

# ATIYAH CLASS AND SHEAF COUNTING ON LOCAL CALABI YAU FOURFOLDS

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## Abstract

We discuss Donaldson-Thomas (DT) invariants of torsion sheaves with 2 dimensional support on a smooth projective surface in an ambient non-compact Calabi Yau fourfold given by the total space of a rank 2 bundle on the surface. We prove that in certain cases, when the rank 2 bundle is chosen appropriately, the universal truncated Atiyah class of these codimension 2 sheaves reduces to one, defined over the moduli space of such sheaves realized as torsion codimension 1 sheaves in a noncompact divisor (threefold) embedded in the ambient fourfold. Such reduction property of universal Atiyah class enables us to relate our fourfold DT theory to a reduced DT theory of a threefold and subsequently then to the moduli spaces of sheaves on the base surface using results in [15, 16]. We finally make predictions about modularity of such fourfold invariants when the base surface is an elliptic K3.

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## 1. Introduction

This paper is motivated by two recent developments in Donaldson-Thomas theory, namely the mathematical theory of Vafa-Witten invariants of complex surfaces constructed in [31, 32], [16, 15] and [19], and the construction [3, 4, 5] of virtual cycle invariants for sheaves on Calabi-Yau fourfolds. In the approach of [15, 31] the Vafa-Witten invariants of a smooth projective surface  $S$  are constructed as reduced virtual cycle invariants of two dimensional sheaves on the total space of the canonical bundle  $K_S$ . In fact, the construction of loc. cit. is more general, including equivariant residual invariants of two dimensional sheaves on the total space  $Y$  of an arbitrary line bundle  $L$  on  $S$ . The starting point for the present study is the observation that the total space  $X$  of the rank two bundle  $L \oplus K_S \otimes L^{-1}$  is a Calabi-Yau fourfold, which is at the same time isomorphic to the total space of the canonical line bundle  $K_Y$ . This leads to the natural question whether the invariants constructed in [15, 31] for  $Y$  are related to fourfold Donaldson-Thomas invariants. In fact such a correspondence has already been proven in [4] for the case where  $Y$  is a compact Fano threefold, using a local Kuranishi map construction for fourfold Donaldson-Thomas invariants. The main goal of this paper is prove an analogous correspondence for more general situations allowing  $Y$  to be a quasi-projective threefold with a suitable torus action. As discussed in detail below, the approach employed here is different, using the truncated Atiyah class formalism of [27]. A second goal is to test the modularity conjecture for two dimensional sheaf counting invariants in the local fourfold framework. This will be

carried out for a particular class of examples, where  $Y$  is the total space of certain line bundles  $L$  over an elliptic K3 surface  $S$ .

The main motivation for this study resides in modular properties of Donaldson-Thomas invariants for sheaves with two dimensional support. This has been a central theme in enumerative geometry starting with the early work of [17, 38, 39, 40, 42, 18] on torsion free sheaves on surfaces. It has been also studied intensively for two dimensional sheaves on Calabi-Yau threefolds [13, 12, 36, 14, 34, 35]. Further results on this topic include [19, 22, 23, 31]. As modularity is a natural consequence of S-duality [37, 7, 6, 11], similar physical arguments predict that it should be also manifest in the Donaldson-Thomas theory of two dimensional sheaves on Calabi-Yau fourfolds. At the moment this question is completely open. This paper essentially consists of two parts, as explained below.

**1.1. General results.** Let  $Y$  be a smooth quasi-projective variety of dimension  $d - 1$ ,  $d \geq 2$ , and let  $X$  be the total space of the canonical bundle  $K_Y$ . Let  $\mathcal{M}$  be a quasi-projective fine moduli scheme of compactly supported stable sheaves  $\mathcal{F}$  on  $X$ . Suppose that all sheaves,  $\mathcal{F}$ , parameterized by  $\mathcal{M}$  are scheme theoretically supported on  $Y$ . The formalism of truncated Atiyah classes developed in [27], yields two natural (unreduced) “deformation-obstruction theories”,  $a_X : \mathbb{E}^\bullet_X \rightarrow \mathbb{L}^\bullet_{\mathcal{M}}$  and  $a_Y : \mathbb{E}^\bullet_Y \rightarrow \mathbb{L}^\bullet_{\mathcal{M}}$  for the moduli space  $\mathcal{M}$ . These are associated respectively to deformations of sheaves, realized, respectively, as modules over the structure rings of  $X$  and  $Y$ . As in [2], in the present paper an obstruction theory will be an object  $\mathbb{E}^\bullet$  of amplitude  $[-1, 0]$  in the derived category of the moduli space, together with a morphism in derived category  $\phi : \mathbb{E}^\bullet \rightarrow \mathbb{L}^\bullet_{\mathcal{M}}$  such that  $h^0(\phi)$  is an isomorphism and  $h^{-1}(\phi)$  is an epimorphism of sheaves. Here  $\mathbb{L}^\bullet_{\mathcal{M}}$  denotes the  $[-1, 0]$  truncation of the full cotangent complex of  $\mathcal{M}$  as in [27]. The first general result proven in this paper is

**Theorem 1.1.** *(Proposition 3.2 and Proposition 3.6). There is a natural splitting in  $\mathcal{D}^b(\mathcal{M})$ :  $\mathbb{E}^\bullet_X \cong \mathbb{E}^\bullet_Y \oplus \mathbb{E}^{\bullet \vee}_Y[1 - d]$ . Moreover, the obstruction theory  $a_X$  factors through  $\mathbb{E}^\bullet_Y$  and there is a direct sum decomposition*

$$a_X = (a_Y, 0).$$

This shows that, under the above assumptions, the fourfold obstruction theory of  $\mathcal{M}$  admits a reduction to the threefold obstruction theory. Although this is a natural statement, the proof of it, using the truncated Atiyah classes, is surprisingly long. In particular, it uses an explicit calculation, Theorem 2.5, for the relative Atiyah class of a product.

Using this result, the proof of Theorem 1.1 is then given subsequently in Sections 3.1 and 3.3, the main intermediate results being Propositions 3.2 and 3.6

respectively. In particular, Propositions 3.6 is a more explicit restatement of Theorem 1.1.

Applying Theorem 1.1, a reduced equivariant Donaldson-Thomas theory is constructed in Section 4 for two dimensional sheaves on certain local fourfolds. In section 4,  $S$  is a smooth projective surface with nef canonical bundle,  $K_S$ , equipped with a polarization  $h = c_1(\mathcal{O}_S(1))$ . It is also assumed that  $H^1(\mathcal{O}_S) = 0$  and the integral cohomology of  $S$  is torsion free. Given an effective divisor  $D$  on  $S$ , a fourfold  $X$  is constructed as the total space of a rank two bundle  $K_S(D) \oplus \mathcal{O}_S(-D)$ . The threefold  $Y$  is the total space of the line bundle  $L = K_S(D)$  on  $S$ , which is canonically embedded in  $X$  as a divisor.

As in [15] the topological invariants of a compactly supported two dimensional sheaf  $\mathcal{F}$  on  $X$  are completely determined by the total Chern class  $c(g_*\mathcal{F})$  of the direct image  $g_*\mathcal{F}$ , where  $g : X \rightarrow S$  is the canonical projection. In turn, the latter is naturally given by a triple

$$(1) \quad \gamma(\mathcal{F}) = (r(\mathcal{F}), \beta(\mathcal{F}), n(\mathcal{F})) \in \mathbb{Z} \oplus H_2(S, \mathbb{Z}) \oplus \mathbb{Z}$$

The entry,  $r(\mathcal{F}) = \text{rk}(g_*\mathcal{F})$ , will be called the rank of  $\mathcal{F}$  in the following. Moreover, since  $D$  is effective nonzero divisor, one easily shows (Corollary 4.3) that any such  $h$ -stable sheaf  $\mathcal{F}$  is scheme theoretically supported on  $Y \subset X$ . This yields a natural identification of the moduli spaces  $\mathcal{M}_h(X, \gamma) \cong \mathcal{M}_h(Y, \gamma)$ . Moreover, as proven in [15, Proposition 2.4] and [31, Theorem 6.5] the latter has a *reduced* perfect obstruction theory  $a_Y^{\text{red}} : \mathbb{E}_Y^{\bullet, \text{red}} \rightarrow \mathbb{L}_{\mathcal{M}}^{\bullet}$ . The detail of this construction is reviewed in Section 4.2 for completeness. Then Theorem 1.1 implies:

**Corollary 1.2.** *The fourfold obstruction theory  $a_X : \mathbb{E}_X^{\bullet} \rightarrow \mathbb{L}_{\mathcal{M}}^{\bullet}$  admits a reduction to the reduced perfect obstruction theory  $a_Y^{\text{red}} : \mathbb{E}_Y^{\bullet, \text{red}} \rightarrow \mathbb{L}_{\mathcal{M}}^{\bullet}$  constructed in [15, Proposition 2.4] and [31, Theorem 6.5].*

Using this result, the reduced equivariant Donaldson-Thomas invariants for two dimensional sheaves on  $X$  are defined by residual virtual integration in Section 4.3. The next two sections, 5 and 6, are focused on explicit computations and modularity conjectures for rank two invariants. As shown in [15, Section 3] and [31, Section 7], there are two types of rank two torus fixed loci. The “*type I*” fixed locus is isomorphic to the moduli space  $\mathcal{M}_h(S, \gamma)$  of torsion free sheaves on  $S$  with topological invariants  $\gamma$ . The “*type II*” fixed loci consist of sheaves  $\mathcal{F}$  with scheme theoretic support on the divisor  $2S \subset Y$ , whose connected components are naturally isomorphic to twisted nested Hilbert schemes, as reviewed in detail in Section 5.1. By analogy with similar previous results [15, 19], universality results for residual virtual integrals of type I and II are proven in Propositions 5.1 and 5.3. These are valid for any pair  $(S, D)$  as above. Explicit results and a modularity conjecture are then obtained in Section 6 for elliptic K3 surfaces with  $D$  a positive

multiple of the fiber class,  $D = mf$ .

**1.2. Rank two results and conjectures for elliptic K3 surfaces.** Let  $S$  be a smooth generic elliptic K3 surface in Weierstrass form. By the genericity assumption, all elliptic fibers are reduced and irreducible, and have at most nodal singularities. Furthermore, the Mordell-Weill group is trivial and the Picard group of  $S$  is freely generated by the fiber class  $\mathcal{F}$  and the canonical section class  $\sigma$ . The Kähler cone of  $S$  then consists of real divisor classes  $h = t\sigma + uf$  with  $t, u \in \mathbb{R}$ ,  $0 < 2t < u$ .

Let  $D = mf$ ,  $m \geq 1$ , be a positive multiple of the elliptic fiber so that the local fourfold  $X$  is the total space of  $\mathcal{O}_S(mf) \oplus \mathcal{O}_S(-mf)$ . Moreover, let the topological invariant (1) be of the form  $\gamma = (2, f, n)$  with  $n \in \mathbb{Z}$ . The Bogomolov inequality, proven in Lemma 6.6, shows that in this case the moduli space  $\mathcal{M}_h(Y, \gamma)$  is empty for  $n < 0$ . Moreover, the global boundedness result proven in Lemma 6.7 shows that there are no marginal stability walls associated to sheaves  $\mathcal{F}$  with invariants  $\gamma$  for

$$(2) \quad 0 < \frac{t}{u} < \frac{1}{1 + 8n}.$$

Therefore for any fixed  $n \geq 0$  the equivariant residual Donaldson-Thomas invariants  $DT_h(X, \gamma)$  reach a stable limit,  $D_\infty(X, \gamma) \in \mathbb{Q}(\mathbf{s})$  as the size of the elliptic fibers decreases, keeping the size of the section fixed. Then let

$$Z_\infty(X, 2, f; q) = \sum_{n \geq 0} DT_\infty(X, 2, f, n) q^{n-2} \in q^{-2} \mathbb{Q}(\mathbf{s})[[q]]$$

be the generating function of limit invariants. Clearly the above partition functions splits into type I and type II contributions denoted by  $Z_\infty(X, 2, f; q)_I$  and  $Z_\infty(X, 2, f; q)_{II}$  respectively. Then the following result is proven in Section 6.3.

