

# THE QUENCHING RATE FOR A NONLOCAL PROBLEM ARISING IN THE MICRO-ELECTRO MECHANICAL SYSTEM

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ABSTRACT. In this paper, we study the quenching rate of the solution for a nonlocal parabolic problem which arises in the study of the micro-electro mechanical system. This question is equivalent to the stabilization of the solution of the transformed problem in the self-similar variables which relies on a suitable Lyapunov function. In order to construct a Lyapunov function, due to the lack of time monotonicity property and the negative power in the reaction term, we first derive some delicate estimates to establish a positive lower bound of the solution. Then, with this Lyapunov function and some a priori estimates, we prove that the quenching rate is self-similar which is the same as the problem without the nonlocal term, except the constant coefficient depends on the solution itself.

## 1. INTRODUCTION

In this paper, we consider the following initial boundary value problem

$$(1.1) \quad u_t = u_{xx} - g(t; u, \lambda)u^{-2}, \quad -1 < x < 1, \quad t > 0,$$

$$(1.2) \quad u(\pm 1, t) = 1, \quad t > 0,$$

$$(1.3) \quad u(x, 0) = u_0(x), \quad x \in [-1, 1],$$

where

$$g(t; u, \lambda) := \lambda \left( 1 + \int_{-1}^1 u^{-1}(\xi, t) d\xi \right)^{-2}.$$

Throughout this paper, we always assume that  $u_0$  is smooth such that

$$u_0(\pm 1) = 1, \quad 0 < u_0(x) \leq 1, \quad u_0(x) = u_0(-x), \quad u_0'(x) \geq 0, \quad u_0''(x) \geq 0 \quad \text{for } 0 \leq x \leq 1.$$

Also, we shall simply denote  $g(t; u, \lambda)$  by  $g(t)$  when there is no confusion.

The problem (1.1)-(1.3) arises in the study of the micro-electro mechanical system. We refer to [23, 24] for the physical background of this model. In fact, equation (1.1) is a special

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case of the following general model

$$(1.4) \quad \varepsilon u_{tt} + u_t = \Delta u - \frac{\lambda f(x)}{u^2 \left(1 + \alpha \int_{\Omega} u^{-1}(\xi, t) d\xi\right)^2}, \quad x \in \Omega, t > 0,$$

where  $u$  represents the distance of the membrane and the ground electrode plate,  $\varepsilon$  is the ratio of the interaction due to the inertial and damping terms,  $\lambda$  is the applied voltage,  $\alpha \geq 0$  is related to the capacitor and  $f(x)$  is the varying dielectric properties of the membrane. The model (1.4) has been studied extensively, see, e.g., [16, 3, 4, 5, 13, 17, 18, 19, 21, 15] for the case  $\varepsilon = 0$  (without inertia) and [20, 14] for the case  $\varepsilon > 0$ . We also refer the reader to a recent survey paper [12] for more details and some open problems.

It is known [13] that, for  $\lambda$  suitably large, solution  $u(x, t)$  of (1.1)-(1.3) quenches in finite time, i.e.,  $u(0, t) = \min_{|x| \leq 1} u(x, t) \rightarrow 0$  as  $t \rightarrow T^-$  for some finite  $T$ . Moreover,  $x = 0$  is the only quenching point. In engineering application, quenching means the touchdown of membrane to the ground plate.

The main purpose of this paper is to study the temporal quenching rate of the nonlocal problem (1.1)-(1.3). In fact, the study of temporal singular rates has been one of the important issues in the formation of singularities (such as blow-up, quenching, extinction and dead-core). This can be traced back to the seminal works of Giga and Kohn ([6, 7]) for the study of blow-up rate. Since then, the study of temporal singular rates has attracted a lot of attentions. The temporal singular rates can be either self-similar or non-self-similar. We refer the reader to the references for various temporal singular rates cited in [11].

In particular, for the study of quenching rate for the following equation

$$(1.5) \quad u_t = u_{xx} - \lambda u^{-p}, \quad p > 0,$$

we refer the reader to [8, 1, 9, 22]. It is shown that the quenching rate for (1.5) is self-similar, i.e., it is proportional to  $(T - t)^{1/(1+p)}$ , which is the same as that of the decay rate for the corresponding spatially independent ordinary differential equation. Note that the constant coefficient in the quenching rate is the same for all solutions. The spatially quenching profile for (1.5) was studied in [2]. However, little is done for quenching rates of nonlocal problems. In [10], the quenching rate for the nonlocal equation

$$u_t = u_{xx} - \lambda \left\{1 - \int_0^1 u(\xi, t) d\xi\right\} u^{-p}, \quad 0 < x < 1, t > 0,$$

was studied for any  $p > 0$ . It is proved that the quenching rate is self-similar which is the same as that of (1.5), except the constant coefficient depends on the solution itself.

Throughout this paper, we always assume that  $\lambda \gg 1$  in (1.1) such that quenching occurs.

To study the quenching rate, we introduce the following self-similar variables

$$y := x/\sqrt{T-t}, \quad s := -\ln(T-t), \quad z(y, s) := (T-t)^{-1/3}u(x, t).$$

Then system (1.1)-(1.3) is equivalent to

$$(1.6) \quad z_s - z_{yy} + \frac{1}{2}yz_y - \frac{1}{3}z + h(s)z^{-2} = 0, \quad |y| < R(s), \quad s > s_0,$$

$$(1.7) \quad z(\pm R(s), s) = e^{s/3}, \quad s > s_0,$$

$$(1.8) \quad z(y, s_0) = T^{-1/3}u_0(T^{1/2}y) \triangleq z_0(y), \quad |y| \leq R(s_0),$$

where

$$h(s) := g(T - e^{-s}), \quad R(s) := e^{s/2}, \quad s_0 := -\ln T.$$

It is well-known that the determination of quenching rate is equivalent to the study of the asymptotic behavior of  $z(y, s)$  as  $s \rightarrow \infty$ . In fact, that quenching rate is self-similar is equivalent to the stabilization limit of  $z$  is nonzero.

A standard method for the study of long time behavior is to construct a suitable Lyapunov function with the help of some a priori estimates. Due to the nonlocal nature and the negative power in the reaction term in (1.6), the construction of a Lyapunov function relies upon an upper and a positive lower bounds of the solution  $z$  of (1.6)-(1.8). In many earlier results such as [8, 1, 9, 10], it is proved that the solution is monotone in time for certain class of initial data. This important property implies the desired estimates for a Lyapunov function. Unfortunately, in our problem (1.1)-(1.3), this time monotonic property is not available, even if we assume that the solution satisfies the monotonicity property initially. In fact, the lack of time monotonic property is a grand challenge for establishing a positive lower bound for the solution in the similarity variables. The main contribution of this paper is to derive all the necessary estimates to overcome this difficulty.

In section 3, with a careful construction of various auxiliary functions, we show that the solution  $z(y, s)$  in its similarity variables will be bounded from below by a positive constant, *provided* an estimate on  $\mathcal{F} \triangleq \int_{s_0}^{\infty} \int_{-R(s)}^{R(s)} e^{-y^2/4} z_s^2(y, s) dy ds$  is available. Note that the lower bound estimate is crucial because our equation contains terms involving negative powers such as  $z^{-1}$  and  $z^{-2}$ . The quantity  $\mathcal{F}$  appears naturally in the Lyapunov function. In order to obtain estimates on  $\mathcal{F}$ , we are required to take on the challenge to construct a Lyapunov function without apriorily obtaining estimates on the lower bound of the solution in the similarity variables.

