LOCALIZED DONALDSON-THOMAS THEORY OF SURFACES

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Abstract. Let S be a projective simply connected complex surface and $\mathcal{L}$ be a line bundle on S. We study the moduli space of stable compactly supported 2-dimensional sheaves on the total spaces of $\mathcal{L}$. The moduli space admits a $\mathbb{C}^*$-action induced by scaling the fibers of $\mathcal{L}$. We identify certain components of the fixed locus of the moduli space with the moduli space of torsion free sheaves and the nested Hilbert schemes on S. We define the localized Donaldson-Thomas invariants of $\mathcal{L}$ by virtual localization in the case that $\mathcal{L}$ twisted by the anti-canonical bundle of S admits a nonzero global section. When $p_g(S) > 0$, in combination with Mochizuki's formulas, we are able to express the localized DT invariants in terms of the invariants of the nested Hilbert schemes defined by the authors in [GSY17a], the Seiberg-Witten invariants of S, and the integrals over the products of Hilbert schemes of points on S. When $\mathcal{L}$ is the canonical bundle of S, the Vafa-Witten invariants defined recently by Tanaka-Thomas, can be extracted from these localized DT invariants. VW invariants are expected to have modular properties as predicted by S-duality.

Contents

1. Introduction

1.1. **Overview.** The Donaldson-Thomas invariants of 2-dimensional sheaves in projective nonsingular (Calabi-Yau) threefolds are expected to have modular properties through S-duality considerations ([DM11, VW94, GS13, GST14]). These invariants are very difficult to compute in general due to lack of control over the singularity of surfaces supporting these sheaves. To make the situation more manageable, we consider the total space of a line bundle $\mathcal{L}$ over a fixed nonsingular projective surface S. We then study the moduli space of h-stable 2-dimensional compactly supported sheaves $\mathcal{E}$ such that $c_1(\mathcal{E}) = r[S]$, where $[S]$ is the class of the 0-section and $h = c_1(\mathcal{O}_S(1))$.

To define DT invariants of $\mathcal{L}$ we have to overcome two main obstacles:

1. Construct a perfect obstruction theory over the moduli space, which contains no trivial factor in its obstruction[1],
2. If $H^0(\mathcal{L}) \neq 0$ then the moduli space is not compact and hence one cannot expect to get a well-defined virtual fundamental class from 1.

For 1, we do not allow strictly semistable sheaves in the moduli space, and we assume that the line bundle $\mathcal{L} \otimes \omega_S^{-1}$ admits a nonzero global section, where ω_S is the canonical bundle of S. The latter condition guarantees that higher obstruction spaces of stable sheaves under consideration either vanish (if $\mathcal{L} \neq \omega_C$) or can be ignored (if $\mathcal{L} = \omega_C$), and in any case [T98, HT10] provide the moduli space with a natural perfect obstruction theory. We assume that S is simply connected and then construct a reduced perfect obstruction theory out of the natural one by removing a trivial factor of rank $p_g(S)$.

For 2, we consider the $\mathbb{C}^*$-action on the moduli space induced by scaling the fibers of $\mathcal{L}$. The fixed set of the moduli space is compact and the fixed part of the reduced perfect obstruction theory above leads to a reduced virtual fundamental class over this fixed set [GP99]. We define two types of Donaldson-Thomas invariants by integrating against this class. The study of these invariants completely boils down to understanding the fixed set of the moduli space and also the fixed and moving parts of the reduced perfect obstruction theory. By restricting to the fixed set of the moduli space, we have much more control over the possible singularities of the supporting surfaces: the only singularities that can occur are the thickenings of the zero section along the fibers of $\mathcal{L}$.

1.2. **Main results.** We fix some symbols and notation before expressing the results. Let S be a nonsingular projective simply connected surface with $h = c_1(\mathcal{O}_S(1))$. Let $\mathrm{q} : \mathcal{L} \to S$ be a line bundle on S so that $H^0(\mathcal{L} \otimes \omega_S^{-1}) \neq 0$. Let

$$v = (r, \gamma, m) \in \oplus_{i=0}^{2} H^{2i}(S, \mathbb{Q})$$

[1] Otherwise, the DT invariants would vanish.

be a Chern character vector with $r \geq 1$, and $\mathcal{M}_h^{\mathcal{L}}(v)$ be the moduli space of compactly supported 2-dimensional stable sheaves $\mathcal{E}$ on $\mathcal{L}$ such that $\mathrm{ch}(\mathrm{q}_* \mathcal{E}) = v$. Here stability is defined by means of the slope of $\mathrm{q}_* \mathcal{E}$ with respect to the polarization h.

The $\mathbb{C}^*$-fixed locus $\mathcal{M}_h^{\mathcal{L}}(v)^{\mathbb{C}^*}$ consists of sheaves supported on S (the zero section of $\mathcal{L}$) and its thickenings. As discussed above, we show that $\mathcal{M}_h^{\mathcal{L}}(v)^{\mathbb{C}^*}$ carries a reduced virtual fundamental class denoted by $[\mathcal{M}_h^{\mathcal{L}}(v)^{\mathbb{C}^*}]^{\mathrm{vir}}_{\mathrm{red}}$ (Proposition 2.4). In this paper we study two types of DT invariants

$$\mathrm{DT}_h^{\mathcal{L}}(v;\alpha) = \int_{[\mathcal{M}_h^{\mathcal{L}}(v)^{\mathbb{C}^*}]^{\mathrm{vir}}_{\mathrm{red}}} \frac{\alpha}{\mathrm{Nor}^{\mathrm{vir}}} \in \mathbb{Q}[\mathbf{s},\mathbf{s}^{-1}] \qquad \alpha \in H^*_{\mathbb{C}^*}(\mathcal{M}_h^{\mathcal{L}}(v)^{\mathbb{C}^*},\mathbb{Q})_{\mathbf{s}},$$
$$\mathrm{DT}_h^{\mathcal{L}}(v) = \chi^{\mathrm{vir}}(\mathcal{M}_h^{\mathcal{L}}(v)^{\mathbb{C}^*}) \in \mathbb{Z},$$

where $\mathrm{Nor}^{\mathrm{vir}}$ is the virtual normal bundle of $\mathcal{M}_h^{\mathcal{L}}(v)^{\mathbb{C}^*} \subset \mathcal{M}_h^{\mathcal{L}}(v)$, $\chi^{\mathrm{vir}}(-)$ is the virtual Euler characteristic [FG10], and $\mathbf{s}$ is the equivariant parameter.

If $\mathcal{L} = \omega_C$ and $\alpha = 1$ then

$$\mathrm{DT}_h^{\omega_S}(v;1) = \mathbf{s}^{-p_g}\,\mathrm{VW}_h(v),$$

where $\mathrm{VW}_h(-)$ is the Vafa-Witten invariant defined by Tanaka and Thomas [TT] and are expected to have modular properties (see Remark 2.10).

We write $\mathcal{M}_h^{\mathcal{L}}(v)^{\mathbb{C}^*}$ as a disjoint union of several types of components, where each type is indexed by a partition of r. Out of these component types, there are two types that are in particular important for us. One of them (we call it type I) is identified with $\mathcal{M}_h(v)$, the moduli space of rank r torsion free stable sheaves on S. The other type (we call it type II) can be identified with the nested Hilbert scheme $S_{\boldsymbol{\beta}}^{[\boldsymbol{n}]}$ for a suitable choice of nonnegative integers $\boldsymbol{n} := n_1, \ldots, n_r$ and effective curve classes $\boldsymbol{\beta} := \beta_1, \ldots, \beta_{r-1}$ in S. Here $S_{\boldsymbol{\beta}}^{[\boldsymbol{n}]}$ is the nested Hilbert scheme on S parameterizing tuples

$$(Z_1, Z_2, \ldots, Z_r), \quad (C_1, \ldots, C_{r-1})$$

where $Z_i \subset S$ is a 0-dimensional subscheme of length n_i, and $C_i \subset S$ is a curve with $[C_i] = \beta_i$, and Z_{i+1} is a subscheme of $Z_i \cup C_i$ for any $i < r$, or equivalently in terms of ideal sheaves

$$I_{Z_i}(-C_i) \subset I_{Z_{i+1}}. \tag{1}$$

If $\beta_1 = \cdots = \beta_{r-1} = 0$, then $S^{[\boldsymbol{n}]} := S_{\boldsymbol{\beta}=0}^{[\boldsymbol{n}]}$ is the Hilbert scheme of points on S. The authors have constructed a perfect obstruction theory over $S_{\boldsymbol{\beta}}^{[\boldsymbol{n}]}$ in [GSY17a] by studying the deformation/obstruction of the natural inclusions (1). As a result $S_{\boldsymbol{\beta}}^{[\boldsymbol{n}]}$ is equipped with a virtual fundamental class denoted by $[S_{\boldsymbol{\beta}}^{[\boldsymbol{n}]}]^{\mathrm{vir}}$. This allows us to define new invariants for S recovering in particular Poincaré invariants of [DKO07], and (after reduction) stable pair invariants of [KT14].

The following Theorems are proven in Propositions 3.2, 3.8 and 3.11:

Theorem 1. *The restriction of* $[\mathcal{M}_h^{\mathcal{L}}(v)^{\mathbb{C}^*}]^{\mathrm{vir}}_{\mathrm{red}}$ *to the type I component* $\mathcal{M}_h(v)$ *is identified with* $[\mathcal{M}_h(v)]_0^{\mathrm{vir}}$ *induced by the natural trace free perfect obstruction theory over* $\mathcal{M}_h(v)$.

Theorem 2. *The restriction of* $[\mathcal{M}_h^{\mathcal{L}}(v)^{\mathbb{C}^*}]^{\mathrm{vir}}_{\mathrm{red}}$ *to a type II component* $S_{\boldsymbol{\beta}}^{[\boldsymbol{n}]}$ *is identified with* $[S_{\boldsymbol{\beta}}^{[\boldsymbol{n}]}]^{\mathrm{vir}}$ *constructed in* [GSY17a].

When $r = 2$ then types I and II components are the only possibilities. This leads us to the following result (Propositions 3.1, 3.2, 3.11):

Theorem 3. *Suppose that* $v = (2, \gamma, m)$. *Then,*

$$\mathrm{DT}_h^{\mathcal{L}}(v;\alpha) = \mathrm{DT}_h^{\mathcal{L}}(v;\alpha)_{\mathrm{I}} + \sum_{n_1,n_2,\beta} \mathrm{DT}_h^{\mathcal{L}}(v;\alpha)_{\mathrm{II},S_\beta^{[n_1,n_2]}},$$
$$\mathrm{DT}_h^{\mathcal{L}}(v) = \chi^{\mathrm{vir}}(\mathcal{M}_h(v)) + \sum_{n_1,n_2,\beta} \chi^{\mathrm{vir}}(S_\beta^{[n_1,n_2]}),$$

where the sum is over all n_1, n_2, β *(depending on* v *as in Definition 3.7) for which* $S_\beta^{[n_1,n_2]}$ *is a type II component of* $\mathcal{M}_h^{\mathcal{L}}(v)^{\mathbb{C}^*}$, *and the indices I and II indicate the contributions of type I and II components to the invariant* $\mathrm{DT}_h^{\mathcal{L}}(v;\alpha)$.

The invariants $\chi^{\mathrm{vir}}(S_\beta^{[n_1,n_2]})$ and $\mathrm{DT}_h^{\mathcal{L}}(v;\alpha)_{\mathrm{II},S_\beta^{[n_1,n_2]}}$ (for a suitable choice of class α e.g. $\alpha = 1$) appearing in Theorem 3 are special types of the invariants

$$\mathsf{N}_S(n_1, n_2, \beta; -)$$

that we have defined in [GSY17a] by integrating against $[S_\beta^{[n_1,n_2]}]^{\mathrm{vir}}$ (Definition 3.5 and Corollary 3.13). One advantage of this viewpoint is that it enables us to apply some of the techniques that we developed in [ibid] to evaluate these invariants in certain cases.

Mochizuki in [M02] expresses certain integrals against the virtual cycle of $\mathcal{M}_h(v)$ in terms of Seiberg-Witten invariants and integrals $\mathsf{A}(\gamma_1, \gamma_2, v; -)$ over the product of Hilbert scheme of points on S (see Section 4). Using this result we are able to find the following expression for our DT invariants (Corollaries 3.13, 3.15, 3.16 and Proposition 4.4):

Theorem 4. *Suppose that* $p_g(S) > 0$, *and* $v = (2, \gamma, m)$ *is such that* $\gamma \cdot h > 2K_S \cdot h$ *and* $\chi(v) := \int_S v \cdot td_S \geq 1$. *Then,*

$$\mathrm{DT}_h^{\mathcal{L}}(v;1) = -\sum_{\substack{\gamma_1+\gamma_2=\gamma \\ \gamma_1\cdot h<\gamma_2\cdot h}} \mathrm{SW}(\gamma_1)\cdot 2^{2-\chi(v)}\cdot \mathsf{A}(\gamma_1,\gamma_2,v;\mathsf{P}_1) + \sum_{n_1,n_2,\beta} \mathsf{N}_S(n_1,n_2,\beta;\mathcal{P}_1).$$
$$\mathrm{DT}_h^{\mathcal{L}}(v) = -\sum_{\substack{\gamma_1+\gamma_2=\gamma \\ \gamma_1\cdot h<\gamma_2\cdot h}} \mathrm{SW}(\gamma_1)\cdot 2^{2-\chi(v)}\cdot \mathsf{A}(\gamma_1,\gamma_2,v;\mathsf{P}_2) + \sum_{n_1,n_2,\beta} \mathsf{N}_S(n_1,n_2,\beta;\mathcal{P}_2).$$

Here $\mathrm{SW}(-)$ *is the Seiberg-Witten invariant of* S*,* P_i *and* $\mathcal{P}_i$ *are certain universally defined (independent of* S*) explicit integrands (see Proposition 4.4), and the second sums in the formulas are over all* n_1, n_2, β *(depending on* v *as in Definition 3.7) for which* $S_\beta^{[n_1,n_2]}$ *is a type II component of* $\mathcal{M}_h^{\mathcal{L}}(v)^{\mathbb{C}^*}$*.*

Moreover, if $\mathcal{L} = \omega_S$ *and* S *is isomorphic to a* $K3$ *surface or one of the five types of generic complete intersections*

$$(5) \subset \mathbb{P}^3,\ (3,3) \subset \mathbb{P}^4,\ (4,2) \subset \mathbb{P}^4,\ (3,2,2) \subset \mathbb{P}^5,\ (2,2,2,2) \subset \mathbb{P}^6,$$

the DT invariants $\mathrm{DT}_h^{\omega_S}(v;1)$ *and* $\mathrm{DT}_h^{\omega_S}(v)$ *can be completely expressed as the sum of integrals over the product of the Hilbert schemes of points on* S*.*

In Theorem 4, we can always replace a given vector v by another vector (without changing the DT invariants in the right hand side of formulas), for which the condition in theorem is satisfied (see Remark 4.3).