**Proposition 1.3.** *The series  $Z_\infty(X, 2, f; q)_I$  belongs to  $\mathbb{Q}(\mathbf{s})[[q]] \subset \mathbb{Q}(\mathbf{s})[[q^{-1}, q]]$  and the following identity holds in the quadratic extension  $\mathbb{Q}(\mathbf{s})[[q^{1/2}]]$ :*

$$(3) \quad Z_\infty(X, 2, f; q)_I = \frac{\mathbf{s}^{-1}}{2} (\Delta^{-1}(q^{1/2}) + \Delta^{-1}(-q^{1/2})),$$

where  $\Delta(q)$  is the discriminant modular form.

Moreover, Proposition 6.15 shows that all type II nonzero contributions vanish for  $m$  even. At the same time for  $m$  odd, all nonzero type II contributions are associated to nested Hilbert schemes without divisorial twists. Then the following conjecture is inferred from the results of [31, Section 8] using the universality result Proposition 5.3.

**Conjecture 1.4.** *Suppose  $m \geq 1$  is odd. Then*

$$(4) \quad Z_\infty(X, 2, f; q)_{II} = \frac{\mathbf{s}^{-1}}{4} \Delta^{-1}(q^2).$$

**Plan of the paper.** The key theorem of the current article is the reduction Theorem 1.1. We approach the problem by essentially proving a statement that the universal (relative) Atiyah class of the (universal) sheaf  $\mathbb{F}$  on the moduli space,  $\mathcal{M}_h(X, \gamma)$  of stable torsion sheaves  $\mathcal{F}$  on  $X$ , with scheme theoretic support on  $Y$ , has a reduction to the universal Atiyah class of the universal sheaf  $\mathbb{G}$  on  $\mathcal{M}_h(Y, \gamma)$  (which is isomorphic to  $\mathcal{M}_h(X, \gamma)$ ) parameterizing stable sheaves  $\mathcal{G}$  on  $Y$  (where for all  $\mathcal{F} \in \mathcal{M}_h(X, \gamma)$  we have  $\mathcal{F} \cong i_* \mathcal{G}, i : Y \hookrightarrow X$ ). This will be done in Proposition 3.2. Our proof of Proposition 3.2 relies heavily on a certain splitting property of the universal relative Atiyah classes on product schemes, as stated in Theorem 2.5. Hence, we start in Section 2 a detailed discussion of Atiyah classes on product schemes which can be read independently from the rest of the paper. Then, in Section 3 we use Theorem 2.5 to prove Proposition 3.2 and later on, as an application of Proposition 3.2, we show in Proposition 3.6 that the reduction of Atiyah classes as above, leads to a reduction in the level of deformation obstruction theories, as stated in Theorem 1.1. The latter proves that in our current geometric setup the Donaldson-Thomas invariants of fourfold  $X$  can be completely recovered from those of threefold  $Y$  which, after being reduced, enjoy modularity property, discussions of which we have included in Sections 4, 5 and 6.

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## 2. Universal relative Atiyah classes on product schemes

As explained in the plan of the paper, this section is a self-contained account of relative truncated Atiyah classes, the main result being Theorem 2.5 below.

**2.1. Truncated Atiyah class of a scheme.** Let  $W$  be an arbitrary scheme of finite type over  $\mathbb{C}$  and  $W \subset A$  a closed embedding in a smooth scheme  $A$ . As in [27, Definition. 2.1], the truncated cotangent complex  $\mathbb{L}_W$  is the two term complex of amplitude  $[-1, 0]$

$$\mathcal{I}_W / \mathcal{I}_W^2 \rightarrow \Omega_A^1|_W,$$

where  $\mathcal{I}_W \subset \mathcal{O}_A$  is the defining ideal sheaf of  $W \subset A$ . Let us denote by  $\Delta_W : W \rightarrow W \times W$  the diagonal morphism, by  $\mathcal{O}_{\Delta_W}$  the extension by zero of the structure sheaf of the diagonal, and by  $\mathcal{I}_{\Delta_W} \subset \mathcal{O}_{W \times W}$  the ideal sheaf of the diagonal. Then the following results are proven in [27, Lemma 2.2, Definition 2.3].

**Lemma 2.1.** (i) *There is an exact sequence of  $\mathcal{O}_{W \times W}$ -modules*

$$(5) \quad 0 \rightarrow \Delta_{W*}(\mathcal{I}_W / \mathcal{I}_W^2) \xrightarrow{\tau_W} \mathcal{I}_{\Delta_A}|_{W \times W} \xrightarrow{\zeta_W} \mathcal{I}_{\Delta_W} \rightarrow 0$$

where the morphism  $\zeta_W$  is the natural projection.

(ii) *Let  $\iota_W : \mathcal{I}_{\Delta_W} \rightarrow \mathcal{O}_{W \times W}$  denote the canonical inclusion. Then there is a commutative diagram of  $\mathcal{O}_{W \times W}$ -modules*

$$(6) \quad \begin{array}{ccccc} \Delta_{W*}(\mathcal{I}_W / \mathcal{I}_W^2) & \xrightarrow{\tau_W} & \mathcal{I}_{\Delta_A}|_{W \times W} & \xrightarrow{\iota_W \circ \zeta_W} & \mathcal{O}_{W \times W} \\ \downarrow \mathbf{1} & & \downarrow r_W & & \\ \Delta_{W*}(\mathcal{I}_W / \mathcal{I}_W^2) & \xrightarrow{\Delta_{W*}(d_{W/A})} & (\mathcal{I}_{\Delta_A} / \mathcal{I}_{\Delta_A}^2)|_{W \times W} & & \end{array}$$

where the right vertical map  $r_W$  is the natural projection and the bottom row is naturally isomorphic to  $\mathbb{L}_W[1]$ .

By construction, the top row of diagram (6) is quasi-isomorphic to  $\mathcal{O}_{\Delta_W}$ . Hence, the Diagram (6) determines a morphism  $\alpha_W : \mathcal{O}_{\Delta_W} \rightarrow \Delta_{W*}\mathbb{L}_W[1]$ . The latter is the truncated universal Atiyah class of  $W$ . According to [27, Definition 2.1], the truncated cotangent complex and truncated Atiyah class are independent of choice of  $A$ . As shown in [27, Definition 2.3], an alternative presentation of the truncated Atiyah class is obtained from the proof of Lemma 2.2 in loc. cit., as follows.

**Lemma 2.2.** (i) *There is a canonical isomorphism*

$$(7) \quad \mathcal{I}_{\Delta_A}|_{W \times A} \cong \mathcal{I}_{\Delta_W \subset W \times A}$$

where the right hand side denotes the defining ideal sheaf for the closed embedding  $\Delta_W \subset W \times A$ .

(ii) The isomorphism (7) determines a further canonical isomorphism of  $\mathcal{O}_{W \times A}$ -modules

$$(8) \quad (\mathcal{O}_W \boxtimes \mathcal{I}_W) \otimes_{W \times A} \mathcal{O}_{\Delta_W} \xrightarrow{\sim} \Delta_{W*}(\mathcal{I}_W / \mathcal{I}_W^2)$$

and a natural injection

$$(9) \quad \mathcal{O}_W \boxtimes \mathcal{I}_W \hookrightarrow \mathcal{I}_{\Delta_A}|_{W \times A}$$

which fit in a commutative diagram of morphisms of  $\mathcal{O}_{W \times A}$ -modules:

$$(10) \quad \begin{array}{ccc} \mathcal{O}_W \boxtimes \mathcal{I}_W & \hookrightarrow & \mathcal{I}_{\Delta_A}|_{W \times A} \\ \downarrow & & \downarrow \\ (\mathcal{O}_W \boxtimes \mathcal{I}_W) \otimes_{W \times A} \mathcal{O}_{\Delta_W} & \xrightarrow{\iota_W} & \mathcal{I}_{\Delta_A}|_{W \times W} \\ \downarrow \wr & \nearrow \tau_W & \downarrow r_W \\ \Delta_{W*}(\mathcal{I}_W / \mathcal{I}_W^2) & \xrightarrow{\Delta_{W*}(d_{W/A})} & (\mathcal{I}_{\Delta_A} / \mathcal{I}_{\Delta_A}^2)|_{W \times W} \end{array}$$

Moreover, the two upper vertical arrows in the above diagram are the natural projections.

**Remark 2.3.** By a slight abuse of notation, in the above lemma  $\Delta_W$  indicates the canonical closed embedding  $W \hookrightarrow W \times A$  as opposed to the diagonal embedding  $W \hookrightarrow W \times W$ . This notation will be consistently used throughout this section, the distinction being clear from the context.

Next note the following immediate corollary of Lemma 2.2.

**Corollary 2.4.** The universal Atiyah class is equivalently defined by the commutative diagram

$$(11) \quad \begin{array}{ccc} (\mathcal{O}_W \boxtimes \mathcal{I}_W) \otimes_{W \times A} \mathcal{O}_{\Delta_W} & \xrightarrow{\iota_W} & \mathcal{I}_{\Delta_A}|_{W \times W} \xrightarrow{\iota_W \circ \xi_W} \mathcal{O}_{W \times W} \\ \downarrow \mathbf{1} & & \downarrow r_W \\ (\mathcal{O}_W \boxtimes \mathcal{I}_W) \otimes_{W \times A} \mathcal{O}_{\Delta_W} & \xrightarrow{r_W \circ \iota_W} & (\mathcal{I}_{\Delta_A} / \mathcal{I}_{\Delta_A}^2)|_{W \times W}. \end{array}$$

where the left column is restricted to  $W \times W$ . As in Lemma 2.1, the right vertical map is the natural projection.

**2.2. Atiyah classes on product schemes.** In most applications to moduli problems  $W$  will be a product scheme  $Z \times T$ , where  $Z$  is a smooth quasi-projective variety and  $T$  is a test scheme, or the moduli space. In the latter case, as shown in [27, Thm. 4.1], the construction of the obstruction theory of the moduli space, uses only the relative component  $\alpha_{W/Z}$  of the Atiyah class (as will be recalled below).



The goal of the section is to prove Theorem 2.5 below, which relates  $\alpha_{W/Z}$  to the Atiyah class  $\alpha_T$  of  $T$ .

To fix the notation, for any product  $W_1 \times W_2$  of schemes over  $\mathbb{C}$  let  $\pi_{W_i}$ ,  $1 \leq i \leq 2$ , denote the canonical projections. Moreover, for any test scheme  $T$  let  $Z_T = Z \times T$  and, given a morphism of schemes  $f : Z_1 \rightarrow Z_2$ , let  $f_T : (Z_1)_T \rightarrow (Z_2)_T$  denote the morphism  $f \times 1_T$ . The subscripts will be suppressed if  $T = \text{Spec}(\mathbb{C})$ . Given two complexes of sheaves  $C_1^\bullet, C_2^\bullet$  on  $W_1, W_2$ , the tensor product  $\pi_{W_1}^* C_1^\bullet \otimes \pi_{W_2}^* C_2^\bullet$  will be denoted by  $C_1^\bullet \boxtimes C_2^\bullet$ . Similarly, the direct sum  $\pi_{W_1}^* C_1^\bullet \oplus \pi_{W_2}^* C_2^\bullet$  will be denoted by  $C_1^\bullet \boxplus C_2^\bullet$ . Some background results needed below have been recorded in the Appendix A.