We shall achieve this goal in section 4. A key step is to show that  $h(s) \frac{d}{ds} K_1(s)$  is integrable over  $(s_0, \infty)$ , where the function  $h(s) = g(T - e^{-s})$  is decomposed as

$$h(s) = \lambda \left( 1 + e^{-s/6} K_1(s) + e^{-s/6} K_2(s) \right)^{-2},$$

$$K_1(s) := \int_{-R(s)}^{R(s)} e^{-y^2/4} z^{-1}(y, s) dy, \quad K_2(s) := \int_{-R(s)}^{R(s)} [1 - e^{-y^2/4}] z^{-1}(y, s) dy.$$

After differentiation in  $s$ ,  $K_1$  involves a possible singularity at  $y = 0$  (for lacking a positive lower bound), but  $K_2$  contains a factor  $y^2$  which will cancel possible singularities at  $y = 0$ . Directly doing integration by parts will not completely move the  $d/ds$  derivatives from  $K_1$  to

$K_2$ . We overcome this difficulty by going to an infinite series, so that we can move *completely* the  $d/ds$  derivatives from  $K_1$  to  $K_2$ . Of course, doing so produces other extra terms that need to be estimated, and we are fortunate enough to be able to derive all the estimates necessary for the extra terms.

Since the limit  $h(\infty)$  exists (see Lemma 2.1 in §2), the natural candidate for the limit of  $z(y, s)$  as  $s \rightarrow \infty$  is the solution to the problem

$$(1.9) \quad w_{yy} - \frac{1}{2}yw_y + \frac{1}{3}w - h(\infty)w^{-2} = 0, \quad y \in \mathbb{R},$$

$$(1.10) \quad w_y(0) = 0 \leq w_y(y), \quad y \geq 0, \quad \text{and } w \text{ grows at most polynomially at } y = \infty.$$

Our problem corresponds to the case  $p = 2$  in (1.5). Since  $p = 2 > 1$ , there is no slow orbit (see [1]) and the only solution to (1.9)–(1.10) with a polynomial bound at  $\infty$  is the constant solution  $w \equiv w_\infty := [3h(\infty)]^{1/3}$ . Note that the constant  $w_\infty$  depends on the solution itself. We now state our main theorem of this paper as follows.

**Theorem 1.** *As  $s \rightarrow \infty$ , the solution  $z(y, s)$  converges uniformly on any compact set to the constant  $w_\infty$ .*

The rest of this paper is organized as follows. In §2, we provide some a priori estimates for problem (1.1)–(1.3). Then we derive some a priori estimates for the transformed problem (1.6)–(1.8) in §3. In particular, we derive the upper bounds of  $z(y, s)$ ; the lower bounds, however, is derived only for the solution away from  $y = 0$ . Finally, in §4, we first derive some very useful and challenging estimates to construct a Lyapunov function for problem (1.6)–(1.8). In particular, a positive lower bound for  $z$  is obtained. With all these estimates in hand, we then prove our main theorem. This shows that the quenching rate for the problem (1.1)–(1.3) is self-similar.

## 2. SOME A PRIORI ESTIMATES FOR PROBLEM (1.1)–(1.3)

In this section, we shall give some a priori estimates for the solution of problem (1.1)–(1.3).

**Lemma 2.1.** *The function  $g(t)$  is continuous for  $0 \leq t \leq T$  and  $\min_{0 \leq t \leq T} g(t) > 0$ .*

*Proof.* By [13], for any  $\beta \in (2, 3)$ , for sufficiently large  $\lambda$  there exists  $c > 0$  such that

$$(2.1) \quad c|x|^{2/\beta} \leq u(x, t) \leq 1 \quad \text{for } |x| \leq 1, \quad 0 \leq t \leq T.$$

It follows that  $\int_{-1}^1 u^{-1}(x, t) dx$  is uniformly integrable and we have the estimate

$$\inf_{0 \leq t \leq T} g(t) > 0.$$

By (2.1), we can apply parabolic estimates in the region  $|x| > 0$  to obtain

$$\lim_{t \rightarrow T^-} u(x, t) = u(x, T) \quad \text{for } |x| > 0.$$

The estimate (2.1) also implies  $u^{-1}(x, t) \leq c^{-1}|x|^{-2/\beta}$ . By Lebesgue Dominate Convergence Theorem,

$$\lim_{t \rightarrow T^-} \int_{-1}^1 u^{-1}(x, t) dx = \int_{-1}^1 u^{-1}(x, T) dx.$$

This shows that  $g(t)$  is continuous at  $t = T$ .  $\square$

A careful examination of the proof in [13] shows that the constant  $c$  in (2.1) is independent of  $\lambda$ . Therefore, we have

$$\lim_{\lambda \rightarrow \infty} g(T; u, \lambda) = \infty.$$

We next establish

**Lemma 2.2.** *It holds*

$$(2.2) \quad (T - t)^{-1/3} u(0, t) \leq \left( \frac{3}{T - t} \int_t^T g(\tau) d\tau \right)^{1/3}.$$

*Proof.* Since  $u(0, t)$  is the minimum,

$$u_t(0, t) \geq -g(t)u^{-2}(0, t).$$

Thus

$$\frac{1}{3} \left( u^3(0, t) \right)_t \geq -g(t).$$

Recalling that  $u(0, T) = 0$ , integrating the above inequality leads to (2.2).  $\square$

It is not difficult to find (see [13]) that

$$\mathcal{E}(t) \triangleq \frac{1}{2} \int_{-1}^1 u_x^2(x, t) dx + \lambda \left( 1 + \int_{-1}^1 u^{-1}(x, t) dx \right)^{-1}$$

satisfies

$$\mathcal{E}'(t) = - \int_{-1}^1 u_t^2(x, t) dt < 0.$$

Integrating this equation over  $[0, T)$ , we obtain  $L^2$  estimates for  $u_t$ :

$$\int_0^T \int_{-1}^1 u_t^2(x, t) dx dt \leq \mathcal{E}(0) < \infty.$$

Using the monotonicity of  $\mathcal{E}(t)$ , we obtain  $L^\infty([0, T], L^2)$  estimate for  $u_x$ :

$$\frac{1}{2} \int_{-1}^1 u_x^2(x, t) dx + \lambda \left( 1 + \int_{-1}^1 u^{-1}(x, t) dx \right)^{-1} \leq \mathcal{E}(0) < \infty.$$

These estimates imply the Hölder continuity of  $u$  on  $[-1, 1] \times [0, T]$  as follows.

**Lemma 2.3.** *The following estimates hold:*

$$(2.3) \quad |u(x_1, t) - u(x_2, t)| \leq C|x_1 - x_2|^{1/2},$$

$$(2.4) \quad |u(x, t_1) - u(x, t_2)| \leq C|t_1 - t_2|^{1/4}.$$

*Proof.* The estimate (2.3) is a direct consequence of Hölder's inequality:

$$\begin{aligned} |u(x_1, t) - u(x_2, t)| &\leq \left| \int_{x_1}^{x_2} u_x(\xi, t) d\xi \right| \leq |x_1 - x_2|^{1/2} \left( \int_{-1}^1 u_x^2(\xi, t) d\xi \right)^{1/2} \\ &\leq \left( 2\mathcal{E}(0) \right)^{1/2} |x_1 - x_2|^{1/2}. \end{aligned}$$

To show (2.4), we note that

$$|u(x, t_1) - u(x, t_2)|^2 \leq \left| \int_{t_1}^{t_2} u_t(x, \tau) d\tau \right|^2 \leq |t_1 - t_2| \int_0^T u_t^2(x, \tau) d\tau.$$