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2. Local reduced Donaldson-Thomas Invariants

Let (S, h) be a pair of a nonsingular projective surface S with $H^1(\mathcal{O}_S) = 0$, and $h := c_1(\mathcal{O}_S(1))$, and let

$$v := (r, \gamma, m) \in H^{\mathrm{ev}}(S, \mathbb{Q}) = H^0(S) \oplus H^2(S) \oplus H^4(S),$$

with $r \geq 1$. We denote by $\mathcal{M}_h(v)$ the moduli space of h-semistable sheaves on S with Chern character v. $\mathcal{M}_h(v)$ is a projective scheme. We always assume v is such that slope semistability implies slope stability with respect h for any sheaf on S with Chern character v. We also assume $\mathcal{M}_h(v)$ admits a universal family[2], denoted by $\mathbb{E}$. For example, if $\gcd(r, \gamma \cdot h) = 1$, these requirements are the case (see [HL10, Corollary 4.6.7]). If p is the projection to the second factor of $S \times \mathcal{M}_h(v)$, by [T98, HT10]

$$\mathbf{R}\mathcal{H}om_p(\mathbb{E}, \mathbb{E})_0[1]$$

[2]The existence of the universal family is not essential in this paper, but we assume it for simplicity.

is the dual of a perfect trace-free obstruction theory on $\mathcal{M}_h(v)$, and hence gives a virtual fundamental class, denoted by $[\mathcal{M}_h(v)]_0^{\mathrm{vir}}$.

Let $\mathcal{L}$ be a line bundle on S such that

$$H^0(\mathcal{L}\otimes\omega_S^{-1})\neq 0, \tag{2}$$

and let

$$X := \mathcal{L}\xrightarrow{\mathrm{q}} S$$

be the total space of the canonical line bundle on S. Note that X is non-compact with canonical bundle $\omega_X\cong \mathrm{q}^*(\mathcal{L}^{-1}\otimes\omega_S)$. In particular X is a Calabi-Yau 3-fold if $\mathcal{L}=\omega_S$. Let $z: S\to X$ be the zero section inclusion.

Notation. *For simplicity, we use the symbols z and q to indicate respectively the inclusion $z\times\mathrm{id}: S\times B\to X\times B$ and the projection $\mathrm{q}\times\mathrm{id}: X\times B\to S\times B$ for any scheme B.*

The one dimensional complex torus $\mathbb{C}^*$ acts on X by the multiplication on the fibers of q, so that

$$\mathrm{q}_*\,\mathcal{O}_X=\bigoplus_{i=0}^{\infty}\mathcal{L}^{-i}\otimes\mathbf{t}^{-i}, \tag{3}$$

where $\mathbf{t}$ denotes the trivial line bundle on S with the $\mathbb{C}^*$-action of weight 1 on the fibers. Let

$$\mathrm{Coh}_c(X)\subset\mathrm{Coh}(X)$$

be the abelian category of coherent sheaves on X whose supports are compact. The slope function μ_h on $\mathrm{Coh}_c(X)\setminus\{0\}$

$$\mu_h(\mathcal{E})=\frac{c_1(\mathrm{q}_*\,\mathcal{E})\cdot h}{\mathrm{Rank}(\mathrm{q}_*\,\mathcal{E})}\in\mathbb{Q}\cup\{\infty\}$$

determines a slope stability condition on $\mathrm{Coh}_c(X)$[3]. Let $\mathcal{M}_h^{\mathcal{L}}(v)$ be the moduli space of μ_h-stable sheaves $\mathcal{E}\in\mathrm{Coh}_c(X)$ with $\mathrm{ch}(\mathrm{q}_*\,\mathcal{E})=v$. For simplicity, we also assume $\mathcal{M}_h^{\mathcal{L}}(v)$ admits a universal family, denoted by $\overline{\mathbb{E}}$. This is again the case if for example $\gcd(r,\gamma\cdot h)=1$ (see [HL10, Corollary 4.6.7]).

We denote by $\overline{p}$ the projection from $X\times\mathcal{M}_h^{\mathcal{L}}(v)$ to $\mathcal{M}_h^{\mathcal{L}}(v)$. By the condition (2) and [T98, HT10], one obtains a natural perfect obstruction theory on $\mathcal{M}_h^{\mathcal{L}}(v)$ given by the complex[4]

$$E^\bullet := \left(\tau^{[1,2]}\mathbf{R}\mathcal{H}om_{\overline{p}}(\overline{\mathbb{E}},\overline{\mathbb{E}})[1]\right)^\vee.$$

[3] If $\mathrm{Rank}(\mathrm{q}_*\,\mathcal{E})=0$, then $\mu_h(\mathcal{E})=\infty$.

[4] $E^\bullet$ is symmetric [B09] if $\mathcal{L}=\omega_S$.

Note that Serre duality and Hirzerbruch-Riemann-Roch hold for the compactly supported coherent sheaves, even though X is not compact. Using the latter we can calculate the rank of $E^\bullet$:

$$\text{Rank}(E^\bullet) = \begin{cases} 0 & \mathcal{L} = \omega_S, \\ r^2 c_1(\mathcal{L}) \cdot (c_1(\mathcal{L}) - \omega_S)/2 + 1 & \mathcal{L} \neq \omega_S. \end{cases} \tag{4}$$

By [GP99], we obtain the $\mathbb{C}^*$-fixed perfect obstruction theory

$$E^{\bullet,\text{fix}} = \left(\tau^{[1,2]}\left(\mathbf{R}\mathcal{H}om_{\overline{p}}(\overline{\mathbb{E}}, \overline{\mathbb{E}})[1]\right)^\vee\right)^{\mathbb{C}^*} \tag{5}$$

over the fixed locus $\mathcal{M}^{\mathcal{L}}_h(v)^{\mathbb{C}^*}$. Since the $\mathbb{C}^*$-fixed set of X (i.e. S) is projective, we conclude that $\mathcal{M}^{\mathcal{L}}_h(v)^{\mathbb{C}^*}$ is projective, therefore $E^{\bullet,\text{fix}}$ gives the virtual fundamental class $[\mathcal{M}^{\mathcal{L}}_h(v)^{\mathbb{C}^*}]^{\text{vir}}$. Define

$$\widehat{\text{DT}}^{\mathcal{L}}_h(v;\alpha) = \int_{[\mathcal{M}^{\mathcal{L}}_h(v)^{\mathbb{C}^*}]^{\text{vir}}} \frac{\alpha}{e((E^{\bullet,\text{mov}})^\vee)} \qquad \alpha \in H^*_{\mathbb{C}^*}(\mathcal{M}^{\mathcal{L}}_h(v), \mathbb{Q})_{\mathbf{s}}, \tag{6}$$

where $E^{\bullet,\text{mov}}$ is the $\mathbb{C}^*$-moving part of $E^\bullet$, and $e(-)$ indicates the equivariant Euler class.

Remark 2.1. *Note that $(E^{\bullet,\text{mov}})^\vee$ is the virtual normal bundle of $\mathcal{M}^{\mathcal{L}}_h(v)^{\mathbb{C}^*}$. If $\mathcal{M}^{\mathcal{L}}_h(v)$ is compact then $\widehat{\text{DT}}^{\mathcal{L}}_h(v;\alpha)$ will be equal to $\int_{[\mathcal{M}^{\mathcal{L}}_h(v)]^{vir}} \alpha$ via the virtual localization formula* [GP99]*. This is the case when $c_1(\mathcal{L}) \cdot h < 0$, as then one can see that all the stable sheaves must be supported (even scheme theoretically!) on the zero section of* $\mathrm{q} : X \to S$*. Note that if $c_1(\mathcal{L}) \cdot h \geq 0$, then $\int_{[\mathcal{M}^{\mathcal{L}}_h(v)]^{vir}} \alpha$ is not defined in general.*

Remark 2.2. *If $\mathcal{L} = \omega_S$ (i.e. X is Calabi-Yau), one can also define the invariants by taking weighted Euler characteristics of the moduli spaces $\int_{\mathcal{M}^{\omega_S}_h(v)} \nu_{\mathcal{M}^{\omega_S}}\, d\chi$, where $\nu_{\mathcal{M}^{\omega_S}}$ is Behrend's constructible function* [B09] *on $\mathcal{M}^{\omega_S}_h(v)$. By localization this coincides with the integration of $\nu_{\mathcal{M}^{\omega_S}}$ over the $\mathbb{C}^*$-fixed locus $\mathcal{M}^{\omega_S}_h(v)^{\mathbb{C}^*}$. These invariants were computed by* [TT] *and were shown to have modular properties in some interesting examples. If $\mathcal{M}^{\omega_S}_h(v)$ is compact e.g. when $K_S \cdot h < 0$ (see Remark 2.1) then these invariants coincide with the invariants $\widehat{\text{DT}}^{\omega_S}_h(v;1)$* [B09].

In the case that $p_g(S) > 0$ the fixed part of the obstruction theory $E^\bullet$ contains a trivial factor which causes the invariants $\widehat{\text{DT}}^{\mathcal{L}}_h(v)$ to vanish; we reduce the obstruction theory $E^\bullet$ as follows. Define

$$\mathbf{R}\mathcal{H}om_{\overline{p}}(\overline{\mathbb{E}}, \overline{\mathbb{E}})_0$$

to be the cone of the composition

$$\mathrm{q}_*\,\mathbf{R}\mathcal{H}om_{X\times\mathcal{M}^{\mathcal{L}}_h(v)}(\overline{\mathbb{E}}, \overline{\mathbb{E}}) \xrightarrow{\mathrm{q}_*} \mathbf{R}\mathcal{H}om_{S\times\mathcal{M}^{\mathcal{L}}_h(v)}(\mathrm{q}_*\,\overline{\mathbb{E}}, \mathrm{q}_*\,\overline{\mathbb{E}}) \xrightarrow{\text{tr}} \mathcal{O}_{S\times\mathcal{M}^{\mathcal{L}}_h(v)},$$

followed by the derived push forward via the projection $S \times \mathcal{M}_h^{\mathcal{L}}(v) \to \mathcal{M}_h^{\mathcal{L}}(v)$. Note that q is an affine morphism and hence $R^i\,\mathrm{q}_* = 0$ for $i > 0$. Then, define

$$E^\bullet_{\mathrm{red}} := \left(\tau^{\leq 2}\mathbf{R}\mathcal{H}om_{\overline{p}}(\mathbb{E},\mathbb{E})_0[1]\right)^\vee, \qquad \mathrm{Rank}(E^\bullet_{\mathrm{red}}) = \mathrm{Rank}(E^\bullet) + p_g(S). \tag{7}$$

Remark 2.3. *Note that by construction this reduction only affects the fixed part of the obstruction theory i.e.* $E^{\bullet,\mathrm{mov}} = E^{\bullet,\mathrm{mov}}_{\mathrm{red}}$. *Since* $H^1(\mathcal{O}_S) = 0$ *by assumption, we see that in case* $p_G(S) = 0$, *we have* $E^\bullet = E^\bullet_{\mathrm{red}}$.

Proposition 2.4. $E^\bullet_{\mathrm{red}}$ *gives a perfect obstruction theory over* $\mathcal{M}_h^{\mathcal{L}}(v)$.

Proof. We closely follow the construction of the perfect obstruction theory in [T98]. Let $B_0 \subset B \subset B_1$ be closed immersions of B_0-schemes over $\mathbb{C}$. We denote the ideals of $B_0 \subset B$, $B_0 \subset B_1$, $B \subset B_1$ by $\mathfrak{n}$, $\mathfrak{m}$ and $\mathfrak{J}$, respectively, and suppose that $\mathfrak{m}\cdot\mathfrak{J} = 0$. Let $\mathcal{G}_0$ be a sheaf on $B_0 \times X$ flat over B_0, and $\mathcal{G}$ be a sheaf on $B \times X$ flat over B extending $\mathcal{G}_0$. Note that q is an affine morphism and hence $R^i\,\mathrm{q}_* = 0$ for $i > 0$, so by flat base change $\mathrm{q}_*\,\mathcal{G}_0$ and $\mathrm{q}_*\,\mathcal{G}$ remain flat and $\mathrm{q}_*\,\mathcal{G}|_{B_0\times S} = \mathrm{q}_*\,\mathcal{G}_0$ (as before, we simply use the symbol q to denote the morphism $\mathrm{id}\times\mathrm{q}$). By [T98, Proposition 3.13], the obstruction for extending $\mathcal{G}$ (respectively $\mathrm{q}_*\,\mathcal{G}$) to a sheaf on $B_1 \times X$ (respectively $B_1 \times S$) flat over B_1 lies in

$$\mathrm{ob}(\mathcal{G}_0) \in \mathrm{Ext}^2_{X\times B_0}(\mathcal{G}_0, \mathcal{G}_0 \otimes \mathfrak{J}), \quad (\text{respectively } \mathrm{ob}(\mathrm{q}_*\,\mathcal{G}_0) \in \mathrm{Ext}^2_{S\times B_0}(\mathrm{q}_*\,\mathcal{G}_0, \mathrm{q}_*\,\mathcal{G}_0 \otimes \mathfrak{J})).$$

We will prove the following lemma after finishing the proof of the proposition:

Lemma 2.5. *Under the natural map*

$$\mathrm{Ext}^2_{X\times B_0}(\mathcal{G}_0, \mathcal{G}_0 \otimes \mathfrak{J}) \xrightarrow{\mathrm{q}_*} \mathrm{Ext}^2_{S\times B_0}(\mathrm{q}_*\,\mathcal{G}_0, \mathrm{q}_*\,\mathcal{G}_0 \otimes \mathfrak{J}),$$

we have $\mathrm{q}_*\,\mathrm{ob}(\mathcal{G}_0) = \mathrm{ob}(\mathrm{q}_*\,\mathcal{G}_0)$.

Next, by [T98, Theorem 3.23], the obstruction for deforming $\det(q_*\mathcal{G}_0)$ is given by $\mathrm{tr}(\mathrm{ob}(\mathrm{q}_*\,\mathcal{G}_0))$ which is equal to $\mathrm{tr}(\mathrm{q}_*\,\mathrm{ob}(\mathcal{G}_0))$, by Lemma 2.5. However, there is no obstruction for deforming line bundles, and therefore $\mathrm{tr}(\mathrm{q}_*\,\mathrm{ob}(\mathcal{G}_0)) = 0$ or

$$\mathrm{q}_*\,\mathrm{ob}(\mathcal{G}_0) \in \mathrm{Ext}^2_{S\times B_0}(\mathrm{q}_*\,\mathcal{G}_0, \mathrm{q}_*\,\mathcal{G}_0 \otimes \mathfrak{J})_0,$$

This means that $\mathrm{ob}(\mathcal{G}_0) \in \ker(\mathrm{tr}\circ\mathrm{q}_*)$. The rest of the proof is similar to proving $E^\bullet$ is perfect obstruction theory as given in [T98, Theorems 3.28, 3.30][5]. $\square$

Proof of Lemma 2.5. Suppose that $\mathcal{G}_1$ is a B_1-flat lift of $\mathcal{G}$. By the proof of [T98, Proposition 3.13] we have short exact sequences $0 \to \mathfrak{J} \to \mathfrak{m} \to \mathfrak{n} \to 0$, and

$$0 \to \mathcal{G}\otimes\mathfrak{n} \to \mathcal{G} \to \mathcal{G}_0 \to 0, \quad 0 \to \mathcal{G}\otimes\mathfrak{m} \to \mathcal{G}_1 \to \mathcal{G}_0 \to 0. \tag{8}$$

Since $R^i\,\mathrm{q}_* = 0$ for $i > 0$, we get the corresponding short exact sequences

$$0 \to \mathrm{q}_*\,\mathcal{G}\otimes\mathfrak{n} \to \mathrm{q}_*\,\mathcal{G} \to \mathrm{q}_*\,\mathcal{G}_0 \to 0, \quad 0 \to \mathrm{q}_*\,\mathcal{G}\otimes\mathfrak{m} \to \mathrm{q}_*\,\mathcal{G}_1 \to \mathrm{q}_*\,\mathcal{G}_0 \to 0, \tag{9}$$

$$0 \to \mathcal{G}_0\otimes\mathfrak{J} \to \mathcal{G}\otimes\mathfrak{m} \to \mathcal{G}\otimes\mathfrak{n} \to 0, \quad 0 \to \mathrm{q}_*\,\mathcal{G}_0\otimes\mathfrak{J} \to \mathrm{q}_*\,\mathcal{G}\otimes\mathfrak{m} \to \mathrm{q}_*\,\mathcal{G}\otimes\mathfrak{n} \to 0.$$

[5] See also [TT, Proposition 3.5] for an alternative less elementary argument.

