\eta) \in \mathbb{Z}^2$.

The proof is complete. □

For a given basic set $\mathcal{B} \subset \{0,1\}^{\mathbb{Z}_{2\times 2}}$, the definition of trace operator $\mathbf{T}_{\gamma;n\times m}$ of $\mathcal{B}$ on $\mathbb{Z}_{\gamma;n\times m}$ has to be justified, since $\mathcal{B}$ is given in a 2×2 square lattice in the $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ directions and $\mathbf{T}_{\gamma;n\times m}$ is defined in the $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} c \\ d \end{pmatrix}$ directions.

For any $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$, the height $h(\gamma)$ of γ is

$$h = h(\gamma) = |a| + |b|, \tag{5.16}$$

and the width $w(\gamma)$ of γ is

$$w = w(\gamma) = |c| + |d|. \tag{5.17}$$

The following lemma determines that the first square lattice that occurs in a parallelogram in the γ-coordinates is proven first.

Lemma 5.3. *For any* $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$, *there exists exactly one square lattice that is determined by a parallelogram with vertices* $(0,0)_\gamma$, $(w,0)_\gamma$, $(0,h)_\gamma$ *and* $(w,h)_\gamma$. *The square lattice has either vertices* $(0,h)_\gamma$ *and* $(w,0)_\gamma$ *or vertices* $(0,0)_\gamma$ *and* $(w,h)_\gamma$.

Proof. The proofs are divided into three cases.
(I) no zero in a, b, c or d: eight subcases.
(II) exactly one zero in a, b, c and d: 16 subcases.
(III) exactly two zeros in a, b, c and d: four subcases.

The proof is given for only a few cases. The proofs for the other cases are analogous and so are omitted.

(I)(i) $a,b,c,d > 0$. Since $ad - bc = 1$, $0 < \frac{b}{a} < \frac{d}{c}$. Let $(p,0)_\gamma$ and $(0,q)_\gamma$ be the two couple of vertices of the first square lattice along γ_1 and γ_2 directions. See Fig 5.2 (i). Then

$$pa - qc = 1 \quad \text{and} \quad -pb + qd = 1,$$

implying

$$p = c + d = w \quad \text{and} \quad q = a + b = h.$$

(I)(ii) $a>0, b>0, c<0, d<0$. Then, $ad-bc=1$ implies $\frac{b}{a}>\frac{d}{c}>0$. See Fig 5.2 (ii). In this case,

$$pa+qc=-1 \quad \text{and} \quad pb+qd=1.$$

Therefore,

$$p=-c-d=|c|+|d|=w \quad \text{and} \quad q=a+b=h.$$

The proof is complete.

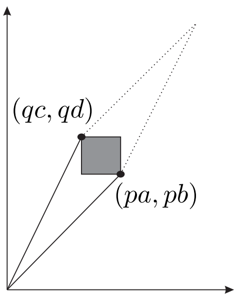


Fig 5.2 (i).

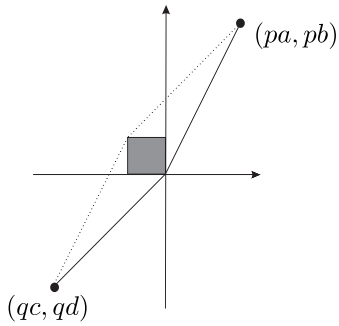


Fig 5.2 (ii).

□

The lemma shows that the existence of the parallelogram contains exactly $n\cdot k$ square lattices, as follows.

Proposition 5.4. *Given* $\gamma=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\in SL_2(\mathbb{Z})$, *for any* $n\geq 1$ *and* $k\geq 1$, *exactly* $n\cdot k$ *square lattices have pairs of vertices that lie on the parallelogram that is determined by* $(0,0)_\gamma$, $(w+n-1,0)_\gamma$, $(0,h+k-1)_\gamma$ *and* $(w+n-1,h+k-1)_\gamma$.

For a given $\mathcal{B}$, $\gamma\in SL_2(\mathbb{Z})$ and $n\geq 1$, the trace operator $\mathbf{T}_{\gamma;n}(\mathcal{B})$ acts exactly on n square lattices which lie in the γ_1-direction.

Therefore, consider $\mathbb{Z}_{\gamma;n+w,h+1}$. From Proposition 5.4, n square lattices have pairs of vertices on $\mathbb{Z}_{\gamma;n+w,h+1}$. The γ_1-periodic patterns with period n and height $h+1$ are denoted by $\mathbf{C}_{\gamma;n+w,h+1}$.

The trace operator $\mathbf{T}_{\gamma;n}=\mathbf{T}_{\gamma;n}(\mathcal{B})=[t_{\gamma;n;i,j}]$, associated with $\mathcal{B}$ (where $\mathcal{B}$ is omitted for brevity later to prevent confusion), is defined by

(5.18)
$$t_{\gamma;n;i,j}=1 \quad \text{if and only if} \quad \text{the pattern in } \mathbf{C}_{\gamma;n+w,h+1;i,j} \text{ is } \mathcal{B}\text{-admissible.}$$

As in another study [3], a recursive formula exists for $\mathbf{T}_{\gamma;n+1}$ in terms of $\mathbf{C}_{\gamma;n+w+1,h+1;i,j}$, $\mathcal{B}$ and γ.

A similar result as in Proposition 2.7 can be obtained; the detailed proof is omitted.

Proposition 5.5. *For* $\gamma=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\in SL_2(\mathbb{Z})$, $n\geq 1$ *and* $k\geq 1$, $\left(\mathbf{T}^k_{\gamma;n}\right)_{i,j}$ *is the number of* $\mathcal{B}$*-admissible patterns of the form*

$$\begin{aligned} & \left[\overline{\beta}_{\gamma;1}\overline{\beta}_{\gamma;2}\cdots\overline{\beta}_{\gamma;h+k}\right] \\ = & \left[\overline{\beta}_{\gamma;1}\cdots\overline{\beta}_{\gamma;h}\right]\bullet\left[\overline{\beta}_{\gamma;2}\cdots\overline{\beta}_{\gamma;h+1}\right]\bullet\cdots\bullet\left[\overline{\beta}_{\gamma;k+1}\cdots\overline{\beta}_{\gamma;h+k}\right], \end{aligned}$$

where

$$i = \psi\left(\left[\overline{\beta}_{\gamma;1}\cdots\overline{\beta}_{\gamma;h}\right]\right) \tag{5.19}$$

and

$$j = \psi\left(\left[\overline{\beta}_{\gamma;k+1}\cdots\overline{\beta}_{\gamma;k+h}\right]\right). \tag{5.20}$$

Now, for any $n \geq 1$, the associated rotational matrix $R_{\gamma;n}$ which is a zero-one $2^{nh} \times 2^{nh}$ matrix is defined by

$$R_{\gamma;n;i,j} = 1 \quad \text{if and only if} \quad j = \sigma_\gamma(i), \tag{5.21}$$

where $1 \leq i \leq 2^{nh}$ is given by (5.19) and $1 \leq \sigma_\gamma(i) \leq 2^{nh}$ is defined by

$$\sigma_\gamma(i) = \psi\left(\left[\sigma_\gamma(\overline{\beta}_{\gamma;1})\sigma_\gamma(\overline{\beta}_{\gamma;2})\cdots\sigma_\gamma(\overline{\beta}_{\gamma;h})\right]\right). \tag{5.22}$$

The equivalent class $C_{\gamma;n}(i)$, the index set $\mathcal{I}_{\gamma;n}$ and the cardinal number $\chi_{\gamma;n}$ of $\mathcal{I}_{\gamma;n}$ can be defined as similar to (4.14)∼(4.16) and are omitted here.

Now, the following is the rotationality of $R_{\gamma;n}$, as Proposition 2.3. The proof is similar to the proof of Proposition 2.3 and omitted here.

Proposition 5.6. *Let* $\mathbf{M} = [M_{i,j}]$ *be a* $2^{nh} \times 2^{nh}$ *matrix, where* $M_{i,j}$ *is a number or pattern or set of patterns. Then,*

$$(R_{\gamma;n}\mathbf{M})_{i,j} = M_{\sigma_\gamma(i),j} \text{ and } (\mathbf{M}R_{\gamma;n})_{i,j} = M_{i,\sigma_\gamma^{-1}(j)}. \tag{5.23}$$

Furthermore, for any $l \geq 1$,

$$(R^l_{\gamma;n}\mathbf{M})_{i,j} = M_{\sigma^l_\gamma(i),j} \text{ and } (\mathbf{M}R^l_{\gamma;n})_{i,j} = M_{i,\sigma_\gamma^{-l}(j)}. \tag{5.24}$$

Also, $\mathbf{T}_{\gamma;n}$ is $R_{\gamma;n}$-symmetric such that the following result holds.

Proposition 5.7. *Given* $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$ *and* $\mathcal{B} \subset \Sigma_{2\times 2}$, *for any* $n \geq 1$,

$$t_{\gamma;n;\sigma_\gamma(i),\sigma_\gamma(j)} = t_{\gamma;n;i,j}$$

for any $1 \leq i, j \leq 2^{nh}$.

Now, the reduced trace operator is defined as follows.

Definition 5.8. *For* $n \geq 1$, *the reduced trace operator* $\tau_{\gamma;n} = [\tau_{\gamma;n;i,j}]$ *of* $\mathbf{T}_{\gamma;n}$ *is a* $\chi_{\gamma;n} \times \chi_{\gamma;n}$ *matrix defined by*

$$\tau_{\gamma;n;i,j} = \sum_{k\in C_{\gamma;n}(j)} t_{\gamma;n;i,k} \tag{5.25}$$

for each $i, j \in \mathcal{I}_{\gamma;n}$.

The eigenvector U of $\mathbf{T}_{\gamma;n}$ is $R_{\gamma;n}$-symmetric if

$$R^l_{\gamma;n}U = U \quad \text{for all } 0 \leq l \leq n-1, \tag{5.26}$$

and anti-symmetric if

$$\sum_{l=0}^{n-1} R^l_{\gamma;n}U = 0. \tag{5.27}$$

See also (3.38) and (3.39). Now, define

$$\mathbf{R}_{\gamma;n} = \sum_{l=0}^{n-1} R^l_{\gamma;n}. \tag{5.28}$$

It is easy to verify that all results also hold for any $\gamma \in GL_2(\mathbb{Z})$ with $\det\gamma = -1$. The main results as in Theorem 2.13 are then obtained.

Theorem 5.9. *Given any $\mathcal{B} \subset \Sigma_{2\times 2}$ and $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{Z})$. Then, for any $n \geq 1$, $k \geq 1$ and $0 \leq l \leq n-1$,*

$$\Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}_\gamma\right) = tr\left(\mathbf{T}^k_{\gamma;n} R^l_{\gamma;n}\right) \tag{5.29}$$

and

$$\sum_{l=0}^{n-1}\Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}_\gamma\right) = tr\left(\mathbf{T}^k_{\gamma;n}\mathbf{R}_{\gamma;n}\right). \tag{5.30}$$

Moreover,

$$\zeta_{\mathcal{B},\gamma,n}(s) = \exp\left(\frac{1}{n}\sum_{k=1}^{\infty}\frac{1}{k} tr\left(\mathbf{T}^k_{\gamma;n}\mathbf{R}_{\gamma;n}\right) s^{nk}\right). \tag{5.31}$$

Finally, by the argument as in sections 3 and 4, the rationality of the n-th order zeta function $\zeta_{\mathcal{B};\gamma;n}$ is established, as in Theorems 3.8, 3.12 and 4.7.

Theorem 5.10. *For any $\mathcal{B} \subset \Sigma_{2\times 2}$ and $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{Z})$,*

$$\zeta_{\mathcal{B};\gamma;n}(s) = \prod_{\lambda\in\Sigma(\mathbf{T}_{\gamma;n}(\mathcal{B}))} (1-\lambda s^n)^{-\chi_{\gamma;n}(\lambda)} \tag{5.32}$$

$$= \left(\det\left(I - s^n \tau_{\gamma;n}\right)\right)^{-1}, \tag{5.33}$$

where the exponent $\chi_{\gamma;n}(\lambda)$ is the number of linearly independent $R_{\gamma;n}$-symmetric eigenvectors of $\mathbf{T}_{\gamma;n}(\mathcal{B})$ with respect to eigenvalue λ. The zeta function of $\mathcal{B}$ with respect to γ-coordinates is

$$\zeta_{\mathcal{B};\gamma}(s) = \prod_{n=1}^{\infty}\left(\det\left(I - s^n\tau_{\gamma;n}\right)\right)^{-1}. \tag{5.34}$$

An immediate consequence of (5.34) is the following result, see Proposition 6.2 and [21].

Theorem 5.11. *For any $\mathcal{B} \subset \Sigma_{2\times 2}$ and $\gamma \in GL_2(\mathbb{Z})$, the Taylor series for $\zeta_{\mathcal{B};\gamma}$ at $s = 0$ has integer coefficients.*

Proof. Since $\tau_{\gamma;n}$ has integer entries for any $n \geq 1$. The result follows. □

We now briefly investigate the zeta functions presented in the lower Hermite normal form. For any $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{Z})$ and $n \geq 1$, define

$$\widehat{\zeta}_{\mathcal{B};\gamma;n}(s) = \exp\left(\frac{1}{n}\sum_{k=1}^{\infty}\sum_{l=0}^{n-1}\frac{1}{k}\Gamma_{\mathcal{B}}\left(\begin{bmatrix} k & 0 \\ l & n \end{bmatrix}_{\gamma}\right)s^{nk}\right) \tag{5.35}$$

and

$$\widehat{\zeta}_{\mathcal{B};\gamma}(s) = \prod_{n=1}^{\infty}\widehat{\zeta}_{\mathcal{B};\gamma;n}(s). \tag{5.36}$$

Denote by

$$\widehat{\gamma} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{5.37}$$

the reflection $\dfrac{\pi}{4}$ with respect to the diagonal axis $y = x$.

Then we have the following results.