Let  $W = Z_T$  where  $Z$  is a smooth complex quasi-projective variety and  $T$  is an arbitrary quasi-projective scheme over  $\mathbb{C}$ . Then there is a canonical splitting

$$(12) \quad \mathbb{L}_{Z_T} \cong \Omega_Z \boxplus \mathbb{L}_T.$$

In the present context this splitting follows from equation (112) proven in Corollary A.5, which shows that for any quasi-projective schemes  $W_1, W_2$  there is a natural isomorphism

$$\bar{\rho}_W : \mathcal{I}_{\Delta_{W_1}} / \mathcal{I}_{\Delta_{W_1}}^2 \boxtimes \mathcal{O}_{\Delta_{W_2}} \oplus \mathcal{O}_{\Delta_{W_1}} \boxtimes \mathcal{I}_{\Delta_{W_2}} / \mathcal{I}_{\Delta_{W_2}}^2 \xrightarrow{\sim} \mathcal{I}_{\Delta_W} / \mathcal{I}_{\Delta_W}^2.$$

Since  $T$  is quasi-projective, there is a closed embedding  $T \subset U$  into a smooth quasi-projective scheme  $U$ . Hence one can pick  $A = Z_U$  in the construction of the Atiyah class, since  $Z$  is smooth. Then, setting  $W_1 = Z$  and  $W_2 = U$  in the above equation yields the splitting (12).

The relative Atiyah class is the component  $\alpha_{Z_T/Z} : \mathcal{O}_{\Delta_{Z_T}} \rightarrow \Delta_{Z_T^*}(\pi_T^* \mathbb{L}_T[1])$  of  $\alpha_{Z_T}$  with respect to this splitting. Using Corollary A.3, this is canonically identified with a morphism

$$\alpha_{Z_T/Z} : \mathcal{O}_{\Delta_Z} \boxtimes \mathcal{O}_{\Delta_T} \rightarrow \mathcal{O}_{\Delta_Z} \boxtimes \mathbb{L}_T[1].$$

The notation can be preserved with no risk of confusion. Note also that the Atiyah class of  $T$  is a morphism  $\alpha_T : \mathcal{O}_{\Delta_T} \rightarrow \Delta_{T^*} \mathbb{L}_T[1]$ . Then the main result of this section is to relate the latter two classes as follows:

**Theorem 2.5.** *Let  $Z, T$  be quasi-projective schemes over  $\mathbb{C}$  with  $Z$  smooth. Then there is a relation*

$$(13) \quad \alpha_{Z_T/Z} = \mathbf{1}_{\mathcal{O}_{\Delta_Z}} \boxtimes \alpha_T.$$

The proof will use diagram (11) in Corollary 2.4 and will proceed in several steps. Since  $T$  is quasi-projective, there is a closed embedding  $T \subset U$  with  $U$  smooth quasi-projective. Moreover, as already noted below equation (12) one can

set  $A = Z_U$  in the construction of the Atiyah class since  $Z$  is smooth. Then diagram (11) reads

$$(14) \quad \begin{array}{ccc} (\mathcal{O}_{Z_T} \boxtimes \mathcal{I}_{Z_T}) \otimes_{Z_T \times Z_U} \mathcal{O}_{\Delta_{Z_T}} & \xrightarrow{v_{Z_T}} & \mathcal{I}_{\Delta_{Z_U}}|_{Z_T \times Z_T} \xrightarrow{i_{Z_T} \circ \xi_{Z_T}} \mathcal{O}_{Z_T \times Z_T} \\ \downarrow \mathbf{1} & & \downarrow r_{Z_T} \\ (\mathcal{O}_{Z_T} \boxtimes \mathcal{I}_{Z_T}) \otimes_{Z_T \times Z_U} \mathcal{O}_{\Delta_{Z_T}} & \xrightarrow{r_{Z_T} \circ v_{Z_T}} & (\mathcal{I}_{\Delta_{Z_U}} / \mathcal{I}_{\Delta_{Z_U}}^2)|_{Z_T \times Z_T}, \end{array}$$

where the left hand column is restricted to  $Z_T \times Z_T$ . The strategy to prove Theorem 2.5 is to construct a suitable quasi-isomorphic model for the rows in the above diagram, making relation (13) manifest. In order to facilitate the exposition, diagrams such as (14) will be regarded in an obvious way as quiver representations in abelian categories of coherent sheaves. A map between two diagrams will be a morphism of quiver representations i.e. a collection of morphisms between the sheaves associated to the respective nodes which intertwine between the maps belonging to the two representations. Then, as a first step, note the following consequence of Corollary A.3.

**Lemma 2.6.** *Diagram (14) is canonically isomorphic to*

$$(15) \quad \begin{array}{ccc} \mathcal{O}_{\Delta_Z} \boxtimes \Delta_{T^*}(\mathcal{I}_T / \mathcal{I}_T^2) & \xrightarrow{v_{Z_T}} & \mathcal{I}_{\Delta_{Z_U}}|_{Z_T \times Z_T} \xrightarrow{i_{Z_T} \circ \xi_{Z_T}} \mathcal{O}_{Z_T \times Z_T} \\ \downarrow \mathbf{1} & & \downarrow r_{Z_T} \\ \mathcal{O}_{\Delta_Z} \boxtimes \Delta_{T^*}(\mathcal{I}_T / \mathcal{I}_T^2) & \xrightarrow{r_{Z_T} \circ v_{Z_T}} & (\mathcal{I}_{\Delta_{Z_U}} / \mathcal{I}_{\Delta_{Z_U}}^2)|_{Z_T \times Z_T} \end{array}$$

Moreover, the morphism  $v_{Z_T}$  in diagram (15) is determined by the commutative diagram

$$(16) \quad \begin{array}{ccc} \mathcal{O}_{Z \times Z} \boxtimes (\mathcal{O}_T \boxtimes \mathcal{I}_T) & \hookrightarrow & \mathcal{I}_{\Delta_{Z_U}}|_{Z_T \times Z_U} \\ \downarrow & & \downarrow \\ \mathcal{O}_{\Delta_Z} \boxtimes \Delta_{T^*}(\mathcal{I}_T / \mathcal{I}_T^2) & \xrightarrow{v_{Z_T}} & \mathcal{I}_{\Delta_{Z_U}}|_{Z_T \times Z_T}. \end{array}$$

where the top horizontal arrow is the natural injection induced by the isomorphism  $\mathcal{I}_{\Delta_{Z_U}}|_{Z_T \times Z_U} \cong \mathcal{I}_{\Delta_{Z_T} \subset Z_T \times Z_U}$  and the vertical arrows are natural projections.

*Proof.* Using equation (108) in Corollary A.3 it is straightforward to check that there is a commutative diagram of morphisms of  $\mathcal{O}_{Z_T \times Z_U}$ -modules

$$(17) \quad \begin{array}{ccc} \mathcal{O}_{Z_T} \boxtimes \mathcal{I}_{Z_T} & \xrightarrow{\sim} & \mathcal{O}_{Z \times Z} \boxtimes (\mathcal{O}_T \boxtimes \mathcal{I}_T) \\ \downarrow & & \downarrow \\ (\mathcal{O}_{Z_T} \boxtimes \mathcal{I}_{Z_T}) \otimes_{Z_T \times Z_U} \mathcal{O}_{\Delta_{Z_T}} & \xrightarrow{\sim} & \mathcal{O}_{\Delta_Z} \boxtimes \Delta_{T*}(\mathcal{I}_T / \mathcal{I}_T^2) \end{array}$$

where the horizontal arrows are canonical isomorphisms and the vertical arrows are natural projections. This follows from the basic properties of tensor products using the canonical isomorphisms

$$\mathcal{I}_{Z_T} \cong \mathcal{O}_Z \boxtimes \mathcal{I}_T, \quad (\mathcal{O}_T \boxtimes \mathcal{I}_T) \otimes_{T \times U} \mathcal{O}_{\Delta_T} \cong \Delta_{T*}(\mathcal{I}_T / \mathcal{I}_T^2)$$

of  $\mathcal{O}_{Z_U}$  and  $\mathcal{O}_{T \times U}$ -modules respectively. The details will be omitted. Moreover, as shown in Lemma 2.2, the morphism  $\nu_{Z_T}$  in diagram (14) is determined by the commutative diagram

$$(18) \quad \begin{array}{ccc} \mathcal{O}_{Z_T} \boxtimes \mathcal{I}_{Z_T} & \hookrightarrow & \mathcal{I}_{\Delta_{Z_U}}|_{Z_T \times Z_U} \\ \downarrow & & \downarrow \\ (\mathcal{O}_{Z_T} \boxtimes \mathcal{I}_{Z_T}) \otimes_{Z_T \times Z_U} \mathcal{O}_{\Delta_{Z_T}} & \xrightarrow{\nu_{Z_T}} & \mathcal{I}_{\Delta_{Z_U}}|_{Z_T \times Z_T} \end{array}$$

where the top horizontal arrow is the natural injection while the vertical arrows are the natural projections. Diagram (16) follows immediately from (18) using (17). q.e.d.

For the next step, recall the splitting (12) is determined by the canonical isomorphism

$$(19) \quad \bar{\rho}_U : \mathcal{I}_{\Delta_Z} / \mathcal{I}_{\Delta_Z}^2 \boxtimes \mathcal{O}_{\Delta_U} \oplus \mathcal{O}_{\Delta_Z} \boxtimes \mathcal{I}_{\Delta_U} / \mathcal{I}_{\Delta_U}^2 \xrightarrow{\sim} \mathcal{I}_{\Delta_{Z_U}} / \mathcal{I}_{\Delta_{Z_U}}^2$$

obtained in Corollary A.5. As also shown in loc. cit., there is a commutative diagram

$$(20) \quad \begin{array}{ccc} \mathcal{I}_{\Delta_Z} / \mathcal{I}_{\Delta_Z}^2 \boxtimes \mathcal{O}_{\Delta_U} \oplus \mathcal{O}_{\Delta_Z} \boxtimes \mathcal{I}_{\Delta_U} / \mathcal{I}_{\Delta_U}^2 & \xrightarrow{\bar{\rho}_U} & \mathcal{I}_{\Delta_{Z_U}} / \mathcal{I}_{\Delta_{Z_U}}^2 \\ \downarrow (0,1) & & \downarrow \bar{\gamma}_{Z_U} \\ \mathcal{O}_{\Delta_Z} \boxtimes \mathcal{I}_{\Delta_U} / \mathcal{I}_{\Delta_U}^2 & \xrightarrow{1} & \mathcal{O}_{\Delta_Z} \boxtimes \mathcal{I}_{\Delta_U} / \mathcal{I}_{\Delta_U}^2 \end{array}$$

where  $\bar{\gamma}_{Z_U} : \mathcal{I}_{\Delta_{Z_U}} / \mathcal{I}_{\Delta_{Z_U}}^2 \rightarrow \mathcal{O}_{\Delta_Z} \boxtimes \mathcal{I}_{\Delta_U} / \mathcal{I}_{\Delta_U}^2$  is determined by the natural projection  $\gamma_{Z_U} : \mathcal{I}_{\Delta_{Z_U}} \rightarrow \mathcal{O}_{\Delta_Z} \boxtimes \mathcal{I}_{\Delta_U}$ .