Thus for any  $x_1 < x_2$ , there exists  $\bar{x} \in [x_1, x_2]$  such that

$$(x_2 - x_1) |u(\bar{x}, t_1) - u(\bar{x}, t_2)|^2 = \int_{x_1}^{x_2} |u(x, t_1) - u(x, t_2)|^2 dx \leq |t_1 - t_2| \int_0^T \int_{-1}^1 u_t^2(x, \tau) dx d\tau,$$

which implies that

$$|u(\bar{x}, t_1) - u(\bar{x}, t_2)| \leq \frac{|t_1 - t_2|^{1/2}}{|x_2 - x_1|^{1/2}} (\mathcal{E}(0))^{1/2}.$$

For any  $x$ , we take  $x_1$  and  $x_2$  such that  $x_1 < x < x_2$ , then  $|x - \bar{x}| \leq |x_2 - x_1|$ , and

$$\begin{aligned} |u(x, t_1) - u(x, t_2)| &\leq |u(\bar{x}, t_1) - u(\bar{x}, t_2)| + |u(x, t_1) - u(\bar{x}, t_1)| + |u(x, t_2) - u(\bar{x}, t_2)| \\ &\leq 2 \left( 2\mathcal{E}(0) \right)^{1/2} |x_1 - x_2|^{1/2} + \frac{|t_1 - t_2|^{1/2}}{|x_2 - x_1|^{1/2}} (\mathcal{E}(0))^{1/2}. \end{aligned}$$

Taking  $|x_1 - x_2| = |t_1 - t_2|^{1/2}$ , we obtain the desired results.  $\square$

As a corollary, we have

**Lemma 2.4.** *It holds*

$$u_x(x, t) \leq Cx^{-1/2}.$$

*Proof.* We only need to get an estimate near  $x = 0$ . By Lemma 2.3, the function

$$\psi(y, s) = \frac{u(ay, a^2s + t^*) - u(0, t^*)}{a^{1/2}}, \quad 1 < y < 4, -1 < s \leq 0, \quad 0 < a < 1/4,$$

is uniformly bounded. It satisfies

$$\psi_s - \psi_{yy} = -a^{3/2}g(a^2s + t^*)u^{-2}(ay, a^2s + t^*) \triangleq J.$$

By Lemma 2.1 (we choose  $\beta$  so that  $8/3 < \beta < 3$ ), the right-hand side of the above equation is estimated by

$$|J| \leq Ca^{3/2}|ay|^{-4/\beta} \leq Ca^{3/2-4/\beta} \leq C, \quad 1 < y < 4, -1 < s \leq 0, \quad 0 < a < 1/4.$$

It follows from interior parabolic estimates that

$$\psi_y(y, s) \leq C, \quad 2 \leq y \leq 3, -1/2 \leq s \leq 0.$$

Taking  $y = 2, s = 0$ , we obtain

$$a^{1/2}u_x(2a, t^*) \leq C, \quad 0 < a < 1/4,$$

which implies the conclusion of the lemma.  $\square$

We next proceed to establish Hölder continuity of  $g(t)$ .

**Lemma 2.5.** *There exists  $\gamma > 0$  such that*

$$|g(t_1) - g(t_2)| \leq C|t_1 - t_2|^\gamma$$

for any  $t_1, t_2 \in [0, T]$ .

*Proof.* It is clear that Hölder continuity of the function  $g(t)$  is the same as that for

$$k(t) \triangleq \int_{-1}^1 u^{-1}(x, t) dx = 2 \int_0^1 u^{-1}(x, t) dx.$$

For any  $t_1, t_2 \in (0, T]$ ,  $t_1 < t_2$ , and any  $0 < \mu < 1/4$ , we have

$$\begin{aligned} |k(t_1) - k(t_2)| &\leq 2 \int_0^{2\mu} |u^{-1}(x, t_1) - u^{-1}(x, t_2)| dx + 2 \int_{2\mu}^1 |u^{-1}(x, t_1) - u^{-1}(x, t_2)| dx \\ &\triangleq I_1 + I_2. \end{aligned}$$

It is clear that (note that  $c^{-1}$  depends on  $\beta$ )

$$I_1 \leq 4c^{-1} \int_0^{2\mu} |x|^{-2/\beta} dx \leq C(\beta)\mu^{1-2/\beta}.$$

To establish an estimate for  $I_2$ , we compute

$$\begin{aligned} I_2 &= 2 \int_{2\mu}^1 \frac{|u(x, t_1) - u(x, t_2)|}{u(x, t_1) \cdot u(x, t_2)} dx \\ &\leq C\mu^{-4/\beta} \int_{2\mu}^1 |u(x, t_1) - u(x, t_2)| dx = C\mu^{-4/\beta} \int_{2\mu}^1 \left| \int_{t_1}^{t_2} u_t(x, t) dt \right| dx \\ &\leq C\mu^{-4/\beta} |t_1 - t_2|^{1/2} \left( \int_{2\mu}^1 \int_{t_1}^{t_2} u_t^2 dx dt \right)^{1/2} \\ &\leq C\mu^{-4/\beta} |t_1 - t_2|^{1/2}. \end{aligned}$$

Combining the estimates for  $I_1$  and  $I_2$ , we obtain

$$|k(t_1) - k(t_2)| \leq C\mu^{1-2/\beta} + C\mu^{-4/\beta} |t_1 - t_2|^{1/2}.$$

Taking  $\mu = |t_1 - t_2|^{\beta/[2(\beta+2)]}$ , we obtain

$$|k(t_1) - k(t_2)| \leq C|t_1 - t_2|^\gamma, \quad \gamma := \frac{\beta - 2}{2(\beta + 2)}.$$

The lemma is proved.  $\square$

Since  $u$  is clearly bounded by 1 and bounded below by a positive constant near  $x = 1$ , we clearly have the following bounds on the derivatives at  $x = 1$ :

$$(2.5) \quad 0 \leq u_x(1, t) \leq C, \quad u_t(1, t) = 0, \quad |u_{xx}(1, t)| \leq C \quad \text{for } 0 \leq t \leq T.$$

In fact, the estimate (2.1) implies that  $u \in C^{1+\alpha, (1+\alpha)/2}([b, 1] \times [0, T])$  for any  $b \in (0, 1)$ .

### 3. SOME A PRIORI ESTIMATES FOR PROBLEM (1.6)-(1.8)

This section is devoted to the derivation of some a priori estimate for the solution  $z$  of problem (1.6)-(1.8).

First, in terms of similarity variables, the estimate (2.2) gives

$$(3.1) \quad z(0, s) \leq C_1, \quad \limsup_{s \rightarrow \infty} z(0, s) \leq \left(3g(T)\right)^{1/3} \triangleq \mathcal{H},$$

where

$$C_1 := \sup_{0 \leq t < T} \left( \frac{3}{T-t} \int_t^T g(\tau) d\tau \right)^{1/3}.$$

To obtain an upper bound for  $z$ , we construct an upper solution as follows. Take  $\mu$  and  $c_1(\mu)$  such that

$$\frac{2}{3} < \mu < 1, \quad c_1(\mu) = \min_{0 < y < \infty} \left\{ (1-\mu)\mu y^{\mu-2} + \left(\frac{1}{2}\mu - \frac{1}{3}\right)y^\mu \right\} > 0.$$

Then the function

$$w = y^\mu + c_1(\mu)$$

satisfies

$$\begin{aligned} & -w_{yy} + \frac{1}{2}yw_y - \frac{1}{3}w \\ &= (1-\mu)\mu y^{\mu-2} + \left(\frac{1}{2}\mu - \frac{1}{3}\right)y^\mu - \frac{1}{3}c_1(\mu) \\ &\geq \frac{2}{3}c_1(\mu) > 0. \end{aligned}$$