Theorem 5.12. *For any $\gamma \in GL_2(\mathbb{Z})$,*

$$\widehat{\zeta}_{\mathcal{B};\gamma;n} = \zeta_{\mathcal{B};\widehat{\gamma}\gamma;n} \tag{5.38}$$

and

$$\widehat{\zeta}_{\mathcal{B};\gamma} = \zeta_{\mathcal{B};\widehat{\gamma}\gamma}. \tag{5.39}$$

In particular,

$$\widehat{\zeta}_{\mathcal{B}} = \zeta_{\mathcal{B};\widehat{\gamma}}. \tag{5.40}$$

Proof. For any $n \geq 1, k \geq 1$ and $0 \leq l \leq n-1$, and $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{Z})$, denote by the lattices

$$\widehat{L} = \begin{bmatrix} k & 0 \\ l & n \end{bmatrix}\mathbb{Z}^2 \text{ and } \widehat{L}_{\gamma} = \widehat{M}_{\gamma}\mathbb{Z}^2, \tag{5.41}$$

where the parallelogram $\widehat{M}_{\gamma}$ is defined by

$$\widehat{M}_{\gamma} = \begin{bmatrix} k & 0 \\ l & n \end{bmatrix}_{\gamma}. \tag{5.42}$$

Hence,

$$\widehat{M}_{\gamma}\begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\gamma} = \begin{pmatrix} k \\ l \end{pmatrix}_{\gamma} = \begin{pmatrix} ka + lc \\ kb + ld \end{pmatrix}$$

and

$$\widehat{M}_{\gamma}\begin{pmatrix} 0 \\ 1 \end{pmatrix}_{\gamma} = \begin{pmatrix} 0 \\ n \end{pmatrix}_{\gamma} = \begin{pmatrix} nc \\ nd \end{pmatrix}.$$

As in (5.7), it is easy to verify

$$\widehat{L}_{\gamma} = \gamma^t \widehat{L}. \tag{5.43}$$

Now, we show that

$$\widehat{L} = L_{\widehat{\gamma}}. \tag{5.44}$$

Indeed, by (5.7),

$$\begin{aligned} L_{\widehat{\gamma}} &= \widehat{\gamma}^t L = \widehat{\gamma} L = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} n & l \\ 0 & k \end{bmatrix} \mathbb{Z}^2 \\ &= \begin{bmatrix} 0 & k \\ n & l \end{bmatrix} \mathbb{Z}^2 = \begin{bmatrix} 0 & k \\ n & l \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathbb{Z}^2 \right) \\ &= \begin{bmatrix} k & 0 \\ l & n \end{bmatrix} \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathbb{Z}^2 \right) = \begin{bmatrix} k & 0 \\ l & n \end{bmatrix} \mathbb{Z}^2 = \widehat{L}. \end{aligned}$$

Similarly,

$$L = \widehat{L}_{\widehat{\gamma}} \tag{5.45}$$

holds.

Therefore,

$$\Gamma_{\mathcal{B}} \left(\begin{bmatrix} k & 0 \\ l & n \end{bmatrix} \right) = \Gamma_{\mathcal{B}} \left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}_{\widehat{\gamma}} \right). \tag{5.46}$$

Hence, (5.46) implies

$$\widehat{\zeta}_{\mathcal{B};n} = \zeta_{\mathcal{B};\widehat{\gamma};n}$$

and

$$\widehat{\zeta}_{\mathcal{B}} = \zeta_{\mathcal{B};\widehat{\gamma}}.$$

Furthermore, we show that

$$\widehat{L}_{\gamma} = L_{\widehat{\gamma}\gamma}.$$

Indeed, by (5.43), (5.45) and (5.7),

$$\begin{aligned} \widehat{L}_{\gamma} &= \gamma^t \widehat{L} = \gamma^t \widehat{\gamma}\widehat{\gamma} \widehat{L} = (\widehat{\gamma}\gamma)^t \widehat{\gamma}^t \widehat{L} \\ &= (\widehat{\gamma}\gamma)^t \widehat{L}_{\widehat{\gamma}} = (\widehat{\gamma}\gamma)^t L = L_{\widehat{\gamma}\gamma}. \end{aligned}$$

Similarly,

$$L_{\gamma} = \widehat{L}_{\widehat{\gamma}\gamma} \tag{5.47}$$

also holds. Therefore, (5.38) and (5.39) follow. The proof is complete. □

Remark 5.13. *From Theorem 5.12, for any $\mathcal{B} \subset \Sigma_{2\times 2}$ there is a family of zeta functions $\{\zeta_{\mathcal{B};\gamma} | \gamma \in GL_2(\mathbb{Z})\} = \left\{\widehat{\zeta}_{\mathcal{B};\gamma} | \gamma \in GL_2(\mathbb{Z})\right\}$. In computation, it is much easier to study $\zeta_{\mathcal{B}}$ and $\widehat{\zeta}_{\mathcal{B}}$, i.e., the rectangular zeta functions. However, for certain $\mathcal{B}$, some other $\gamma \in GL_2(\mathbb{Z})$ may give a better description, see Example 7.4.*

Remark 5.14. *For any $\mathcal{B} \subset \Sigma_{2\times 2}$ and $\gamma \in GL_2(\mathbb{Z})$, $\zeta_{\mathcal{B};\gamma}$ in (5.34), which is an infinite product of rational function, is a rearrangement of $\zeta^0_{\mathcal{B}}$ in (1.6), which is a triple series. In deriving the rationality of $\zeta_{\mathcal{B};\gamma;n}$, the basic formula used is the power series*

$$\sum_{k=1}^{\infty} \frac{t^k}{k} = -\log(1-t). \tag{5.48}$$

The other rearrangements of $\zeta^0_{\mathcal{B}}$ *may not have the form as in (5.34). For example, for any* $m \geq 1$*, denote by*

$$f_{\mathcal{B};m}(s) = \exp\left(\sum_{n|m}\sum_{l=0}^{n-1}\frac{1}{m}\Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & \frac{m}{n}\end{bmatrix}\right)s^m\right) \tag{5.49}$$

and

$$f_{\mathcal{B}}(s) = \prod_{m=1}^{\infty} f_{\mathcal{B};m}(s). \tag{5.50}$$

In general, $f_{\mathcal{B};m}(s)$ *is not a rational function of the form as in (1.3). It is also not clear how to identify the poles or natural boundary of* $f_{\mathcal{B}}(s)$ *from (5.49) and (5.50), see Section 6.*

6. Analyticity and meromorphic extensions of zeta functions

This section studies the analyticity and meromorphisms of zeta functions obtained in the previous sections. Application to number theory is also considered. For simplicity, only $\mathcal{B} \subset \Sigma_{2\times 2}$ with two symbols are considered. The general cases can be treated analogously.

6.1. **Analyticity of zeta functions.** Recall the analyticity results of Lind [21]. Given an admissible set $\mathcal{B} \subset \Sigma_{2\times 2}$, the analytic region found by Lind is related to quantity $g(\mathcal{B})$, which specifies the growth rate of admissible periodic patterns.

Given an admissible set $\mathcal{B} \subset \Sigma_{2\times 2}$,

$$\begin{aligned} g(\mathcal{B}) &\equiv \limsup_{[L]\to\infty}\tfrac{1}{[L]}\log\Gamma_{\mathcal{B}}(L) \\ &= \lim_{n\to\infty}\sup_{[L]\geq n}\frac{\log\Gamma_{\mathcal{B}}(L)}{[L]}. \end{aligned} \tag{6.1}$$

Remark 6.1. *Since* $\Gamma_{\mathcal{B}}(L) \leq 2^{[L]}$,

$$g(\mathcal{B}) \leq \log 2. \tag{6.2}$$

In particular,

$$\exp\left(-g(\mathcal{B})\right) \geq \frac{1}{2}. \tag{6.3}$$

Recall the results of Lind [21] that are related to analyticity of zeta functions.

Proposition 6.2. *According to Lind,* [21]

(i) *The zeta function*

$$\zeta^0_{\mathcal{B}}(s) = \exp\left(\sum_{L\in\mathcal{L}_2}\frac{\Gamma_{\mathcal{B}}(L)}{[L]}s^{[L]}\right) \tag{6.4}$$

has radius of convergence $\exp(-g(\mathcal{B}))$ *and is analytic in* $|s| < \exp(-g(\mathcal{B}))$.

(ii) $\zeta^0_{\mathcal{B}}$ *satisfies the product formula,*

$$\zeta^0_{\mathcal{B}}(s) = \prod_{\alpha}\pi_2(s^{|\alpha|}), \tag{6.5}$$

where the product is taken over all admissible periodic patterns α with respect to $\mathcal{B}$, and

$$\pi_2(s) = \sum_{n=1}^{\infty} P(n)s^n, \tag{6.6}$$

where $P(n)$ is the partition function.

(iii) *The Taylor series for $\zeta^0_{\mathcal{B}}(s)$ has integer coefficients: for $|s| < \exp(-g(\mathcal{B}))$,*

$$\zeta^0_{\mathcal{B}}(s) = \sum_{n=0}^{\infty} a_n(\mathcal{B})s^n \tag{6.7}$$

with

$$a_n(\mathcal{B}) \in \mathbb{Z}. \tag{6.8}$$

Remark 6.3. *The L that appears in the sum of (6.4) is taken within each equivalent class of $\mathcal{L}_2$. In particular, if the upper (or lower) Hermite normal form is adopted, then*

$$\zeta^0_{\mathcal{B};\gamma} = \exp\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=0}^{n-1}\frac{1}{nk}\Gamma_{\mathcal{B}}\left(\left[\begin{array}{cc} n & l \\ 0 & k \end{array}\right]_{\gamma}\right)s^{nk}\right) \tag{6.9}$$

or

$$\widehat{\zeta}^0_{\mathcal{B};\gamma} = \exp\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=0}^{n-1}\frac{1}{nk}\Gamma_{\mathcal{B}}\left(\left[\begin{array}{cc} k & 0 \\ l & n \end{array}\right]_{\gamma}\right)s^{nk}\right). \tag{6.10}$$

Equations (6.9) and (6.10) are triple sums and can be treated as a double series in n and k after taking the summation in l.

If U is an $L_\gamma = \left[\begin{array}{cc} n & l \\ 0 & k \end{array}\right]_{\gamma}$-periodic pattern, then U is a $\left[\begin{array}{cc} na & la+kc \\ nb & lb+kd \end{array}\right]$-periodic pattern. By the Hermite normal form theorem, U is $\left[\begin{array}{cc} n_1 & l_1 \\ 0 & k_1 \end{array}\right]$- periodic and $\left[\begin{array}{cc} k_2 & 0 \\ l_2 & n_2 \end{array}\right]$-periodic with $n_1k_1 = n_2k_2 = nk$, $0 \leq l_1 \leq n_1 - 1$ and $0 \leq l_2 \leq n_2 - 1$. Therefore, $\zeta^0_{\mathcal{B};\gamma}$ and $\widehat{\zeta}^0_{\mathcal{B};\gamma}$ are rearrangements of $\zeta^0_{\mathcal{B}}$ and $\widehat{\zeta}^0_{\mathcal{B}}$, respectively.

Now, Propositions 6.2 and 5.1 imply

Theorem 6.4. *For any admissible set $\mathcal{B} \subset \Sigma_{2\times 2}$ and $\gamma \in GL_2(\mathbb{Z})$,*

$$\zeta^0_{\mathcal{B}}(s) = \zeta_{\mathcal{B};\gamma}(s) = \widehat{\zeta}_{\mathcal{B};\gamma}(s) \tag{6.11}$$

for $|s| < \exp(-g(\mathcal{B}))$. Moreover, $\zeta_{\mathcal{B};\gamma}$ and $\widehat{\zeta}_{\mathcal{B};\gamma}$ have the same (integer) coefficients in their Taylor series around $s = 0$: if

$$\zeta_{\mathcal{B};\gamma} = \sum_{n=0}^{\infty} a_{\gamma;n}(\mathcal{B})s^n \tag{6.12}$$

and

$$\widehat{\zeta}_{\mathcal{B};\gamma} = \sum_{n=0}^{\infty} \widehat{a}_{\gamma;n}(\mathcal{B})s^n, \tag{6.13}$$

then

$$a_{\gamma;n}(\mathcal{B}) = \widehat{a}_{\gamma;n}(\mathcal{B}) = a_n(\mathcal{B}) \tag{6.14}$$

for each $\gamma \in GL_2(\mathbb{Z})$ *and* $n \geq 0$.

Proof. Since

$$\sum_{L\in\mathcal{L}_2} \frac{\log \Gamma_{\mathcal{B}}(L)}{[L]} s^{[L]}$$

is absolutely convergent in $|s| < \exp(-g(\mathcal{B}))$, for each $\gamma \in GL_2(\mathbb{Z})$,

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=0}^{n-1} \frac{1}{nk}\Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}_{\gamma}\right) s^{nk}$$

and

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=0}^{n-1} \frac{1}{nk}\Gamma_{\mathcal{B}}\left(\begin{bmatrix} k & 0 \\ l & n \end{bmatrix}_{\gamma}\right) s^{nk}$$

are absolutely convergent in $|s| < \exp(-g(\mathcal{B}))$. Hence (6.11) holds. (6.14) follows from (6.11) and Proposition 6.2 (iii) or Theorem 5.11.

The proof is complete. □

To express the Taylor series of $\zeta_{\mathcal{B};\gamma}$ and $\widehat{\zeta}_{\mathcal{B};\gamma}$ explicitly, consider the general infinite product

$$\zeta(s) = \prod_{n=1}^{\infty} \prod_{j=1}^{J_n} (1 - \lambda_{n,j} s^n)^{-\chi_{n,j}}, \tag{6.15}$$

where J_n and $\chi_{n,j}$ are positive integers and $\lambda_{n,j} \in \mathbb{C}$.