Now let

$$p_{Z_T} = \bar{\gamma}_{Z_U}|_{Z_T \times Z_T} : (\mathcal{I}_{\Delta_{Z_U}} / \mathcal{I}_{\Delta_{Z_U}}^2)|_{Z_T \times Z_T} \rightarrow \mathcal{O}_{\Delta_Z} \boxtimes (\mathcal{I}_{\Delta_U} / \mathcal{I}_{\Delta_U}^2)|_{T \times T}.$$

and let  $q_{Z_T} : \mathcal{I}_{\Delta_{Z_U}}|_{Z_T \times Z_T} \rightarrow \mathcal{O}_{\Delta_Z} \boxtimes (\mathcal{I}_{\Delta_U} / \mathcal{I}_{\Delta_U}^2)|_{T \times T}$  be the composite map

$$\begin{array}{ccc} \mathcal{I}_{\Delta_{Z_U}}|_{Z_T \times Z_T} & \xrightarrow{r_{Z_T}} & (\mathcal{I}_{\Delta_{Z_U}} / \mathcal{I}_{\Delta_{Z_U}}^2)|_{Z_T \times Z_T} \\ & \searrow q_{Z_T} & \downarrow p_{Z_T} \\ & & \mathcal{O}_{\Delta_Z} \boxtimes (\mathcal{I}_{\Delta_U} / \mathcal{I}_{\Delta_U}^2)|_{T \times T}. \end{array}$$

Here  $r_{Z_T}$  is the natural projection used in diagram (15). Then note that the canonical projection  $\mathbb{L}_{Z_T} \rightarrow \mathcal{O}_Z \boxtimes \mathbb{L}_T$  obtained from the splitting (12) is determined by the commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{\Delta_Z} \boxtimes \Delta_{T^*}(\mathcal{J}_T / \mathcal{J}_T^2) & \xrightarrow{r_{Z_T} \circ v_{Z_T}} & (\mathcal{I}_{\Delta_{Z_U}} / \mathcal{I}_{\Delta_{Z_U}}^2)|_{Z_T \times Z_T} \\ \downarrow \mathbf{1} & & \downarrow p_{Z_T} \\ \mathcal{O}_{\Delta_Z} \boxtimes \Delta_{T^*}(\mathcal{J}_T / \mathcal{J}_T^2) & \xrightarrow{q_{Z_T} \circ v_{Z_T}} & \mathcal{O}_{\Delta_Z} \boxtimes (\mathcal{I}_{\Delta_U} / \mathcal{I}_{\Delta_U}^2)|_{T \times T}. \end{array}$$

As a result, Lemma 2.6 immediately implies:

**Corollary 2.7.** *The relative Atiyah class  $\alpha_{Z_T/Z}$  is determined by the commutative diagram*

$$(21) \quad \begin{array}{ccccc} \mathcal{O}_{\Delta_Z} \boxtimes \Delta_{T^*}(\mathcal{J}_T / \mathcal{J}_T^2) & \xrightarrow{v_{Z_T}} & \mathcal{I}_{\Delta_{Z_U}}|_{Z_T \times Z_T} & \xrightarrow{\iota_{Z_T} \circ \xi_{Z_T}^2} & \mathcal{O}_{Z_T \times Z_T} \\ \downarrow \mathbf{1} & & \downarrow q_{Z_T} & & \\ \mathcal{O}_{\Delta_Z} \boxtimes \Delta_{T^*}(\mathcal{J}_T / \mathcal{J}_T^2) & \xrightarrow{q_{Z_T} \circ v_{Z_T}} & \mathcal{O}_{\Delta_Z} \boxtimes (\mathcal{I}_{\Delta_U} / \mathcal{I}_{\Delta_U}^2)|_{T \times T} & & \end{array}$$

Note also that the Atiyah class of  $T$  is obtained by setting  $Z = \text{Spec}(\mathbb{C})$  in Corollary 2.4. This yields the following commutative diagram

$$(22) \quad \begin{array}{ccccc} \Delta_{T^*}(\mathcal{J}_T / \mathcal{J}_T^2) & \xrightarrow{v_T} & \mathcal{I}_{\Delta_U}|_{T \times T} & \xrightarrow{\iota_T \circ \xi_T^2} & \mathcal{O}_{T \times T} \\ \downarrow \mathbf{1} & & \downarrow r_T & & \\ \Delta_{T^*}(\mathcal{J}_T / \mathcal{J}_T^2) & \xrightarrow{r_T \circ v_T} & \mathcal{I}_{\Delta_U} / \mathcal{I}_{\Delta_U}^2|_{T \times T} & & \end{array}$$

where the bottom row is isomorphic to  $\mathbb{L}_T[1]$ . The map  $v_T$  is determined by the commutative diagram

$$(23) \quad \begin{array}{ccc} \mathcal{O}_T \boxtimes \mathcal{I}_T & \hookrightarrow & \mathcal{I}_{\Delta_U}|_{T \times U} \\ \downarrow & & \downarrow \\ \Delta_{T*}(\mathcal{I}_T / \mathcal{I}_T^2) & \xrightarrow{v_T} & \mathcal{I}_{\Delta_U}|_{T \times T} \end{array}$$

where the top horizontal map is the natural inclusion induced by the isomorphism  $\mathcal{I}_{\Delta_U}|_{T \times U} \cong \mathcal{I}_{\Delta_T \subset T \times U}$ , and the vertical maps are natural projections.

As the next step in the proof of Theorem 2.5, it will be shown below in Lemma 2.9 that the domain of the relative Atiyah class is quasi-isomorphic to the three term complex

$$(24) \quad \mathcal{O}_{\Delta_Z} \boxtimes \Delta_{T*}(\mathcal{I}_T / \mathcal{I}_T^2) \xrightarrow{\mathbf{1} \boxtimes v_T} \mathcal{O}_{\Delta_Z} \boxtimes \mathcal{I}_{\Delta_U}|_{T \times T} \xrightarrow{\mathbf{1} \boxtimes (\iota_T \circ \xi_T)} \mathcal{O}_{\Delta_Z} \boxtimes \mathcal{O}_{T \times T}.$$

This is an important step in the proof since it provides a direct connection with the domain of the  $\alpha_T$  in diagram (23).

The starting point is the exact sequence of  $\mathcal{O}_{Z_U \times Z_U}$ -modules

$$(25) \quad 0 \rightarrow \mathcal{I}_{\Delta_Z} \boxtimes \mathcal{O}_{U \times U} \xrightarrow{j_{Z_U}} \mathcal{I}_{\Delta_{Z_U}} \xrightarrow{\gamma_{Z_U}} \mathcal{O}_{\Delta_Z} \boxtimes \mathcal{I}_{\Delta_U} \rightarrow 0,$$

which follows from Corollary A.4. The key idea is to gain some control over the restriction  $\mathcal{I}_{\Delta_{Z_U}}|_{Z_T \times Z_T}$  by proving:

**Lemma 2.8.** *The exact sequence (25) remains exact upon restriction to  $Z_T \times Z_T$ .*

*Proof.* First note that Corollary A.4 yields a second exact sequence of  $\mathcal{O}_{Z_U \times Z_U}$ -modules

$$(26) \quad 0 \rightarrow \mathcal{I}_{\Delta_Z} \boxtimes \mathcal{I}_{\Delta_U} \xrightarrow{\delta_{Z_U}} \mathcal{I}_{\Delta_Z} \boxplus \mathcal{I}_{\Delta_U} \xrightarrow{\rho_{Z_U}} \mathcal{I}_{\Delta_{Z_U}} \rightarrow 0.$$

As shown below, this sequence remains exact upon restriction to  $Z_T \times Z_T$ . The restriction of (26) to  $Z_T \times Z_T$  is the exact sequence

$$(27) \quad \mathcal{I}_{\Delta_Z} \boxtimes (\mathcal{I}_{\Delta_U}|_{T \times T}) \rightarrow \mathcal{I}_{\Delta_Z} \boxplus (\mathcal{I}_{\Delta_U}|_{T \times T}) \rightarrow \mathcal{I}_{\Delta_{Z_U}}|_{Z_T \times Z_T} \rightarrow 0.$$

The second component of the first map from the left is

$$(28) \quad \iota_Z \boxtimes \mathbf{1} : \mathcal{I}_{\Delta_Z} \boxtimes \mathcal{I}_{\Delta_U}|_{T \times T} \rightarrow \mathcal{O}_{Z \times Z} \boxtimes \mathcal{I}_{\Delta_U}|_{T \times T}$$

where  $\iota_Z : \mathcal{I}_{\Delta_Z} \rightarrow \mathcal{O}_{Z \times Z}$  is the canonical injection. This is injective, as shown in Lemma A.2 hence (27) is also left exact.

Now, the restriction of (25) to  $Z_T \times Z_T$  is the exact sequence

$$(29) \quad \mathcal{I}_{\Delta_Z} \boxtimes \mathcal{O}_{T \times T} \xrightarrow{\phi} \mathcal{I}_{\Delta_{Z_U}}|_{T \times T} \longrightarrow \mathcal{O}_{\Delta_Z} \boxtimes \mathcal{I}_{\Delta_U}|_{T \times T} \rightarrow 0.$$

where  $\phi = j_{Z_U}|_{Z_T \times Z_T}$ . Furthermore, by construction there is a commutative diagram

$$\begin{array}{ccccccc} & & \mathcal{I}_{\Delta_Z} \boxtimes \mathcal{O}_{T \times T} & \xrightarrow{\mathbf{1}} & \mathcal{I}_{\Delta_Z} \boxtimes \mathcal{O}_{T \times T} & & \\ & & \downarrow \left( \begin{array}{c} \mathbf{1} \\ 0 \end{array} \right) & & \downarrow \phi & & \\ 0 & \longrightarrow & \mathcal{I}_{\Delta_Z} \boxtimes (\mathcal{I}_{\Delta_U}|_{T \times T}) & \longrightarrow & \mathcal{I}_{\Delta_Z} \boxplus (\mathcal{I}_{\Delta_U}|_{T \times T}) & \longrightarrow & \mathcal{I}_{\Delta_{Z_U}}|_{T \times T} \longrightarrow 0. \end{array}$$

where the bottom row is the restriction of (26) to  $Z_T \times Z_T$ . Since the latter is exact, using equation (28), the snake lemma yields a long exact sequence

$$0 \rightarrow \text{Ker}(\phi) \rightarrow \mathcal{I}_{\Delta_Z} \boxtimes (\mathcal{I}_{\Delta_U}|_{T \times T}) \xrightarrow{\iota_Z \boxtimes \mathbf{1}} \mathcal{O}_{Z \times Z} \boxtimes (\mathcal{I}_{\Delta_U}|_{T \times T}) \rightarrow \cdots.$$

As noted above,  $\iota_Z \boxtimes \mathbf{1}$  is injective, hence (29) is also exact on the left. q.e.d.

Now recall that the domain of the Atiyah class in diagram (21) is the complex

$$\mathcal{O}_{\Delta_Z} \boxtimes \Delta_{T^*}(\mathcal{I}_T / \mathcal{I}_T^2) \xrightarrow{\nu_{Z_T}} \mathcal{I}_{\Delta_{Z_U}}|_{Z_T \times Z_T} \xrightarrow{\iota_{Z_T} \circ \xi_{Z_T}} \mathcal{O}_{Z \times Z} \boxtimes \mathcal{O}_{T \times T}.$$

Let

$$(30) \quad 0 \rightarrow \mathcal{I}_{\Delta_Z} \boxtimes \mathcal{O}_{T \times T} \xrightarrow{\epsilon_{Z_T}} \mathcal{I}_{\Delta_{Z_U}}|_{Z_T \times Z_T} \xrightarrow{\eta_{Z_T}} \mathcal{O}_{\Delta_Z} \boxtimes \mathcal{I}_{\Delta_U}|_{T \times T} \rightarrow 0$$

denote the restriction of (25) to  $Z_T \times Z_T$  and let

$$(31) \quad 0 \rightarrow \mathcal{I}_{\Delta_Z} \xrightarrow{\iota_Z} \mathcal{O}_{Z \times Z} \xrightarrow{f_Z} \mathcal{O}_{\Delta_Z} \rightarrow 0.$$

be the canonical exact sequence of the diagonal in  $Z \times Z$ .

As claimed above equation (24), the next result proves that the domain of the relative Atiyah class is indeed quasi-isomorphic to the complex (24).