Take  $C^*$  such that  $C^*c_1(\mu) \geq C_1$ ,  $C^*e^{\mu s_0/2} \geq e^{s_0/3}$  and  $C^*c_1(\mu) \geq \|z_0\|_{L^\infty}$ . Then it follows from (3.1) and the maximum principle that

$$z(y, s) \leq C^*w(y) \quad \text{for } 0 \leq y \leq e^{s/2}, \quad s \geq s_0.$$

We have established

**Lemma 3.1.** *For any  $\frac{2}{3} < \mu < 1$ , there exists a  $C^* = C^*(\mu) > 0$  such that*

$$(3.2) \quad z(y, s) \leq C^*(1 + |y|^\mu) \quad \text{for } 0 \leq y \leq e^{s/2}, \quad s_0 \leq s < \infty.$$



Recall (2.5). In terms of variables  $(z, y, s)$ , we have

$$(3.3) \quad 0 \leq -z_y(-R(s), s) = z_y(R(s), s) \leq CR^{-1/3}(s).$$

Differentiate the boundary condition (1.7), we obtain

$$\frac{1}{2}yz_y(y, s) + z_s(y, s) = \frac{1}{3}|y|^{2/3} \quad \text{for } y = \pm R(s).$$

Using (3.3), we conclude

$$(3.4) \quad |z_s(\pm R(s), s)| \leq CR^{2/3}(s).$$

To derive a lower bound for  $z(y, s)$ , we rewrite (1.6) as

$$(3.5) \quad \rho z_s = \left(\rho \cdot z_y\right)_y + \rho \cdot \left(\frac{1}{3}z - h(s)z^{-2}\right),$$

where  $\rho(y) := e^{-y^2/4}$ . For any  $\alpha \geq 0$ , we compute

$$(3.6) \quad \frac{d}{ds} \int_{-R(s)}^{R(s)} \rho(y) z^{1+\alpha}(y, s) dy = 2\rho(R(s))R'(s)z^{1+\alpha}(R(s), s) + \int_{-R(s)}^{R(s)} \rho(y)(z^{1+\alpha})_s(y, s) dy.$$

It follows from (3.6) with  $\alpha = 0$  and (3.5) that

$$(3.7) \quad \frac{d}{ds} \int_{-R(s)}^{R(s)} \rho(y) z(y, s) dy = J_0 + \int_{-R(s)}^{R(s)} \rho(y) \left(\frac{1}{3}z - h(s)z^{-2}\right) dy,$$

where

$$J_0 = 2\rho(R(s))R'(s)z(R(s), s) + 2\rho(R(s))z_y(R(s), s).$$

Clearly,

$$|J_0| \leq C \exp\left(-\frac{1}{8}e^{s/2}\right).$$

And, from Lemma 3.1,

$$(3.8) \quad \sup_{s > s_0} \int_{-R(s)}^{R(s)} \rho(y) z(y, s) dy < \infty.$$

By integrating (3.7), we obtain

$$(3.9) \quad \left| \int_{s_1}^{s_2} \int_{-R(s)}^{R(s)} \rho(y) \left(\frac{1}{3}z - h(s)z^{-2}\right) dy ds \right| \leq C^* < \infty \quad \text{for } s_2 > s_1 \geq s_0,$$

where  $C^*$  is independent of  $s_2$  and  $s_1$ . It follows from (3.8) and (3.9) that

$$(3.10) \quad \int_{s_1}^{s_1+1} \int_{-R(s)}^{R(s)} \rho(y) z^{-2}(y, s) dy ds \leq C \quad \text{for any } s_1 > s_0.$$

We claim that

$$(3.11) \quad \int_{s_1}^{s_1+1} \int_{-R(s)}^{R(s)} \rho(y) z^{-2+\alpha}(y, s) dy ds \leq C \quad \text{for any } s_1 > s_0$$

for any  $\alpha > 0$ . For  $\alpha \geq 2$ , (3.11) follows from (3.2). For  $\alpha \in (0, 2)$ , it follows from (3.10) and Hölder's inequality with  $p = 2/\alpha$  and  $q = 1/(1 - \alpha/2)$  that

$$\begin{aligned} & \int_{s_1}^{s_1+1} \int_{-R(s)}^{R(s)} \rho(y) z^{-2+\alpha}(y, s) dy ds \\ &= \int_{s_1}^{s_1+1} \int_{-R(s)}^{R(s)} \rho^{1/p}(y) \rho^{1/q}(y) z^{-2+\alpha}(y, s) dy ds \\ &\leq \left( \int_{s_1}^{s_1+1} \int_{-R(s)}^{R(s)} \rho(y) dy ds \right)^{1/p} \left( \int_{s_1}^{s_1+1} \int_{-R(s)}^{R(s)} \rho(y) z^{-2}(y, s) dy ds \right)^{1/q} \\ &\leq C \quad \text{for any } s_1 > s_0. \end{aligned}$$

Hence (3.11) holds for any  $\alpha > 0$ .

Multiplying the equation (3.5) by  $(1+\alpha)z^\alpha$ , using (3.6) and the above estimates, we obtain

$$(3.12) \quad \int_{s_1}^{s_1+1} \int_{-R(s)}^{R(s)} \rho(y) \left\{ \left( z^{(1+\alpha)/2} \right)_y \right\}^2 dy ds \leq C_\alpha \quad \text{for any } s_1 > s_0.$$

By Hölder's inequality, for  $y > 0$ ,

$$z^{(1+\alpha)/2}(y, s) - z^{(1+\alpha)/2}(0, s) \leq \sqrt{y} \left\{ \int_0^y \left\{ \left( z^{(1+\alpha)/2} \right)_y \right\}^2 dy \right\}^{1/2}.$$

It follows from (3.12) that

$$(3.13) \quad \int_{s_1}^{s_1+1} G(s) ds \leq C_\alpha, \quad \text{where } G(s) := \sup_{0 < y \leq 1} \frac{[z^{(1+\alpha)/2}(y, s) - z^{(1+\alpha)/2}(0, s)]^2}{y}.$$

Since  $z_y(0, s) = 0$ , we have

$$\lim_{y \rightarrow 0} \frac{[z^{(1+\alpha)/2}(y, s) - z^{(1+\alpha)/2}(0, s)]^2}{y} = 0.$$

Hence the "sup" in (3.13) is actually achieved and  $G(s)$  is a continuous function in  $s$ .

Combining (3.10) and (3.13), we obtain

$$\int_{s_1}^{s_1+1} \left\{ \sup_{0 < y \leq 1} \frac{[z^{(1+\alpha)/2}(y, s) - z^{(1+\alpha)/2}(0, s)]^2}{y} + \int_0^{R(s)} \rho(y) z^{-2}(y, s) dy \right\} ds \leq C.$$

By the mean value theorem, for each  $s_1 > s_0$ , there exists  $\tau_1 \in [s_1, s_1 + 1]$  such that

$$\begin{aligned} & \sup_{0 < y \leq 1} \frac{[z^{(1+\alpha)/2}(y, \tau_1) - z^{(1+\alpha)/2}(0, \tau_1)]^2}{y} + \int_0^{R(\tau_1)} \rho(y) z^{-2}(y, \tau_1) dy \\ (3.14) \quad &= \int_{s_1}^{s_1+1} \left\{ \sup_{0 < y \leq 1} \frac{[z^{(1+\alpha)/2}(y, s) - z^{(1+\alpha)/2}(0, s)]^2}{y} + \int_0^{R(\tau_1)} \rho(y) z^{-2}(y, s) dy \right\} ds \\ &\leq C. \end{aligned}$$