Applying $\mathrm{Hom}(\mathcal{G}_0,-)$ and $\mathrm{Hom}(\mathrm{q}_*\mathcal{G}_0,-)$ to the last two sequences above and using the functoriality of q_* we get the following commutative diagram with exact rows:

$$\begin{tikzcd}
\mathrm{Ext}^1(\mathcal{G}_0,\mathcal{G}\otimes\mathfrak{m}) \arrow[r] \arrow[d, "\mathrm{q}_*"] & \mathrm{Ext}^1(\mathcal{G}_0,\mathcal{G}\otimes\mathfrak{n}) \arrow[r, "\partial"] \arrow[d, "\mathrm{q}_*"] & \mathrm{Ext}^2(\mathcal{G}_0,\mathcal{G}_0\otimes\mathfrak{J}) \arrow[d, "\mathrm{q}_*"] \\
\mathrm{Ext}^1(\mathrm{q}_*\mathcal{G}_0,\mathrm{q}_*\mathcal{G}\otimes\mathfrak{m}) \arrow[r] & \mathrm{Ext}^1(\mathrm{q}_*\mathcal{G}_0,\mathrm{q}_*\mathcal{G}\otimes\mathfrak{n}) \arrow[r, "\partial"] & \mathrm{Ext}^2(\mathrm{q}_*\mathcal{G}_0,\mathrm{q}_*\mathcal{G}_0\otimes\mathfrak{J}).
\end{tikzcd}$$

In particular we get $\partial\circ\mathrm{q}_* = \mathrm{q}_*\circ\partial$. Let $e\in\mathrm{Ext}^1(\mathcal{G}_0,\mathcal{G}\otimes\mathfrak{n})$ be the class of the first extension in (8), and $e'\in\mathrm{Ext}^1(\mathrm{q}_*\mathcal{G}_0,\mathrm{q}_*\mathcal{G}\otimes\mathfrak{n})$ be the class of the first extension in (9). By the naturality of q_* we have $e'=\mathrm{q}_*(e)$. By the proof of [T98, Proposition 3.13] $\mathrm{ob}(\mathcal{G}_0)=\partial(e)$ and

$$\mathrm{ob}(\mathrm{q}_*\mathcal{G}_0)=\partial(e')=\partial(\mathrm{q}_*(e))=\mathrm{q}_*(\partial(e))=\mathrm{q}_*(\mathrm{ob}(\mathcal{G}_0)).$$

□

In particular, by [GP99], we get

Corollary 2.6. *$E^{\bullet,\mathrm{fix}}_{\mathrm{red}}$ gives a perfect obstruction theory over $\mathcal{M}^{\mathcal{L}}_h(v)^{\mathbb{C}^*}$, and hence a virtual fundamental class*

$$[\mathcal{M}^{\mathcal{L}}_h(v)^{\mathbb{C}^*}]^{\mathrm{vir}}_{\mathrm{red}}\in A_*(\mathcal{M}^{\mathcal{L}}_h(v)^{\mathbb{C}^*}).$$

□

In the rest of the paper, we will study the invariants defined below:

Definition 2.7. *We can define two types of DT invariants*

$$\mathrm{DT}^{\mathcal{L}}_h(v;\alpha):=\int_{[\mathcal{M}^{\mathcal{L}}_h(v)^{\mathbb{C}^*}]^{\mathrm{vir}}_{\mathrm{red}}}\frac{\alpha}{e((E^{\bullet,\mathrm{mov}})^\vee)}\in\mathbb{Q}[\mathbf{s},\mathbf{s}^{-1}]\qquad \alpha\in H^*_{\mathbb{C}^*}(\mathcal{M}^{\mathcal{L}}_h(v),\mathbb{Q})_{\mathbf{s}},$$

$$\mathrm{DT}^{\mathcal{L}}_h(v):=\int_{[\mathcal{M}^{\mathcal{L}}_h(v)^{\mathbb{C}^*}]^{\mathrm{vir}}_{\mathrm{red}}}c((E^{\bullet,\mathrm{fix}}_{\mathrm{red}})^\vee)\in\mathbb{Z}.$$

Here $e(-)$ denotes the equivariant Euler class, $\mathbf{s}$ is the equivariant parameter, and $c(-)$ denotes the total Chern class. Note that $E^{\bullet,\mathrm{mov}}=E^{\bullet,\mathrm{mov}}_{\mathrm{red}}$ by Remark 2.3.

Remark 2.8. *The invariant $\mathrm{DT}^{\mathcal{L}}_h(v;\alpha)$ is the reduced version of the invariant $\widehat{\mathrm{DT}}^{\mathcal{L}}_h(v;\alpha)$ given in (6). If $\alpha=1$ then it can be seen easily that*

$$\mathrm{DT}^{\mathcal{L}}_h(v;1)\cdot\mathbf{s}^{\mathrm{Rank}(E^\bullet_{\mathrm{red}})}\in\mathbb{Q},$$

where $\mathrm{Rank}(E^\bullet_{\mathrm{red}})$ is given by (4) and (7). In particular, if $\mathcal{L}=\omega_S$ then $\mathrm{Rank}(E^\bullet_{\mathrm{red}})=p_g(S)$.

Remark 2.9. *The definition of the invariant $\mathrm{DT}^{\mathcal{L}}_h(v)$ is motivated by Fantechi-Göttsche's virtual Euler characteristic* [FG10]. *$\mathrm{DT}^{\mathcal{L}}_h(v)$ is the virtual Euler number of $\mathcal{M}^{\mathcal{L}}_h(v)^{\mathbb{C}^*}$:*

$$\mathrm{DT}^{\mathcal{L}}_h(v)=\chi^{\mathrm{vir}}(\mathcal{M}^{\mathcal{L}}_h(v)^{\mathbb{C}^*}).$$

If $\mathcal{M}_h^{\mathcal{L}}(v)^{\mathbb{C}^*}$ is nonsingular with expected dimension, then $\mathrm{DT}_h^{\mathcal{L}}(v)$ coincides with the topological Euler characteristic of $\mathcal{M}_h^{\mathcal{L}}(v)^{\mathbb{C}^*}$.

Remark 2.10. *Motivated by Vafa-Witten equation and S-duality conjecture* [VW94], *Tanaka and Thomas* [TT] *define Vafa-Witten invariants by constructing a delicate symmetric perfect obstruction theory over the moduli space of Higgs pairs (G,ϕ) on S with fixed determinant $\det(G)$ and such that $\mathrm{tr}(\phi)=0$. When $p_g=0$ (note that in this paper $H^1(\mathcal{O}_S)=0$), their moduli space and its perfect obstruction theory are the same as $\mathcal{M}_h^{\omega_S}(v)$ and $E^{\bullet}_{\mathrm{red}}$, respectively.*

The moduli space of Higgs pairs is equipped with a $\mathbb{C}^*$-action obtained by scaling ϕ. The corresponding $\mathbb{C}^*$-fixed locus of the moduli space of Higgs pairs and the fixed part of Tanaka-Thomas' obstruction theory always coincides with $\mathcal{M}_h^{\omega_S}(v)^{\mathbb{C}^*}$ and $E^{\bullet,\mathrm{fix}}_{\mathrm{red}}$, respectively (even though the moduli spaces and the moving parts of obstruction theories could be different in general). The moduli space of Higgs pairs is not compact. Tanaka and Thomas define Vafa-Witten invariants $\mathrm{VW}_h(v)\in\mathbb{Q}$ by taking the $\mathbb{C}^*$-equivariant residue of the class of 1. Tanaka and Thomas have computed the invariants $\mathrm{VW}_h(v)$ in some interesting examples and express the generating functions of the invariants of certain components of the $\mathbb{C}^*$-fixed locus in terms of algebraic functions. They were also able to match the invariants $\mathrm{VW}_h(v)$ (after adding the contributions of all $\mathbb{C}^*$-fixed loci) with the few first terms of the modular forms of [VW94] *through heavy calculations. Their calculation provides compelling evidence that the invariants $\mathrm{VW}_h(v)$ have modular properties that match with S-duality predictions. By analyzing the moving part of the obstruction theory of Tanaka-Thomas, and $E^{\bullet,\mathrm{mov}}_{\mathrm{red}}$ one can show that if we choose $\alpha=1$ in Definition 2.7 (see Remark 2.8):*

$$\mathrm{DT}_h^{\omega_S}(v;1)=\mathbf{s}^{-p_g}\,\mathrm{VW}_h(v).$$

3. Description of the fixed locus of moduli space

We continue this section by giving a precise description of the components of $\mathcal{M}_h^{\mathcal{L}}(v)^{\mathbb{C}^*}$. Suppose that $\mathcal{E}$ is a closed point of $\mathcal{M}_h^{\mathcal{L}}(v)^{\mathbb{C}^*}$. Because $\mathcal{E}$ is a pure $\mathbb{C}^*$-equivariant sheaf, up to tensoring with a power of $\mathbf{t}$, we can assume that, for some partition $\lambda\vdash r$, with $\lambda=(\lambda_1\le\cdots\le\lambda_{\ell(\lambda)})$, we have

$$\mathrm{q}_*\,\mathcal{E}=\bigoplus_{i=0}^{\ell(\lambda)-1}E_{-i}\otimes\mathbf{t}^{-i},$$

where E_{-i} is a rank λ_{i+1} torsion free sheaf on S, and the $\mathcal{O}_X$-module structure on $\mathcal{E}$ is given by a collection of injective maps of $\mathcal{O}_S$-modules (using (3)):

$$\psi_i:E_{-i}\to E_{-i-1}\otimes\mathcal{L},\qquad i=0,\dots,\ell(\lambda)-1.$$

Let $\mathcal{E}_i:=z_*E_{-i}$, for any i and let $\mathcal{E}'_0:=\mathcal{E}$. Define $\mathcal{E}'_i$ for $i>0$ inductively by

$$0\to\mathcal{E}'_{i+1}\otimes\mathbf{t}^{-1}\to\mathcal{E}'_i\to\mathcal{E}_i\to 0. \tag{10}$$

Therefore, we get a filtration (forgetting the equivariant structures)

$$\mathcal{E}'_{\ell(\lambda)-1} \subset \cdots \subset \mathcal{E}'_1 \subset \mathcal{E}'_0 = \mathcal{E},$$

and the stability of $\mathcal{E}$ imposes the following conditions:

$$\mu_h(\mathcal{E}'_i) < \mu_h(\mathcal{E}) \qquad i = 1, \dots, \ell(\lambda). \tag{11}$$

Note that for all j we have

$$\mathrm{q}_* \, \mathcal{E}'_j \otimes \mathbf{t}^{-j} = \bigoplus_{i=j}^{\ell(\lambda)-1} E_{-i} \otimes \mathbf{t}^{-i},$$

and hence (11) imposes some restrictions on the ranks and degrees of E_{-i}'s.

This construction also works well for the B-points of the moduli space $\mathcal{M}_h^{\mathcal{L}}(v)^{\mathbb{C}^*}$ for any $\mathbb{C}$-schemes B. As a result, one gets a decomposition of the $\mathbb{C}^*$-fixed locus $\mathcal{M}_h^{\mathcal{L}}(v)^{\mathbb{C}^*}$ into connected components

$$\mathcal{M}_h^{\mathcal{L}}(v)^{\mathbb{C}^*} = \coprod_{\lambda \vdash r} \mathcal{M}_h^{\mathcal{L}}(v)_\lambda^{\mathbb{C}^*},$$

where in the level of the universal families

$$\begin{aligned} &\mathrm{q}_* \left(\overline{\mathbb{E}}|_{X \times \mathcal{M}_h^{\mathcal{L}}(v)_\lambda^{\mathbb{C}^*}} \right) = \bigoplus_{i=0}^{\ell(\lambda)-1} \mathbb{E}_{-i} \otimes \mathbf{t}^{-i}, \\ &\Psi_i : \mathbb{E}_{-i} \to \mathbb{E}_{-i-1} \otimes \mathcal{L}, \qquad i = 0, \dots, \ell(\lambda) - 1, \\ &\overline{\mathbb{E}}'_{\ell(\lambda)-1} \subset \cdots \subset \overline{\mathbb{E}}'_1 \subset \overline{\mathbb{E}}'_0 := \overline{\mathbb{E}}, \end{aligned} \tag{12}$$

in which $\mathbb{E}_{-i}$ is a flat family[6] of rank λ_i torsion free sheaves on $S \times \mathcal{M}_h^{\mathcal{L}}(v)_\lambda^{\mathbb{C}^*}$, Ψ_i is a family of fiberwise injective maps over $\mathcal{M}_h^{\mathcal{L}}(v)_\lambda^{\mathbb{C}^*}$, $\overline{\mathbb{E}}_i := z_* \mathbb{E}_{-i}$, and $\overline{\mathbb{E}}'_i$ for $i > 0$ are inductively defined by

$$0 \to \overline{\mathbb{E}}'_{i+1} \otimes \mathbf{t}^{-1} \to \overline{\mathbb{E}}'_i \to \overline{\mathbb{E}}_i \to 0. \tag{13}$$

In the rest of the paper, we only study the two extreme cases $\lambda = (r)$ and $\lambda = (1^r)$. By the construction, it is clear that the former case coincides set theoretically with the moduli space $\mathcal{M}_h(v)$; as we will see in the next section the latter case is related to the nested Hilbert schemes on S. Note that when $r = 2$, these cases are the only possibilities, and hence

Proposition 3.1. *Suppose that $r = 2$, then*

$$[\mathcal{M}_h^{\mathcal{L}}(v)^{\mathbb{C}^*}]^{\mathrm{vir}}_{\mathrm{red}} = [\mathcal{M}_h^{\mathcal{L}}(v)^{\mathbb{C}^*}_{(2)}]^{\mathrm{vir}}_{\mathrm{red}} + [\mathcal{M}_h^{\mathcal{L}}(v)^{\mathbb{C}^*}_{(1^2)}]^{\mathrm{vir}}_{\mathrm{red}}$$

□

[6] Since q is an affine morphism, $\mathrm{q}_* \, \overline{\mathbb{E}}$ is flat over $\mathcal{M}_h^{\mathcal{L}}(v)_\lambda^{\mathbb{C}^*}$, and hence each weight space $\mathbb{E}_{-i}$ is flat over $\mathcal{M}_h^{\mathcal{L}}(v)_\lambda^{\mathbb{C}^*}$.

3.1. **Moduli space of stable torsion free sheaves as fixed locus.**

Notation. *We sometimes use* $-\cdot\mathbf{t}^a$ *instead of* $-\otimes\mathbf{t}^a$ *to make the formulas shorter. We also let* $\mathbf{s}=c_1(\mathbf{t})$.