Proposition 6.5. *Given an infinite product (6.15), its Taylor series at* $s = 0$ *is given by*

$$\zeta(s) = \sum_{n=0}^{\infty} P_{\zeta}(n) s^n, \tag{6.16}$$

where

$$P_{\zeta}(n) = \sum_{\substack{n_1+2n_2+\cdots+ln_l=n \\ n_i\in\mathbb{N}\cup\{0\}}} \prod_{m=1}^{l} a_{m,n_m}, \tag{6.17}$$

and

$$a_{n,i} \equiv \sum_{\kappa_n=i} \prod_{j=1}^{J_n} \lambda_{n,j}^{|K_{n,j}|}, \tag{6.18}$$

where

$$K_{n,j} = (k_{n,j;1}, \cdots, k_{n,j;\chi_{n,j}}), \tag{6.19}$$

$$|K_{n,j}| = \sum_{l=1}^{\chi_{n,j}} k_{n,j;l}, \tag{6.20}$$

$k_{n,j;l}$ *is a non-negative integer, and*

$$\kappa_n = \sum_{j=1}^{J_n} |K_{n,j}| = \sum_{j=1}^{J_n} \sum_{l=1}^{\chi_{n,j}} k_{n,j;l}. \tag{6.21}$$

Proof. It is easy to verify that

$$\begin{aligned} & \prod_{j=1}^{J_n} (1-\lambda_{n,j} s^n)^{-\chi_{n,j}} \\ = & \prod_{j=1}^{J_n} \left(\sum_{k=0}^{\infty} \lambda_{n,j}^k s^{kn} \right)^{\chi_{n,j}} \\ = & \sum_{p=0}^{\infty} \left(\sum_{\kappa_n = p} \prod_{j=1}^{J_n} \lambda_{n,j}^{|K_{n,j}|} \right) s^{pn}. \end{aligned}$$

Therefore, (6.16)∼(6.21) follow. The proof is complete. □

Remark 6.6. $P_\zeta(n)$ *is a general partition function where* n *is partitioned three times. Indeed, if* $\mathbf{H}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$*, as shown by Lind* [21], $J_n = \lambda_{n,j} = \chi_{n,j} = 1$ *and*

$$\zeta(s) = \prod_{n=1}^{\infty} (1-s^n)^{-1} = \sum_{n=0}^{\infty} P(n) s^n,$$

where $P(n)$ *is the typical partition function. In this case,* $P_\zeta(n) = P(n)$.

The rest of subsection discusses the meromorphicity of zeta function $\zeta_{\mathcal{B};\gamma}$. We need the following notations.

Definition 6.7.

(i) *Given any* $\mathcal{B} \subset \Sigma_{2\times 2}$ *and* $\gamma \in GL_2(\mathbb{Z})$. *The meromorphic domain* $\mathcal{M}_{\mathcal{B};\gamma}$ *of* $\zeta_{\mathcal{B};\gamma}$ *is defined by*

$$\mathcal{M}_{\mathcal{B};\gamma} = \{ s \in \mathbb{C} | \zeta_{\mathcal{B};\gamma}(s) \text{ is meromorphic at } s \}. \tag{6.22}$$

(ii) *The pole set* $\mathcal{P}_{\mathcal{B};\gamma}$ *of* $\zeta_{\mathcal{B};\gamma}$ *is defined by*

$$\begin{aligned} \mathcal{P}_{\mathcal{B};\gamma} &= \{ s \in \mathbb{C} | 1 - \lambda s^n = 0, \text{ where } \lambda \in \Sigma(\mathbf{T}_{\gamma;n}(\mathcal{B})), \chi_{\gamma;n}(\lambda) \geq 1 \text{ and } n \geq 1 \} \\ &= \{ s \in \mathbb{C} | 1 - \lambda s^n = 0, \text{ where } \lambda \in \Sigma(\tau_{\mathcal{B};\gamma;n}) \text{ and } n \geq 1 \}. \end{aligned} \tag{6.23}$$

(iii) $\zeta_{\mathcal{B};\gamma}$ *has a natural boundary* $\partial\mathcal{M}_{\mathcal{B};\gamma}$ *if every point in* $\partial\mathcal{M}_{\mathcal{B};\gamma}$ *is singular.*

Remark 6.8.

(i) *From (6.3) and Proposition 6.2 (i),*

$$\mathcal{M}_{\mathcal{B};\gamma} \supseteq \{ s \in \mathbb{C} | \, |s| < \exp(-g(\mathcal{B})) \} \supseteq \left\{ s \in \mathbb{C} | \, |s| < \frac{1}{2} \right\}. \tag{6.24}$$

(ii) $\zeta_{\mathcal{B};\gamma}$ *has a natural boundary if*

$$\overline{\mathcal{P}}_{\mathcal{B};\gamma} \supseteq \partial\mathcal{M}_{\mathcal{B};\gamma}. \tag{6.25}$$

In studying the infinite products $\zeta_{\mathcal{B};\gamma}(s)$, the associated infinite series

$$\xi_{\mathcal{B};\gamma}(s) \equiv \sum_{n=1}^{\infty} \left(\sum_{\lambda\in\Sigma(\mathbf{T}_{\gamma;n})} \lambda\chi_{\gamma;n}(\lambda) \right) s^n \tag{6.26}$$

is useful. Denote by

$$\lambda^*_{\mathcal{B};\gamma} \equiv \limsup_{n\to\infty} \left(\sum_{\lambda\in\Sigma(\mathbf{T}_{\gamma;n})} |\lambda|\chi_{\gamma;n}(\lambda) \right)^{\frac{1}{n}}. \tag{6.27}$$

Let

$$S^*_{\mathcal{B};\gamma} \equiv \left(\lambda^*_{\mathcal{B};\gamma}\right)^{-1}. \tag{6.28}$$

Therefore, $\xi_{\mathcal{B};\gamma}$ absolutely converges for $|s| < S^*_{\mathcal{B};\gamma}$.

Furthermore, the reciprocal of $\zeta_{\mathcal{B};\gamma}$,

$$\zeta^{-1}_{\mathcal{B};\gamma} \equiv \prod_{n=1}^{\infty} \prod_{\lambda\in\Sigma(\mathbf{T}_{\gamma;n})} (1-\lambda s^n)^{\chi_{\gamma;n}(\lambda)} \tag{6.29}$$

is absolutely convergent in $|s| < S^*_{\mathcal{B};\gamma}$. The similar notations can also be introduced to $\widehat{\zeta}_{\mathcal{B};\gamma}$, the details are omitted here.

Accordingly, zeta functions $\zeta_{\mathcal{B};\gamma}$ have the following meromorphic property.

Theorem 6.9. *Given an admissible set* $\mathcal{B} \subset \Sigma_{2\times 2}$ *and* $\gamma \in GL_2(\mathbb{Z})$. *Then zeta function* $\zeta_{\mathcal{B};\gamma}$ *is meromorphic in* $|s| < \mathcal{S}^*_{\mathcal{B};\gamma}$ *and may have poles in* $\mathcal{P}_{\mathcal{B};\gamma} \cap \left\{s \in \mathbb{C} \mid |s| < \mathcal{S}^*_{\mathcal{B};\gamma}\right\}$, *i.e.,* $\left\{s \in \mathbb{C} \mid |s| < \mathcal{S}^*_{\mathcal{B};\gamma}\right\} \subset \mathcal{M}_{\mathcal{B};\gamma}$.

Proof. For each $s \notin \mathcal{P}_{\mathcal{B};\gamma}$ and $|s| < S^*_{\mathcal{B};\gamma}$, $\zeta_{\mathcal{B};\gamma}$ is convergent and has an isolated pole in $\mathcal{P}_{\mathcal{B};\gamma}$ when $|s| < \mathcal{S}^*_{\mathcal{B};\gamma}$, and then is meromorphic in $|s| < \mathcal{S}^*_{\mathcal{B};\gamma}$.

The proof is complete. □

Theorem 6.10. *Given admissible set* $\mathcal{B} \subset \Sigma_{2\times 2}$. *For any* γ *and* γ' *in* $GL_2(\mathbb{Z})$, *the zeta functions* $\zeta_{\mathcal{B};\gamma} = \zeta_{\mathcal{B};\gamma'}$ *in* $|s| < \min(\mathcal{S}^*_{\mathcal{B};\gamma}, \mathcal{S}^*_{\mathcal{B};\gamma'})$.

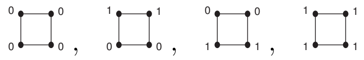
Proof. Since $\zeta_{\mathcal{B};\gamma}$ and $\zeta_{\mathcal{B};\gamma'}$ are meromorphic functions and are equal to $\zeta^0_{\mathcal{B}}$ on $|s| < \exp(-g(\mathcal{B}))$, by uniqueness theorem of meromorphic functions [31], they are equal on $|s| < \min(\mathcal{S}^*_{\mathcal{B};\gamma}, \mathcal{S}^*_{\mathcal{B};\gamma'})$. □

Remark 6.11. *Given* $\mathcal{B} \subset \Sigma_{2\times 2}$, *can we find a* $\gamma \in GL_2(\mathbb{Z})$ *such that* $\zeta_{\mathcal{B};\gamma}$ *is the maximum meromorphic extension of* $\zeta^0_{\mathcal{B}}$, *i.e, for any meromorphic extension* $\zeta'_{\mathcal{B}}$ *of* $\zeta^0_{\mathcal{B}}$, $\zeta_{\mathcal{B};\gamma}$ *is a meromorphic extension of* $\zeta'_{\mathcal{B}}$. *In particular, for any* $\gamma' \in GL_2(\mathbb{Z})$, $\mathcal{M}_{\mathcal{B};\gamma'} \subseteq \mathcal{M}_{\mathcal{B};\gamma}$? *Furthermore, is there* $\gamma \in GL_2(\mathbb{Z})$ *such that* $\zeta_{\mathcal{B};\gamma}$ *admits a natural boundary? These two problems are closely related. The answers are not clear. See examples studied in subsection 6.2 and section 7.*

6.2. **EXAMPLES.** This subsection presents some examples to elucidate the methods described above.

Example 6.12. Consider

(6.30) $$\mathcal{B} = \left\{ \quad , \quad , \quad , \quad \right\}.$$



Clearly,

(6.31) $$\mathbf{H}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{V}_2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

First, $\Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\right)$ and $\Gamma_{\mathcal{B}}\left(\begin{bmatrix} k & 0 \\ l & n \end{bmatrix}\right)$ are computed directly. Indeed, $\mathcal{B}$-admissible patterns have the same symbols in each row of the lattice, as presented in Fig 6.1. Then,

(6.32) $$\Gamma_{\mathcal{B}}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\right) = 2^k \quad \text{for any} \quad 0 \le l \le n-1$$

and

(6.33) $$\Gamma_{\mathcal{B}}\left(\begin{bmatrix} k & 0 \\ l & n \end{bmatrix}\right) = 2^{(n,l)} \quad \text{for any} \quad 1 \le l \le n-1,$$

where (n, l) is the greatest common divisor of n and l, are easily verified.

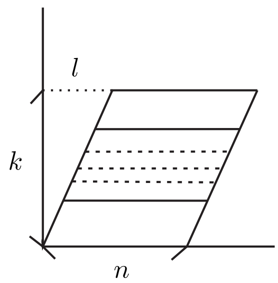


Fig 6.1 (a).

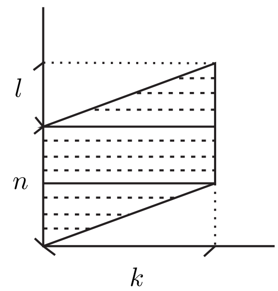


Fig 6.1 (b).

Consequently, for any $n \ge 1$,

$$\begin{aligned} \zeta_n(s) &= \exp\left(\frac{1}{n}\sum_{k=1}^{\infty}\frac{n2^k}{k}s^{kn}\right) \\ &= (1-2s^n)^{-1} \end{aligned} \tag{6.34}$$

and the zeta function $\zeta(s) = \prod_{n=1}^{\infty}(1-2s^n)^{-1}$ with $S^* = 1$, which was obtained by Lind in [21].

However, (6.33) implies

$$\widehat{\zeta}_n(s) = \exp\left(\left(\frac{1}{n}\sum_{l=1}^{n} 2^{(n,l)}\right)\sum_{k=1}^{\infty}\frac{s^{kn}}{k}\right)$$

$$= (1-s^n)^{-\widehat{\chi}_n}, \tag{6.35}$$

and the zeta function $\widehat{\zeta}(s) = \prod_{n=1}^{\infty}(1-s^n)^{-\widehat{\chi}_n}$, where

$$\widehat{\chi}_n = \frac{1}{n}\sum_{l=1}^{n} 2^{(n,l)}. \tag{6.36}$$

Now, it is easy to check that $\lim_{n\to\infty}(\widehat{\chi}_n)^{\frac{1}{n}} = 2$. Therefore, $\widehat{S}^* = \frac{1}{2}$ as in (6.27) and (6.28) for $\widehat{\zeta}(s)$.

Theorem 6.4 implies that the zeta function $\zeta^0_{\mathcal{B}}(s)$ of $\mathcal{B}$ given by (6.30) is

$$\zeta^0_{\mathcal{B}}(s) = \prod_{n=1}^{\infty}(1-2s^n)^{-1} = \prod_{n=1}^{\infty}(1-s^n)^{-\widehat{\chi}_n} \tag{6.37}$$

in $|s| < \frac{1}{2}$.

The natural boundary of (6.37) is $|s| = 1$ and ζ has poles

$$\left\{2^{-\frac{1}{n}}e^{2\pi ij/n} : 0 \le j \le n-1, n \ge 1\right\},$$

as described elsewhere [21].

However, (6.31) implies

$$\mathbf{T}_2 = \mathbf{V}_2 \quad \text{and} \quad \widehat{\mathbf{T}}_2 = \mathbf{H}_2.$$

Furthermore,

$$\mathbf{T}_n = \begin{bmatrix} 1 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 1 \end{bmatrix}_{2^n \times 2^n} \tag{6.38}$$

and

$$\widehat{\mathbf{T}}_n = I_{2^n}, \tag{6.39}$$

where I_{2^n} is the $2^n \times 2^n$ identity matrix.

From (6.38), $\lambda_{n,1} = 2$ and $\lambda_{n,j} = 0$, $2 \le j \le 2^n$. Then, the R_n-symmetry eigenvector of $\lambda_{n,1}$ can be chosen as $(1, 0, \cdots, 0, 1)^t$. Therefore, $\chi(\lambda_{n,1}) = 1$. Hence, (6.34) follows.