This implies that, for  $0 < y \leq 1$ ,

$$z^{(1+\alpha)/2}(y, \tau_1) \leq z^{(1+\alpha)/2}(0, \tau_1) + C\sqrt{y},$$

and so (assuming that  $\alpha < 1$ ),

$$z^2(y, \tau_1) \leq C[z^{1+\alpha}(0, \tau_1) + y]^{2/(1+\alpha)}.$$

Substituting into (3.14), we obtain

$$\frac{1+\alpha}{1-\alpha} z(0, \tau_1)^{-(1-\alpha)} \leq \int_0^1 \frac{dy}{[z^{1+\alpha}(0, \tau_1) + y]^{2/(1+\alpha)}} \leq C.$$

Since we take  $\alpha < 1$ , this implies that

$$z(0, \tau_1) \geq c_0$$

for some  $c_0 > 0$  independent of  $s_1$ . Let  $k$  to be the first integer bigger than  $s_0$ , and take  $s_1$  to be  $k, k+1, k+2, k+3, \dots$ . We have proved:

**Lemma 3.2.** *There exists  $\tau_j \in [k+j, k+j+1]$  with  $0 \leq \tau_{j+1} - \tau_j \leq 2$  such that*

$$(3.15) \quad z(0, \tau_j) \geq c_0$$

for some positive constant  $c_0$  independent of  $s_1$ .

Based on Lemma 3.2, we can establish the following lemma on the lower bound of  $z$ , provided some  $L^2$  estimates on  $z_s^-$  is available.

**Lemma 3.3.** *Let  $c_0$  be given as in (3.15) and let*

$$\Lambda = \|h\|_{L^\infty[-\ln T, \infty)}, \quad A = \Lambda \left(\frac{c_0}{2}\right)^{-2}, \quad y^* = \sqrt{\frac{c_0}{2A}}.$$

If

$$\frac{1}{y^*} \int_{\tau_j}^{\tau} \int_0^{y^*} [(z_s)^-]^2 dy ds \leq \frac{c_0^2}{32}$$

for some  $\tau \in (\tau_j, \tau_j + 2]$ , then

$$(3.16) \quad z(0, s) \geq \frac{c_0}{2} \quad \text{for } \tau_j \leq s \leq \tau.$$

*Proof.* We already established

$$z(0, \tau_j) \geq c_0 > 0, \quad \tau_j \rightarrow \infty, \quad 0 < \tau_{j+1} - \tau_j \leq 2.$$

Since  $y = 0$  is the minimum of  $z(y, s)$ , we have

$$z(y, \tau_j) \geq z(0, \tau_j) \geq c_0.$$

We write

$$\mathcal{L}[z] \triangleq z_s - z_{yy} + \frac{1}{2}yz_y - \frac{1}{3}z + h(s)z^{-2} = 0.$$

Since  $A = \Lambda(c_0/2)^{-2}$ , we have

$$\begin{aligned} \mathcal{L}\left[\frac{c_0}{2} + \frac{A}{2}y^2\right] &= -A + \frac{A}{2}y^2 - \frac{1}{3}\left(\frac{c_0}{2} + \frac{A}{2}y^2\right) + h(s)\left(\frac{c_0}{2} + \frac{A}{2}y^2\right)^{-2} \\ &\leq -A + \Lambda\left(\frac{c_0}{2}\right)^{-2} = 0 \quad \text{for } |y| < \sqrt{\frac{c_0}{2A}} = y^*. \end{aligned}$$

It is also clear that with our choice of  $y^*$  we also have

$$\frac{c_0}{2} + \frac{A}{2}y^2 \leq \frac{3c_0}{4} \quad \text{for } |y| \leq y^*.$$

By our assumption,

$$\frac{1}{y^*} \int_{\tau_j}^{\tau} \int_0^{y^*} [(z_s)^-]^2 dy ds \leq c_0^2/32.$$

By the mean value theorem, there exists  $y_1^* \in [0, y^*]$  such that

$$\int_{\tau_j}^{\tau} [(z_s(y_1^*, s))^-]^2 ds \leq c_0^2/32.$$

Hence

$$\begin{aligned} -[z(y_1^*, s) - z(y_1^*, \tau_j)] &\leq \sqrt{\tau - \tau_j} \left( \int_{\tau_j}^{\tau} [(z_s(y_1^*, s))^-]^2 ds \right)^{1/2} \\ &\leq \sqrt{2} \sqrt{\frac{c_0^2}{32}} = \frac{c_0}{4} \quad \text{for } \tau_j \leq s \leq \tau, \end{aligned}$$

which implies

$$z(y_1^*, s) \geq z(y_1^*, \tau_j) - \frac{c_0}{4} \geq c_0 - \frac{c_0}{4} \geq \frac{3c_0}{4} \quad \text{for } \tau_j \leq s \leq \tau.$$

By symmetry, we also have  $z(-y_1^*, s) = z(y_1^*, s)$ .

If  $y_1^* = 0$ , we then already have  $z(0, s) \geq 3c_0/4$  for  $\tau_j \leq s \leq \tau$ . In the case  $y_1^* > 0$ , we can then apply the comparison principle in the region  $\{(y, s); |y| < y_1^*, \tau_j < s < \tau\}$  to conclude

$$z(y, s) \geq \frac{c_0}{2} + \frac{A}{2}y^2 \quad \text{for } |y| < y_1^*, \tau_j \leq s \leq \tau.$$

Hence (3.16) follows and this concludes the proof of this lemma.  $\square$

**Remark 3.1.** Lemmas 3.1, 3.2 and 3.3 are valid for any function  $h(s)$  bounded by positive constants from above and below, with  $c_0$  and  $\tau_j$  depending only on these two bounds.

Next, we show the following lemma which is very useful in the proof of Lemma 4.3 in §4.

**Lemma 3.4.** *There exist constants  $L_1 > 0$  and  $c_1 > 0$  such that*

$$(3.17) \quad z(L_1, s) \geq c_1 \quad \text{for } s \gg 1.$$

*Proof.* Multiplying the equation (3.5) by  $z^2$  and integrating, we obtain

$$\frac{d}{ds} \int_{-R(s)}^{R(s)} \frac{1}{3} \rho z^3 dy = -2 \int_{-R(s)}^{R(s)} \rho z z_y^2 dy + \int_{-R(s)}^{R(s)} \frac{1}{3} \rho z^3 - h(s) \int_{-R(s)}^{R(s)} \rho dy + J(s),$$

where  $J(s)$  comes from various terms generated by integration by parts. By (1.7), (3.2), (3.3) and (3.4), we have

$$|J(s)| \leq C \exp\left(-\frac{1}{8}e^{s/2}\right).$$

Thus, for any small  $\varepsilon > 0$ , we can take  $S_1 \gg 1$  so that

$$(3.18) \quad \frac{d}{ds} \int_{-R(s)}^{R(s)} \frac{1}{3} \rho z^3 dy < \int_{-R(s)}^{R(s)} \frac{1}{3} \rho z^3 - (h(\infty) - \varepsilon) \int_{-R(s)}^{R(s)} \rho dy := I(s)$$

for  $s \gg S_1$ . If  $I(s^*) < 0$  for some  $s^* > S_1$ , then (3.18) implies that  $I(s) < 0$  for all  $s > s^*$ . Therefore,  $\int_{-R(s)}^{R(s)} \frac{1}{3} \rho z^3$  must reach zero in a finite time, which is a contradiction. Thus we must have

$$\int_{-R(s)}^{R(s)} \frac{1}{3} \rho z^3 \geq (h(\infty) - \varepsilon) \int_{-R(s)}^{R(s)} \rho dy \quad \text{for } s > S_1.$$