Proposition 3.2. *We have the isomorphism of schemes* $\mathcal{M}_h^{\mathcal{L}}(v)^{\mathbb{C}^*}_{(r)} \cong \mathcal{M}_h(v)$. *Moreover, under this identification, we have the following isomorphisms*

$$\begin{aligned} E^{\bullet,\mathrm{fix}}_{\mathrm{red}}|_{\mathcal{M}_h^{\mathcal{L}}(v)^{\mathbb{C}^*}_{(r)}} &\cong \big(\mathbf{R}\mathcal{H}om_p(\mathbb{E},\mathbb{E})_0[1]\big)^\vee, \\ E^{\bullet,\mathrm{mov}}|_{\mathcal{M}_h^{\mathcal{L}}(v)^{\mathbb{C}^*}_{(r)}} &\cong \big(\tau^{\leq 1}\mathbf{R}\mathcal{H}om_p(\mathbb{E},\mathbb{E}\otimes\mathcal{L}\cdot\mathbf{t})\big)^\vee. \end{aligned}$$

In particular, $E^{\bullet,\mathrm{fix}}_{\mathrm{red}}|_{\mathcal{M}_h^{\mathcal{L}}(v)^{\mathbb{C}^*}_{(r)}}$ *is identified with the natural perfect trace-free obstruction theory over* $\mathcal{M}_h(v)$, *and hence* $[\mathcal{M}_h^{\mathcal{L}}(v)^{\mathbb{C}^*}_{(r)}]^{\mathrm{vir}}_{\mathrm{red}} = [\mathcal{M}_h(v)]^{\mathrm{vir}}_0$.

Proof. The first claim follows by the description above and noting that in this case (12) and (13) give

$$\overline{\mathbb{E}}|_{X\times\mathcal{M}_h^{\mathcal{L}}(v)^{\mathbb{C}^*}_{(r)}} \cong z_*(\mathbb{E}\boxtimes\mathcal{N}), \qquad \mathrm{q}_*\left(\overline{\mathbb{E}}|_{X\times\mathcal{M}_h^{\mathcal{L}}(v)^{\mathbb{C}^*}_{(r)}}\right) = \mathbb{E}\boxtimes\mathcal{N},$$

for a line bundle $\mathcal{N}$ on $\mathcal{M}_h(v)$, by the universal properties of the moduli spaces. For the second part, by [H06, Corollary 11.4], we have the following natural exact triangle

$$\mathbb{E}\boxtimes\mathcal{N}\otimes\mathcal{L}^{-1}\otimes\mathbf{t}^{-1}[1] \to \mathbf{L}z^*\overline{\mathbb{E}} \to \mathbb{E}\boxtimes\mathcal{N},$$

which implies, by adjunction, the exact triangle

$$z_*\,\mathbf{R}\mathcal{H}om(\mathbb{E},\mathbb{E}) \to \mathbf{R}\mathcal{H}om(\overline{\mathbb{E}},\overline{\mathbb{E}}) \to z_*\,\mathbf{R}\mathcal{H}om(\mathbb{E},\mathbb{E}\otimes\mathcal{L}\cdot\mathbf{t})[-1].$$

Taking the trace free part, shifting by 1, pushing forward, dualizing, and taking the $\mathbb{C}^*$-fixed part of this exact triangle, we get the first isomorphism; pushing forward, applying the truncation $\tau\leq 1$, and taking the $\mathbb{C}^*$-moving part of this exact triangle, we get the second isomorphism. ☐

Corollary 3.3.

$$\begin{aligned} \mathrm{DT}_h^{\mathcal{L}}(v;\alpha)_{(r)} &= \int_{[\mathcal{M}_h(v)]^{\mathrm{vir}}_0} \frac{\mathbf{s}^\kappa\cdot\alpha}{e\left(\mathbf{R}\mathcal{H}om_p(\mathbb{E},\mathbb{E}\otimes\mathcal{L}\cdot\mathbf{t})\right)}, \\ \mathrm{DT}_h^{\mathcal{L}}(v)_{(r)} = \chi^{\mathrm{vir}}(\mathcal{M}_h(v)) &= \int_{[\mathcal{M}_h(v)]^{\mathrm{vir}}_0} c_d\left(\mathbf{R}\mathcal{H}om_p(\mathbb{E},\mathbb{E})_0[1]\right) \\ &= \int_{[\mathcal{M}_h(v)]^{\mathrm{vir}}_0} c_d\left(\mathcal{E}xt^1_p(\mathbb{E},\mathbb{E}) - \mathcal{E}xt^2_p(\mathbb{E},\mathbb{E})_0\right), \end{aligned}$$

where d *is the virtual dimension of* $\mathcal{M}_h(v)$, *and* $\kappa=1$ *if* $\mathcal{L}=\omega_S$, *otherwise* $\kappa=0$.

Proof. To see the first formula, by Proposition 3.2 we can write

$$\begin{aligned}
\mathrm{DT}_h^{\mathcal{L}}(v;\alpha)_{(r)} &= \int_{[\mathcal{M}_h^{\mathcal{L}}(v)_{(r)}^{\mathbb{C}^*}]^{\mathrm{vir}}_{\mathrm{red}}} \frac{\alpha}{e((E^{\bullet,\mathrm{mov}})^\vee)} \\
&= \int_{[\mathcal{M}_h(v)]_0^{\mathrm{vir}}} \frac{\alpha}{e\left(\tau^{\leq 1}\mathbf{R}\mathcal{H}om_p(\mathbb{E},\mathbb{E}\otimes\mathcal{L}\cdot\mathbf{t})\right)} \\
&= \int_{[\mathcal{M}_h(v)]_0^{\mathrm{vir}}} \frac{\mathbf{s}^{\kappa}\cdot\alpha}{e\left(\mathbf{R}\mathcal{H}om_p(\mathbb{E},\mathbb{E}\otimes\mathcal{L}\cdot\mathbf{t})\right)}.
\end{aligned}$$

For the last equality, note that the trace map and Grothendieck-Verdier duality induces

$$\mathcal{E}xt^2_p(\mathbb{E},\mathbb{E}\otimes\mathcal{L}\cdot\mathbf{t}) \cong \mathbf{R}^2p_*(\mathcal{L}\cdot\mathbf{t}) \cong \left(p_*(\mathcal{L}^{-1}\otimes\omega_S\cdot\mathbf{t}^{-1})\right)^*,$$

and by (2), $p_*(\mathcal{L}^{-1}\otimes\omega_S\cdot\mathbf{t}^{-1}) = 0$ unless $\mathcal{L}=\omega_S$ in which case it is $\mathcal{O}_{\mathcal{M}}\otimes\mathbf{t}^{-1}$.

The second formula in corollary follows directly from Proposition 3.2, by noting that $\mathcal{E}xt^1_p(-,-)_0 = \mathcal{E}xt^1_p(-,-)$ by the assumption $H^1(\mathcal{O}_S)=0$, and that $\mathcal{H}om_p(\mathbb{E},\mathbb{E})_0 = 0$ by the simplicity of the fibers of $\mathbb{E}$. ☐

Corollary 3.4. *If $\mathcal{L}=\omega_S$ and $\alpha=1$ then* $\mathrm{DT}_h^{\omega_S}(v;1)_{(r)} = (-1)^d\mathbf{s}^{-p_g}\,\mathrm{DT}_h^{\omega_S}(v)_{(r)}$.

Proof. By Corollary 3.3,

$$\begin{aligned}
\mathrm{DT}_h^{\omega_S}(v;1)_{(r)} &= \int_{[\mathcal{M}_h(v)]_0^{\mathrm{vir}}} \frac{\mathbf{s}}{e\left(\mathbf{R}\mathcal{H}om_p(\mathbb{E},\mathbb{E}\otimes\omega_S\cdot\mathbf{t})\right)} \\
&= \int_{[\mathcal{M}_h(v)]_0^{\mathrm{vir}}} \frac{(-1)^{-1+d-p_g}\mathbf{s}}{e\left(\mathbf{R}\mathcal{H}om_p(\mathbb{E},\mathbb{E}\cdot\mathbf{t}^{-1})\right)} \\
&= \int_{[\mathcal{M}_h(v)]_0^{\mathrm{vir}}} (-1)^d\mathbf{s}^{-p_g}c_d\left(\mathcal{E}xt^1_p(\mathbb{E},\mathbb{E}) - \mathcal{E}xt^2_p(\mathbb{E},\mathbb{E})_0\right),
\end{aligned}$$

and then use Corollary 3.3 again. Here we used Grothendieck-Verdier duality in the second equality and the identities

$$e\left(\mathcal{H}om_p(\mathbb{E},\mathbb{E}\cdot\mathbf{t}^{-1})\right) = e(\mathcal{O}_{\mathcal{M}}\cdot\mathbf{t}^{-1}) = -\mathbf{s},$$

$$e\left(\mathcal{E}xt^2_p(\mathbb{E},\mathbb{E}\cdot\mathbf{t}^{-1})\right) = e\left(\mathbf{R}^2p_*\mathcal{O}\cdot\mathbf{t}^{-1}\right)\cdot e\left(\mathcal{E}xt^2_p(\mathbb{E},\mathbb{E}\cdot\mathbf{t}^{-1})_0\right) = (-\mathbf{s})^{p_g}e\left(\mathcal{E}xt^2_p(\mathbb{E},\mathbb{E}\cdot\mathbf{t}^{-1})_0\right),$$

$$\mathrm{Rank}\left[\mathbf{R}\mathcal{H}om_p\left(\mathbb{E},\mathbb{E}\cdot\mathbf{t}^{-1}\right)\right] = 1-d+p_g,$$

in the third equality.

☐

3.2. Nested Hilbert schemes on S.

3.2.1. *Review of the results of* [GSY17a]. Let $S^{[\boldsymbol{n}]}_{\boldsymbol{\beta}}$ be the nested Hilbert scheme as in Section 1.2. When $r=2$, and so $\boldsymbol{n}=n_1,n_2$, $\boldsymbol{\beta}=\beta_1$, we have the following well-known special cases:

1. $n_2=0$, $\beta_1=0$. The Hilbert scheme of n_1 points on S, denoted by $S^{[n_1]}$. It is nonsingular of dimension $2n_1$.

2. $n_1 = n_2 = 0,\ \beta_1 \neq 0$. The Hilbert scheme of divisors in class β_1, denoted by S_{β_1}. It is nonsingular if $H^{i\geq 1}(L) = 0$ for any line bundle L with $c_1(L) = \beta_1$.
3. $n_2 = 0$. Then $S^{[n_1]}_{\beta_1} = S^{[n_1]} \times S_{\beta_1}$. This is the Hilbert scheme of 1-dimensional subschemes $Z \subset S$ such that $[Z] = \beta_1, \quad c_2(I_Z) = n_1$.

Notation. *We will denote the universal ideal sheaves of* $S^{[m]}_\beta$, $S^{[m]}$, *and* S_β *respectively by* $\mathcal{I}^{[m]}_{-\beta}$, $\mathcal{I}^{[m]}$, *and* $\mathcal{I}_{-\beta}$, *and the corresponding universal subschemes respectively by* $\mathcal{Z}^{[m]}_\beta$, $\mathcal{Z}^{[m]}$, *and* $\mathcal{Z}_\beta$. *We will use the same symbol for the pull backs of* $\mathcal{I}^{[m]}$ *and* $\mathcal{I}_{-\beta} = \mathcal{O}(-\mathcal{Z}_\beta)$ *via* $\mathrm{id}\times\mathrm{pts}$ *and* $\mathrm{id}\times\mathrm{div}$ *to* $S \times S^{[m]}_\beta$. *We will also write* $\mathcal{I}^{[m]}_\beta$ *for* $\mathcal{I}^{[m]} \otimes \mathcal{O}(\mathcal{Z}_\beta)$. *Using the universal property of the Hilbert scheme, it can be seen that* $\mathcal{I}^{[m]}_{-\beta} \cong \mathcal{I}^{[m]} \otimes \mathcal{O}(-\mathcal{Z}_\beta)$, *and hence it is consistent with the chosen notation. Let* $\pi : S \times S^{[m]}_\beta \to S^{[m]}_\beta$ *be the projection, we denote the derived functor* $\mathbf{R}\pi_*\mathbf{R}\mathcal{H}om$ *by* $\mathbf{R}\mathcal{H}om_\pi$ *and its i-th cohomology sheaf by* $\mathcal{E}xt^i_\pi$.

The tangent bundle of $S^{[m]}$ is identified with

$$T_{S^{[m]}} \cong \mathcal{H}om_\pi\left(\mathcal{I}^{[m]}, \mathcal{O}_{\mathcal{Z}^{[m]}}\right) \cong \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[m]}, \mathcal{I}^{[m]}\right)_0[1] \cong \mathcal{E}xt^1_\pi\left(\mathcal{I}^{[m]}, \mathcal{I}^{[m]}\right).$$

The nested Hilbert scheme is realized as the closed subscheme

$$\iota : S^{[\boldsymbol{n}]}_{\boldsymbol{\beta}} \hookrightarrow S^{[n_1]}_{\beta_1} \times \cdots \times S^{[n_{r-1}]}_{\beta_{r-1}} \times S^{[n_r]}. \tag{14}$$

The inclusions in (1) in the level of universal ideal sheaves give the universal inclusions

$$\Phi_i : \mathcal{I}^{[n_i]} \to \mathcal{I}^{[n_{i+1}]}_{\beta_i} \qquad 1 \leq i < r$$

defined over $S \times S^{[\boldsymbol{n}]}_{\boldsymbol{\beta}}$.

Notation. *Let* pr_i *be the closed immersion* (14) *followed by the projection to the i-th factor, and let* $\pi : S \times S^{[\boldsymbol{n}]}_{\boldsymbol{\beta}} \to S^{[\boldsymbol{n}]}_{\boldsymbol{\beta}}$ *be the projection. Then we have the fibered square*

$$\begin{array}{ccc} S \times S^{[\boldsymbol{n}]}_{\boldsymbol{\beta}} & \xhookrightarrow{\iota'} & S \times S^{[n_1]}_{\beta_1} \times \cdots \times S^{[n_{r-1}]}_{\beta_{r-1}} \times S^{[n_r]} \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi'} \\ S^{[\boldsymbol{n}]}_{\boldsymbol{\beta}} & \xhookrightarrow{\iota} & S^{[n_1]}_{\beta_1} \times \cdots \times S^{[n_{r-1}]}_{\beta_{r-1}} \times S^{[n_r]} \end{array} \tag{15}$$

where π' *is the projection and* $\iota' = \mathrm{id} \times \iota$.

Convention. *We slightly abuse the notation and use the same symbol for the universal objects (which are flat) on Hilbert schemes or line bundles on S and their pullbacks to the products of the Hilbert schemes via projections and other natural morphisms defined above, possibly followed by the restriction to the nested Hilbert schemes embedded in the product. This convention makes the notation much simpler.*

Applying the functors $\mathbf{R}\mathcal{H}om_\pi\left(-,\mathcal{I}^{[n_{i+1}]}_{\beta_i}\right)$ and $\mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_i]},-\right)$ to the universal map Φ_i, we get the following morphisms of the derived category

$$\mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_{i+1}]},\mathcal{I}^{[n_{i+1}]}\right) \xrightarrow{\Xi_i} \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_i]},\mathcal{I}^{[n_{i+1}]}_{\beta_i}\right),$$

$$\mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_i]},\mathcal{I}^{[n_i]}\right) \xrightarrow{\Xi'_i} \mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_i]},\mathcal{I}^{[n_{i+1}]}_{\beta_i}\right).$$

The following theorem is one of the main results of [GSY17a]:

Theorem 5 ([GSY17a, Theorem 1 and Proposition 2.5])**.** *$S^{[\boldsymbol{n}]}_{\boldsymbol{\beta}}$ is equipped with the perfect absolute obstruction theory $F^\bullet$ with the derived dual*

$$F^{\bullet\vee} = \mathrm{Cone}\left(\left[\bigoplus_{i=1}^{r}\mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_i]},\mathcal{I}^{[n_i]}\right)\right]_0 \to \bigoplus_{i=1}^{r-1}\mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_i]},\mathcal{I}^{[n_{i+1}]}_{\beta_i}\right)\right),$$

where the map above is naturally induced from all the maps Ξ_i and Ξ'_i, and $[-]_0$ means the trace-free part. As a result, $S^{[\boldsymbol{n}]}_{\boldsymbol{\beta}}$ carries a natural virtual fundamental class

$$[S^{[\boldsymbol{n}]}_{\boldsymbol{\beta}}]^{\mathrm{vir}} \in A_d(S^{[\boldsymbol{n}]}_{\boldsymbol{\beta}}), \qquad d = n_1 + n_r + \frac{1}{2}\sum_{i=1}^{r-1}\beta_i\cdot(\beta_i - K_S),$$

where K_S is the canonical divisor of S.