As for $\widehat{\mathbf{T}}_n$, $\lambda_{n,j} = 1$ for $1 \le j \le 2^n$. Furthermore, for each $i \in \mathcal{I}_n$, define eigenvector $U_i = (u_{i,j})^t$ where $u_{i,j} = 1$ if $j \in C_n(i)$ and $u_{i,j} = 0$ if $j \notin C_n(i)$. Clearly, U_i is R_n-symmetric and $\chi(\lambda_{n,i}) = 1$. Therefore,

$$\widehat{\zeta}_n(s) = (1-s^n)^{-\chi_n}, \tag{6.40}$$

where χ_n is the cardinal number of $\mathcal{I}_n$. Now, (6.36) and (6.40) imply $\widehat{\chi}_n = \chi_n$, i.e.,

$$\chi_n = \frac{1}{n}\sum_{l=1}^{n} 2^{(n,l)}. \tag{6.41}$$

Moreover, (6.39) implies

$$\begin{aligned}\widehat{\zeta}_n(s) &= \exp\left(\frac{1}{n}\sum_{k=1}^{\infty}\frac{tr(\mathbf{R}_n)}{k}s^{kn}\right)\\ &= \exp\left(\frac{1}{n}tr(\mathbf{R}_n)\sum_{k=1}^{\infty}\frac{s^{kn}}{k}\right)\\ &= \frac{1}{n}tr(\mathbf{R}_n)(1-s^n)^{-1}.\end{aligned}$$

Therefore, (6.40) implies

$$\frac{1}{n}tr(\mathbf{R}_n) = \chi_n. \tag{6.42}$$

Hence,

$$tr(\mathbf{R}_n) = \sum_{l=1}^{n} 2^{(n,l)}. \tag{6.43}$$

The equality (6.14) of the Taylor series of $\prod_{n=1}^{\infty}(1-2s^n)^{-1}$ and $\prod_{n=1}^{\infty}(1-s^n)^{-\chi_n}$ yield some identity for χ_n. Indeed, let

$$\prod_{n=1}^{\infty}(1-s^n)^{-\chi_n} = \sum_{n=0}^{\infty}\widehat{a}_n s^n$$

and

$$\prod_{n=1}^{\infty}(1-2s^n)^{-1} = \sum_{n=0}^{\infty}a_n s^n.$$

Now,

$$\widehat{a}_n = a_n \text{ for any } n \geq 0. \tag{6.44}$$

The expressions for $\widehat{a}_n$ and a_n are omitted here.

The following example can also be solved explicitly and is helpful in elucidating the natural boundary and location of the poles of the zeta function.

Example 6.13. Consider

$$\mathbf{H}_2 = \begin{bmatrix} 1 & 1 & 1 & 0\\ 1 & 1 & 1 & 0\\ 1 & 1 & 1 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{6.45}$$

Then,

$$\mathbf{V}_2 = \begin{bmatrix} 1 & 1 & 1 & 1\\ 1 & 0 & 1 & 0\\ 1 & 1 & 0 & 0\\ 1 & 0 & 0 & 0 \end{bmatrix} = G \otimes G, \tag{6.46}$$

where

$$G = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \tag{6.47}$$

is the one-dimensional golden-mean matrix, which has eigenvalues

$$g = \tfrac{1+\sqrt{5}}{2} \quad \text{and} \quad \overline{g} = \tfrac{1-\sqrt{5}}{2} = -g^{-1}. \tag{6.48}$$

Now,

$$\widetilde{\mathbf{H}}_2 = \mathbf{V}_2 \quad \text{and} \quad \widetilde{\mathbf{V}}_2 = \mathbf{H}_2. \tag{6.49}$$

Then,

$$\mathbf{T}_2 = \mathbf{V}_2 \circ \widetilde{\mathbf{H}}_2 = \mathbf{V}_2 = G \otimes G$$

can be verified, and for any $n \geq 2$,

$$\mathbf{T}_n = G \underbrace{\otimes G \otimes \cdots \otimes G \otimes}_{n-1 \text{ times} \otimes} G = \overset{n-1}{\otimes} G, \tag{6.50}$$

which is the $n-1$ times Kronecker product of G.

The eigenvalues of $\mathbf{T}_n$ are given by

$$\lambda_{\overline{\varepsilon}} = \left(\prod_{j=1}^{n} \varepsilon_j \right) g^{\varepsilon_1} g^{\varepsilon_2} \cdots g^{\varepsilon_n}, \tag{6.51}$$

where $\overline{\varepsilon} = \varepsilon_1 \varepsilon_2 \cdots \varepsilon_n$ is an n-sequence with $\varepsilon_j \in \{-1, 1\}$. The corresponding eigenvector of (6.51) is

$$U_{\overline{\varepsilon}} = \begin{pmatrix} \varepsilon_1 g^{\varepsilon_1} \\ 1 \end{pmatrix} \otimes \begin{pmatrix} \varepsilon_2 g^{\varepsilon_2} \\ 1 \end{pmatrix} \otimes \cdots \begin{pmatrix} \varepsilon_n g^{\varepsilon_n} \\ 1 \end{pmatrix}. \tag{6.52}$$

Clearly, $\{U_{\overline{\varepsilon}}\}$ are linearly independent.

The total number of $\overline{\varepsilon}$ is 2^n and the spectrum of $\mathbf{T}_n$ is

$$\Sigma(\mathbf{T}_n) = \{ g^{n-j} \cdot \overline{g}^j | 0 \leq j \leq n \}, \tag{6.53}$$

which has n+1 members. In fact, $\overline{\varepsilon}$ has

$$j = \sum_{i=1}^{n} \frac{1}{2}(1 - \varepsilon_i) \tag{6.54}$$

many $\overline{g}$ and $n-j$ many g.

For $\overline{\varepsilon}$, the counting function

$$i = \widetilde{\psi}(\overline{\varepsilon}) = \sum_{k=1}^{n} \frac{1}{2}(1 - \varepsilon_k) 2^{n-k} \tag{6.55}$$

is defined when $1 \leq i \leq 2^n$ and denoted by $U_{n;i} = U_{\overline{\varepsilon}}$.

Define

$$\sigma(\overline{\varepsilon})=\varepsilon_2\varepsilon_3\cdots\varepsilon_n\varepsilon_1;$$

then,

$$\sigma(i)=\widetilde{\psi}(\sigma(\overline{\varepsilon})) \tag{6.56}$$

for any $1\leq i\leq 2^n$, where $\sigma(i)$ is defined in (2.35).

For any $i\in\mathcal{I}_n$,

$$\sum_{i=0}^{n-1}U_{n,\sigma^l(i)}=\sum_{i=0}^{n-1}R_n^lU_{n,i}\neq 0.$$

Then, Lemma 3.8 implies

$$\chi(\lambda_{n,i})=1$$

for $i\in\mathcal{I}_n$. Therefore,

$$\begin{aligned}\chi_{n,j}= &\ \chi(g^{n-j}\overline{g}^j)\\ = &\sum_{\substack{i\in\mathcal{I}_n\\ \lambda_{n,i}=g^{n-j}\overline{g}^j}}1.\end{aligned} \tag{6.57}$$

Clearly, $\chi_{n,0}=\chi_{n,n}=1$. Furthermore for any $1\leq j\leq n-1$, by Burnside's Lemma,

$$\chi_{n,j}=\frac{1}{n}\sum_{d|(j,n-j)}\phi((j,n-j)/d)C^{nd/(j,n-j)}_{jd/(j,n-j)}, \tag{6.58}$$

where ϕ is the Euler totient function (2.40). The detailed proof of (6.58) is omitted for brevity. Therefore,

$$\zeta_n(s)=\prod_{j=0}^{n}\left(1-g^{n-j}\overline{g}^js^n\right)^{-\chi_{n,j}} \tag{6.59}$$

and

$$\begin{aligned}\zeta(s)=&\prod_{n=1}^{\infty}\zeta_n(s)\\ =&\prod_{n=1}^{\infty}(1-g^ns^n)^{-1}\prod_{j=1}^{n}\left(1-g^{n-j}\overline{g}^js^n\right)^{-\chi_{n,j}}.\end{aligned} \tag{6.60}$$

Since $\lambda_{n,1}=g^n$ is the maximum eigenvalue of $\mathbf{T}_n$,

$$\lim_{n\to\infty}\lambda_{n,1}^{\frac{1}{n}}=g. \tag{6.61}$$

From (6.58),

$$\limsup_{n\to\infty}\max_{0\leq j\leq n}\left(\left|g^{n-j}\overline{g}^j\chi_{n,j}\right|\right)^{\frac{1}{n}}=2,$$

which implies $S^*=\frac{1}{2}$ in (6.28) and $S^*<g^{-1}$.

Now, consider $\widehat{\mathbf{T}}_n$ and the associated zeta function $\widehat{\zeta}(s)$.
Clearly,

$$\widehat{\mathbf{T}}_2 = \mathbf{H}_2 \circ \widetilde{V}_2 = \mathbf{H}_2.$$

To study higher-order $\widehat{\mathbf{T}}_n$, $n \geq 3$, the recursive formula of $\mathbf{H}_n$ must be obtained.
Let

$$\mathbf{H}_n = \begin{bmatrix} H_{n;1} & H_{n;2} \\ H_{n;3} & H_{n;4} \end{bmatrix}. \tag{6.62}$$

Now,

$$\begin{aligned}
\mathbf{H}_{n+1} &= \begin{bmatrix} H_{n+1;1} & H_{n+1;2} \\ H_{n+1;3} & H_{n+1;4} \end{bmatrix} \\
&= \begin{bmatrix} H_{n;1} & H_{n;2} & H_{n;1} & 0 \\ H_{n;3} & H_{n;4} & H_{n;3} & 0 \\ H_{n;1} & H_{n;2} & H_{n;1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
&= \left[\begin{array}{cc} \multicolumn{2}{c}{\mathbf{H}_n} & \begin{array}{cc} \mathbf{H}_{n-1} & 0 \\ H_{n;3} & 0 \end{array} \\ \begin{array}{cc} \mathbf{H}_{n-1} & H_{n;2} \\ 0 & 0 \end{array} & \begin{array}{cc} \mathbf{H}_{n-1} & 0 \\ 0 & 0 \end{array} \end{array} \right].
\end{aligned} \tag{6.63}$$

If the zero rows and columns are deleted from $\mathbf{H}_n$, then clearly the remaining matrix is a $r_n \times r_n$ full matrix E_{r_n}, where r_n is the sum of the entries of the first row of $\mathbf{H}_n$. The maximum eigenvalue $\rho(\mathbf{H}_n)$ equals r_n.
Therefore, (6.63) implies

$$r_{n+1} = r_n + r_{n-1}, \tag{6.64}$$

where $r_2 = 3$ and $r_3 = 5$.
Furthermore, for $n \geq 2$,

$$\widehat{\mathbf{T}}_n = \begin{bmatrix} H_{n;1} & H_{n;2} \\ H_{n;3} & H_{n;4} \end{bmatrix} \circ \begin{bmatrix} \overset{n-2}{\otimes} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} & \overset{n-2}{\otimes} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \\ \overset{n-2}{\otimes} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} & \overset{n-2}{\otimes} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix}. \tag{6.65}$$

The remaining matrix of $\widehat{\mathbf{T}}_n$ can be verified to be a full matrix $E_{\widehat{r}_n}$ after the zero rows and columns have been deleted, where $\widehat{r}_n$ is the sum of entries in the first row of $\widehat{\mathbf{T}}_n$. Hence, the maximum eigenvalue $\widehat{\lambda}_n$ of $\widehat{\mathbf{T}}_n$ equals $\widehat{r}_n$, the other eigenvalues are zeros.
Clearly, $\widehat{r}_2 = 3$,

$$\widehat{\mathbf{T}}_3 = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and $\widehat{r}_3 = 4$. Since

$$H_{n+1;2} = \begin{bmatrix} \mathbf{H}_{n-1} & 0 \\ H_{n;3} & 0 \end{bmatrix},$$

the first row of $H_{n+1;2} \overset{n-1}{\otimes} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ can be verified to be the same as the first row in $\mathbf{H}_{n-2}$. Hence,

$$\widehat{r}_{n+1} = r_n + r_{n-2}. \tag{6.66}$$

Combining (6.64) with (6.66) and

$$\widehat{\lambda}_{n+1} = \widehat{\lambda}_n + \widehat{\lambda}_{n-1} \tag{6.67}$$

with

$$\widehat{\lambda}_2 = 3 \text{ and } \widehat{\lambda}_3 = 4 \tag{6.68}$$

yields

$$\widehat{\zeta}_n(s) = (1 - \widehat{\lambda}_n s^n)^{-1} \tag{6.69}$$

and

$$\widehat{\zeta}(s) = \prod_{n=1}^{\infty} (1 - \widehat{\lambda}_n s^n)^{-1}. \tag{6.70}$$

Now, $\widehat{\lambda}_n$ and g^n must be compared. Let

$$g^n = \alpha_n g + \beta_n$$

with $\alpha_2 = \beta_2 = 1$. Then, $\alpha_{n+1} = \alpha_n + \beta_n$ and $\beta_{n+1} = \alpha_n$, or $\alpha_{n+1} = \alpha_n + \alpha_{n-1}$ with $\alpha_3 = 2$. That

$$\widehat{\lambda}_n = \alpha_n + 2\beta_n$$

can be verified and

$$\widehat{\lambda}_{n+1} - g^{n+1} = -\left(\frac{(\sqrt{5}-1)\alpha_{n+1} + 2\beta_n}{(\sqrt{5}-1)\alpha_n + 2\beta_{n-1}} \right) (\widehat{\lambda}_n - g^n). \tag{6.71}$$

Equation (6.71) implies

$$\widehat{\lambda}_{2n}^{-\frac{1}{2n}} < g^{-1} < \widehat{\lambda}_{2n+1}^{-\frac{1}{2n+1}}. \tag{6.72}$$

Equation (6.72) implies that the meromorphic extension $\widehat{\zeta}$ of $\zeta^0_{\mathcal{B}}$ satisfies $\widehat{S}^* = g^{-1}$ and has poles on $\left\{\widehat{\lambda}_{2n}^{-\frac{1}{2n}} e^{\pi ij/n} : 0 \leq j \leq 2n-1, n \geq 1\right\}$ with the natural boundary $|s| = g^{-1}$.

7. Equations on $\mathbb{Z}^2$ with numbers in a finite field

This section briefly discusses the equations on $\mathbb{Z}^2$ with numbers in a finite field, see [15, 21, 33]. The problems can be studied by applying the methods that were developed in the previous sections. Consider first the following example.