**Lemma 2.9.** *There is a commutative diagram*  
 (32)

$$\begin{array}{ccccc}
 & & 0 & & 0 \\
 & & \downarrow & & \downarrow \\
 & & \mathcal{I}_{\Delta_Z} \boxtimes \mathcal{O}_{T \times T} & \xrightarrow{\mathbf{1}} & \mathcal{I}_{\Delta_Z} \boxtimes \mathcal{O}_{T \times T} \\
 & & \downarrow \epsilon_{Z_T} & & \downarrow i_Z \boxtimes \mathbf{1} \\
 \mathcal{O}_{\Delta_Z} \boxtimes \Delta_{T^*} \mathcal{I}_T / \mathcal{I}_T^2 & \xrightarrow{\nu_{Z_T}} & \mathcal{I}_{\Delta_{Z_U}}|_{Z_T \times Z_T} & \xrightarrow{\iota_{Z_T} \circ \xi_{Z_T}} & \mathcal{O}_{Z \times Z} \boxtimes \mathcal{O}_{T \times T} \\
 \downarrow \mathbf{1} & & \downarrow \eta_{Z_T} & & \downarrow f_Z \boxtimes \mathbf{1} \\
 \mathcal{O}_{\Delta_Z} \boxtimes \Delta_{T^*} (\mathcal{I}_T / \mathcal{I}_T^2) & \xrightarrow{\mathbf{1} \boxtimes \nu_T} & \mathcal{O}_{\Delta_Z} \boxtimes \mathcal{I}_{\Delta_U}|_{T \times T} & \xrightarrow{\mathbf{1} \boxtimes (\iota_T \circ \xi_T)} & \mathcal{O}_{\Delta_Z} \boxtimes \mathcal{O}_{T \times T} \\
 & & \downarrow & & \downarrow \\
 & & 0 & & 0
 \end{array}$$

with exact columns.

*Proof.* The middle column is the exact sequence (30). The exactness of the right column follows from the exactness of (31) using Lemma A.2. The lower right square is commutative since all the arrows in it are natural restrictions.

In order to prove that the lower left square is commutative, note that the maps  $\nu_{Z_T}, \nu_T$  are determined by the commutative diagrams (17) and (23) respectively. These are displayed again below for convenience:

$$(33) \quad \begin{array}{ccc}
 \mathcal{O}_{Z \times Z} \boxtimes (\mathcal{O}_T \boxtimes \mathcal{I}_T) & \xrightarrow{k_{Z_T}} & \mathcal{I}_{\Delta_{Z_U}}|_{Z_T \times Z_U} \\
 \downarrow a_{Z_T} & & \downarrow b_{Z_T} \\
 \mathcal{O}_{\Delta_Z} \boxtimes \Delta_{T^*} (\mathcal{I}_T / \mathcal{I}_T^2) & \xrightarrow{\nu_{Z_T}} & \mathcal{I}_{\Delta_{Z_U}}|_{Z_T \times Z_T}
 \end{array}$$

$$\begin{array}{ccc}
 \mathcal{O}_T \boxtimes \mathcal{I}_T & \xrightarrow{k_T} & \mathcal{I}_{\Delta_U}|_{T \times U} \\
 \downarrow a_T & & \downarrow b_T \\
 \Delta_{T^*} (\mathcal{I}_T / \mathcal{I}_T^2) & \xrightarrow{\nu_T} & \mathcal{I}_{\Delta_U}|_{T \times T}
 \end{array}$$

In each case the vertical arrows are canonical projections while the upper horizontal arrow is a natural injection. Obviously, the second' diagram in (33) yields

a commutative diagram of  $\mathcal{O}_{Z_T \times Z_U}$ -modules

$$(34) \quad \begin{array}{ccc} \mathcal{O}_{\Delta_Z} \boxtimes (\mathcal{O}_T \boxtimes \mathcal{I}_T) & \xrightarrow{\mathbf{1} \boxtimes k_T} & \mathcal{O}_{\Delta_Z} \boxtimes (\mathcal{I}_{\Delta_U}|_{T \times U}) \\ \downarrow \mathbf{1} \boxtimes a_T & & \downarrow \mathbf{1} \boxtimes b_T \\ \mathcal{O}_{\Delta_Z} \boxtimes \Delta_{T^*}(\mathcal{I}_T / \mathcal{I}_T^2) & \xrightarrow{\mathbf{1} \boxtimes v_T} & \mathcal{O}_{\Delta_Z} \boxtimes (\mathcal{I}_{\Delta_U}|_{T \times T}) \end{array}$$

Then the from Diagram (34) and the left diagram in (33) one obtains a diagram of the form

$$(35) \quad \begin{array}{ccccc} & & \mathcal{O}_{Z \times Z} \boxtimes (\mathcal{O}_T \boxtimes \mathcal{I}_T) & \xrightarrow{k_{Z_T}} & \mathcal{I}_{\Delta_{Z_U}}|_{Z_T \times Z_U} \\ & \nearrow f_Z \boxtimes \mathbf{1} & \downarrow a_{Z_T} & & \downarrow b_{Z_T} \\ & & \mathcal{O}_{\Delta_Z} \boxtimes \Delta_{T^*}(\mathcal{I}_T / \mathcal{I}_T^2) & \xrightarrow{v_{Z_T}} & \mathcal{I}_{\Delta_{Z_U}}|_{Z_T \times Z_T} \\ & & & & \searrow \gamma_{Z_U}|_{Z_T \times Z_U} \\ \mathbf{1} & \searrow & \mathcal{O}_{\Delta_Z} \boxtimes (\mathcal{O}_T \boxtimes \mathcal{I}_T) & \xrightarrow{\mathbf{1} \boxtimes k_T} & \mathcal{O}_{\Delta_Z} \boxtimes (\mathcal{I}_{\Delta_U}|_{T \times U}) \\ & & \downarrow \mathbf{1} \boxtimes a_T & & \downarrow \mathbf{1} \boxtimes b_T \\ & & \mathcal{O}_{\Delta_Z} \boxtimes \Delta_{T^*}(\mathcal{I}_T / \mathcal{I}_T^2) & \xrightarrow{\mathbf{1} \boxtimes v_T} & \mathcal{O}_{\Delta_Z} \boxtimes (\mathcal{I}_{\Delta_U}|_{T \times T}) \\ & & & & \nearrow \eta_{Z_T} \end{array}$$

where  $\gamma_U : \mathcal{I}_{\Delta_{Z_U}} \rightarrow \mathcal{O}_{\Delta_Z} \boxtimes \mathcal{I}_{\Delta_U}$  is the projection in (25). Since  $k_{Z_T}, k_T$  are natural injections, the following relations hold

$$\gamma_{Z_U}|_{Z_T \times Z_U} \circ k_{Z_T} = (\mathbf{1} \boxtimes k_T) \circ (f_Z \boxtimes \mathbf{1})$$

Moreover, since  $\eta_{Z_T} = \gamma_U|_{Z_T \times Z_T}$  and  $b_{Z_T}, b_T$  are natural projections, one also has

$$(\mathbf{1} \boxtimes b_T) \circ \gamma_U|_{Z_U \times Z_T} = \eta_{Z_T} \circ b_{Z_T}.$$

Since  $a_T, a_{Z_T}$  are surjections, and two straight squares in the diagram (35) are commutative, this implies

$$\eta_{Z_T} \circ v_{Z_T} = (\mathbf{1} \boxtimes v_T).$$

Finally, in order to prove that the upper right square is commutative, first note that Lemma A.4 yields an exact sequence

$$0 \rightarrow \mathcal{I}_{\Delta_Z} \boxtimes \mathcal{O}_{T \times T} \xrightarrow{\delta_{Z_T}} \mathcal{I}_{\Delta_{Z_T}} \xrightarrow{\gamma_{Z_T}} \mathcal{O}_{\Delta_Z} \boxtimes \mathcal{I}_{\Delta_T} \rightarrow 0.$$



Then note the diagram

$$(36) \quad \begin{array}{ccc} \mathcal{I}_{\Delta_Z} \boxtimes \mathcal{O}_{T \times T} & \xrightarrow{\mathbf{1}} & \mathcal{I}_{\Delta_Z} \boxtimes \mathcal{O}_{T \times T} \\ \epsilon_{Z_T} \downarrow & & \delta_{Z_T} \downarrow \\ \mathcal{I}_{\Delta_{Z_U}}|_{Z_T \times Z_T} & \xrightarrow{\xi_{Z_T}} & \mathcal{I}_{\Delta_{Z_T}} \\ & \searrow \iota_{Z_T} \circ \tilde{\xi}_{Z_T} & \searrow \iota_{Z_T} \\ & & \mathcal{O}_{Z \times Z} \boxtimes \mathcal{O}_{T \times T} \end{array} \quad \begin{array}{l} \nearrow \iota_Z \boxtimes \mathbf{1} \\ \nearrow \iota_{Z_T} \end{array}$$

Since all the maps involved are natural injections or projections, it is straightforward to check that the square and the triangles are commutative. This implies that the outer polygon is also commutative. q.e.d.

Finally, the next lemma provides a relation between the target of the relative Atiyah class  $\alpha_{Z_T/Z}$  and the target of  $\alpha_T$ .

**Lemma 2.10.** *There is a further commutative diagram*

$$(37) \quad \begin{array}{ccc} \mathcal{O}_{\Delta_Z} \boxtimes \Delta_{T^*}(\mathcal{I}_T / \mathcal{I}_T^2) & \xrightarrow{v_{Z_T}} & \mathcal{I}_{\Delta_{Z_U}}|_{Z_T \times Z_T} \\ \downarrow \mathbf{1} & & \downarrow r_{Z_T} \\ \mathcal{O}_{\Delta_Z} \boxtimes \Delta_{T^*}(\mathcal{I}_T / \mathcal{I}_T^2) & \xrightarrow{q_{Z_T} \circ v_{Z_T}} & \mathcal{O}_{\Delta_Z} \boxtimes (\mathcal{I}_{\Delta_U} / I_{\Delta_U}^2)|_{T \times T} \\ \downarrow \mathbf{1} & & \downarrow \mathbf{1} \boxtimes r_T \\ \mathcal{O}_{\Delta_Z} \boxtimes \Delta_{T^*}(\mathcal{I}_T / \mathcal{I}_T^2) & \xrightarrow{\mathbf{1} \boxtimes v_T} & \mathcal{O}_{\Delta_Z} \boxtimes \mathcal{I}_{\Delta_U}|_{T \times T} \\ \downarrow \mathbf{1} & & \downarrow \mathbf{1} \boxtimes r_T \\ \mathcal{O}_{\Delta_Z} \boxtimes \Delta_{T^*}(\mathcal{I}_T / \mathcal{I}_T^2) & \xrightarrow{\mathbf{1} \boxtimes s_T} & \mathcal{O}_{\Delta_Z} \boxtimes (\mathcal{I}_{\Delta_U} / I_{\Delta_U}^2)|_{T \times T} \end{array} \quad \begin{array}{l} \nearrow \eta_{Z_T} \\ \nearrow \mathbf{1} \end{array}$$

where  $s_T = r_T \circ v_T$ , and  $\eta_{Z_T}$  is the third map from the left in the exact sequence (30). In particular, the following relation holds:

$$(38) \quad q_{Z_T} \circ v_{Z_T} = \mathbf{1} \boxtimes (r_T \circ v_T).$$

*Proof.* The relation  $\eta_{Z_T} \circ v_{Z_T} = \mathbf{1} \boxtimes v_T$  has already been proven in Lemma 2.8. The rest follows from the fact that the  $r_{Z_T}$ ,  $r_T$  and  $\eta_{Z_T}$  are natural projections. q.e.d.