It follows that

$$\frac{2L_1}{3} z^3(L_1) \geq \int_{-L_1}^{L_1} \frac{1}{3} z^3 dy \geq \int_{-L_1}^{L_1} \frac{1}{3} \rho z^3 dy \geq (h(\infty) - \varepsilon) \int_{-R(s)}^{R(s)} \rho dy - \int_{|y| \geq L_1} \frac{1}{3} \rho z^3 dy.$$

Since  $z \leq C(1 + |y|^\mu)$ , we can take  $L_1$  suitable large so that the right-hand side of the above inequality is uniformly positive, and this implies the conclusion of the lemma.  $\square$

#### 4. PROOF OF THE MAIN THEOREM

In this section, we first construct a Lyapunov function. Let

$$(4.1) \quad E(s) = \frac{1}{2} \int_{-R(s)}^{R(s)} \rho(y) z_y^2(y, s) dy - \frac{1}{6} \int_{-R(s)}^{R(s)} \rho(y) z^2(y, s) dy.$$

Then  $E(s)$  is bounded from below, due to (3.2). We compute, using (3.5),

$$\begin{aligned} \frac{d}{ds} E(s) &= \int_{-R(s)}^{R(s)} \rho z_y z_{ys} dy - \frac{1}{3} \int_{-R(s)}^{R(s)} \rho z z_s dy + J_{11}(s) \\ &= - \int_{-R(s)}^{R(s)} (\rho z_y)_y z_s dy - \frac{1}{3} \int_{-R(s)}^{R(s)} \rho z z_s dy + J_{11}(s) + J_{12}(s) \\ &= - \int_{-R(s)}^{R(s)} \rho z_s^2 dy + h(s) \frac{d}{ds} \left\{ \int_{-R(s)}^{R(s)} \rho z^{-1} dy \right\} + J_{11}(s) + J_{12}(s) + J_{13}(s), \end{aligned}$$

where

$$\begin{aligned} J_{11}(s) &:= \rho(R(s))z_y^2(R(s), s)R'(s) - \frac{1}{3}\rho(R(s))z^2(R(s), s)R'(s), \\ J_{12}(s) &:= 2\rho(R(s))z_y(R(s), s)z_s(R(s), s), \\ J_{13}(s) &:= -2h(s)\rho(R(s))z^{-1}(R(s), s)R'(s). \end{aligned}$$

Set  $J_1(s) := J_{11}(s) + J_{12}(s) + J_{13}$ . Then we have

$$(4.2) \quad \frac{d}{ds}E(s) = - \int_{-R(s)}^{R(s)} \rho z_s^2 dy + h(s) \frac{d}{ds}K_1(s) + J_1(s),$$

where

$$(4.3) \quad K_1(s) := \int_{-R(s)}^{R(s)} \rho(y)z^{-1}(y, s)dy.$$

Note that, by (3.2), (1.7), (3.3) and (3.4), we have

$$(4.4) \quad |J_1(s)| \leq C \exp\left(-\frac{1}{8}e^{s/2}\right).$$

Therefore, in order to verify the function  $E(s)$  defined by (4.1) is a Lyapunov function, we need to derive some useful estimates for  $h(s)$  and the derivative of the function  $K_1(s)$  defined by (4.3) to ensure the integrability of  $h(s)K_1'(s)$  over  $[s_0, \infty)$ .

**4.1. Estimates for  $h(s)$ .** Recall that  $h(s) = g(T - e^{-s})$ . From the definition of  $h(s)$ , we derive

$$\begin{aligned} \sqrt{\lambda} h^{-1/2}(s) - 1 &= \int_{-1}^1 u^{-1}(x, T - e^{-s})dx = e^{-s/6} \int_{-R(s)}^{R(s)} z^{-1}(y, s)dy \\ &= e^{-s/6} \left\{ \int_{-R(s)}^{R(s)} \rho(y)z^{-1}(y, s)dy + \int_{-R(s)}^{R(s)} [1 - \rho(y)]z^{-1}(y, s)dy \right\} \\ &\triangleq e^{-s/6} \{K_1(s) + K_2(s)\}. \end{aligned}$$

Differentiating the equality  $(1 + w)^{-1} = \sum_{j=0}^{\infty} (-1)^j w^j$  in  $w$  (for  $|w| < 1$ ), we obtain

$$(1 + w)^{-2} = - \sum_{j=0}^{\infty} (-1)^j j w^{j-1} = \sum_{j=0}^{\infty} (-1)^j (j + 1) w^j.$$

It follows that

$$(4.5) \quad \begin{aligned} \lambda^{-1}h(s) &= (1 + e^{-s/6}K_2)^{-2} \left(1 + \frac{e^{-s/6}K_1}{1 + e^{-s/6}K_2}\right)^{-2} \\ &= (1 + e^{-s/6}K_2)^{-2} \sum_{j=0}^{\infty} b_j \left(\frac{e^{-s/6}K_1}{1 + e^{-s/6}K_2}\right)^j, \end{aligned}$$

where  $b_j = (-1)^j(j + 1)$ ,  $j \geq 0$ . The series is uniformly convergent whenever  $\left|\frac{e^{-s/6}K_1}{1 + e^{-s/6}K_2}\right| < 1$ .

We first establish

**Lemma 4.1.** *Let  $\beta \in (2, 3)$  close to 3 and let*

$$u_* := \lim_{t \rightarrow T} \int_{-1}^1 u^{-1}(x, t) dx > 0.$$

*Then, for  $\gamma_1 = \frac{1}{2} - \frac{1}{\beta} > 0$  and  $\gamma > 0$  given in Lemma 2.5, we have*

$$(4.6) \quad |e^{-s/6} K_1(s)| \leq C e^{-\gamma_1 s}, \quad |e^{-s/6} K_2(s) - u_*| \leq C e^{-\gamma s}.$$

*for all  $s \geq s_0$ .*

*Proof.* By (2.1),  $z(y, s) \geq c e^{s/3-s/\beta} |y|^{2/\beta}$ , so that

$$(4.7) \quad e^{-s/6} z^{-1}(y, s) \leq c^{-1} e^{-s/6+s/\beta-s/3} |y|^{-2/\beta}, \quad \forall |y| \leq R(s), \quad s \geq s_0.$$

Since  $2 < \beta < 3$ , we have  $-\frac{1}{6} + \frac{1}{\beta} - \frac{1}{3} = -\gamma_1 < 0$ . It follows that  $|e^{-s/6} K_1(s)| \leq C e^{-\gamma_1 s}$  for all  $s \geq s_0$ . Using this in the relation

$$\int_{-1}^1 u^{-1}(x, T - e^{-s}) dx = e^{-s/6} (K_1 + K_2),$$

applying also Lemma 2.5, we immediately obtain the second estimate in (4.6). This proves the lemma.  $\square$

Recall from (3.11) that

$$\int_s^{s+1} \int_{-R(s)}^{R(s)} \rho(y) z^{-k}(y, s) dy ds \leq C$$

for any  $s \geq s_0$  for any  $1 \leq k \leq 2$ . This actually implies

**Lemma 4.2.** *For any  $\alpha > 0$*

$$(4.8) \quad \int_{s_0}^{\infty} e^{-\alpha s} K_1(s) ds \leq C, \quad \int_{s_0}^{\infty} e^{-\alpha s} K_1^2(s) ds \leq C.$$

*Proof.* We break the integral into an infinite series:

$$\begin{aligned} \int_{s_0}^{\infty} e^{-\alpha s} K_1(s) ds &= \sum_{j=0}^{\infty} \int_{s_0+j}^{s_0+j+1} e^{-\alpha s} K_1(s) ds \\ &\leq \sum_{j=0}^{\infty} e^{-\alpha(s_0+j)} \int_{s_0+j}^{s_0+j+1} K_1(s) ds \\ &\leq C \sum_{j=0}^{\infty} e^{-\alpha(s_0+j)} \leq C. \end{aligned}$$