Example 7.1. Let $F_2 = \{0, 1\}$ be the field with two elements and

$$\mathbb{X} = \left\{ x \in F_2^{\mathbb{Z}^2} : x_{i,j} + x_{i+1,j} + x_{i,j+1} + x_{i+1,j+1} = 0 \text{ for all } i, j \in \mathbb{Z} \right\}. \tag{7.1}$$

Then, $\mathbb{X}$ is a compact group with coordinate-wise operations, and it is invariant under the natural $\mathbb{Z}^2$-shift action σ.

The equation

$$x_{i,j} + x_{i+1,j} + x_{i,j+1} + x_{i+1,j+1} = 0 \tag{7.2}$$

is now interpreted as a pattern generation problem on $\mathbb{Z}_{2\times 2}$. The solutions of (7.2) are clearly given by

$$\mathcal{B} = \left\{ \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}, \begin{array}{cc} 0 & 1 \\ 0 & 1 \end{array}, \begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array}, \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}, \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}, \begin{array}{cc} 0 & 0 \\ 1 & 1 \end{array}, \begin{array}{cc} 1 & 0 \\ 1 & 0 \end{array}, \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right\}, \tag{7.3}$$

which consists of all even patterns of $\mathbb{Z}_{2\times 2}$.

$\mathcal{B}$ is the basic set of admissible local patterns determined by (7.2). The set of all global patterns $\Sigma(\mathcal{B})$ generated by $\mathcal{B}$ is exactly $\mathbb{X}$:

$$\Sigma(\mathcal{B}) = \mathbb{X}. \tag{7.4}$$

Hence, the zeta function $\zeta_{\mathcal{B}}(s)$ of $\mathcal{B}$ can be derived as in (3.50).

$$\mathbf{H}_2 = \mathbf{H}_2(\mathcal{B}) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} I_2 & J_2 \\ J_2 & I_2 \end{bmatrix}.$$

That $\mathbf{H}_2 = \mathbf{V}_2 = \widetilde{\mathbf{H}}_2 = \widetilde{\mathbf{V}}_2$ can be easily checked; then $\mathbf{T}_n = \widehat{\mathbf{T}}_n$ for all $n \geq 1$. For $n \geq 1$,

$$I_{2^n} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}_{2^n \times 2^n} \quad \text{and} \quad J_{2^n} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}_{2^n \times 2^n}.$$

Then, $\mathbf{T}_n = I_{2^n} + J_{2^n}$ for all $n \geq 1$. Moreover, $\sigma(\mathbf{T}_n) = \{2, 0\}$ for $n \geq 1$, and the algebraic multiplicities of $\lambda = 2$ and $\lambda = 0$ of $\mathbf{T}_n$ are both equal to 2^{n-1}.

For $n \geq 1$, consider the $2^n \times 2^n$ matrix

$$\mathbf{U}_n = [U_1, \cdots, U_{2^n}] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & & 0 & 0 & & 0 & 0 & 1 \\ 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 1 & & 0 & 0 & & 1 & 0 & 0 \\ & \vdots & & & \vdots & \vdots & & & \vdots & \\ 0 & 0 & 0 & & 1 & 1 & & 0 & 0 & 0 \\ 0 & 0 & 0 & & 1 & -1 & & 0 & 0 & 0 \\ & \vdots & & & \vdots & \vdots & & & \vdots & \\ 0 & 0 & 1 & & 0 & 0 & & -1 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & -1 & 0 \\ 1 & 0 & 0 & & 0 & 0 & & 0 & 0 & -1 \end{bmatrix}.$$

Then, $\mathbf{T}_n$ is expressed in Jordan canonical form as $\mathbf{T}_n = \mathbf{U}_n \mathbf{J}_n \mathbf{U}_\mathbf{n}$, where

$$\begin{cases} \mathbf{J}_{n;i,i} = 2 & \text{for } 1 \leq i \leq 2^{n-1}, \\ \mathbf{J}_{n;i,j} = 0 & \text{otherwise.} \end{cases}$$

From (3.10),

$$\begin{aligned} \chi_n(2) &= \frac{1}{n} \sum_{j=1}^{2^{n-1}} |\mathbf{R}_n \circ U_j U_j^t| \\ &= \frac{1}{2n} |\mathbf{R}_n \circ (I_{2^n} + J_{2^n})| \\ &= \frac{1}{2n} tr(\mathbf{R}_n) + \frac{1}{2n} |\mathbf{R}_n \circ J_{2^n}|. \end{aligned}$$

Since $\frac{1}{2n} tr(\mathbf{R}_n) = \frac{\chi_n}{2}$ by (6.42) and $\frac{1}{2n} |\mathbf{R}_n \circ J_{2^n}| \geq 0$,

$$\lim_{n \to \infty} \chi_n(2)^{\frac{1}{n}} = 2. \tag{7.5}$$

In fact,

$$\chi_n(2) = \sum_{d|n} \frac{\phi(2d) 2^{n/d}}{2n}, \tag{7.6}$$

where $\phi(n)$ is the Euler totient function.

Then,

$$\zeta_{\mathcal{B}}(s) = \prod_{n=1}^{\infty} \frac{1}{(1 - 2s^n)^{\chi_n(2)}}. \tag{7.7}$$

Equation (7.6) implies that $S^* = \frac{1}{2}$ and $\zeta_{\mathcal{B}}$ is analytic in $|s| < \frac{1}{2}$. However, it is not clear whether there is a $\gamma \in GL_2(\mathbb{Z})$ such that $\zeta_{\mathcal{B};\gamma}$ has poles in $|s| \geq \frac{1}{2}$ and has the natural boundary. Further investigation must be performed later.

Lind [21] considered the following example.

Example 7.2. Consider $F_2 = \{0, 1\}$ and

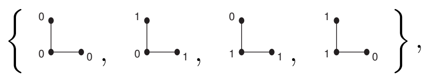
$$\mathbb{X} = \left\{ x \in F_2^{\mathbb{Z}^2} : x_{i,j} + x_{i+1,j} + x_{i,j+1} = 0 \text{ for all } i, j \in \mathbb{Z} \right\}. \tag{7.8}$$

In this case, $\mathbb{X}$ is also a compact group with coordinate-wise operations, and it is invariant under the natural $\mathbb{Z}^2$-shift action σ.

The equation

$$x_{i,j} + x_{i+1,j} + x_{i,j+1} = 0 \tag{7.9}$$

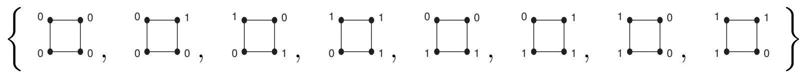
can be interpreted as a pattern generation problem on L-shape lattices: , as in Lin and Yang [**?**]. Indeed, the solutions of (7.9) are given by

(7.10) $\mathcal{B}(\mathbb{L}) =$ 

,

which consists of all even patterns on L-shape lattices.

$\mathcal{B}(\mathbb{L})$ can be extended to $\mathbb{Z}_{2\times 2}$ as

(7.11)
$\mathcal{B} =$ 

.

That

$$\Sigma(\mathcal{B}) = \mathbb{X} \tag{7.12}$$

can be easily verified.

Therefore,

$$\mathbf{H}_2 = \mathbf{H}_2(\mathcal{B}) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} = \mathbf{V}_2 \tag{7.13}$$

and

$$\widetilde{\mathbf{H}}_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} = \widetilde{\mathbf{V}}_2. \tag{7.14}$$

According to (2.58),

$$\mathbf{T}_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \zeta_1(s) = \tfrac{1}{1-s}.$$

For $\mathbf{T}_2 = \mathbf{V}_2 \circ \widetilde{\mathbf{H}}_2$, (7.13) and (7.14) imply

$$\mathbf{T}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{7.15}$$

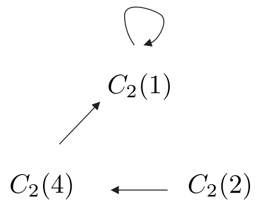
The eigenvalue λ of $\mathbf{T}_2$ with its $\chi_2(\lambda)$ is found and the reduced trace operator τ_2, introduced in (3.52), is applied as follows.

Let $V = (v_1, v_2, v_3, v_4)^t$ be an eigenvector of $\mathbf{T}_2$ with eigenvalue λ. Then,

$$\begin{cases} v_1 = \lambda v_1 \\ v_4 = \lambda v_2 \\ v_4 = \lambda v_3 \\ v_1 = \lambda v_4. \end{cases} \tag{7.16}$$

If V is a rotationally symmetric eigenvector as in Example 2.2 (i), then $v_2 = v_3$. The equivalent classes are $C_2(1) = \{1\}$, $C_2(2) = \{2, 3\}$ and $C_2(4) = \{4\}$.

Let $C_2(j)$, $j = 1, 2, 4$, be vertices; write $v_j = \lambda v_i$ as $C_2(i) \to C_2(j)$, which describes an edge from $C_2(i)$ to $C_2(j)$. Then (7.16) can be plotted as in Fig 7.1. τ_2 is the reduced trace operator of $\mathbf{T}_2$.



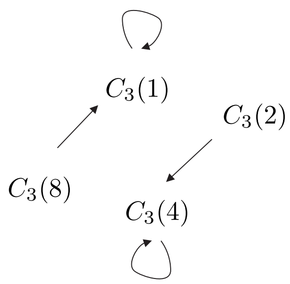
and $\tau_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

Fig 7.1.

Figure 7.1 includes only one cycle $C_2(1) \to C_2(1)$ with period 1. Then, $\lambda = 1$ is an eigenvalue with $\chi_2(1) = 1$. Hence,

$$\zeta_2(s) = \frac{1}{1 - s^2}.$$

For $n = 3$, the equivalent classes $C_3(1) = \{1\}$, $C_3(2) = \{2, 3, 5\}$, $C_3(4) = \{4, 7, 6\}$ and $C_3(8) = \{8\}$ are vertices. $\mathbf{T}_3$ generates the graph in Fig 7.2 and the reduced trace operator τ_3.



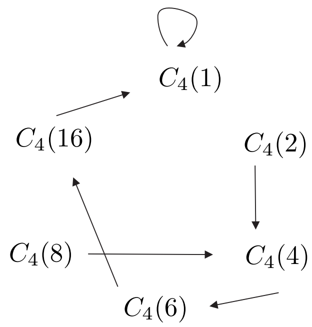
and $\tau_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

Fig 7.2.

Figure 7.2 includes two cycles with period 1. Therefore,

$$\zeta_3(s) = \frac{1}{(1 - s^3)^2}.$$

For $n = 4$, $\mathbf{T}_4$ generates the graph in Fig 7.3 and the reduced trace operator τ_4.



and

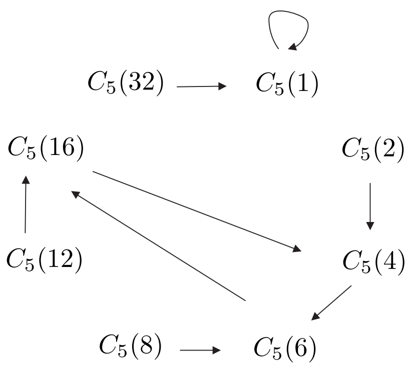
$$\tau_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Fig 7.3.

Figure 7.3 includes only one cycle with period 1. Hence,

$$\zeta_4(s) = \frac{1}{1 - s^4}.$$

For $n = 5$, $\mathbf{T}_5$ generates the graph in Fig 7.4 and the reduced trace operator τ_5.



and

$$\tau_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Fig 7.4.

Figure 7.4 only includes one cycle with period 1 and one cycle with period 3. Therefore,

$$\zeta_5(s) = \frac{1}{(1 - s^5)(1 - s^5)(1 - \rho_3 s^5)(1 - \rho_3^2 s^5)} = \frac{1}{(1 - s^5)(1 - s^{15})}, \tag{7.17}$$

where $\rho_3 = e^{\frac{2\pi i}{3}}$, ($\rho_3^3 = 1$).

In general, for any $n \geq 1$, induction can be used to show that each row of $\mathbf{T}_n$ has exactly a single 1 and each column has either two 1s or all 0s.

Therefore, the eigenvalue λ of $\mathbf{T}_n$ is $|\lambda| = 1$ or $\lambda = 0$. By a similar argument as for $n = 2$ to 5, for the eigenvalue λ with a rotationally symmetric eigenvector, $\mathbf{T}_n$ generates the graph with equivalent classes $C_n(i)$ as vertices and has $m(n)$ disjoint cycles; each cycle has period $p_{n,k} \geq 1$, $1 \leq k \leq m(n)$. In computing, it is more efficient to compute $\lambda \in \Sigma(\tau_n)$ with algebraic multiplicity $\chi(\lambda)$.

The following can be demonstrated

$$\zeta_n(s) = \prod_{k=1}^{m(n)} \frac{1}{(1-\rho_{n,k}s^n)\cdots(1-\rho_{n,k}^{\chi_{n,k}-1}s^n)(1-s^n)} = \prod_{k=1}^{m(n)} \frac{1}{1-s^{np_{n,k}}}, \tag{7.18}$$

where $\rho_{n,k} = e^{\frac{2\pi i}{p_{n,k}}}$.

Hence,

$$\zeta(s) = \prod_{n=1}^{\infty} \prod_{k=1}^{m(n)} \frac{1}{1-s^{np_{n,k}}}. \tag{7.19}$$

For $n = 1$ to 20, the numbers and periods of cycles are listed in Table 7.1.

n	1	2	3	4	5		6		7		8	9		10		
p	1	1	1	1	1	3	1	2	1	7	1	1	7	1	3	6
q	1	1	2	1	1	1	2	1	3	1	1	2	4	1	1	4

n	11		12			13		14				15			16
p	1	31	1	2	4	1	63	1	2	7	14	1	3	15	1
q	1	3	2	1	5	1	5	3	4	1	20	4	4	72	1

n	17			18				19		20			
p	1	5	15	1	2	7	14	1	511	1	3	6	12
q	1	3	256	2	1	4	259	1	27	1	1	4	272

p : the period of cycle.
$q = q(p)$: the number of cycles with period p.
Table 7.1.