**2.3. Proof of Theorem 2.5.** To summarize, Lemma 2.6 shows that the relative Atiyah class  $\alpha_{Z_T/Z}$  is determined by the commutative diagram

$$\begin{array}{ccccc} \mathcal{O}_{\Delta_Z} \boxtimes \Delta_{T^*}(\mathcal{I}_T / \mathcal{I}_T^2) & \xrightarrow{v_{Z_T}} & \mathcal{I}_{\Delta_{Z_U}}|_{Z_T \times Z_T} & \xrightarrow{\iota_{Z_T} \circ \xi_{Z_T}} & \mathcal{O}_{Z_T \times Z_T} \\ \downarrow \mathbf{1} & & \downarrow q_{Z_T} & & \\ \mathcal{O}_{\Delta_Z} \boxtimes \Delta_{T^*}(\mathcal{I}_T / \mathcal{I}_T^2) & \xrightarrow{q_{Z_T} \circ v_{Z_T}} & \mathcal{O}_{\Delta_Z} \boxtimes (\mathcal{I}_{\Delta_U} / \mathcal{I}_{\Delta_U}^2)|_{Z_T \times Z_T} & & \end{array}$$

Let  $\mathcal{A}_{Z_T}$  and  $\mathcal{C}_{Z_T}$  denote the top and the bottom row respectively. At the same time the Atiyah class of  $T$  is determined by the diagram

$$\begin{array}{ccccc} \Delta_{T^*}(\mathcal{I}_T / \mathcal{I}_T^2) & \xrightarrow{v_T} & \mathcal{I}_{\Delta_U}|_{T \times T} & \xrightarrow{\iota_T \circ \xi_T} & \mathcal{O}_{T \times T} \\ \downarrow \mathbf{1} & & \downarrow r_T & & \\ \Delta_{T^*}(\mathcal{I}_T / \mathcal{I}_T^2) & \xrightarrow{r_T \circ v_T} & \mathcal{I}_{\Delta_U} / \mathcal{I}_{\Delta_U}^2|_{T \times T} & & \end{array}$$

Again, let  $\mathcal{A}_T$  and  $\mathcal{C}_T$  denote the top and the bottom row respectively.

Then Lemmas 2.8 and 2.10 show that there is a commutative diagram of complexes

$$\begin{array}{ccc} \mathcal{A}_{Z_T} & \xrightarrow{\sim} & \mathcal{O}_{\Delta_Z} \boxtimes \mathcal{A}_T \\ \downarrow \alpha_{Z_T/Z} & & \downarrow \mathbf{1} \boxtimes \alpha_T \\ \mathcal{C}_{Z_T} & \xrightarrow{\sim} & \mathcal{O}_{\Delta_Z} \boxtimes \mathcal{C}_T \end{array}$$

where the top horizontal arrow is a quasi-isomorphism and the bottom row is a canonical isomorphism. This proves the claim in Theorem 2.5. q.e.d.

### 3. Obstruction theory for sheaves on local Calabi-Yau varieties

This section serves as an interesting application of Theorem 2.5, providing the foundations for a sheaf counting theory of local Calabi-Yau fourfolds. To begin with, we start by discussing the particular moduli theory we are interested in this article. Let  $Y$  be a smooth quasi-projective variety and let  $X$  be the total space of the canonical bundle  $K_Y$ . Let  $i : Y \rightarrow X$  denote the zero section. Let  $\mathcal{M}$  be a quasi-projective fine moduli scheme of stable sheaves,  $\mathcal{F}$ , on  $X$  with proper support. Suppose all sheaves  $\mathcal{F}$  parameterized by  $\mathcal{M}$  are scheme-theoretically supported on  $Y$ . Then the goal of this section is to prove that the obstruction theory of  $\mathcal{M}$  as a moduli space of sheaves on  $X$  coincides with the obstruction theory of  $\mathcal{M}$  as a moduli space of sheaves on  $Y$ . The precise statement is Proposition 3.6, which relies on Proposition 3.2 and Theorem 2.5. The obstruction theory will be defined

as in the work of Behrend and Fantechi [2] and the proof will use the formalism of truncated Atiyah classes, which was discussed in the previous section.

In order to facilitate the exposition, the truncated Atiyah class will be referred to simply as the Atiyah class below since no other variant will be used throughout the paper. To fix notation, let  $\mathcal{Y} = Y \times \mathcal{M}$ ,  $\mathcal{X} = X \times \mathcal{M}$ , and let us denote the canonical embedding  $i_{\mathcal{Y}} := i \times 1_{\mathcal{M}} : \mathcal{Y} \rightarrow \mathcal{X}$ . Let  $\mathbb{F}$  denote the universal sheaf on  $\mathcal{X}$ <sup>1</sup>. Under the current assumptions,  $\mathbb{F}$  is the extension by zero,  $\mathbb{F} = i_{\mathcal{Y}*}\mathbb{G}$ , of a sheaf  $\mathbb{G}$  on  $\mathcal{Y}$ . The latter is also a universal sheaf on  $\mathcal{Y}$ . Following the framework of Section 2.2, as shown in [27], the relative Atiyah class  $\alpha_{\mathcal{X}/X}$  of  $\mathcal{X}$ , is given by the morphism in the derived category

$$(39) \quad \alpha_{\mathcal{X}/X} : \mathcal{O}_{\Delta_{\mathcal{X}}} \rightarrow \Delta_{\mathcal{X}*}(\pi_{\mathcal{M}}^* \mathbb{L}_{\mathcal{M}}^{\bullet}[1]),$$

which determines a natural transformation between the Fourier-Mukai functors  $\mathcal{D}^b(\mathcal{X}) \rightarrow \mathcal{D}^b(\mathcal{X})$  with kernels  $\mathcal{O}_{\Delta_{\mathcal{X}}}$  and  $\Delta_{\mathcal{X}*}(\pi_{\mathcal{M}}^* \mathbb{L}_{\mathcal{M}}^{\bullet}[1])$  respectively. Therefore, given a sheaf  $\mathbb{F}$  on  $\mathcal{X}$ , the map (39) determines, by Fourier-Mukai transform, a morphism

$$\alpha_{\mathcal{X}/X}(\mathbb{F}) : \mathbb{F} \rightarrow \mathbb{F} \otimes^L \pi_{\mathcal{M}}^* \mathbb{L}_{\mathcal{M}}^{\bullet}[1].$$

This is the universal relative Atiyah class of  $\mathbb{F}$ .

**3.1. Comparison of relative Atiyah classes.** Now let  $\pi_X : \mathcal{X} \rightarrow X$ ,  $\pi_{\mathcal{M}} : \mathcal{X} \rightarrow \mathcal{M}$ ,  $\rho_Y : \mathcal{Y} \rightarrow Y$  and  $\rho_{\mathcal{M}} : \mathcal{Y} \rightarrow \mathcal{M}$  denote the canonical projections.

**Notation.** For any scheme  $W$  over  $\mathbb{C}$  the bounded derived category of  $W$  will be denoted by  $\mathcal{D}^b(W)$ . For any two objects  $C_1, C_2$  of  $\mathcal{D}^b(W)$  let  $\mathbf{RHom}_{\mathcal{D}^b(W)}(C_1, C_2)$ , and  $\mathbf{R}\mathcal{H}om_{\mathcal{D}^b(W)}(C_1, C_2)$  denote global and local derived Hom functors respectively. Moreover, for a morphism of schemes  $f : W \rightarrow V$  let  $\mathbf{R}\mathcal{H}om_f(C_1, C_2)$  denote the derived pushforward  $\mathbf{R}f_* \mathbf{R}\mathcal{H}om_{\mathcal{D}^b(W)}(C_1, C_2)$ .

The main goal of this section is to compare the relative Atiyah classes of  $\mathbb{F}$  and  $\mathbb{G}$ ,

$$\begin{aligned} \alpha_{\mathcal{X}/X}(\mathbb{F}) &\in \mathbf{RHom}_{\mathcal{D}^b(\mathcal{X})}(\mathbb{F}, \mathbb{F} \otimes^L \pi_{\mathcal{M}}^* \mathbb{L}_{\mathcal{M}}^{\bullet}[1]), \\ \alpha_{\mathcal{Y}/Y}(\mathbb{G}) &\in \mathbf{RHom}_{\mathcal{D}^b(\mathcal{Y})}(\mathbb{G}, \mathbb{G} \otimes^L \rho_{\mathcal{M}}^* \mathbb{L}_{\mathcal{M}}^{\bullet}[1]), \end{aligned}$$

<sup>1</sup>The moduli space  $\mathcal{M}$  is in fact the coarse moduli space of an Artin moduli stack of  $\mathfrak{M}$  of stable sheaves. The latter is a  $\mathbb{C}^*$ -gerbe over  $\mathcal{M}$  since all stable objects have  $\mathbb{C}^*$ -stabilizers, and hence a universal sheaf  $\mathbb{F}$  might not exist on  $X \times \mathcal{M}$  globally. As explained in [33], below Theorem 3.30, this issue can be ignored, as such a sheaf  $\mathbb{F}_i$  exists locally on open subsets  $X \times \mathcal{U}_i \subset X \times \mathcal{M}$  and different local choices for  $\mathbb{F}_i$  differ by twisting by line bundles  $\mathcal{L}_i$  pulled back from  $\mathcal{U}_i$ . However the direct image  $\mathbf{R}\mathcal{H}om_{\pi_{\mathcal{U}_i}}(\mathbb{F}_i, \mathbb{F}_i)$  used in our construction remains invariant under twisting by  $\mathcal{L}_i$ , and hence there is a unique global object  $\mathbf{R}\mathcal{H}om_{\pi_{\mathcal{M}}}(\mathbb{F}, \mathbb{F})$  on  $X \times \mathcal{M}$ , independent of any choices.

defined in Section 2.2. First, note the following relations in the derived category of  $\mathcal{X}$ .

**Lemma 3.1.** *Let  $Q$  be an object of  $\mathcal{D}^b(\mathcal{M})$ . Then there is a natural isomorphism*

$$(40) \quad \begin{aligned} \mathbf{R}\mathcal{H}om_{\mathcal{D}^b(\mathcal{X})}(\mathbb{F}, \mathbb{F} \otimes^L \pi_{\mathcal{M}}^* Q)[1] &\cong i_{Y*} \mathbf{R}\mathcal{H}om_{\mathcal{D}^b(\mathcal{Y})}(\mathbb{G}, \mathbb{G} \otimes^L \rho_{\mathcal{M}}^* Q)[1] \oplus \\ &i_{Y*} \mathbf{R}\mathcal{H}om_{\mathcal{D}^b(\mathcal{Y})}(\mathbb{G} \otimes \rho_Y^* K_Y^{-1}, \mathbb{G} \otimes^L \rho_{\mathcal{M}}^* Q) \end{aligned}$$

*Proof.* The projection formula [24, Prop. 5.6, Ch. II.5] yields an isomorphism

$$\mathbb{F} \otimes^L \pi_{\mathcal{M}}^* Q \cong i_{Y*}(\mathbb{G} \otimes^L Li_Y^* \pi_{\mathcal{M}}^* Q),$$

where  $i_{Y*} \mathcal{O}_Y \cong \pi_X^* i_* \mathcal{O}_Y$ , hence Lemma A.2 implies that

$$(41) \quad Li_Y^* \pi_{\mathcal{M}}^* Q \cong i_Y^* \pi_{\mathcal{M}}^* Q \cong \rho_{\mathcal{M}}^* Q$$

One is then left with an isomorphism

$$\mathbb{F} \otimes^L \pi_{\mathcal{M}}^* Q \cong i_{Y*}(\mathbb{G} \otimes^L \rho_{\mathcal{M}}^* Q),$$

and the claim follows from Lemma A.1.

q.e.d.