□

The following definition is taken from [GSY17a]:

Definition 3.5. *Suppose that $r = 2$ and $M \in \mathrm{Pic}(S)$. Define the following elements in $K(S^{[n_1,n_2]}_\beta)$:*

$$\mathsf{K}^{n_1,n_2}_{\beta;M} := [\mathbf{R}\pi_* M(\mathcal{Z}_\beta)] - \left[\mathbf{R}\mathcal{H}om_\pi(\mathcal{I}^{[n_1]},\mathcal{I}^{[n_2]}_\beta\otimes M)\right], \quad \mathsf{G}_{\beta;M} := \left[\mathbf{R}\pi_* M(\mathcal{Z}_\beta)|_{\mathcal{Z}_\beta}\right].$$

We also define the twisted tangent bundles in $K(S^{[n_i]})$ (and will use the same symbols for their pullbacks to $S^{[n_1,n_2]}$):

$$\mathsf{T}^M_{S^{[n_i]}} := [\mathbf{R}\pi_* M] - \left[\mathbf{R}\mathcal{H}om_\pi(\mathcal{I}^{[n_i]},\mathcal{I}^{[n_i]}\otimes M)\right].$$

Note $\mathsf{T}^{\mathcal{O}_S}_{S^{[n_i]}} = [T_{S^{[n_i]}}]$.

Let $\mathcal{P} := \mathcal{P}(M,\beta,n_1,n_2)$ be a polynomial in the Chern classes of $\mathsf{K}^{n_1,n_2}_{\beta;M}$, $\mathsf{G}_{\beta;M}$, and $\mathsf{T}^M_{S^{[n_i]}}$, then, we can define the invariant

$$\mathsf{N}_S(n_1,n_2,\beta;\mathcal{P}) := \int_{[S^{[n_1,n_2]}_\beta]^{\mathrm{vir}}}\mathcal{P}.$$

Definition 3.6. *Let $M \in \mathrm{Pic}(S)$. Define the class in $K(S^{[n_1]}\times S^{[n_2]})$:*

$$\mathsf{E}^{n_1,n_2}_M := [\mathbf{R}\pi'_* M] - \left[\mathbf{R}\mathcal{H}om_{\pi'}(\mathcal{I}^{[n_1]},\mathcal{I}^{[n_2]}\otimes M)\right].$$

If $M = \mathcal{O}_S$ then we will drop it from the notation.

The following results are proven in [GSY17a]:

Theorem 6 ([GSY17a, Theorem 6]). *Suppose the class $\mathcal{P} \in H^*(S^{[n_1,n_2]})$ in Definition 3.5 is well-behaved under good degenerations of S (in the sense of* [GSY17a, Remark 5.10, Proposition 5.11]*), and is the pullback of a class $\mathcal{P}'$ from $H^*(S^{[n_1]} \times S^{[n_2]}_\beta)$ via ι^*. Then*

$$\mathsf{N}(n_1, n_2, 0; \mathcal{P}) = \int_{S^{[n_1]} \times S^{[n_2]}} c_{n_1+n_2}(\mathsf{E}^{n_1,n_2}) \cup \mathcal{P}'.$$

□

Theorem 7 ([GSY17a, Proposition 2.9]). *Suppose that $p_g(S) > 0$ and*

$$|L| \neq \emptyset \quad \& \quad |\omega_S \otimes L^{-1}| = \emptyset$$

for any line bundle L with $c_1(L) = \beta$. Then $[S^{[n_1,n_2]}_\beta]^{vir} = 0$. In particular, in this case $\mathsf{N}_S(n_1, n_2, \beta; \mathcal{P}) = 0$ for any choice of the class $\mathcal{P}$.

□

3.2.2. *Nested Hilbert schemes as fixed locus.* Suppose that $\mathcal{E}$ is a closed point of $\mathcal{M}^{\mathcal{L}}_h(v)^{\mathbb{C}^*}_{(1^r)}$. By what we said above, $\mathcal{E}$ determines the rank 1 torsion free sheaves $E_0, \dots, E_{-r+1}$ on S together with the $\mathcal{O}_S$-module injections $\psi_1, \dots, \psi_{r-1}$. Since S is nonsingular, there exist line bundles $L_1, \dots, L_r$ and the ideal sheaves $I_1, \dots, I_r$ of zero dimensional subschemes $Z_1, \dots, Z_r$ such that $E_{-i+1} \cong I_i \otimes L_i$. We can rewrite the maps ψ_i as

$$\phi_i : I_i \to I_{i+1} \otimes M_i,$$

where $M_i := L_i^{-1} \otimes L_{i+1} \otimes \mathcal{L}$. The double dual $\phi_i^{**} : \mathcal{O}_S \to M_i$ defines a nonzero section of M_i and hence either $M_i \cong \mathcal{O}_S$ or $|M_i| \neq \emptyset$.

Let

$$n_i := c_2(I_i), \quad \beta_i := c_1(\mathcal{L}) + c_1(L_{i+1}) - c_1(L_i),$$

and let $\mathrm{d}(G) := c_1(G) \cdot h$ for any torsion free sheaf G on S. By construction we have the following two conditions:

- By the injectivity of ϕ_i, β_i is an effective curve class, in particular,

$$\mathrm{d}(M_i) > 0 \quad \text{or} \quad \mathrm{d}(M_i) = 0 \ \ \& \ \ n_{i+1} \leq n_i.$$

- By the stability of $\mathcal{E}$, using (11),

$$i \sum_{j=i+1}^{r} \mathrm{d}(L_j) < (r-i) \sum_{j=1}^{i} \mathrm{d}(L_j) \qquad i = 1, \dots, r-1.$$

Definition 3.7. *We say*

$$\boldsymbol{n} := n_1, n_2, \dots, n_r, \quad \boldsymbol{\beta} := \beta_1, \dots, \beta_{r-1},$$

are compatible with the vector $v = (r, \gamma, m)$, *if the above two conditions are satisfied, and moreover,*

$$\gamma = \sum_{i=1}^{r} c_1(L_i), \qquad m = \sum_{i=1}^{r} c_1(L_i)^2/2 - n_i.$$

Conversely, given L_i and I_i as above with the numerical invariants $\boldsymbol{n}$ and $\boldsymbol{\beta}$ compatible with the vector v, and the injective maps ϕ_i, one can recover a unique closed point of $\mathcal{M}_h^{\mathcal{L}}(v)^{\mathbb{C}^*}_{(1^r)}$. In fact, since q is an affine morphism, the collection of $E_{-i+1} = L_i \otimes I_i$ and the maps ϕ_i determine a pure $\mathbb{C}^*$-equivariant coherent sheaf $\mathcal{E}$ on X with $\operatorname{ch}(\mathrm{q}_* \mathcal{E}) = v$ (see [H77, Ex. II.5.17]). It remains to show that $\mathcal{E}$ is μ_h-stable. By [K11, Proposition 3.19], it suffices to show that $\mu_h(\mathcal{F}) < \mu_h(\mathcal{E})$ for any pure $\mathbb{C}^*$-equivariant subsheaf $0 \neq \mathcal{F} \subsetneq \mathcal{E}$. Suppose $\operatorname{Rank}(\mathrm{q}_* \mathcal{F}) = s$, so this means that $\mathcal{F} \subseteq \mathcal{E}'_{r-s}$, and hence

$$\mu_h(\mathcal{F}) \leq \mu_h(\mathcal{E}'_{r-s}) < \mu_h(\mathcal{E}),$$

where the first inequality is because $\operatorname{Rank}(\mathrm{q}_* \mathcal{F}) = \operatorname{Rank}(\mathrm{q}_* \mathcal{E}'_{r-s}) = s$ and the second inequality is because of (11).

Proposition 3.8. *For any connected component $\mathcal{T} \subset \mathcal{M}_h^{\mathcal{L}}(v)^{\mathbb{C}^*}_{(1^r)}$, there exist $\boldsymbol{n}$ and $\boldsymbol{\beta}$ compatible with the vector v, such that $\mathcal{T} \cong S^{[\boldsymbol{n}]}_{\boldsymbol{\beta}}$ as schemes.*

Proof. In this case, (12) gives

$$\mathrm{q}_* \left(\overline{\mathbb{E}}|_{X \times \mathcal{M}_h^{\mathcal{L}}(v)^{\mathbb{C}^*}_{(1^r)}} \right) = \bigoplus_{i=1}^{r} \mathbb{E}_{-i} \otimes \mathbf{t}^{-i}, \quad \Psi_i : \mathbb{E}_{-i} \to \mathbb{E}_{-i-1} \otimes \mathcal{L}, \qquad i = 1, \ldots, r-1.$$

where $\mathbb{E}_{-i}$ is a flat family of rank 1 torsion free sheaves on $S \times \mathcal{M}_h^{\mathcal{L}}(v)^{\mathbb{C}^*}_{(1^r)}$. By [K90, Lemma 6.14] the double duals $\mathbb{E}_{-i}^{**}$ are locally free, and hence for each i we get a morphism from $\mathcal{M}_h^{\mathcal{L}}(v)^{\mathbb{C}^*}_{(1^r)}$ to $\operatorname{Pic}(S)$. But $H^1(\mathcal{O}_S) = 0$ so $\operatorname{Pic}(S)$ is a union of discrete reduced points and hence this morphism is constant on connected components of $\mathcal{M}_h^{\mathcal{L}}(v)^{\mathbb{C}^*}_{(1^r)}$. Pulling back a Poincaré line bundle shows that $\mathbb{E}_{-i}^{**}$ restricted to a connected component $\mathcal{T} \subset \mathcal{M}_h^{\mathcal{L}}(v)^{\mathbb{C}^*}_{(1^r)}$ is isomorphic to $L_i \boxtimes \mathcal{N}_i$ for some line bundle L_i on S and $\mathcal{N}_i$ on $\mathcal{T}$. Therefore, the restriction of $\mathbb{E}_{-i} \subset \mathbb{E}_{-i}^{**}$ to $\mathcal{T}$ is of the form

$$(\mathcal{I}_{\mathcal{Z}_i} \otimes L_i) \boxtimes \mathcal{N}_i \tag{16}$$

for some subscheme $\mathcal{Z}_i \subset S \times \mathcal{T}$, which must be flat over $\mathcal{T}$ by the flatness of $\mathbb{E}_{-i}$. Let n_i be the fiberwise length of the subscheme $\mathcal{Z}_i$ over $\mathcal{T}$, which is well-defined by the flatness of $\mathcal{Z}_i$. Let $\beta_i := c_1(\mathcal{L}) + c_1(L_{i+1}) - c_1(L_i)$. Define

$$\boldsymbol{n} := n_1, n_2, \ldots, n_r, \quad \boldsymbol{\beta} := \beta_1, \ldots, \beta_{r-1}.$$

Then, $\boldsymbol{n}, \boldsymbol{\beta}$ are clearly compatible with the vector v. Let $M_i := L_i^{-1} \otimes L_{i+1} \otimes \mathcal{L}$. Since the maps

$$\Psi_i : \left(\mathcal{I}_{\mathcal{Z}_i} \otimes M_i^{-1}\right) \boxtimes \left(\mathcal{N}_i \otimes \mathcal{N}_{i+1}^{-1}\right) \to \mathcal{I}_{\mathcal{Z}_{i+1}}$$

are fiberwise injective over $\mathcal{T}$, there exist subschemes $\mathcal{Z}'_i$ flat over $\mathcal{T}$ such that

$$\mathcal{I}_{\mathcal{Z}'_i} = \left(\mathcal{I}_{\mathcal{Z}_i} \otimes M_i^{-1}\right) \boxtimes \left(\mathcal{N}_i \otimes \mathcal{N}_{i+1}^{-1}\right),$$

and the maps Ψ_i induce the injective maps

$$\mathcal{I}_{\mathcal{Z}'_i} \to \mathcal{I}_{\mathcal{Z}_{i+1}}.$$

Thus, we obtain a classifying morphism $f : \mathcal{T} \to S_{\boldsymbol{\beta}}^{[\boldsymbol{n}]}$.

Conversely, starting with $S_{\boldsymbol{\beta}}^{[\boldsymbol{n}]}$, where $\boldsymbol{n}, \boldsymbol{\beta}$ are as in the previous paragraph, we have the universal objects

$$\Phi_i : \mathcal{I}^{[n_i]} \to \mathcal{I}_{\beta_i}^{[n_{i+1}]} \qquad 1 \leq i < r$$

over $S \times S_{\boldsymbol{\beta}}^{[\boldsymbol{n}]}$. Taking double dual we get the sections

$$\Phi_i^{**} : \mathcal{O}_{S \times S_{\boldsymbol{\beta}}^{[\boldsymbol{n}]}} \to \mathcal{O}_{S \times S_{\boldsymbol{\beta}}^{[\boldsymbol{n}]}}(\mathcal{Z}_{\beta_i}).$$

By the same argument as in the previous paragraph, using $H^1(\mathcal{O}_S) = 0$, we can find the line bundles $M_1, \ldots, M_{r-1}$ on S and $\mathcal{N}'_1, \ldots, \mathcal{N}'_{r-1}$ on $\mathcal{T}$ such that $\mathcal{O}(\mathcal{Z}_{\beta_i}) \cong M_i \boxtimes \mathcal{N}'_i$, where as before M_i and $\mathcal{N}'_i$ can be written as

$$M_i = L_i^{-1} \otimes L_{i+1} \otimes \mathcal{L}, \quad \mathcal{N}'_i = \mathcal{N}_i^{-1} \otimes \mathcal{N}_{i+1},$$

and hence Φ_i is equivalent to

$$\Phi_i : \left(\mathcal{I}^{[n_i]} \otimes L_i\right) \boxtimes \mathcal{N}_i \to \left(\mathcal{I}^{[n_{i+1}]} \otimes L_{i+1} \otimes \mathcal{L}\right) \boxtimes \mathcal{N}_{i+1}, \qquad 1 \leq i < r. \tag{17}$$

By the discussion before the proposition, and the compatibility of $\boldsymbol{n}, \boldsymbol{\beta}$ with v, the maps (17) determine a flat family $\mathcal{E}$ of stable $\mathbb{C}^*$-equivariant sheaves on $X \times S_{\boldsymbol{\beta}}^{[\boldsymbol{n}]}$, and hence an $S_{\boldsymbol{\beta}}^{[\boldsymbol{n}]}$-valued point of $\mathcal{M}_h^{\mathcal{L}}(v)_{(1^r)}^{\mathbb{C}^*}$. Thus, we obtain a classifying morphism $g : S_{\boldsymbol{\beta}}^{[\boldsymbol{n}]} \to \mathcal{M}_h^{\mathcal{L}}(v)_{(1^r)}^{\mathbb{C}^*}$ with the image into the component $\mathcal{T}$ (by the choice of L_i). One can see by inspection that f and g are inverse of each other. $\square$

Remark 3.9. *Given $\boldsymbol{n}, \boldsymbol{\beta}$ compatible with the vector v there could be several components of $\mathcal{T} \subset \mathcal{M}_h^{\mathcal{L}}(v)_{(1^r)}^{\mathbb{C}^*}$ which are isomorphic to $S_{\boldsymbol{\beta}}^{[\boldsymbol{n}]}$. However, if* $\operatorname{Pic}(S) \cong \mathbb{Z}$, *then, given such $\boldsymbol{n}, \boldsymbol{\beta}$, then $S_{\boldsymbol{\beta}}^{[\boldsymbol{n}]}$ is isomorphic to a unique component of $\mathcal{M}_h^{\mathcal{L}}(v)_{(1^r)}^{\mathbb{C}^*}$.*

The following definition is motivated by the proof of Proposition 3.8.