From Table 7.1, ζ_n can be written for $1 \leq n \leq 20$. For example,

$$\zeta_{13} = \frac{1}{(1-s^{13})(1-s^{819})^5}$$

and

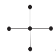
$$\zeta_{14} = \frac{1}{(1-s^{14})^3(1-s^{28})^4(1-s^{98})(1-s^{196})^{20}}.$$

Up to $n = 20$, the Taylor expansion of (7.19) at $s = 0$, which recovers Lind's result [21] (p.438), is

$$\begin{aligned} \zeta_{\mathcal{B}}(s) = {} & 1 + s + 2s^2 + 4s^3 + 6s^4 + 9s^5 + 16s^6 + 24s^7 + 35s^8 + 54s^9 \\ & + 78s^{10} + 110s^{11} + 162s^{12} + 226s^{13} + 317s^{14} + 446s^{15} + 612s^{16} \\ & + 834s^{17} + 1146s^{18} + 1543s^{19} + 2071s^{20} + \cdots. \end{aligned} \tag{7.20}$$

Further investigation is needed to understand τ_n and $p_{n,k}$ for large n. The results will appear elsewhere.

Lind [21] showed that the zeta function ζ^0 defined by (7.9) is analytic in $|s| < 1$. By (7.19), all poles of ζ appear on $|s| = 1$. Therefore, ζ is analytic in $|s| < 1$ with natural boundary $|s| = 1$.

In the following example, the harmonic patterns on square-cross lattice $\mathbb{L}$: , which were studied by Ledrappier [15], are investigated.

Example 7.3. Let $F_2 = \{0,1\}$ and

$$\mathbb{X} = \left\{ x \in F_2^{\mathbb{Z}^2} : x_{i,j} = x_{i-1,j} + x_{i,j-1} + x_{i+1,j} + x_{i,j+1} \text{ for all } i,j \in \mathbb{Z} \right\}. \tag{7.21}$$

Clearly, (7.21) can be written as

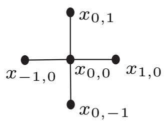
$$\left\{ x \in F_2^{\mathbb{Z}^2} : x_{i,j} + x_{i-1,j} + x_{i,j-1} + x_{i+1,j} + x_{i,j+1} = 0 \text{ for all } i,j \in \mathbb{Z} \right\}. \tag{7.22}$$

Then, $\mathbb{X}$ is also a compact group with coordinate-wise operations, and it is invariant under the natural $\mathbb{Z}^2$-shift action σ.

The equation

$$x_{i,j} + x_{i-1,j} + x_{i,j-1} + x_{i+1,j} + x_{i,j+1} = 0 \tag{7.23}$$

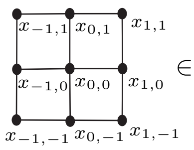
can be interpreted as a pattern generation problem on a square-cross lattice. Indeed, the basic set of (7.23) on $\mathbb{L}$ is

(7.24) $\mathcal{B}(\mathbb{L}) = \Bigg\{$ 

$\in F_2^{\mathbb{L}} : x_{0,0} + x_{-1,0} + x_{0,-1} + x_{1,0} + x_{0,1} = 0 \Bigg\}$,

which consists of all even patterns on a square-cross lattice.

$\mathcal{B}(\mathbb{L})$ can be extended to $\mathbb{Z}_{3\times 3}$ as

(7.25) $\mathcal{B} = \Bigg\{$ 

$\in F_2^{\mathbb{Z}_{3\times 3}} : x_{0,0} + x_{-1,0} + x_{0,-1} + x_{1,0} + x_{0,1} = 0 \Bigg\}$.

Then, that

$$\Sigma(\mathcal{B}) = \mathbb{X} \tag{7.26}$$

can be easily verified.

Now, by (4.6), the associated trace operator $\mathbf{T}_{n\times 3}(\mathcal{B})$ can be constructed for $n \geq 1$. Furthermore, the rotational matrix $R_{n\times 2}$ is defined by (4.12) with

$$\sigma_n(i) = \begin{cases} 1 + 4(i-1) & \text{for } 1 \leq i \leq 2^{2n-2}, \\ 2 + 4(i - 2^{2n-2} - 1) & \text{for } 2^{2n-2} + 1 \leq i \leq 2^{2n-1}, \\ 3 + 4(i - 2^{2n-1} - 1) & \text{for } 2^{2n-1} + 1 \leq i \leq 3 \cdot 2^{2n-2}, \\ 4 + 4(i - 3 \cdot 2^{2n-2} - 1) & \text{for } 3 \cdot 2^{2n-2} + 1 \leq i \leq 2^{2n}. \end{cases} \tag{7.27}$$

The number $\chi_{n\times 2}$ of the equivalent classes of $R_{n\times 2}$ can be shown to be the number of n-bead necklaces with four colors. The formulae for $\chi_{n\times 2}$, $n \geq 1$, is given by

$$\chi_{n\times 2} = \frac{1}{n} \sum_{d|n} \phi(d) 4^{n/d}. \tag{7.28}$$

See [34].

As in Example 7.2, the reduced trace operator $\tau_{n\times 3}$ of $\mathbf{T}_{n\times 3}$ is more convenient for computing the n-th order zeta function ζ_n. The definition and results of the reduced trace operator for more symbols on larger lattices are similar to Definition 3.9 and Theorem 3.12.

For $n=1$, $\mathbf{T}_{1\times 3}$ generates the of graph of equivalent classes in Fig 7.5 and the reduced trace operator $\tau_{1\times 3}$.

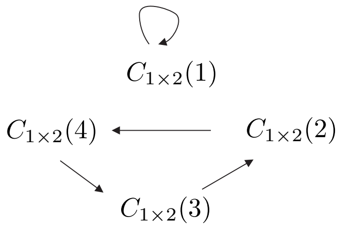


Fig 7.5.

and $\tau_{1\times 3}=\mathbf{T}_{1\times 3}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

The graph in Fig 7.5 has one cycle of period 1 and one cycle of period 3. Hence,

$$\zeta_1(s)=\frac{1}{(1-s)(1-s^3)}.$$

For $n=2$, $\mathbf{T}_{2\times 3}$ generates the of graph of equivalent classes in Fig 7.6 and the reduced trace operator $\tau_{2\times 3}$.

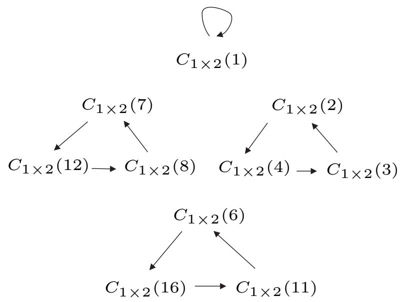


Fig 7.6.

and $\tau_{2\times 3}=\begin{bmatrix} 1&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0 \end{bmatrix}$.

The graph in Fig 7.6 has one cycle of period 1 and three cycles of period 3. Hence,

$$\zeta_2(s)=\frac{1}{(1-s^2)(1-s^6)^3}.$$

For $n=3$, the reduced trace operator $\tau_{3\times 3}$ is a 24×24 matrix and can be used to indicate that the graph of equivalent classes generated by $\mathbf{T}_{3\times 3}$ has two cycles of period 1, two cycles of period 2, two cycles of period 3 and two cycles of period 6. Hence,

$$\zeta_3(s)=\frac{1}{(1-s^3)^2(1-s^6)^2(1-s^9)^2(1-s^{18})^2}.$$

From (7.22), if $x_{i,j-1}$ and $x_{i,j}$ are given for some $j\in\mathbb{Z}$ and for all $i\in\mathbb{Z}$, $x_{i,j+1}$ is determined for all $i\in\mathbb{Z}$. Therefore, the trace operator $\mathbf{T}_{n\times 3}$ is a permutation matrix. Furthermore, the reduced trace operator $\tau_{n\times 3}$ of $\mathbf{T}_{n\times 3}$ is also a permutation matrix. Hence, $|\lambda| = 1$ for all $\lambda \in \Sigma(\tau_{n\times 3})$.

By the same argument as in Example 7.2, let the graph generated by $\mathbf{T}_{n\times 3}$ have $m(n)$ disjoint cycles, each of period $p_{n,k} \geq 1$, for $1 \leq k \leq m(n)$. Then, the n-th order zeta function can be represented as

$$\zeta_n(s) = \prod_{k=1}^{m(n)} \frac{1}{1 - s^{np_{n,k}}}. \tag{7.29}$$

Hence,

$$\zeta(s) = \prod_{n=1}^{\infty}\prod_{k=1}^{m(n)} \frac{1}{1 - s^{np_{n,k}}}. \tag{7.30}$$

Table 7.2 presents the numbers and periods of cycles of $\mathbf{T}_{n\times 3}$. For brevity, only $n = 1$ to 10 are listed.

n	1		2		3				4			5			
p	1	3	1	3	1	2	3	6	1	3	6	1	3	5	15
q	1	1	1	3	2	2	2	2	1	7	8	7	7	9	9

n	6						7			8			
p	1	2	3	4	6	12	1	3	9	1	3	6	12
q	2	6	6	8	10	48	1	1	260	1	7	88	640

n	9							
p	1	2	3	6	7	14	21	42
q	2	2	2	2	260	390	260	390

n	10							
p	1	2	3	5	6	10	15	30
q	7	24	21	9	120	648	27	3240

p : the period of cycle.
$q = q(p)$: the number of cycles with period p.
Table 7.2.

Up to $n = 16$, the Taylor expansion of (7.30) at $s = 0$ is

$$\zeta_{\mathcal{B}}(s) = 1 + s + 2s^2 + 5s^3 + 7s^4 + 17s^5 + 32s^6 + 46s^7 + 84s^8 + 140s^9 + 229s^{10} + 384s^{11} + 615s^{12} + 938s^{13} + 1483s^{14} + 2353s^{15} + 3563s^{16} + \cdots. \tag{7.31}$$

The analyticity and the natural boundary of the zeta function in (7.30) need further investigation. The results will appear elsewhere.

In the following example, we study the equation on the diagonal lattice $\mathbb{L}$:  and show that the rectangular zeta function $\zeta = \widehat{\zeta}$ fails to describe poles and natural boundary of ζ^0 but ζ_γ works well with $\gamma = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Example 7.4. Let $F_2 = \{0, 1\}$ and

$$\mathbb{X} = \left\{ x \in F_2^{\mathbb{Z}^2} : x_{i,j} + x_{i+1,j+1} = 0 \text{ for all } i, j \in \mathbb{Z} \right\}. \tag{7.32}$$

It is clear that the solutions of $x_{i,j} + x_{i+1,j+1} = 0$ mod 2 are given by
(7.33)

$$\mathcal{B} = \left\{ \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}, \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}, \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}, \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}, \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}, \begin{array}{cc} 1 & 1 \\ 1 & 0 \end{array}, \begin{array}{cc} 0 & 1 \\ 1 & 1 \end{array}, \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right\}.$$

Now,

$$\mathbf{H}_2 = \mathbf{V}_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \tag{7.34}$$

and

$$\widetilde{\mathbf{H}}_2 = \widetilde{\mathbf{V}}_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}. \tag{7.35}$$

It is easy to verify

$$\mathbf{T}_1 = \widehat{\mathbf{T}}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = R_1^t \tag{7.36}$$

and

$$\mathbf{T}_2 = \widehat{\mathbf{T}}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = R_2^t. \tag{7.37}$$

Furthermore, for $n \geq 3$, we show that

$$\mathbf{T}_n = \widehat{\mathbf{T}}_n = R_n^t. \tag{7.38}$$

Indeed, by the recursive formula of $\mathbf{V}_n$, it can be verified that $\mathbf{V}_{n;i,j} = 1$ if and only if

$$\begin{cases} i = 2j - 1 \text{ and } 2j & \text{for } 1 \leq j \leq 2^{n-1}, \\ i = 2(i - 2^{n-1}) - 1 \text{ and } 2(i - 2^{n-1}) & \text{for } 2^{n-1} + 1 \leq j \leq 2^n. \end{cases} \tag{7.39}$$

Therefore, by applying (2.30), $\mathrm{T}_n = [t_{n;i,j}]$ with $t_{n;i,j} = 1$ if and only if

$$\begin{cases} i = 2j - 1 & \text{for } 1 \leq j \leq 2^{n-1}, \\ i = 2(i - 2^{n-1}) & \text{for } 2^{n-1} + 1 \leq j \leq 2^n. \end{cases} \tag{7.40}$$

Hence,

$$\mathbf{T}_n = R_n^t. \tag{7.41}$$

Therefore,

$$\zeta_n(s) = \frac{1}{(1 - s^n)^{\chi_n}}, \tag{7.42}$$

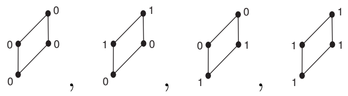
where χ_n is the cardinal number of $\mathcal{I}_n$, and

$$\zeta(s) = \prod_{n=1}^{\infty} \frac{1}{(1-s^n)^{\chi_n}}. \tag{7.43}$$

As in Example 6.12, $\lim_{n\to\infty} \chi_n^{\frac{1}{n}} = 2$ and then $S^* = \frac{1}{2}$.

On the other hand, consider

(7.44) $\mathcal{B}' = \Bigg\{$



$\Bigg\}.$

Then,

$$\Sigma(\mathcal{B}') = \Sigma(\mathcal{B}). \tag{7.45}$$

In particular,

$$\Gamma_{\mathcal{B}'}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}_\gamma\right) = 2^k. \tag{7.46}$$

Therefore, as in Example 6.12,

$$\zeta_{\gamma;n} = \frac{1}{1-2s^n}.$$

We can also use the construction of $\mathbf{T}_{\gamma;n}$ in Section 5 to study $\zeta_{\gamma;n}$. Indeed, by Figure 5.1, it is easy to see that

$$\mathbf{T}_{\gamma;1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}. \tag{7.47}$$

Therefore,

$$\zeta_{\gamma;1} = \frac{1}{1-2s}. \tag{7.48}$$

Furthermore, for any $n \geq 2$, after deleting the zero columns and rows of $\mathbf{T}_{\gamma;n}$, $\mathbf{T}_{\gamma;n}$ is reduced to $\mathbf{T}_{\gamma;1}$. Therefore,

$$\zeta_{\gamma;n} = \frac{1}{1-2s^n}. \tag{7.49}$$

Hence,

$$\zeta_\gamma = \prod_{n=1}^{\infty} \frac{1}{1-2s^n}. \tag{7.50}$$

ζ_γ has natural boundary with $|s| = 1$ and has poles

$$\left\{2^{-\frac{1}{n}} e^{2\pi ij/n} : 0 \leq j \leq n-1, n \geq 1\right\}.$$

Motivated by examples 7.1∼7.4, given a finite field F and a set of finite lattice points $\mathbb{L} \subset \mathbb{Z}^2$, consider the equation

$$\sum_{(i,j)\in\mathbb{L}} x_{i,j} = 0 \quad \text{in } F. \tag{7.51}$$

Then, denote the solution set of (7.51) on $\mathbb{Z}^2$ by

$$\mathbb{X}(\mathbb{L}) = \left\{ x \in F^{\mathbb{Z}^2} : \sum_{(i,j)\in\mathbb{L}} x_{i+k,j+l} = 0, (k,l) \in \mathbb{Z}^2 \right\}. \tag{7.52}$$

Denoted by

$$\mathcal{B}(\mathbb{L}) = \left\{ x : \mathbb{L} \to F : \sum_{(i,j)\in\mathbb{L}} x_{i,j} = 0 \right\}, \tag{7.53}$$

$\mathcal{B}(\mathbb{L}) \subset F^{\mathbb{L}}$ is the set of admissible local patterns.