In particular, setting  $Q = \mathbb{L}_{\mathcal{M}}^{\bullet}$  in Lemma 3.1, and applying the derived functor  $\mathbf{R}\Gamma_{\mathcal{D}^b(\mathcal{X})}$  to both sides of equation (40), one obtains a decomposition

$$(42) \quad \begin{aligned} \mathbf{R}\mathcal{H}om_{\mathcal{D}^b(\mathcal{X})}(\mathbb{F}, \mathbb{F} \otimes^L \pi_{\mathcal{M}}^* \mathbb{L}_{\mathcal{M}}^{\bullet})[1] &\cong \\ &\mathbf{R}\mathcal{H}om_{\mathcal{D}^b(\mathcal{Y})}(\mathbb{G}, \mathbb{G} \otimes^L \rho_{\mathcal{M}}^* \mathbb{L}_{\mathcal{M}}^{\bullet})[1] \oplus \\ &\mathbf{R}\mathcal{H}om_{\mathcal{D}^b(\mathcal{Y})}(\mathbb{G} \otimes \mathcal{K}^{-1}, \mathbb{G} \otimes^L \rho_{\mathcal{M}}^* \mathbb{L}_{\mathcal{M}}^{\bullet}), \end{aligned}$$

where  $\mathcal{K} = \rho_Y^* K_Y$ . The first main goal of this section is to prove:

**Proposition 3.2.** *The components of  $\alpha_{\mathcal{X}/\mathcal{X}}(\mathbb{F})$  with respect to the direct sum decomposition (42) are*

$$(43) \quad (\alpha_{Y/Y}(\mathbb{G}), 0).$$

**Remark 3.3.** *A similar result has been proven by Kuznetsov and Markusevich in [28, Theorem 3.2]. For completeness we will recall their theorem in here.*

**Theorem.** [28, Theorem 3.2] *Let  $i : Y \rightarrow M$  be a locally complete intersection.*

- (i) *For any  $\mathcal{F} \in \mathcal{D}^b(\text{Coh}(Y))$ , the linkage class  $\epsilon_{\mathcal{F}} \in \text{Ext}^2(\mathcal{F}, \mathcal{F} \otimes \mathcal{N}_{Y/M}^{\vee})$  is the product of the Atiyah class  $\text{At}_{\mathcal{F}} \in \text{Ext}^1(\mathcal{F}, \mathcal{F} \otimes \Omega_Y)$  with  $v_{Y/M} \in \text{Ext}^1(\Omega_Y, \mathcal{N}_{Y/M}^{\vee})$ . In other words  $\epsilon_{\mathcal{F}} = \left(1_{\mathcal{F} \otimes v_{Y/M}}\right) \circ \text{At}_{\mathcal{F}}$ .*

(ii) For any  $\mathcal{G} \in \mathcal{D}^b(\mathrm{Coh}(M))$  we have

$$\mathrm{At}_{Li^*\mathcal{G}} \cong \rho_*((\mathrm{At}_{\mathcal{G}})|_Y),$$

where  $\rho_* : \mathrm{Ext}^1(Li^*\mathcal{G}, Li^*\mathcal{G} \otimes \Omega_M|_Y) \rightarrow \mathrm{Ext}^1(Li^*\mathcal{G}, Li^*\mathcal{G} \otimes \Omega_Y)$  is the pushout via  $\rho : \Omega_M|_Y \rightarrow \Omega_Y$ .

(iii) For any  $\mathcal{F} \in \mathcal{D}^b(\mathrm{Cor}(Y))$  the image of the Atiyah class  $\mathrm{At}_{i_*\mathcal{F}} \in \mathrm{Ext}^1(i_*\mathcal{F}, i_*\mathcal{F} \otimes \Omega_M)$  in  $\mathbf{R}\mathrm{Hom}^1(Li^*i_*\mathcal{F}, i_*\mathcal{F} \otimes \Omega_M) = H^0(M, i_*(\mathcal{F}^\vee) \otimes \mathcal{F} \otimes \mathcal{N}_{Y/M} \otimes \Omega_M|_Y)$  equals  $1_{\mathcal{F}} \otimes k$  where  $k = k_{Y/M} : \mathcal{N}_{Y/M}^\vee \rightarrow \Omega_M|_Y$ .

Note that the above theorem is a point-wise statement which uses the classical Atiyah classes instead of universal Atiyah class over the moduli space. In this section we aim at providing an analogous proof of part (iii), with the relative truncated Atiyah class  $\alpha_{\mathcal{X}/\mathcal{Y}}$  in place of the classical Atiyah class. The authors believe that lifting the proof of Proposition 3.2 to the one using the full un-truncated cotangent complex, is possible although, it might be a big step involving difficult technicalities.

As in [28, Theorem 3.2], the proof will use the following commutative diagram, where both squares are cartesian:

$$(44) \quad \begin{array}{ccccc} \mathcal{Y} & \xrightarrow{1} & \mathcal{Y} & \xrightarrow{i_{\mathcal{Y}}} & \mathcal{X} \\ \downarrow \Delta_{\mathcal{Y}} & & \downarrow \Gamma_{\mathcal{Y}} & & \downarrow \Delta_{\mathcal{X}} \\ \mathcal{Y} \times \mathcal{Y} & \xrightarrow{i_{\mathcal{Y}} \times 1} & \mathcal{X} \times \mathcal{Y} & \xrightarrow{1 \times i_{\mathcal{Y}}} & \mathcal{X} \times \mathcal{X}. \end{array}$$

As shown in detail below, the successive application of the projection formula for the maps  $1 \times i_{\mathcal{Y}}$  and  $i_{\mathcal{Y}} \times 1$  will yield Equation (43). A central element in the proof of Proposition 3.2 will be Theorem 2.5 proven in Section 2.2, which provides an explicit formula for the universal relative Atiyah class of  $\mathcal{X}$  in terms of the universal Atiyah class of  $\mathcal{M}$ . To begin with, note the following restriction Lemmas.

**Lemma 3.4.** (i) Let  $Q$  be an object of  $\mathcal{D}^b(\mathcal{M})$ . Then the natural morphism  $L(1 \times i_{\mathcal{Y}})^* \Delta_{\mathcal{X}} \pi_{\mathcal{M}}^* Q \rightarrow (1 \times i_{\mathcal{Y}})^* \Delta_{\mathcal{X}} \pi_{\mathcal{M}}^* Q$  is an isomorphism in  $\mathcal{D}^b(\mathcal{X} \times \mathcal{Y})$ . Moreover there is a further isomorphism

$$(45) \quad (1 \times i_{\mathcal{Y}})^* \Delta_{\mathcal{X}} \pi_{\mathcal{M}}^* Q \cong \Gamma_{\mathcal{Y}} \rho_{\mathcal{M}}^* Q.$$

(ii) Let  $r_{\mathcal{Y}} : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Y}$  and  $p_{i,\mathcal{Y}} : \mathcal{Y} \times \mathcal{Y} \rightarrow \mathcal{Y}$ ,  $1 \leq i \leq 2$ , denote the canonical projections. There is a natural isomorphism

$$(46) \quad L(i_{\mathcal{Y}} \times 1)^* r_{\mathcal{Y}}^* \mathbb{G} \cong p_{2,\mathcal{Y}}^* \mathbb{G}.$$

*Proof.* (i). Consider Diagram (47). Recall that  $\mathcal{X}$  is the total space of the line bundle  $\mathcal{K} = \rho_Y^* K_Y$  on  $\mathcal{Y}$ . Let  $q_{\mathcal{Y}} : \mathcal{X} \rightarrow \mathcal{Y}$  denote the natural projection, and let  $\zeta \in H^0(\mathcal{X}, q_{\mathcal{Y}}^* \mathcal{K})$  denote the tautological section. Then  $\mathcal{X} \times \mathcal{X}$  is the total space

of the line bundle  $r_{\mathcal{Y}}^* \mathcal{K}$  on  $\mathcal{X} \times \mathcal{Y}$ , such that the canonical projection is  $1 \times q_{\mathcal{Y}} : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{Y}$ .

$$(47) \quad \begin{array}{ccc} \mathcal{X} \times \mathcal{X} & \xrightarrow{1 \times q_{\mathcal{Y}}} & \mathcal{X} \times \mathcal{Y} \\ \downarrow \pi_{1, \mathcal{X}} & & \downarrow r_{\mathcal{Y}} \\ \mathcal{X} & \xrightarrow{q_{\mathcal{Y}}} & \mathcal{Y} \\ \downarrow \pi_{\mathcal{X}} & \swarrow \pi_{\mathcal{M}} \quad \searrow \rho_{\mathcal{M}} & \downarrow \rho_{\mathcal{Y}} \\ & \mathcal{M} & \\ \downarrow & & \downarrow \\ X & \xrightarrow{q} & Y \end{array}$$

Moreover, there is a canonical isomorphism  $(1 \times q_{\mathcal{Y}})^* r_{\mathcal{Y}}^* \mathcal{K} \cong \pi_{1, \mathcal{X}}^* q_{\mathcal{Y}}^* \mathcal{K}$  which identifies the tautological section of the former with  $\pi_{1, \mathcal{X}}^* \zeta$ . Finally, the image of the closed embedding  $1 \times i_{\mathcal{Y}} : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{X} \times \mathcal{X}$  coincides scheme theoretically with the zero locus of the tautological section  $\pi_{1, \mathcal{X}}^* \zeta$ . This implies that the sheaf  $(1 \times i_{\mathcal{Y}})_* \mathcal{O}_{\mathcal{X} \times \mathcal{Y}}$  has a two term locally free resolution

$$\pi_{1, \mathcal{X}}^* q_{\mathcal{Y}}^* \mathcal{K}^{-1} \xrightarrow{\pi_{1, \mathcal{X}}^* \zeta} \mathcal{O}_{\mathcal{X} \times \mathcal{X}}$$

Taking the tensor product with  $\Delta_{\mathcal{X}^*} \pi_{\mathcal{M}}^* Q$ , one obtains the two term complex

$$\Delta_{\mathcal{X}^*} (q_{\mathcal{Y}}^* \mathcal{K}^{-1} \otimes \pi_{\mathcal{M}}^* Q) \xrightarrow{\Delta_{\mathcal{X}^*} (\zeta \otimes \mathbf{1})} \Delta_{\mathcal{X}^*} (\pi_{\mathcal{M}}^* Q)$$

However, the morphism  $\zeta \otimes \mathbf{1} : q_{\mathcal{Y}}^* \mathcal{K}^{-1} \otimes \pi_{\mathcal{M}}^* Q \rightarrow \pi_{\mathcal{M}}^* Q$  is injective ( as a morphism in the category complexes,  $K(\mathcal{X})$ ) since it has an injective restriction to each fiber  $X_m$ ,  $m \in \mathcal{M}$ . This implies that all higher local Tor sheaves, resulting from derived pullback of  $\Delta_{\mathcal{X}^*} \pi_{\mathcal{M}}^* Q$  to  $\mathcal{X} \times \mathcal{Y}$  vanish, and the natural projection  $L(1 \times i_{\mathcal{Y}})^* \Delta_{\mathcal{X}^*} \pi_{\mathcal{M}}^* Q \rightarrow (1 \times i_{\mathcal{Y}})^* \Delta_{\mathcal{X}^*} \pi_{\mathcal{M}}^* Q$  is an isomorphism as claimed.

In order to prove (45), note the canonical isomorphism

$$\pi_{\mathcal{M}}^* Q \cong \Delta_{\mathcal{X}}^* \Delta_{\mathcal{X}^*} \pi_{\mathcal{M}}^* Q.$$

This yields

$$\rho_{\mathcal{M}}^* Q \cong i_{\mathcal{Y}}^* \pi_{\mathcal{M}}^* Q \cong i_{\mathcal{Y}}^* \Delta_{\mathcal{X}}^* \Delta_{\mathcal{X}^*} \pi_{\mathcal{M}}^* Q \cong \Gamma_{\mathcal{Y}}^* (1 \times i_{\mathcal{Y}})^* \Delta_{\mathcal{X}^*} \pi_{\mathcal{M}}^* Q.$$

Hence

$$\Gamma_{\mathcal{Y}^*} \rho_{\mathcal{M}}^* Q \cong \Gamma_{\mathcal{Y}^*} \Gamma_{\mathcal{Y}}^* (1 \times i_{\mathcal{Y}})^* \Delta_{\mathcal{X}^*} \pi_{\mathcal{M}}^* Q.$$

The right hand side of the above relation is further isomorphic to

$$(1 \times i_{\mathcal{Y}})^* \Delta_{\mathcal{X}^*} \pi_{\mathcal{M}}^* Q / I_{\Gamma_{\mathcal{Y}}} \cdot (1 \times i_{\mathcal{Y}})^* \Delta_{\mathcal{X}^*} \pi_{\mathcal{M}}^* Q$$

where  $I_{\Gamma_{\mathcal{Y}}} \subset \mathcal{O}_{\mathcal{X} \times \mathcal{Y}}$  is the ideal sheaf associated to the closed embedding  $\Gamma_{\mathcal{Y}}$ . However, since the right square in (44) is cartesian, the pull-back  $(1 \times i_{\mathcal{Y}})^* \Delta_{\mathcal{X}^*} \pi_{\mathcal{M}}^* Q$  is annihilated by  $I_{\Gamma_{\mathcal{Y}}}$ , hence one obtains an isomorphism

$$\Gamma_{\mathcal{Y}^*} \rho_{\mathcal{M}}^* Q \cong (1 \times i_{\mathcal{Y}})^* \Delta_{\mathcal{X}^*} \pi_{\mathcal{M}}^* Q.$$

(ii). As shown in Lemma A.2, the pull-back  $r_{\mathcal{Y}}^* \mathbf{G}$  is flat over  $\mathcal{X}$ . This implies the claim.

q.e.d.