Definition 3.10. *Suppose that $S_{\boldsymbol{\beta}}^{[\boldsymbol{n}]}$ is a component $\mathcal{T}$ of $\mathcal{M}_h^{\mathcal{L}}(v)^{\mathbb{C}^*}$ as in Proposition 3.8. If $\mathcal{O}(\mathcal{Z}_{\beta_i}) \cong M_i \boxtimes \mathcal{N}_i$ for $i = 1, \ldots, r-1$, where $M_i \in$* $\operatorname{Pic}(S)$ *and $\mathcal{N}_i \in$* $\operatorname{Pic}(S_{\boldsymbol{\beta}}^{[\boldsymbol{n}]})$*, then there are line bundles $L_i \in$* $\operatorname{Pic}(S)$ *(determined by $\mathcal{T}$) such that $M_i = L_i^{-1} \otimes L_{i+1} \otimes \mathcal{L}$. Let $\mathcal{N}_0 = \mathcal{O}_{S_{\boldsymbol{\beta}}^{[\boldsymbol{n}]}}$ and define*

$$\mathbb{J}_i := \left(\mathcal{I}^{[n_i]} \otimes L_i\right) \boxtimes \mathcal{N}_{i-1} \quad 1 \leq i \leq r. \tag{18}$$

By the proof of Proposition 3.8, the maps $\mathbb{J}_i \to \mathbb{J}_{i+1} \otimes \mathcal{L}$ induced by the universal maps Φ_i over $S_{\boldsymbol{\beta}}^{[n]}$ give rise to a universal family of stable $\mathbb{C}^*$-equivariant sheaves over $X \times \mathcal{T}$.

Notation. *For any coherent sheaves $\mathcal{F}$, $\mathcal{G}$ on $S \times B$ flat over a scheme B, and a nonzero integer a, we define*

$$\langle \mathcal{F}, \mathcal{G} \cdot \mathbf{t}^a \rangle := e\big(\mathbf{R}\mathcal{H}om_\pi(\mathcal{F}, \mathcal{G} \cdot \mathbf{t}^a)\big),$$

where π is the projection to the second factor of $S \times B$, and $e(-)$ denotes the equivariant Euler class.

In the following proposition we compare the restriction of the $\mathbb{C}^*$-fixed complex $E^{\bullet,\mathrm{fix}}_{\mathrm{red}}$ in (7) to the component $\mathcal{T} \cong S^{[\boldsymbol{n}]}_{\boldsymbol{\beta}}$ with the obstruction theory of $F^\bullet$ of Theorem 5. We also find an explicit expression for the moving part of $E^{\bullet,\mathrm{fix}}_{\mathrm{red}}$ restricted to $\mathcal{T}$ in terms of the universal object over $S^{[\boldsymbol{n}]}_{\boldsymbol{\beta}}$.

Proposition 3.11. *Using the isomorphism in Proposition 3.8, we have $E^{\bullet,\mathrm{fix}}_{\mathrm{red}}|_{\mathcal{T}} \cong F^\bullet$ (of Theorem 5). As a result,*

$$[\mathcal{M}^{\mathcal{L}}_h(v)_{(1^r)}]^{\mathrm{vir}}_{\mathrm{red}} = \sum_{\substack{\mathcal{T} \cong S^{[\boldsymbol{n}]}_{\boldsymbol{\beta}} \\ \text{is a conn. comp. of} \\ \mathcal{M}^{\mathcal{L}}_h(v)^{\mathbb{C}^*}_{(1^r)}}} [S^{[\boldsymbol{n}]}_{\boldsymbol{\beta}}]^{\mathrm{vir}}.$$

Moreover,

$$e\big((E^{\bullet,\mathrm{mov}})^\vee|_{\mathcal{T}}\big) = \frac{\prod_{\substack{1 \le i,j \le r \\ i \neq j-1}} \langle \mathbb{J}_i \cdot \mathbf{t}^{-i}, \mathbb{J}_j \otimes \mathcal{L} \cdot \mathbf{t}^{-j+1} \rangle}{\mathbf{s}^\kappa \prod_{\substack{1 \le i,j \le r \\ i \neq j}} \langle \mathbb{J}_i \cdot \mathbf{t}^{-i}, \mathbb{J}_j \cdot \mathbf{t}^{-j} \rangle},$$

where $\mathbb{J}_i$ are given in (18)*, and $\kappa = 1$ if $\mathcal{L} = \omega_S$, otherwise $\kappa = 0$.*

Proof. **Step 1:** *($r = 2$, fixed part of the obstruction theory)* We first prove the case $r = 2$. By the proof of Proposition 3.8, the short exact sequence (13) gives

$$0 \to z_*\mathbb{J}_2 \otimes \mathbf{t}^{-1} \to \overline{\mathbb{E}}|_{\mathcal{T}\times X} \to z_*\mathbb{J}_1 \to 0, \tag{19}$$

in which $\mathbb{J}_i$ (defined in (18)) carries no $\mathbb{C}^*$-weights. Applying

$$\mathbf{R}\mathcal{H}om(\overline{\mathbb{E}}, -), \quad \mathbf{R}\mathcal{H}om(-, z_*\mathbb{J}_1), \quad \mathbf{R}\mathcal{H}om(-, z_*\mathbb{J}_2 \cdot \mathbf{t}^{-1})$$

to (19), we get the exact triangles in $D^b(X \times \mathcal{T})$ filling respectively the middle row, and the third and the first columns of the following diagram:

$$\begin{array}{ccccc}
\mathbf{R}\mathcal{H}om(z_*\mathbb{J}_1, z_*\mathbb{J}_2 \cdot \mathbf{t}^{-1}) & & & & \mathbf{R}\mathcal{H}om(z_*\mathbb{J}_1, z_*\mathbb{J}_1) \\
\downarrow & & & & \downarrow \\
\mathbf{R}\mathcal{H}om(\overline{\mathbb{E}}, z_*\mathbb{J}_2 \cdot \mathbf{t}^{-1}) & \longrightarrow & \mathbf{R}\mathcal{H}om(\overline{\mathbb{E}}, \overline{\mathbb{E}}) & \longrightarrow & \mathbf{R}\mathcal{H}om(\overline{\mathbb{E}}, z_*\mathbb{J}_1) \\
\downarrow & & & & \downarrow \\
\mathbf{R}\mathcal{H}om(z_*\mathbb{J}_2 \cdot \mathbf{t}^{-1}, z_*\mathbb{J}_2 \cdot \mathbf{t}^{-1}) & & & & \mathbf{R}\mathcal{H}om(z_*\mathbb{J}_2 \cdot \mathbf{t}^{-1}, z_*\mathbb{J}_1)
\end{array} \tag{20}$$

For any coherent sheaf $\mathcal{F}$ on S, by [H06, Corollary 11.4], we have the following natural exact triangle

$$\mathcal{F} \otimes \mathcal{L}^{-1} \cdot \mathbf{t}^{-1}[1] \to \mathbf{L}z^* z_* \mathcal{F} \to \mathcal{F},$$

which for any other sheaf $\mathcal{G}$ on S, by adjunction, implies the exact triangle

$$z_* \, \mathbf{R}\mathcal{H}om_S(\mathcal{F}, \mathcal{G}) \to \mathbf{R}\mathcal{H}om_X(z_* \, \mathcal{F}, z_* \, \mathcal{G}) \to z_* \, \mathbf{R}\mathcal{H}om_S(\mathcal{F}, \mathcal{G} \otimes \mathcal{L} \cdot \mathbf{t})[-1].$$

Using this and taking the $\mathbb{C}^*$-fixed part of the diagram (20), we get the diagram

$$\begin{array}{ccccc}
z_* \, \mathbf{R}\mathcal{H}om(\mathbb{J}_1, \mathbb{J}_2 \otimes \mathcal{L})[-1] & & & & z_* \mathbf{R}\mathcal{H}om(\mathbb{J}_1, \mathbb{J}_1) \\
\downarrow & & & & \downarrow \\
\mathbf{R}\mathcal{H}om(\overline{\mathbb{E}}, z_* \, \mathbb{J}_2 \cdot \mathbf{t}^{-1})^{\mathrm{fix}} & \longrightarrow & \mathbf{R}\mathcal{H}om(\overline{\mathbb{E}}, \overline{\mathbb{E}})^{\mathrm{fix}} & \longrightarrow & \mathbf{R}\mathcal{H}om(\overline{\mathbb{E}}, z_* \, \mathbb{J}_1)^{\mathrm{fix}} \\
\downarrow & & & & \downarrow \\
z_* \mathbf{R}\mathcal{H}om(\mathbb{J}_2, \mathbb{J}_2) & & & & 0
\end{array}$$

in which the middle row and the 1st and 2nd columns are exact triangles. We conclude that

$$\mathbf{R}\mathcal{H}om(\overline{\mathbb{E}}, z_* \, \mathbb{J}_1)^{\mathrm{fix}} \cong z_* \mathbf{R}\mathcal{H}om(\mathbb{J}_1, \mathbb{J}_1)$$

$$\mathbf{R}\mathcal{H}om(\overline{\mathbb{E}}, z_* \, \mathbb{J}_2 \cdot \mathbf{t}^{-1})^{\mathrm{fix}} \cong \mathrm{Cone}\left(z_* \, \mathbf{R}\mathcal{H}om(\mathbb{J}_2, \mathbb{J}_2) \to z_* \mathbf{R}\mathcal{H}om(\mathbb{J}_1, \mathbb{J}_2 \otimes \mathcal{L})\right)[-1].$$

From this, and noting that the induced map $z_* \mathbf{R}\mathcal{H}om(\mathbb{J}_1, \mathbb{J}_1)[-1] \to z_* \mathbf{R}\mathcal{H}om(\mathbb{J}_2, \mathbb{J}_2)$ in the diagram is zero, we see that

$$\mathbf{R}\mathcal{H}om(\overline{\mathbb{E}}, \overline{\mathbb{E}})^{\mathrm{fix}} \cong \mathrm{Cone}\left(z_* \mathbf{R}\mathcal{H}om(\mathbb{J}_1, \mathbb{J}_1)[-1] \to \mathbf{R}\mathcal{H}om(\overline{\mathbb{E}}, z_* \, \mathbb{J}_2 \cdot \mathbf{t}^{-1})^{\mathrm{fix}}\right) \cong$$

$$\mathrm{Cone}\Big(z_* \mathbf{R}\mathcal{H}om(\mathbb{J}_1, \mathbb{J}_1) \oplus z_* \, \mathbf{R}\mathcal{H}om(\mathbb{J}_2, \mathbb{J}_2) \to z_* \mathbf{R}\mathcal{H}om(\mathbb{J}_1, \mathbb{J}_2 \otimes \mathcal{L})\Big)[-1].$$

Taking trace free part as in (7), applying $\mathbf{R}p_*$, and shifting by 1, we get

$$(E^{\bullet,\mathrm{fix}})^\vee|_{\mathcal{T}} \cong$$

$$\mathrm{Cone}\Big([\mathbf{R}\mathcal{H}om_p(\mathbb{J}_1, \mathbb{J}_1) \oplus \mathbf{R}\mathcal{H}om_p(\mathbb{J}_2, \mathbb{J}_2)]_0 \to \mathbf{R}\mathcal{H}om_p(\mathbb{J}_1, \mathbb{J}_2 \otimes \mathcal{L})\Big) \cong$$

$$\mathrm{Cone}\Big(\big[\mathbf{R}\mathcal{H}om_\pi(\mathcal{I}^{[n_1]}, \mathcal{I}^{[n_1]}) \oplus \mathbf{R}\mathcal{H}om_\pi(\mathcal{I}^{[n_2]}, \mathcal{I}^{[n_2]})\big]_0 \to \mathbf{R}\mathcal{H}om_\pi(\mathcal{I}^{[n_1]}, \mathcal{I}_\beta^{[n_2]})\Big) \cong F^{\bullet\vee}.$$

This proves the claim about the fixed part of the obstruction theory when $r = 2$.

Step 2: *($r = 2$, moving part of the obstruction theory)* We use diagram (20) in Step 1 again, but this time we take the moving parts:

$$\begin{array}{ccccc}
z_* \, \mathbf{R}\mathcal{H}om(\mathbb{J}_1, \mathbb{J}_2 \cdot \mathbf{t}^{-1}) & & & & z_* \mathbf{R}\mathcal{H}om(\mathbb{J}_1, \mathbb{J}_1 \otimes \mathcal{L} \cdot \mathbf{t})[-1] \\
\downarrow & & & & \downarrow \\
\mathbf{R}\mathcal{H}om(\overline{\mathbb{E}}, z_* \, \mathbb{J}_2 \cdot \mathbf{t}^{-1})^{\mathrm{mov}} & \longrightarrow & \mathbf{R}\mathcal{H}om(\overline{\mathbb{E}}, \overline{\mathbb{E}})^{\mathrm{mov}} & \longrightarrow & \mathbf{R}\mathcal{H}om(\overline{\mathbb{E}}, z_* \, \mathbb{J}_1)^{\mathrm{mov}} \\
\downarrow & & & & \downarrow \\
z_* \mathbf{R}\mathcal{H}om(\mathbb{J}_2, \mathbb{J}_2 \otimes \mathcal{L} \cdot \mathbf{t})[-1] & & & & A^\bullet
\end{array}$$

in which

$$A^\bullet := \mathrm{Cone}\Big(z_*\,\mathbf{R}\mathcal{H}om(\mathbb{J}_2,\mathbb{J}_1\otimes\mathcal{L}\cdot\mathbf{t}^2)[-2]\to z_*\,\mathbf{R}\mathcal{H}om(\mathbb{J}_2,\mathbb{J}_1\cdot\mathbf{t})\Big),$$

and the middle row and the 1st and 2nd columns are exact triangles. We conclude that

$$\mathbf{R}\mathcal{H}om(\overline{\mathbb{E}},z_*\,\mathbb{J}_1)^{\mathrm{mov}}\cong \mathrm{Cone}\left(A^\bullet\to z_*\mathbf{R}\mathcal{H}om(\mathbb{J}_1,\mathbb{J}_1\otimes\mathcal{L}\cdot\mathbf{t})\right)[-1]$$

$$\mathbf{R}\mathcal{H}om(\overline{\mathbb{E}},z_*\,\mathbb{J}_2\cdot\mathbf{t}^{-1})^{\mathrm{mov}}\cong \mathrm{Cone}\left(z_*\,\mathbf{R}\mathcal{H}om(\mathbb{J}_2,\mathbb{J}_2\otimes\mathcal{L}\cdot\mathbf{t})[-2]\to z_*\mathbf{R}\mathcal{H}om(\mathbb{J}_1,\mathbb{J}_2\cdot\mathbf{t}^{-1})\right).$$

Pushing forward, shifting by 1, and taking the equivariant Euler class, we get

$$e(\mathbf{R}\mathcal{H}om_{\overline{p}}(\overline{\mathbb{E}},\overline{\mathbb{E}})^{\mathrm{mov}}[1]|_{\mathcal{T}})=\frac{\langle\mathbb{J}_1,\mathbb{J}_1\otimes\mathcal{L}\cdot\mathbf{t}\rangle\cdot\langle\mathbb{J}_2,\mathbb{J}_2\otimes\mathcal{L}\cdot\mathbf{t}\rangle\cdot\langle\mathbb{J}_2,\mathbb{J}_1\otimes\mathcal{L}\cdot\mathbf{t}^2\rangle}{\langle\mathbb{J}_1,\mathbb{J}_2\cdot\mathbf{t}^{-1}\rangle\cdot\langle\mathbb{J}_2,\mathbb{J}_1\cdot\mathbf{t}\rangle}.$$

Also note that

$$e(\mathcal{E}xt^3_{\overline{p}}(\overline{\mathbb{E}},\overline{\mathbb{E}}))=\begin{cases}1 & \mathcal{L}\neq\omega_S\\ e(\mathcal{O}_{\mathcal{M}^{\omega_S}}\cdot\mathbf{t})=\mathbf{s} & \mathcal{L}=\omega_S.\end{cases}$$

This proves the claim about the moving part of the obstruction theory when $r=2$.