Let $\mathbb{Z}_{m\times m}$ be the smallest rectangular lattice that contains $\mathbb{L}$. Let $\mathcal{B}$ be the set of all admissible patterns on $\mathbb{Z}_{m\times m}$ that can be generated from $\mathcal{B}(\mathbb{L})$. Then, the following can be easily verified;

$$\mathbb{X}(\mathbb{L}) = \Sigma(\mathcal{B}). \tag{7.54}$$

The results presented in previous sections apply to $\Sigma(\mathcal{B})$ and then to $\mathbb{X}(\mathbb{L})$. The above method can also be applied to any finite set of equations defined on $\mathbb{L}$ with numbers in F, since the solution set $\mathcal{B}(\mathbb{L}) \subset F^{\mathbb{L}}$ and can be extended to a unique admissible set $\mathcal{B} \subseteq F^{\mathbb{Z}_{m\times m}}$.

8. Square lattice Ising model with finite range interaction

This section extends the results presented in previous sections to the thermodynamic zeta function for a square lattice Ising model with finite range interaction, see Ruelle [30] and Lind [21]. For simplicity, the square lattice Ising model with nearest neighbor interaction is considered.

The square lattice Ising model with external field $\mathcal{H}$, the coupling constant $\mathcal{J}$ in the horizontal direction, and the coupling constant $\mathcal{J}'$ in the vertical direction is now considered. Each site (i,j) of the square lattice $\mathbb{Z}^2$ has a spin $u_{i,j}$ with two possible values, $+1$ or -1. First, assume that the state space is $\{+1,-1\}^{\mathbb{Z}^2}$. Given a state $U = \{u_{i,j}\}_{i,j\in\mathbb{Z}}$ in $\{+1,-1\}^{\mathbb{Z}^2}$, denoted by $U_{m\times n} = U\big|_{\mathbb{Z}_{m\times n}} = \{u_{i,j}\}_{0\leq i\leq m-1, 0\leq j\leq n-1}$.

Define the Hamiltonian (energy) $\mathcal{E}(U_{m\times n})$ for $U_{m\times n}$ by

$$\mathcal{E}(U_{m\times n}) = -\mathcal{J} \sum_{\substack{0\leq i\leq m-2\\ 0\leq j\leq n-1}} u_{i,j}u_{i+1,j} - \mathcal{J}' \sum_{\substack{0\leq i\leq m-1\\ 0\leq j\leq n-2}} u_{i,j}u_{i,j+1} - \mathcal{H} \sum_{\substack{0\leq i\leq m-1\\ 0\leq j\leq n-1}} u_{i,j}. \tag{8.1}$$

Therefore, the partition function $\mathcal{Z}_{m\times n}$ is defined by

(8.2)

$$\mathcal{Z}_{m\times n} = \sum_{U_{m\times n}\in\{+1,-1\}^{\mathbb{Z}_{m\times n}}} \exp\left[\mathbf{K}\sum_{\substack{0\le i\le m-2\\0\le j\le n-1}} u_{i,j}u_{i+1,j} + \mathbf{L}\sum_{\substack{0\le i\le m-1\\0\le j\le n-2}} u_{i,j}u_{i,j+1} + \mathbf{h}\sum_{\substack{0\le i\le m-1\\0\le j\le n-1}} u_{i,j}\right],$$

where $\mathbf{K} = \mathcal{J}/k_BT$,$\mathbf{L} = \mathcal{J}'/k_BT$, $\mathbf{h} = \mathcal{H}/k_BT$, k_B is Boltzmann's constant and T is the temperature.

To the thermodynamic zeta function, given $L = \begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\mathbb{Z}^2 \in \mathcal{L}_2$, the partition function for the $\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}$-periodic states is defined by

(8.3)

$$\begin{aligned}\mathcal{Z}_L &= \mathcal{Z}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\right)\\ &= \sum_{U\in fix_L\left(\{+1,-1\}^{\mathbb{Z}^2}\right)} \exp\left[\mathbf{K}\sum_{\substack{0\le i\le n-1\\0\le j\le k-1}} u_{i,j}u_{i+1,j} + \mathbf{L}\sum_{\substack{0\le i\le n-1\\0\le j\le k-1}} u_{i,j}u_{i,j+1} + \mathbf{h}\sum_{\substack{0\le i\le n-1\\0\le j\le k-1}} u_{i,j}\right],\end{aligned}$$

where $u_{n,j} = u_{0,j}$, $0 \le j \le k-1$ and $u_{i,k} = u_{i,0}$, $0 \le i \le n-1$.

Then, the thermodynamic zeta function for the square lattice Ising model with nearest neighbor interaction can be defined by

$$\zeta^0(s) \equiv \zeta^0_{Ising}(s) \equiv \exp\left(\sum_{L\in\mathcal{L}_2} \mathcal{Z}_L \frac{s^{[L]}}{[L]}\right)$$

$$= \exp\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=0}^{n-1} \frac{1}{nk}\mathcal{Z}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\right) s^{nk}\right). \tag{8.4}$$

To simplify the notation, the subscript Ising is omitted in this section whenever such omission will not cause confusion.

As (1.8) and (1.9), for any $n \ge 1$, define the n-th order thermodynamic zeta function $\zeta_{Ising;n}(s)$ as

$$\zeta_n(s) \equiv \zeta_{Ising;n}(s) \equiv \exp\left(\frac{1}{n}\sum_{k=1}^{\infty}\sum_{l=0}^{n-1} \frac{1}{k}\mathcal{Z}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\right) s^{nk}\right); \tag{8.5}$$

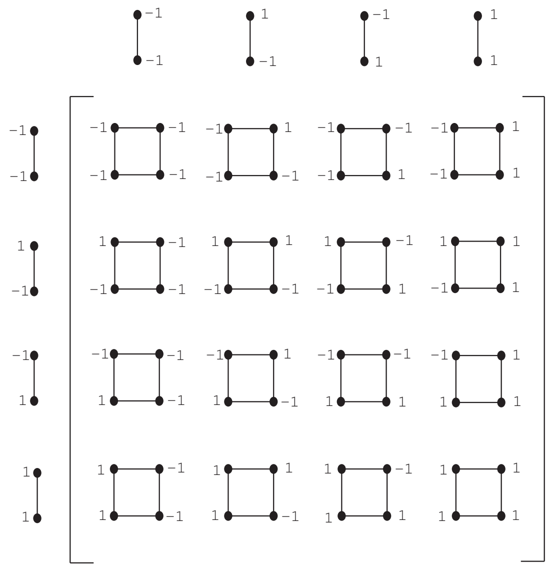
the thermodynamic zeta function $\zeta_{Ising}(s)$ is given by

$$\zeta(s) \equiv \zeta_{Ising}(s) \equiv \prod_{n=1}^{\infty} \zeta_n(s). \tag{8.6}$$

Since the discussion of $\zeta_n(s)$ is similar to that in sections 2 and 3, only the parts of the arguments that differ are emphasized. The results are outlined here and the details are left to the reader.

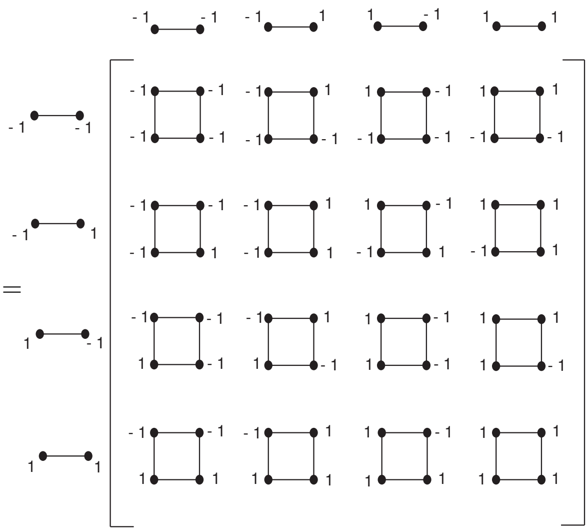
According to the spin $u_{i,j} \in \{+1, -1\}$ for $i, j \in \mathbb{Z}$,replacing all the symbols "0" in (2.1) and (2.2) with the symbol "-1" yields the ordering matrices $\mathbf{X}_{Ising;2\times 2}$ and $\mathbf{Y}_{Ising;2\times 2}$ as follows.

(8.7) $\mathbf{X}_{Ising;2\times 2} =$



and

(8.8) $\mathbf{Y}_{Ising;2\times 2} =$



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The ordering matrix $\mathbf{X}_{Ising;n\times 2}$, $\mathbf{Y}_{Ising;n\times 2}$ and the cylindrical ordering matrix $\mathbf{C}_{Ising;n\times 2}$ can be obtained in the same way. The recursive formulae for generating $\mathbf{Y}_{Ising;n\times 2}$ form $\mathbf{Y}_{Ising;2\times 2}$ are as in (2.14).

Given $L \in \mathcal{L}_2$, (8.3) yields

$$\mathcal{Z}_L = \sum_{U \in fix_L\left(\{+1,-1\}^{\mathbb{Z}^2}\right)} \prod_{\substack{0 \le i \le n-1 \\ 0 \le j \le k-1}} \exp\left[u_{i,j}\left(\mathbf{K}u_{i+1,j} + \mathbf{L}u_{i,j+1} + \mathbf{h}\right)\right]. \tag{8.9}$$

Based on (8.7), (8.8) and (8.9), the associated horizontal transition matrix $\mathbf{H}_{Ising;2} = [a_{I;i,j}]_{4\times4}$ and the vertical transition matrix $\mathbf{V}_{Ising;2} = [b_{I;i,j}]_{4\times4}$ are defined as

$$\mathbf{H}_{Ising;2} = \begin{bmatrix} e^{\mathbf{K}+\mathbf{L}-\mathbf{h}} & e^{-\mathbf{K}-\mathbf{L}-\mathbf{h}} & e^{\mathbf{K}-\mathbf{L}-\mathbf{h}} & e^{-\mathbf{K}+\mathbf{L}-\mathbf{h}} \\ e^{-\mathbf{K}+\mathbf{L}-\mathbf{h}} & e^{\mathbf{K}-\mathbf{L}-\mathbf{h}} & e^{-\mathbf{K}-\mathbf{L}-\mathbf{h}} & e^{\mathbf{K}+\mathbf{L}-\mathbf{h}} \\ e^{\mathbf{K}+\mathbf{L}+\mathbf{h}} & e^{-\mathbf{K}-\mathbf{L}+\mathbf{h}} & e^{\mathbf{K}-\mathbf{L}+\mathbf{h}} & e^{-\mathbf{K}+\mathbf{L}+\mathbf{h}} \\ e^{-\mathbf{K}+\mathbf{L}+\mathbf{h}} & e^{\mathbf{K}-\mathbf{L}+\mathbf{h}} & e^{-\mathbf{K}-\mathbf{L}+\mathbf{h}} & e^{\mathbf{K}+\mathbf{L}+\mathbf{h}} \end{bmatrix} = [a_{I;i,j}]_{4\times4}, \tag{8.10}$$

and

$$\mathbf{Y}_{Ising;2} = \begin{bmatrix} e^{\mathbf{K}+\mathbf{L}-\mathbf{h}} & e^{-\mathbf{K}-\mathbf{L}-\mathbf{h}} & e^{-\mathbf{K}+\mathbf{L}-\mathbf{h}} & e^{\mathbf{K}-\mathbf{L}-\mathbf{h}} \\ e^{\mathbf{K}-\mathbf{L}-\mathbf{h}} & e^{-\mathbf{K}+\mathbf{L}-\mathbf{h}} & e^{-\mathbf{K}-\mathbf{L}-\mathbf{h}} & e^{\mathbf{K}+\mathbf{L}-\mathbf{h}} \\ e^{\mathbf{K}+\mathbf{L}+\mathbf{h}} & e^{-\mathbf{K}-\mathbf{L}+\mathbf{h}} & e^{-\mathbf{K}+\mathbf{L}+\mathbf{h}} & e^{\mathbf{K}-\mathbf{L}+\mathbf{h}} \\ e^{\mathbf{K}-\mathbf{L}+\mathbf{h}} & e^{-\mathbf{K}+\mathbf{L}+\mathbf{h}} & e^{-\mathbf{K}-\mathbf{L}+\mathbf{h}} & e^{\mathbf{K}+\mathbf{L}+\mathbf{h}} \end{bmatrix} = [b_{I;i,j}]_{4\times4}, \tag{8.11}$$

respectively. Similar to (2.21) and (2.22), the associated column matrices $\widetilde{\mathbf{H}}_{Ising;2}$ of $\mathbf{H}_{Ising;2}$ and $\widetilde{\mathbf{V}}_{Ising;2}$ of $\mathbf{V}_{Ising;2}$ are defined as

$$\widetilde{\mathbf{H}}_{Ising;2} = \begin{bmatrix} a_{I;1,1} & a_{I;2,1} & a_{I;1,2} & a_{I;2,2} \\ a_{I;3,1} & a_{I;4,1} & a_{I;3,2} & a_{I;4,2} \\ a_{I;1,3} & a_{I;2,3} & a_{I;1,4} & a_{I;2,4} \\ a_{I;3,3} & a_{I;4,3} & a_{I;3,4} & a_{I;4,4} \end{bmatrix} \tag{8.12}$$

and

$$\widetilde{\mathbf{V}}_{Ising;2} = \begin{bmatrix} b_{I;1,1} & b_{I;2,1} & b_{I;1,2} & b_{I;2,2} \\ b_{I;3,1} & b_{I;4,1} & b_{I;3,2} & b_{I;4,2} \\ b_{I;1,3} & b_{I;2,3} & b_{I;1,4} & b_{I;2,4} \\ b_{I;3,3} & b_{I;4,3} & b_{I;3,4} & b_{I;4,4} \end{bmatrix}. \tag{8.13}$$

Therefore, the trace operators $\mathbf{T}_{Ising;2}$ and $\widehat{\mathbf{T}}_{Ising;2}$ are defined as

$$\mathbf{T}_{Ising;2} = \mathbf{V}_{Ising;2} \circ \widetilde{\mathbf{H}}_{Ising;2} \quad \text{and} \quad \widehat{\mathbf{T}}_{Ising;2} = \mathbf{H}_{Ising;2} \circ \widetilde{\mathbf{V}}_{Ising;2}. \tag{8.14}$$

The recursive formulas for $\mathbf{T}_{Ising;n}$ and $\widehat{\mathbf{T}}_{Ising;n}$ are similar to (2.30). Constructing $\mathbf{T}_{Ising;2}$ and the rotational matrix R_n yield a similar result to that of Theorem 2.12 for $\mathcal{Z}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\right)$.