By Hölder's inequality

$$K_1^2(s) = \left( \int_{-R(s)}^{R(s)} \rho z^{-1} dy \right)^2 \leq C \int_{-R(s)}^{R(s)} \rho z^{-2} dy,$$

so that the second inequality can be established in a similar manner.  $\square$

4.2. **Estimates for the derivative of  $K_2$ .** Our key lemma is as follows

**Lemma 4.3.** *There exists  $\gamma_2 > 0$  such that*

$$(4.9) \quad \left| \frac{d}{ds} (e^{-s/6} K_2(s)) \right| \leq C e^{-\gamma_2 s} \left( 1 + \int_{-R(s)}^{R(s)} \rho |z_s| dy \right).$$

*Proof.* We write  $K_2$  as  $2 \int_0^{R(s)} (1 - \rho) z^{-1} dy$ . Then for any  $L > 0$ , using (1.7), we have

$$\begin{aligned} \frac{dK_2}{ds} &= -2 \int_0^{R(s)} (1 - \rho) z^{-2} z_s dy + 2(1 - \rho(R(s)) z^{-1}(R(s), s) R'(s)) \\ &= 2 \int_0^L \rho(1 - \rho^{-1}) z^{-2} z_s dy + (1 - \rho(e^{s/2})) e^{s/6} + 2 \int_L^{R(s)} \rho z^{-2} z_s dy - 2 \int_L^{R(s)} z^{-2} z_s dy \\ &\triangleq M_1 + M_2 + M_3 + M_4. \end{aligned}$$

By (2.1),  $z^{-2} \leq C |y|^{-4/\beta} e^{(2/3-2/\beta)s}$ . It is also clear that, for  $|y| \leq L$ ,  $|1 - \rho^{-1}| \leq C |y|^2$ . It follows that the first term on the right-hand side is estimated by

$$\begin{aligned} |M_1| &= \left| 2 \int_0^L \rho(1 - \rho^{-1}) z^{-2} z_s dy \right| \\ &\leq C \int_0^L \rho |y|^{2-4/\beta} e^{(-2/3+2/\beta)s} |z_s| dy \leq C e^{(-2/3+2/\beta)s} \int_0^L \rho |z_s| ds, \end{aligned}$$

so that, if we take  $0 < \gamma_2 \leq \frac{1}{6} + \left( \frac{2}{3} - \frac{2}{\beta} \right)$  (this is possible if we take  $\beta$  close to 3),

$$(4.10) \quad e^{-s/6} |M_1| \leq C e^{-\gamma_2 s} \int_0^L \rho |z_s| ds.$$

It is clear that the second term

$$(4.11) \quad |e^{-s/6} M_2 - 1| = |\rho(e^{s/2})| \leq C \exp \left( -\frac{1}{8} e^{s/2} \right) \leq C e^{-\gamma_2 s}.$$

We assume that  $L > L_1$ . Since  $z(y, s) \geq c_0$  for  $y > L$ , by (3.17) and  $z_y \geq 0$ , the third term is bounded by

$$(4.12) \quad e^{-s/6} |M_3| = \left| 2 e^{-s/6} \int_L^{R(s)} \rho z^{-2} z_s dy \right| \leq C e^{-s/6} \int_L^{R(s)} \rho |z_s| ds.$$



It remains to estimate  $M_4$ . Using the equation (1.6), we get

$$\begin{aligned}
M_4 &= -2 \int_L^{R(s)} z^{-2} z_s ds = 2 \int_L^{R(s)} z^{-2} \left\{ -z_{yy} + \frac{1}{2} y z_y - \frac{1}{3} z + h(s) z^{-2} \right\} ds \\
&= 2 \left\{ -2 \int_L^{R(s)} z^{-3} z_y^2 dy + \frac{1}{2} \int_L^{R(s)} z^{-1} dy + \left[ -z^{-2} z_y \Big|_L^{R(s)} - \frac{1}{2} y z^{-1} \Big|_L^{R(s)} \right] \right. \\
&\quad \left. - \frac{1}{3} \int_L^{R(s)} z^{-1} dy + h(s) \int_L^{R(s)} z^{-4} dy \right\} \\
&= \frac{1}{3} \int_L^{R(s)} z^{-1} dy + \left( -e^{s/6} + L z^{-1}(L) - 2 z^{-2} z_y \Big|_L^{R(s)} \right) \\
&\quad - 4 \int_L^{R(s)} z^{-3} z_y^2 dy + 2 h(s) \int_L^{R(s)} z^{-4} dy \\
&\triangleq M_{41} + M_{42} + M_{43} + M_{44}.
\end{aligned}$$

Clearly,

$$(4.13) \quad |e^{-s/6} M_{42} + 1| \leq C e^{-s/6}.$$

To estimate  $M_{43}$ , we use Lemma 2.4 to derive

$$|z_y(y, s)| \leq C e^{-s/6} |y e^{-s/2}|^{-1/2} \leq C e^{s/12} |y|^{-1/2} \leq C e^{s/12} \quad \text{for } y \geq L.$$

It follows that

$$|M_{43}| \leq C \int_L^{R(s)} e^{s/12} z^{-2} z_y dy \leq C e^{s/12} z^{-1}(L, s) \leq C e^{s/12},$$

by using (3.17) twice, and so

$$(4.14) \quad e^{-s/6} |M_{43}| \leq C e^{-s/12}.$$

To estimate  $M_{44}$ , by (2.1), we have

$$z^{-4}(y, s) \leq c^{-4} e^{(4/\beta - 4/3)s} |y|^{-8/\beta}.$$

Hence, if we choose  $\beta$  so that  $0 < \gamma_2 \leq \frac{1}{6} - \frac{4}{\beta} + \frac{4}{3}$ , we get

$$(4.15) \quad e^{-s/6} |M_{44}| \leq C e^{-s/6 + (4/\beta - 4/3)s} \int_L^\infty |y|^{-8/\beta} dy \leq C e^{-\gamma_2 s}.$$

Combining all these estimates, we find

$$\left| \frac{d}{ds} \left( e^{-s/6} K_2 \right) \right| \leq C e^{-\gamma_2 s} \left( 1 + \int_{-R(s)}^{R(s)} \rho |z_s| dy \right) + \left| -\frac{1}{6} e^{-s/6} K_2 + \frac{1}{3} e^{-s/6} \int_L^{R(s)} z^{-1} dy \right|.$$

The last term is estimated by

$$\begin{aligned} \left| -\frac{1}{6}e^{-s/6}K_2 + \frac{1}{3}e^{-s/6} \int_L^{R(s)} z^{-1} dy \right| &= \frac{1}{6}e^{-s/6} \left| K_2 - \int_{|y| \geq L} z^{-1} dy \right| \\ &\leq \frac{1}{6}e^{-s/6} \left( \int_{|y| \leq L} z^{-1} dy + \int_{-R(s)}^{R(s)} \rho z^{-1} dy \right) \\ &\leq Ce^{-\gamma_1 s}, \end{aligned}$$

where (4.7) was used in the last inequality. Therefore, the lemma is proved.  $\square$

**4.3. The conclusion.** We shall now always assume  $s \gg 1$  so that  $0 < e^{-s/6}K_1 < \frac{1}{2}$  and  $e^{-s/6}K_2 \geq \frac{1}{2}u_* > 0$ . This is possible by using (4.6). Therefore, the series defined in (4.5) is uniformly convergent.