Step 3: *($r>2$)* Again by the proof of Proposition 3.8, the short exact sequence (13) gives

$$0\to\overline{\mathbb{E}}_1'|_{\mathcal{T}\times X}\otimes\mathbf{t}^{-1}\to\overline{\mathbb{E}}|_{\mathcal{T}\times X}\to z_*\,\mathbb{J}_1\to 0,$$

One can then repeat the argument of Step 1 and Step 2, by replacing $z_*\,\mathbb{J}_2$ with $\overline{\mathbb{E}}'|_{\mathcal{T}\times X}$, and use the induction on r to complete the proof of the proposition. □

Corollary 3.12. *Suppose that $r=2$, $\mathcal{D}=c_1(\mathcal{L})$ then*

$$e\big((E^{\bullet,\mathrm{mov}})^\vee|_{\mathcal{T}}\big)=\frac{\langle\mathcal{I}^{[n_1]},\mathcal{I}^{[n_1]}(\mathcal{D})\cdot\mathbf{t}\rangle\cdot\langle\mathcal{I}^{[n_2]},\mathcal{I}^{[n_2]}(\mathcal{D})\cdot\mathbf{t}\rangle\cdot\langle\mathcal{I}^{[n_2]}_\beta,\mathcal{I}^{[n_1]}(2\mathcal{D})\cdot\mathbf{t}^2\rangle}{\mathbf{s}^\kappa\cdot\langle\mathcal{I}^{[n_1]},\mathcal{I}^{[n_2]}_\beta(-\mathcal{D})\cdot\mathbf{t}^{-1}\rangle\cdot\langle\mathcal{I}^{[n_2]}_\beta,\mathcal{I}^{[n_1]}(\mathcal{D})\cdot\mathbf{t}\rangle}.$$

Proof. By Definition 3.10, $\mathcal{I}^{[n_2]}_\beta=\mathcal{I}^{[n_2]}\otimes\mathcal{L}\otimes L_2\otimes L_1^{-1}\otimes p^*\mathcal{N}_1$. The result then follows from the formula in Proposition 3.11 when $r=2$. □

Corollary 3.13. *Suppose that $r=2$. Using the notation of Propositions 3.8 and 3.11, Corollary 3.12 and Definition 3.5, we have*

$$\mathrm{DT}^{\mathcal{L}}_h(v;\alpha)_{(1^2)}=\sum_{\mathcal{T}\cong S^{[n_1,n_2]}_\beta}\frac{(-1)^{-\mathcal{D}\cdot\beta-K_S\cdot\mathcal{D}/2+3\mathcal{D}^2/2-\kappa}}{2^{\chi(\mathcal{L}^2)}(-\mathbf{s})^{\chi(\mathcal{L}^2)+\chi(\mathcal{L})-\chi(\mathcal{L}^{-1})-\kappa}}\int_{[S^{[n_1,n_2]}_\beta]^{\mathrm{vir}}}\alpha\cup\mathcal{Q}_{\mathcal{T}}.$$

$$\mathrm{DT}^{\mathcal{L}}_h(v)_{(1^2)}=\sum_{\mathcal{T}\cong S^{[n_1,n_2]}_\beta}\chi^{\mathrm{vir}}(S^{[n_1,n_2]}_\beta)=\sum_{\mathcal{T}\cong S^{[n_1,n_2]}_\beta}\mathsf{N}_S(n_1,n_2,\beta;\mathcal{P}_{\mathcal{T}}),$$

where all sums are over the connected components $\mathcal{T} \cong S_\beta^{[n_1,n_2]}$ *of* $\mathcal{M}_h^{\mathcal{L}}(v)^{\mathbb{C}^*}_{(1^2)}$, *and for any such* n_1, n_2 *and* β,

$$\mathcal{Q}_{\mathcal{T}} := e(\mathsf{T}^{\mathcal{L}\cdot\mathbf{t}}_{S^{[n_1]}}) \cdot e(\mathsf{T}^{\mathcal{L}\cdot\mathbf{t}}_{S^{[n_2]}}) \cdot \frac{e(\mathsf{G}_{\beta;\omega_S\otimes\mathcal{L}^{-1}} \cdot \mathbf{t}^{-1}) \cdot e(\mathsf{G}_{\beta;\mathcal{L}^{-1}} \cdot \mathbf{t}^{-1}) \cdot e(\mathsf{K}^{n_1,n_2}_{\beta;\omega_S\otimes\mathcal{L}^{-2}} \cdot \mathbf{t}^{-2})}{e(\mathsf{K}^{n_1,n_2}_{\beta;\omega_S\otimes\mathcal{L}^{-1}} \cdot \mathbf{t}^{-1}) \cdot e(\mathsf{K}^{n_1,n_2}_{\beta;\mathcal{L}^{-1}} \cdot \mathbf{t}^{-1}) \cdot e(\mathsf{G}_{\beta;\omega_S\otimes\mathcal{L}^{-2}} \cdot \mathbf{t}^{-2})},$$

$$\mathcal{P}_{\mathcal{T}} := c\left(T_{S^{[n_1]}}\right) \cup c\left(T_{S^{[n_2]}}\right) \cup \frac{c\left(\mathsf{G}_{\beta;\mathcal{O}_S}\right)}{c\left(\mathsf{K}^{n_1,n_2}_{\beta;\mathcal{O}_S}\right)}.$$

Proof. The formulas are the direct corollary of Propositions 3.8 and 3.11 and Corollary 3.12. The first formula follows from the following identities:

1. By Grothendieck-Verdier duality, for any coherent sheaves $\mathcal{F}$, $\mathcal{G}$ on $S \times S_\beta^{[n_1,n_2]}$ flat over $S_\beta^{[n_1,n_2]}$ we have
$$\langle \mathcal{F}, \mathcal{G} \cdot \mathbf{t}^a \rangle = (-1)^v \langle \mathcal{G}, \mathcal{F} \otimes \omega_S \cdot \mathbf{t}^{-a} \rangle,$$
where v is the rank of the complex $\mathbf{R}\mathcal{H}om_\pi(\mathcal{F},\mathcal{G})$ and $0 \neq a \in \mathbb{Z}$.
2. For any $0 \neq a \in \mathbb{Z}$ and and $M \in \mathrm{Pic}(S)$,
$$\frac{e(\mathsf{K}^{n_1,n_2}_{\beta;M} \otimes \mathbf{t}^a)}{e(\mathsf{G}_{\beta;M} \otimes \mathbf{t}^a)} = \frac{(a\,\mathbf{s})^{\chi(M)}}{\langle \mathcal{I}^{[n_1]}, \mathcal{I}^{[n_2]}_\beta \otimes M \cdot \mathbf{t}^a \rangle}, \qquad \frac{e(\mathsf{T}^{M\cdot\mathbf{t}}_{S^{[n_i]}})}{\mathbf{s}^{\chi(M)}} = \frac{1}{\langle \mathcal{I}^{[n_i]}, \mathcal{I}^{[n_i]} \otimes M \cdot \mathbf{t} \rangle}.$$

For the second formula note that by definition

$$\mathrm{DT}_h^{\mathcal{L}}(v)_{(1^2)} = \sum_{\mathcal{T} \cong S_\beta^{[n_1,n_2]}} \int_{[S_\beta^{[n_1,n_2]}]^{\mathrm{vir}}} \frac{c\left(\mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_1]}, \mathcal{I}^{[n_2]}_\beta\right)\right)}{c\left(\mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_1]}, \mathcal{I}^{[n_1]}\right)\right) \cdot c\left(\mathbf{R}\mathcal{H}om_\pi\left(\mathcal{I}^{[n_2]}, \mathcal{I}^{[n_2]}\right)\right)}.$$

Then we use

$$T_{S^{[n_i]}} \cong \mathcal{E}xt^1_\pi\left(\mathcal{I}^{[n_i]}, \mathcal{I}^{[n_i]}\right), \quad c\left(\mathcal{E}xt^{j\neq 1}_\pi\left(\mathcal{I}^{[n_i]}, \mathcal{I}^{[n_i]}\right)\right) = c\left(\mathcal{E}xt^{j\neq 1}_\pi\left(\mathcal{I}^{[n_i]}, \mathcal{I}^{[n_i]}\right)_0\right) = 1.$$

∎

3.3. **Generic Complete Intersections.** Suppose that $S \subset \mathbb{P}^{k+2}$ with $k \geq 1$ is a generic complete intersection of type $(d_1, \dots, d_k) \neq (2), (3), (2,2)$ and $d_i > 1$. Let $r = 2$, $h = \mathcal{O}_S(1)$, and $\boldsymbol{n} = n_1, n_2$ and β be compatible with the vector v as defined in Definition 3.7. Then, $\omega_S = \mathcal{O}_S(-k-3+d_1+\cdots+d_k)$ and by the genericity $\mathrm{Pic}\, S = \mathbb{Z}$ (see [L21]). If $\mathcal{L} = \mathcal{O}(\ell)$ and $L_i = \mathcal{O}(l_i)$ for $i = 1, 2$ we must have (by the conditions before Definition 3.7 and (2))

$$\ell \geq -k-3+d_1+\cdots+d_k, \qquad l_1 > l_2, \qquad \ell + l_2 \geq l_1.$$

Therefore, we get

$$0 < l_1 - l_2 \leq \ell. \tag{21}$$

Note that in this case $\beta = c_1(\mathcal{O}_S(\ell + l_2 - l_1))$, and that $\beta^D := K_S - \beta$ is effective if

$$-k-3+d_1+\cdots+d_k - \ell + l_1 - l_2 \geq 0. \tag{22}$$

These observations lead to the following proposition:

Proposition 3.14. *Suppose that $v = (r = 2, \gamma = c_1(\mathcal{O}(2g+1)), m)$.*
1. If $\ell \leq 0$ then $\mathcal{M}_h^{\mathcal{L}}(v)^{\mathbb{C}^*}_{(1^2)} = \emptyset$.
2. If $\ell = 1$ then $l_1 = g+1,\ l_2 = g$ and

$$\mathcal{M}_h^{\mathcal{L}}(v)^{\mathbb{C}^*}_{(1^2)} = \coprod_{\substack{n_1+n_2=-m+b \\ n_1 \geq n_2}} S^{[n_1 \geq n_2]},$$

where $b = d_1 \cdots d_k\,(g^2+g+1/2)$.
3. Suppose that $d_1+\cdots+d_k \geq k+3$, $\ell > 0$ and l_1, l_2 are so that $2g+1 = l_1+l_2$, condition (21) is satisfied, but condition (22) is not satisfied. If $S_\beta^{[n_1,n_2]}$ is a nonempty component of $\mathcal{M}_h^{\mathcal{L}}(v)^{\mathbb{C}^*}_{(1^2)}$ with $\beta = c_1(\mathcal{O}_S(\ell+l_2-l_1))$ and $n_1+n_2 = (l_1^2/2+l_2^2/2)d_1\cdots d_k - m$, then $[S_\beta^{[n_1,n_2]}]^{\mathrm{vir}} = 0$.

Proof. Part 1 follows immediately from (21). Part 2 follows from (21) and Definition 3.7. Part 3 follows from Theorem 7.

□

Corollary 3.15. *In the notation of Proposition 3.14 if $\mathcal{L} = \omega_S$ then*
1. If S is a Fano complete intersection or a $K3$ surface i.e. when $d_1+\cdots+d_k \leq k+3$ then $\mathcal{M}_h^{\omega_S}(v)^{\mathbb{C}^*}_{(1^2)} = \emptyset$.
2. If S is isomorphic to one of the following five complete intersection types

$$(5),\ (3,3),\ (4,2),\ (3,2,2),\ (2,2,2,2)$$

then $\mathcal{M}_h^{\omega_S}(v)^{\mathbb{C}^*}_{(1^2)}$ is a disjoint union of the nested Hilbert schemes of points as in Proposition 3.14 item (2).
3. If $d_1+\cdots+d_k \geq k+4$ and if $S_\beta^{[n_1,n_2]}$ is a nonempty component of $\mathcal{M}_h^{\omega_S}(v)^{\mathbb{C}^*}_{(1^2)}$, then condition (22) is always satisfied.

□

Corollary 3.16. *If S is isomorphic to one of the five types of generic complete intersections in part (2) of Proposition 3.14, then,*
1. Suppose that the conditions of Remark [GSY17a, Remark 5.10] *hold for the class α (e.g. $\alpha = 1$), then,*

$$\mathrm{DT}_h^{\omega_S}(v;\alpha)_{(1^2)} = \sum_{\substack{n_1+n_2=-m+b \\ n_2 \leq n_1}} \frac{(-1)^{d_1\cdots d_k - 1}}{2^{\chi(\mathcal{O}_S(2))}(-\mathbf{s})^{\chi(\mathcal{O}_S(1))-1}}$$

$$\int_{S^{[n_1]}\times S^{[n_2]}} \frac{\alpha \cup c_{n_1+n_2}(\mathsf{E}^{n_1,n_2}) \cup e(\mathsf{T}_{S^{[n_1]}}^{\mathcal{O}_S(1)\cdot\mathbf{t}}) \cdot e(\mathsf{T}_{S^{[n_2]}}^{\mathcal{O}_S(1)\cdot\mathbf{t}}) \cdot e(\mathsf{E}^{n_1,n_2}_{\mathcal{O}_S(-1)}\cdot\mathbf{t}^{-2})}{e(\mathsf{E}^{n_1,n_2}\cdot\mathbf{t}^{-1})\cdot e(\mathsf{E}^{n_1,n_2}_{\mathcal{O}_S(-1)}\cdot\mathbf{t}^{-1})},$$

2.

$$\mathrm{DT}_h^{\omega_S}(v)_{(1^2)} =$$
$$\sum_{\substack{n_1+n_2=-m+b \\ n_2\le n_1}} \int_{S^{[n_1]}\times S^{[n_2]}} \frac{c_{n_1+n_2}(\mathsf{E}^{n_1,n_2})\cup c(T_{S^{[n_1]}})\cup c(T_{S^{[n_2]}})}{c(\mathsf{E}^{n_1,n_2})},$$

where $b = d_1\cdots d_k\,(g^2+g+1/2)$.