Theorem 8.1. *Given* $n \ge 2, 0 \le l \le n-1, k \ge 1$,

$$\mathcal{Z}\left(\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\right) = tr\left(\mathbf{T}^k_{Ising;n} R^l_n\right). \tag{8.15}$$

Furthermore, let

$$\mathbf{T}_{Ising;1} = \begin{bmatrix} a_{I;1,1}a_{I;1,1} & a_{I;2,2}a_{I;2,2} \\ a_{I;3,3}a_{I;3,3} & a_{I;4,4}a_{I;4,4} \end{bmatrix};$$

then

$$\mathcal{Z}\left(\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}\right) = tr\left(\mathbf{T}^{k}_{Ising;1}\right) \quad \textit{for } k \geq 1.$$

From Theorem 8.1, the n-th order thermodynamic zeta function $\zeta_{Ising;n}$ can now be obtained as follows.

Theorem 8.2. *For any $n \geq 1$,*

$$\zeta_{Ising;n} = \exp\left(\frac{1}{n}\sum_{k=1}^{\infty} tr\left(\mathbf{T}^{k}_{Ising;n}\mathbf{R}_n\right) s^{nk}\right). \tag{8.16}$$

The R_n-symmetric property of $\mathbf{T}_{Ising;n}$ is essential to the rationality of n-th order thermodynamic zeta function $\zeta_{Ising;n}$.

Proposition 8.3. *For any $n \geq 1$,*

$$\mathbf{T}_{Ising;n;\sigma^l(i),\sigma^l(j)} = \mathbf{T}_{Ising;n;i,j} \tag{8.17}$$

for all $1 \leq i,j \leq 2^n$ and $0 \leq l \leq n-1$.

Similarly, the associated reduced trace operator $\tau_{Ising;n}$ can be defined as in (3.52). Finally, by the arguments presented in section 3, the rationality of the n-th order thermodynamic zeta function $\zeta_{Isng;n}$ is established as follows.

Theorem 8.4. *For $n \geq 1$,*

$$\zeta_{Ising;n}(s) = \prod_{\lambda\in\Sigma(\mathbf{T}_{Ising;n})} (1-\lambda s^n)^{-\chi(\lambda)} \tag{8.18}$$

$$= \left(\det\left(I - s^n\tau_{Ising;n}\right)\right)^{-1}, \tag{8.19}$$

where $\chi(\lambda)$ is the number of linear independent symmetric eigenvectors and generalized eigenvectors of $\mathbf{T}_{Ising;n}$ with eigenvalue λ. Furthermore,

$$\zeta_{Ising}(s) = \prod_{n=1}^{\infty}\left(\det\left(I - s^n\tau_{Ising;n}\right)\right)^{-1}. \tag{8.20}$$

The state space $\{+1,-1\}^{\mathbb{Z}^2}$ is extended to the shift of finite type given by $\mathcal{B} \subseteq \{+1,-1\}^{\mathbb{Z}_{2\times 2}}$.

Given $\mathcal{B} \subseteq \{+1,-1\}^{\mathbb{Z}_{2\times 2}}$ and $L = \begin{bmatrix} n & l \\ 0 & k \end{bmatrix}\mathbb{Z}^2 \in \mathcal{L}_2$, the partition function for $\mathcal{B}$ with $\begin{bmatrix} n & l \\ 0 & k \end{bmatrix}$-periodic patterns is defined as

(8.21)

$$\begin{aligned}\mathcal{Z}_L(\mathcal{B}) &= \mathcal{Z}_{\mathcal{B}}\left(\left[\begin{array}{cc} n & l \\ 0 & k \end{array}\right]\right) \\ &= \sum_{U\in fix_L(\Sigma(\mathcal{B}))} \exp\left[\mathbf{K}\sum_{\substack{0\leq i\leq n-1\\ 0\leq j\leq k-1}} u_{i,j}u_{i+1,j} + \mathbf{L}\sum_{\substack{0\leq i\leq n-1\\ 0\leq j\leq k-1}} u_{i,j}u_{i,j+1} + \mathbf{h}\sum_{\substack{0\leq i\leq n-1\\ 0\leq j\leq k-1}} u_{i,j}\right],\end{aligned}$$

where $u_{n,j} = u_{0,j}$, $0 \leq j \leq k-1$ and $u_{i,k} = u_{i,0}$, $0 \leq i \leq n-1$.

Hence, the thermodynamic zeta function is defined by

$$\zeta^0_{Ising;\mathcal{B}}(s) \equiv \exp\left(\sum_{L\in\mathcal{L}_2} \mathcal{Z}_L(\mathcal{B})\frac{s^{[L]}}{[L]}\right)$$

$$= \exp\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=0}^{n-1}\frac{1}{nk}\mathcal{Z}_{\mathcal{B}}\left(\left[\begin{array}{cc} n & l \\ 0 & k \end{array}\right]\right)s^{nk}\right). \tag{8.22}$$

Similar to (8.5) and (8.6), for any $n \geq 1$, the n-th order thermodynamic zeta function $\zeta_{Ising;\mathcal{B};n}(s)$ is defined as

$$\zeta_{Ising;\mathcal{B};n}(s) \equiv \exp\left(\frac{1}{n}\sum_{k=1}^{\infty}\sum_{l=0}^{n-1}\frac{1}{k}\mathcal{Z}_{\mathcal{B}}\left(\left[\begin{array}{cc} n & l \\ 0 & k \end{array}\right]\right)s^{nk}\right) \tag{8.23}$$

and the thermodynamic zeta function $\zeta_{Ising;\mathcal{B}}(s)$ is given by

$$\zeta_{Ising;\mathcal{B}}(s) \equiv \prod_{n=1}^{\infty}\zeta_{Ising;\mathcal{B};n}(s). \tag{8.24}$$

Equations (2.17), (2.18), (8.10) and (8.11) are combined to define the associated horizontal transition matrix and vertical transition matrix as follows.

$$\mathbf{H}_{Ising;2}(\mathcal{B}) = \mathbf{H}_{Ising;2} \circ \mathbf{H}_2(\mathcal{B}) \tag{8.25}$$

and

$$\mathbf{V}_{Ising;2}(\mathcal{B}) = \mathbf{V}_{Ising;2} \circ \mathbf{V}_2(\mathcal{B}). \tag{8.26}$$

Therefore, the trace operator $\mathbf{T}_{Ising;n}(\mathcal{B})$ and the associated reduced trace operator $\tau_{Ising;n}(\mathcal{B})$ can be defined for all $n \geq 1$ as above. Since all arguments for $\zeta_{Ising;\mathcal{B};n}$ are similar to those above; the final result is as follows.

Theorem 8.5. *For $n \geq 1$,*

$$\zeta_{Ising;\mathcal{B};n}(s) = \prod_{\lambda\in\Sigma(\mathbf{T}_{Ising;n}(\mathcal{B}))}(1-\lambda s^n)^{-\chi(\lambda)} \tag{8.27}$$

$$= \left[\det\left(I - s^n\tau_{Ising;n}(\mathcal{B})\right)\right]^{-1}, \tag{8.28}$$

where $\chi(\lambda)$ is the number of linear independent symmetric eigenvectors and generalized eigenvectors of $\mathbf{T}_{Ising;n}(\mathcal{B})$ with eigenvalue λ. Moreover,

$$\zeta_{Ising;\mathcal{B}}(s) = \prod_{n=1}^{\infty} \left[\det\left(I - s^n \tau_{Ising;n}(\mathcal{B})\right)\right]^{-1}. \tag{8.29}$$

Remark 8.6. *The results in this section hold for any model with finite range interaction.*

References

[1] M. Artin and B. Mazur, *On periodic points*, Annals Math. **81** (1965), 82-99.
[2] J.C. Ban and S.S. Lin, *Patterns generation and transition matrices in multi-dimensional lattice models*, Discrete Contin. Dyn. Syst. **13**, (2005), no. 3, 637-658.
[3] J.C. Ban, S.S. Lin and Y.H. Lin, *Patterns generation and spatial entropy in two dimensional lattice models*, Asian J. Math. **11**, (2007), no. 3, 497-534.
[4] R. J. Baxter, *Eight-vertex model in lattice statistics.*, Phys. Rev. Lett. **26**, (1971) 832-833.
[5] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic Press, 1982.
[6] R. Bowen and O. Lanford, *Zeta functions of restrictions of the shift transformation*, Proc. AMS Symp. Pure Math. **14** (1970), 43-49.
[7] S.N. Chow, J. Mallet-paret and E.S. Van Vleck, *Pattern formation and spatial chaos in spatially discrete evolution equations*, Random Comput. Dynam. **4**, (1996) 109-178.
[8] D. Fried, *Finitely presented dynamical systems*, Ergodic Th. and Dynam. Syst. **7** (1987), 489-507.
[9] F.R. Gantmacher, *The theory of matrices* (2 vols. Chelsea, New York, 1959)
[10] G. H. Hardy and S. Ramanujan, *Asympototic formulae for the distribution of integers of various types*, Proc. Royal Soc. **A 95**, 144-155.
[11] G. H. Hardy and E. M. Wright, *An introduction to the Theory of Numbers*, Fifth Ed., Oxford, 1988.
[12] R.A. Horn and C.R. Johnson, *Matrix analysis* (Cambridge University Press, Cambridge, 1990)
[13] J. Juang and S.S. Lin, *Cellular Neural Networks: Mosaic pattern and spatial chaos*, SIAM J. Appl. Math. **60**, (2000) 891-915.
[14] B. Kitchens and K. Schmidt, *Automorphisms of compact groups*, Ergodic Th. and Dynam. Syst. **9** (1989), 691-735.
[15] F. Ledrappier, *Un champ markovien peut être d'entropie nulle et mélangeant*, C. R. Acad. Sc. Paris Ser. A **287** (1978), 561-562.
[16] E.H. Lieb, *Exact solution of the problem of the entropy of two-dimensional ice*, Phys. Rev. Lett. **18**, (1967) 692-694.
[17] E.H. Lieb, *Exact solution of the f model of an antiferroelectric*, Phys. Rev. Lett. **18**, (1967) 1046-1048.
[18] E.H. Lieb, *Exact Solution of the Two-Dimensional Slater KDP Model of a Ferroelectric*, Phys. Rev. Lett. **19**, (1967) 108-110.
[19] E.H. Lieb, *The Residual Entropy of Square Ice*, Phys. Rev. **162**, (1967) 162-172.
[20] E.H. Lieb, *Ice, ferro- and antiferroelectrics, in methods and problems in theoretical physics*, in honour of R.E. Peierls, proceedings of the 1967 Birmingham conference, (North-Holland, 1970) 21-28.
[21] D. Lind, *A zeta function for $\mathbb{Z}^d$-actions*, London Math. Soc. Lecture Note Series **228**, Cambridge Univ. Press, (1996) 433-450.
[22] D. Lind and B. Marcus, *An introduction to symbolic dynamics and coding*, Cambridge University Press, Cambridge, (1995).
[23] D. Lind, K. Schmidt, and T. Ward, *Mahler measure and entropy for commuting automorphisms of compact groups*, Inventiones Math. **101** (1990), 593-629.
[24] C. MacDuffie, *The Theory of Matrices*, Chelsea, New York, 1956.
[25] A. Manning, *Axiom A diffeomorphisms have rational zeta functions*, Bull. London Math. Soc. **3** (1971), 215-220.

[26] N.G. Markley and M.E. Paul, *Maximal measures and entropy for Z^ν subshifts of finite type.* Classical mechanics and dynamical systems (Medford, Mass., 1979) 135-157, Lecture Notes in Pure and Appl. Math., **70**, Dekker, New York, (1981).
[27] N.G. Markley and M.E. Paul, *Matrix subshifts for Z^ν symbolic dynamics*, Proc. London Math. Soc., (3) **43**, (1981) 251-272.
[28] L. Onsager, *Crystal statistics. I. A two-dimensional model with an order-disorder transition.*, Phys. Rev. **65**, (1944) 117-149.
[29] M. Pollicott and K. Schmidt, *Ergodic theory of $\mathbb{Z}^d$ actions*, London Math. Soc. Lecture Note Series. **228**. Cambridge University Press, 1996.
[30] D. Ruelle, *Thermodynamic Formalism*, Encyclopedia of Math and its Appl, Vol 5. Addison-Wesley, Reading, Massachussetts, 1978.
[31] S. Saks and A. Zygmund; tr. by E. J. Scott, *Analytic functions*, Elsevier Science Ltd, 1952.
[32] K. Schmidt, *Automorphisms of compact groups and affine varieties*, Proc. London Math. Soc. **61** (1990), 480-496.
[33] K. Schmidt, *Dynamical systems of algebraic origin*, Basel, Birkhäuser Verlag, 1995.
[34] N. J. A. Sloane and S. Plouffe, *The encyclopedia of integer sequences*, Academic Press, 1995.

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