We now proceed to estimate

$$\lambda^{-1}h(s)\frac{dK_1}{ds} = (1 + e^{-s/6}K_2)^{-2} \sum_{j=0}^{\infty} (-1)^j \left( \frac{e^{-s/6}}{1 + e^{-s/6}K_2} \right)^j \frac{dK_1^{j+1}}{ds}.$$

Applying integration by parts, we have

$$\begin{aligned} \lambda^{-1} \int_{s_1}^{s_2} h(s) \frac{dK_1}{ds} ds &= \left[ \sum_{j=0}^{\infty} (-1)^j \frac{e^{-js/6}}{(1 + e^{-s/6}K_2)^{j+2}} K_1^{j+1} \right] \Big|_{s_1}^{s_2} \\ &\quad - \sum_{j=0}^{\infty} (-1)^j \int_{s_1}^{s_2} K_1^{j+1}(s) \frac{d}{ds} \left\{ \frac{e^{-js/6}}{(1 + e^{-s/6}K_2)^{j+2}} \right\} ds \\ &\triangleq K_{21} + K_{22}. \end{aligned}$$

First, we write  $K_{21}$  as

$$\left[ K_1 \frac{1}{(1 + e^{-s/6}K_2)^2} \sum_{j=0}^{\infty} (-1)^j \left( \frac{e^{-s/6}K_1}{1 + e^{-s/6}K_2} \right)^j \right] \Big|_{s_1}^{s_2};$$

the quantity  $\frac{1}{(1 + e^{-s/6}K_2)^2} \sum_{j=0}^{\infty} (-1)^j \left( \frac{e^{-s/6}K_1}{1 + e^{-s/6}K_2} \right)^j$  is uniformly bounded, by (4.6). Therefore,

$$|K_{21}| \leq C(|K_1(s_2)| + |K_1(s_1)|).$$

To estimate  $K_{22}$ , we compute

$$\begin{aligned} &\frac{d}{ds} \left\{ \frac{e^{-js/6}}{(1 + e^{-s/6}K_2)^{j+2}} \right\} \\ &= -\frac{j}{6} \frac{e^{-js/6}}{(1 + e^{-s/6}K_2)^{j+2}} - \frac{e^{-js/6}(j+2)}{(1 + e^{-s/6}K_2)^{j+3}} \frac{d}{ds} (e^{-s/6}K_2), \end{aligned}$$

so that

$$\begin{aligned}
K_{22} &= \sum_{j=1}^{\infty} (-1)^j \frac{j}{6} \int_{s_1}^{s_2} K_1^{j+1}(s) \frac{e^{-js/6}}{(1 + e^{-s/6} K_2)^{j+2}} ds \\
&\quad + \sum_{j=0}^{\infty} (-1)^j \int_{s_1}^{s_2} K_1^{j+1}(s) \frac{e^{-js/6} (j+2)}{(1 + e^{-s/6} K_2)^{j+3}} \frac{d}{ds} \left( e^{-s/6} K_2 \right) ds \\
&\triangleq K_{221} + K_{222}.
\end{aligned}$$

Clearly,

$$\begin{aligned}
&\left| \sum_{j=1}^{\infty} (-1)^j \frac{j}{6} K_1^{j+1} \frac{e^{-js/6}}{(1 + e^{-s/6} K_2)^{j+2}} \right| \\
&= e^{-s/6} K_1^2 \left| \sum_{j=1}^{\infty} (-1)^j \frac{j}{6} \frac{1}{(1 + e^{-s/6} K_2)^3} \left( \frac{e^{-s/6} K_1}{(1 + e^{-s/6} K_2)} \right)^{j-1} \right| \\
&\leq C e^{-s/6} K_1^2,
\end{aligned}$$

so that, by Lemma 4.2,

$$|K_{221}| \leq C \int_{s_1}^{s_2} e^{-s/6} K_1^2(s) ds \leq C.$$

Similarly,

$$\begin{aligned}
\left| \sum_{j=0}^{\infty} (-1)^j K_1^{j+1} \frac{e^{-js/6} (j+2)}{(1 + e^{-s/6} K_2)^{j+3}} \right| &= \left| \sum_{j=0}^{\infty} (-1)^j (j+2) \frac{K_1}{(1 + e^{-s/6} K_2)^3} \left( \frac{e^{-s/6} K_1}{(1 + e^{-s/6} K_2)} \right)^j \right| \\
&\leq C K_1,
\end{aligned}$$

so that, by Lemma 4.3, for small  $\varepsilon > 0$ ,

$$\begin{aligned}
|K_{222}| &\leq C \int_{s_1}^{s_2} e^{-\gamma_2 s} K_1(s) \left( 1 + \int_{-R(s)}^{R(s)} \rho |z_s| dy \right) ds \\
&\leq C + C \int_{s_1}^{s_2} e^{-\gamma_2 s} K_1(s) \left( \int_{-R(s)}^{R(s)} \rho |z_s| dy \right) ds \\
&\leq C + \varepsilon \int_{s_1}^{s_2} \int_{-R(s)}^{R(s)} \rho |z_s|^2 dy + C \varepsilon^{-1} \int_{s_1}^{s_2} e^{-2\gamma_2 s} K_1^2(s) ds \\
&\leq C \varepsilon^{-1} + \varepsilon \int_{s_1}^{s_2} \int_{-R(s)}^{R(s)} \rho |z_s|^2 dy.
\end{aligned}$$

Combining these estimates, we find

$$|K_{22}| \leq C \varepsilon^{-1} + \varepsilon \int_{s_1}^{s_2} \int_{-R(s)}^{R(s)} \rho |z_s|^2 dy.$$

Since  $\int_{s_1}^{s_1+1} K_1(s)ds < C$ , we can choose  $s_j \rightarrow \infty$  such that  $K_1(s_j) < C$ . Therefore, we have proved

**Lemma 4.4.** *There exists  $s_j \rightarrow \infty$  such that*

$$(4.16) \quad \lambda^{-1} \left| \int_{s_1}^{s_j} h(s) \frac{dK_1}{ds} ds \right| \leq C\varepsilon^{-1} + \varepsilon \int_{s_1}^{s_j} \int_{-R(s)}^{R(s)} \rho |z_s|^2 dy$$

for some small positive constant  $\varepsilon$ .

With Lemma 4.4, we are ready to verify that  $E(s)$  is actually a Lyapunov function as follows. Indeed, by integrating (4.2) from  $s_1$  to any  $s_j$  and using (4.16), it follows that

$$\begin{aligned} \int_{s_1}^{s_j} \int_{-R(s)}^{R(s)} \rho(y) z_s^2 dy ds &= E(s_1) - E(s_j) + \int_{s_1}^{s_j} h(s) \frac{dK_1(s)}{ds} ds + \int_{s_1}^{s_j} J_1(s) ds \\ &\leq C\varepsilon^{-1} + \lambda\varepsilon \int_{s_1}^{s_j} \int_{-R(s)}^{R(s)} \rho(y) z_s^2 dy ds. \end{aligned}$$

Taking a fixed  $\varepsilon \in (0, \lambda^{-1}/2)$ , we obtain

**Lemma 4.5.** *It holds*

$$(4.17) \quad \int_{s_1}^{\infty} \int_{-R(s)}^{R(s)} \rho(y) z_s^2 dy ds < \infty.$$

Applying (4.17) to Lemma 3.3, we find that now that  $z(y, s)$  is uniformly bounded from below by a positive constant, we can immediately apply parabolic estimates to derive uniform  $C^{2+\alpha, 1+\alpha/2}$  estimates. The standard method implies that  $z(y, s)$  converges uniformly on any compact set to the constant  $w_\infty$  (depending on the solution itself). This completes the proof of Theorem 1.

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