Proof. This follows from Theorem 6 by noting that all the (equivariant) classes in the integrands satisfy conditions of [GSY17a, Remark 5.10]. Also note that for any $M\in \operatorname{Pic}(S)$, $\mathsf{G}_{0;M}=0$ and $\iota^*\mathsf{E}_M^{n_1,n_2}=\mathsf{K}_{0;M}^{n_1,n_2}$. □

4. Mochizuki's result and proof of Theorem 4

In this section, we assume that $p_g(S)>0$, for instance, any generic hyperplane section of a quintic 3-fold satisfies this assumption. The perfect obstruction theory (see Corollary 3.3)

$$(\mathbf{R}\mathcal{H}om_p(\mathbb{E},\mathbb{E})_0[1])^\vee$$

gives the virtual cycle $[\mathcal{M}_h(v)]^{\mathrm{vir}}$ whose virtual dimension d is

$$d=\gamma^2-4m-3\chi(\mathcal{O}_S).$$

Let $\mathsf{P}(\mathbb{E})$ be a polynomial in the slant products $\mathrm{ch}_i(\mathbb{E})/b$ for elements $b\in H^*(S)$ and $i\in\mathbb{Z}_{\ge 0}$. By the wall-crossing argument using the master space, Mochizuki describes the invariant

$$\int_{[\mathcal{M}_h(v)]^{\mathrm{vir}}}\mathsf{P}(\mathbb{E})$$

in terms of Seiberg-Witten invariants and certain integration over the Hilbert schemes of points on S. The SW invariants are defined as follows: for a curve class $\mathsf{c}\in H^2(S,\mathbb{Z})$, let L be the line bundle on S with $c_1(L)=\mathsf{c}$, which is uniquely determined by the assumption $H^1(\mathcal{O}_S)=0$. Let S_{c} be the Hilbert scheme of curves in class c or equivalently the moduli space of non-zero morphisms $\mathcal{O}_S\to L$, that is isomorphic to $\mathbb{P}(H^0(L))$. By the discussion in item (3) of [GSY17a, Section 3], $(\mathbf{R}\pi_*\mathcal{O}_{\mathcal{Z}_c}(\mathcal{Z}_c))^\vee$ gives a perfect obstruction theory $S_{\mathsf{c}}\cong\mathbb{P}(H^0(L))$. Under this identification, it is easy to see that the tangent and the obstruction bundles $\mathsf{T}(\mathsf{c})$ and $\mathrm{ob}(\mathsf{c})$ naturally sit in the exact sequences on $\mathbb{P}(H^0(L))$:

$$0\to H^0(\mathcal{O}_S)\otimes\mathcal{O}\to H^0(L)\otimes\mathcal{O}(1)\to\mathsf{T}(\mathsf{c})\to 0,$$

$$0\to H^1(L)\otimes\mathcal{O}(1)\to\mathrm{ob}(\mathsf{c})\to H^2(\mathcal{O}_S)\otimes\mathcal{O}\to H^2(L)\otimes\mathcal{O}(1)\to 0.$$

By [BF97, Proposition 5.6], the $[S_{\mathsf{c}}]^{\mathrm{vir}}=e(\mathrm{ob}(\mathsf{c}))\cap[S_{\mathsf{c}}]$. Since by our assumption $p_g>0$ a simple argument (cf. [M02, Proposition 6.3.1]) shows that the only way that $e(\mathrm{ob}(c)) \neq 0$ is that $h^1(L) - h^2(L) < 0$ in which case, $\mathrm{Rank}(\mathrm{ob}(\mathsf{c})) = \mathrm{Rank}(T(\mathsf{c}))$, i.e. the virtual dimension of S_{c} is 0. Then, by a simple calculation

$$\mathrm{SW}(\mathsf{c}) := \int_{[S_{\mathsf{c}}]^{\mathrm{vir}}} 1 = (-1)^{h^0(L)-1} \begin{pmatrix} p_g - 1 \\ h^0(L) - 1 \end{pmatrix}.$$

Consider the decomposition

$$\gamma_1 + \gamma_2 = \gamma, \ \gamma_i \in H^2(S, Z) \text{ are effective curve classes,}$$

and let L_{γ_i} be the line bundle on S with $c_1(L_{\gamma_i}) = \gamma_i$, and define $\mathcal{I}^{[n_i]}_{L_{\gamma_i}} := \mathcal{I}^{[n_i]} \otimes L_{\gamma_i}$. Recall that we use the symbol π' to denote all the projections

$$S \times S^{[n_i]} \to S^{[n_i]}, \qquad S \times S^{[n_1]} \times S^{[n_2]} \to S^{[n_1]} \times S^{[n_2]}.$$

Notation. *Let* $\mathbf{t}'$ *is the trivial line bundle on* S *with the* $\mathbb{C}^*$*-action of weight 1 on the fibers*[7]*, and let* $\mathbf{s}' := c_1(\mathbf{t}')$*. We also consider the rank* n *tautological vector bundle on* $S^{[n_i]}$*, given by*

$$V^{[n_i]}_{L_{\gamma_i}} := \pi'_* \left(\mathcal{O}_{\mathcal{Z}^{[n_i]}} \otimes L_{\gamma_i} \right).$$

Following Mochizuki, we define

$$\mathsf{A}(\gamma_1, \gamma_2, v; \mathsf{P}) :=$$
$$\sum_{\substack{n_1+n_2= \\ \gamma^2/2-m-\gamma_1\cdot\gamma_2}} \int_{S^{[n_1]}\times S^{[n_2]}} \mathrm{Res}_{\mathbf{s}'=0} \left(\frac{e\left(V^{[n_1]}_{L_{\gamma_1}}\right) \cdot \mathsf{P}\left(\mathcal{I}^{[n_1]}_{L_{\gamma_1}} \cdot \mathbf{t}'^{-1} \oplus \mathcal{I}^{[n_2]}_{L_{\gamma_2}} \cdot \mathbf{t}'\right) \cdot e\left(V^{[n_2]}_{L_{\gamma_2}} \cdot \mathbf{t}'^2\right)}{(2s')^{n_1+n_2-p_g} \cdot \mathsf{Q}\left(\mathcal{I}^{[n_1]}_{L_{\gamma_1}} \cdot \mathbf{t}'^{-1}, \mathcal{I}^{[n_2]}_{L_{\gamma_2}} \cdot \mathbf{t}'\right)} \right).$$

where

$$\mathsf{Q}\left(\mathcal{I}^{[n_1]}_{L_{\gamma_1}} \cdot \mathbf{t}'^{-1}, \mathcal{I}^{[n_2]}_{L_{\gamma_2}} \cdot \mathbf{t}'\right) =$$
$$e\left(-\mathbf{R}\mathcal{H}om_{\pi'}\left(\mathcal{I}^{[n_1]}_{L_{\gamma_1}} \cdot \mathbf{t}'^{-1}, \mathcal{I}^{[n_2]}_{L_{\gamma_2}} \cdot \mathbf{t}'\right) - \mathbf{R}\mathcal{H}om_{\pi'}\left(\mathcal{I}^{[n_2]}_{L_{\gamma_2}} \cdot \mathbf{t}', \mathcal{I}^{[n_1]}_{L_{\gamma_1}} \cdot \mathbf{t}'^{-1}\right)\right).$$

The following result was obtained by Mochizuki:

Proposition 4.1. (Mochizuki [M02, Theorem 1.4.6]) *Assume that* $\gamma \cdot h > 2K_S \cdot h$ *and* $\chi(v) := \int_S v \cdot td_S \geq 1$*. Then we have the following formula:*

$$\frac{1}{2}\int_{\mathcal{M}_h(v)} \mathsf{P}(\mathbb{E}) = -\sum_{\substack{\gamma_1+\gamma_2=\gamma \\ \gamma_1\cdot h<\gamma_2\cdot h}} \mathrm{SW}(\gamma_1) \cdot 2^{1-\chi(v)} \cdot \mathsf{A}(\gamma_1, \gamma_2, v; \mathsf{P}).$$

□

[7]Here we use the symbol $\mathbf{t}'$ to distinguish this line bundle from the equivariant trivial line bundle $\mathbf{t}$ defined before with respect to a different $\mathbb{C}^*$-action.

Remark 4.2. *The factor $1/2$ in the left hand side of the formula above comes from the difference between Mochizuki's convention and ours. Mochizuki used the moduli stack of oriented stable sheaves, which is a μ_2-gerb over our moduli space $\mathcal{M}_h(v)$.*

Remark 4.3. *The assumptions $\gamma \cdot h > 2K_S \cdot h$ and $\chi(v) \geq 1$ are satisfied if we replace v by $v \cdot L_{kh}$ for $k \gg 0$. Note that tensoring with a bundle does not affect the isomorphism class of $\mathcal{M}_h^{\mathcal{L}}(v)$, and hence in particular the DT invariants $\mathrm{DT}_h^{\mathcal{L}}(v;\alpha), \mathrm{DT}_h^{\mathcal{L}}(v)$ remain unchanged.*

Recall that the $\mathbb{C}^*$-fixed locus $\overline{\mathcal{M}}_h(v)^{\mathbb{C}^*}$ decomposes into components

$$\mathcal{M}_h^{\mathcal{L}}(v)^{\mathbb{C}^*} = \mathcal{M}_h^{\mathcal{L}}(v)^{\mathbb{C}^*}_{(2)} \coprod \mathcal{M}_h^{\mathcal{L}}(v)^{\mathbb{C}^*}_{(1^2)},$$

and by Proposition 3.1,

$$\mathrm{DT}_h(v;\alpha) = \mathrm{DT}_h(v;\alpha)_{(2)} + \mathrm{DT}_h(v;\alpha)_{(1^2)}, \quad \mathrm{DT}_h(v) = \mathrm{DT}_h(v)_{(2)} + \mathrm{DT}_h(v)_{(1^2)}.$$

Recall form Corollary 3.3

$$\mathrm{DT}_h^{\mathcal{L}}(v;\alpha)_{(2)} = \int_{[\mathcal{M}_h(v)]^{\mathrm{vir}}} \mathbf{s}^{\kappa} \cdot e\left(-\mathbf{R}\mathcal{H}om_p(\mathbb{E}, \mathbb{E} \otimes \mathcal{L} \cdot t)\right) \cup \alpha,$$

$$\mathrm{DT}_h^{\mathcal{L}}(v)_{(2)} = \int_{[\mathcal{M}_h(v)]^{\mathrm{vir}}} c_d\left(-\mathbf{R}\mathcal{H}om_p(\mathbb{E}, \mathbb{E})_0\right),$$

Suppose that the class α can be written as a polynomial in $\mathrm{ch}_i(\mathbb{E})/b$ for $b \in H^*(S)$. Since both $e\left(-\mathbf{R}\mathcal{H}om_p(\mathbb{E}, \mathbb{E} \otimes \mathcal{L} \cdot t)\right)$ and $c_d\left(-\mathbf{R}\mathcal{H}om_p(\mathbb{E}, \mathbb{E})_0\right)$ can be written as polynomials P_1 and P_2 of $\mathrm{ch}_i(\mathbb{E})/b$ for $b \in H^*(S)$, by the Grothendieck-Riemann-Roch theorem, we can apply Proposition 4.1 to write $\mathrm{DT}_h(v)_{(2)}$ in terms of SW invariants and the integration over the Hilbert schemes of points. Therefore, by Corollaries 3.13 and 3.16 we have

Proposition 4.4. *Under the assumption of Proposition 4.1, we have the identity*

$$\mathrm{DT}_h^{\mathcal{L}}(v;\alpha) = -\sum_{\substack{\gamma_1+\gamma_2=\gamma \\ \gamma_1\cdot h<\gamma_2\cdot h}} \mathrm{SW}(\gamma_1) \cdot 2^{2-\chi(v)} \cdot \mathsf{A}(\gamma_1,\gamma_2,v;\mathsf{P}_1 \cup \alpha)$$
$$+ \sum_{\substack{\mathcal{T}\cong S_\beta^{[n_1,n_2]} \\ \text{is a conn. comp. of} \\ \mathcal{M}_h^{\mathcal{L}}(v)^{\mathbb{C}^*}_{(1^2)}}} \frac{(-1)^{-\mathcal{D}\cdot\beta - K_S\cdot\mathcal{D}/2 + 3\mathcal{D}^2/2 - \kappa}}{2^{\chi(\mathcal{L}^2)}(-\mathbf{s})^{\chi(\mathcal{L}^2)+\chi(\mathcal{L})-\chi(\mathcal{L}^{-1})-\kappa}} \int_{[S_\beta^{[n_1,n_2]}]^{\mathrm{vir}}} \alpha \cup \mathcal{Q}_{\mathcal{T}}.$$

$$\mathrm{DT}_h^{\mathcal{L}}(v) = -\sum_{\substack{\gamma_1+\gamma_2=\gamma \\ \gamma_1\cdot h<\gamma_2\cdot h}} \mathrm{SW}(\gamma_1) \cdot 2^{2-\chi(v)} \cdot \mathsf{A}(\gamma_1,\gamma_2,v;\mathsf{P}_2) + \sum_{\substack{\mathcal{T}\cong S_\beta^{[n_1,n_2]} \\ \text{is a conn. comp. of} \\ \mathcal{M}_h^{\mathcal{L}}(v)^{\mathbb{C}^*}_{(1^2)}}} \mathsf{N}_S(n_1,n_2,\beta;\mathcal{P}_{\mathcal{T}}).$$

In particular, when $\mathcal{L} = \omega_S$ and S is isomorphic to one of five types generic complete intersections $(5) \subset \mathbb{P}^3$, $(3,3) \subset \mathbb{P}^4$, $(4,2) \subset \mathbb{P}^4$, $(3,2,2) \subset \mathbb{P}^5$, $(2,2,2,2) \subset \mathbb{P}^6$, *and the class α additionally satisfies the requirements of* [GSY17a, Remark 5.10] *(e.g. $\alpha = 1$) then,*

$$\mathrm{DT}_h^{\omega_S}(v;\alpha) = -\sum_{\substack{\gamma_1+\gamma_2=\gamma \\ \gamma_1\cdot h<\gamma_2\cdot h}} \mathrm{SW}(\gamma_1)\cdot 2^{2-\chi(v)}\cdot \mathsf{A}(\gamma_1,\gamma_2,v;\mathsf{P}_1\cup\alpha) + \frac{(-1)^{d_1\cdots d_k-1}}{2^{\chi(\mathcal{O}_S(2))}(-\mathbf{s})^{\chi(\mathcal{O}_S(1))-1}}$$

$$\sum_{\substack{n_1+n_2=-m+b \\ n_2\leq n_1}} \int_{S^{[n_1]}\times S^{[n_2]}} \frac{\alpha\cup c_{n_1+n_2}(\mathsf{E}^{n_1,n_2})\cup e(\mathsf{T}_{S^{[n_1]}}^{\mathcal{O}_S(1)}\cdot \mathbf{t}^{-1})\cdot e(\mathsf{T}_{S^{[n_2]}}^{\mathcal{O}_S(1)}\cdot\mathbf{t}^{-1})\cdot e(\mathsf{E}^{n_1,n_2}_{\mathcal{O}_S(-1)}\cdot\mathbf{t}^{-2})}{e(\mathsf{E}^{n_1,n_2}\cdot\mathbf{t}^{-1})\cdot e(\mathsf{E}^{n_1,n_2}_{\mathcal{O}_S(-1)}\cdot\mathbf{t}^{-1})},$$

$$\begin{aligned}\mathrm{DT}_h^{\omega_S}(v) = &-\sum_{\substack{\gamma_1+\gamma_2=\gamma \\ \gamma_1\cdot h<\gamma_2\cdot h}} \mathrm{SW}(\gamma_1)\cdot 2^{2-\chi(v)}\cdot\mathsf{A}(\gamma_1,\gamma_2,v;\mathsf{P}_2) \\ &+\sum_{\substack{n_1+n_2=-m+b \\ n_2\leq n_1}} \int_{S^{[n_1]}\times S^{[n_2]}} \frac{c_{n_1+n_2}(\mathsf{E}^{n_1,n_2})\cup c(T_{S^{[n_1]}})\cup c(T_{S^{[n_2]}})}{c(\mathsf{E}^{n_1,n_2})},\end{aligned}$$

where $\gamma = c_1(\mathcal{O}_S(2g+1))$ and $b = d_1\cdots d_k\,(g^2+g+1/2)$.

□

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