**Lemma 3.5.** *Let*

$$\pi_{i, \mathcal{X}} : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}, \quad p_{i, \mathcal{Y}} : \mathcal{Y} \times \mathcal{Y} \rightarrow \mathcal{Y}, \quad 1 \leq i \leq 2,$$

and

$$r_{\mathcal{X}} : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{X}, \quad r_{\mathcal{Y}} : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Y}$$

denote the canonical projections. Let  $Q$  be an object of  $\mathcal{D}^b(\mathcal{M})$ . Then there are isomorphisms

$$(48) \quad p_{2, \mathcal{X}}^* i_{\mathcal{Y}^*} \mathbf{F} \otimes^L \Delta_{\mathcal{X}^*} (\pi_{\mathcal{M}}^* Q) \cong (1 \times i_{\mathcal{Y}})_* (r_{\mathcal{Y}}^* \mathbf{G} \otimes^L \Gamma_{\mathcal{Y}^*} (\rho_{\mathcal{M}}^* Q))$$

in  $\mathcal{D}^b(\mathcal{X} \times \mathcal{X})$ , respectively

$$(49) \quad r_{\mathcal{Y}}^* \mathbf{G} \otimes^L \Gamma_{\mathcal{Y}^*} (\rho_{\mathcal{M}}^* Q) \cong (i_{\mathcal{Y}} \times 1)_* (p_{2, \mathcal{Y}}^* \mathbf{G} \otimes^L \Delta_{\mathcal{Y}^*} (\rho_{\mathcal{M}}^* Q)).$$

in  $\mathcal{D}^b(\mathcal{X} \times \mathcal{Y})$ .

*Proof.* Note the commutative diagram

$$(50) \quad \begin{array}{ccc} \mathcal{X} \times \mathcal{Y} & \xrightarrow{1 \times i_{\mathcal{Y}}} & \mathcal{X} \times \mathcal{X} \\ \left. \begin{array}{c} \uparrow \\ \downarrow \end{array} \right\} r_{\mathcal{Y}} & \Gamma_{\mathcal{Y}} & \left. \begin{array}{c} \uparrow \\ \downarrow \end{array} \right\} p_{2, \mathcal{X}} \\ \mathcal{Y} & \xrightarrow{i_{\mathcal{Y}}} & \mathcal{X} \\ \left. \begin{array}{c} \searrow \\ \swarrow \end{array} \right\} \rho_{\mathcal{M}} & & \left. \begin{array}{c} \searrow \\ \swarrow \end{array} \right\} \pi_{\mathcal{M}} \\ & & \mathcal{M} \end{array}$$

where both squares are cartesian. Since  $\mathbf{F} = i_{\mathcal{Y}^*} \mathbf{G}$  and  $p_{2, \mathcal{X}}$  is flat, there is an isomorphism

$$p_{2, \mathcal{X}}^* \mathbf{F} \cong (1 \times i_{\mathcal{Y}})_* r_{\mathcal{Y}}^* \mathbf{G}.$$

Then the projection formula [24, Prop. II.5.6] yields an isomorphism

$$(51) \quad p_{2, \mathcal{X}}^* i_{\mathcal{Y}^*} \mathbf{F} \otimes^L \Delta_{\mathcal{X}^*} (\pi_{\mathcal{M}}^* Q) \cong (1 \times i_{\mathcal{Y}})_* (r_{\mathcal{Y}}^* \mathbf{G} \otimes^L L(1 \times i_{\mathcal{Y}})^* \Delta_{\mathcal{X}^*} (\pi_{\mathcal{M}}^* Q)).$$

As shown in Lemma 3.4.i, the derived restriction  $L(1 \times i_y)^* \Delta_{\mathcal{X}*}(\pi_{\mathcal{M}}^* Q)$  coincides with the ordinary restriction and it is furthermore isomorphic to  $\Gamma_{\mathcal{Y}*}(\rho_{\mathcal{M}}^* Q)$ . Therefore (51) yields

$$(52) \quad p_{2,\mathcal{X}}^* \mathbb{F} \otimes^L \Delta_{\mathcal{X}*}(\pi_{\mathcal{M}}^* Q) \cong (1 \times i_y)_*(r_{\mathcal{Y}}^* \mathbb{G} \otimes^L \Gamma_{\mathcal{Y}*}(\rho_{\mathcal{M}}^* Q)).$$

Next note that

$$(53) \quad \Gamma_{\mathcal{Y}*}(\rho_{\mathcal{M}}^* Q) \cong (i_y \times 1)_* \Delta_{\mathcal{Y}*}(\rho_{\mathcal{M}}^* Q)$$

since  $\Gamma_{\mathcal{Y}} = (i_y \times 1) \circ \Delta_{\mathcal{Y}}$ . Using again the projection formula and Lemma 3.4.ii, this further yields the isomorphism (49). q.e.d.

**3.2. Proof of Proposition 3.2.** By Theorem 2.5, the relative Atiyah class  $\alpha_{\mathcal{X}/X}$  is given by

$$\alpha_{\mathcal{X}/X} = \Delta_{\mathcal{X}*}(\pi_{\mathcal{M}}^* \alpha_{\mathcal{M}}) : \Delta_{\mathcal{X}*}(\pi_{\mathcal{M}}^* \mathcal{O}_{\mathcal{M}}) \rightarrow \Delta_{\mathcal{X}*}(\pi_{\mathcal{M}}^* \mathbb{L}_{\mathcal{M}}^{\bullet}[1]).$$

Using isomorphisms (48) and (49), and the functorial properties of the projection formula, one obtains the commutative diagrams

$$(54) \quad \begin{array}{ccc} p_{2,\mathcal{X}}^* \mathbb{F} \otimes^L \Delta_{\mathcal{X}*}(\pi_{\mathcal{M}}^* \mathcal{O}_{\mathcal{M}}) & \xrightarrow{\mathbf{1} \otimes \alpha_{\mathcal{X}/X}} & p_{2,\mathcal{X}}^* \mathbb{F} \otimes^L \Delta_{\mathcal{X}*}(\pi_{\mathcal{M}}^* \mathbb{L}_{\mathcal{M}}^{\bullet}[1]) \\ \downarrow \wr & & \downarrow \wr \\ (1 \times i_y)_*(r_{\mathcal{Y}}^* \mathbb{G} \otimes^L \Gamma_{\mathcal{Y}*} \rho_{\mathcal{M}}^* \mathcal{O}_{\mathcal{M}}) & \xrightarrow{\mathbf{f}_1} & (1 \times i_y)_*(r_{\mathcal{Y}}^* \mathbb{G} \otimes^L \Gamma_{\mathcal{Y}*} \rho_{\mathcal{M}}^* \mathbb{L}_{\mathcal{M}}^{\bullet}[1]) \end{array}$$

and

$$(55) \quad \begin{array}{ccc} r_{\mathcal{Y}}^* \mathbb{G} \otimes^L \Gamma_{\mathcal{Y}*}(\rho_{\mathcal{M}}^* \mathcal{O}_{\mathcal{M}}) & \xrightarrow{(\mathbf{1} \otimes \Gamma_{\mathcal{Y}})_*(\rho_{\mathcal{M}}^* \alpha_{\mathcal{M}})} & r_{\mathcal{Y}}^* \mathbb{G} \otimes^L \Gamma_{\mathcal{Y}*}(\rho_{\mathcal{M}}^* \mathbb{L}_{\mathcal{M}}^{\bullet}[1]) \\ \downarrow \wr & & \downarrow \wr \\ (i_y \times 1)_*(p_{1,\mathcal{Y}}^* \mathbb{G} \otimes^L \Delta_{\mathcal{Y}*} \rho_{\mathcal{M}}^* \mathcal{O}_{\mathcal{M}}) & \xrightarrow{\mathbf{f}_2} & (i_y \times 1)_*(p_{2,\mathcal{Y}}^* \mathbb{G} \otimes^L \Delta_{\mathcal{Y}*} \rho_{\mathcal{M}}^* \mathbb{L}_{\mathcal{M}}^{\bullet}[1]) \end{array}$$

where in  $\mathbf{f}_1, \mathbf{f}_2$  in the above diagrams denote the morphisms

$$\mathbf{f}_1 = (1 \times i_y)_*((\mathbf{1} \otimes \Gamma_{\mathcal{Y}})_*(\rho_{\mathcal{M}}^* \alpha_{\mathcal{M}})), \quad \mathbf{f}_2 = (i_y \times 1)_*((\mathbf{1} \otimes \Delta_{\mathcal{Y}})_*(\rho_{\mathcal{M}}^* \alpha_{\mathcal{M}})).$$

Furthermore, the left and right vertical arrows in the above diagrams are specializations of the isomorphisms (48) and (49) to  $Q = \mathcal{O}_{\mathcal{M}}$  and  $Q = \mathbb{L}_{\mathcal{M}}^{\bullet}[1]$  respectively. Note also that  $\Delta_{\mathcal{Y}*}(\rho_{\mathcal{M}}^* \alpha_{\mathcal{M}}) = \alpha_{\mathcal{Y}/Y}$  according to Theorem 2.5.



Now consider the commutative diagrams

$$(56) \quad \begin{array}{ccc} \mathcal{Y} \times \mathcal{Y} & \xrightarrow{i_{\mathcal{Y} \times 1}} & \mathcal{X} \times \mathcal{Y} \\ p_{1,\mathcal{Y}} \downarrow & & \downarrow r_{\mathcal{X}} \\ \mathcal{Y} & \xrightarrow{i_{\mathcal{Y}}} & \mathcal{X} \end{array} \quad \begin{array}{ccc} \mathcal{X} \times \mathcal{Y} & \xrightarrow{1 \times i_{\mathcal{Y}}} & \mathcal{X} \times \mathcal{X} \\ r_{\mathcal{X}} \downarrow & & \downarrow p_{1,\mathcal{X}} \\ \mathcal{X} & \xrightarrow{1} & \mathcal{X} \end{array}$$

Using again isomorphisms (48) and (49), one obtains successively

$$(57) \quad \begin{aligned} \mathbb{F} \otimes^L \pi_{\mathcal{M}}^* Q &\cong \mathbf{R}p_{1,\mathcal{X}*}(p_{2,\mathcal{X}}^* \mathbb{F} \otimes^L \Delta_{\mathcal{X}*}(\pi_{\mathcal{M}}^* Q)) \\ &\cong \mathbf{R}p_{1,\mathcal{X}*}(r_{\mathcal{Y}}^* \mathbf{G} \otimes^L \Gamma_{\mathcal{Y}*} \rho_{\mathcal{M}}^* Q) \\ &\cong \mathbf{R}p_{1,\mathcal{X}*}(i_{\mathcal{Y}} \times 1)_*(p_{2,\mathcal{Y}}^* \mathbf{G} \otimes^L \Delta_{\mathcal{Y}*}(\rho_{\mathcal{M}}^* Q)) \\ &\cong i_{\mathcal{Y}*} \mathbf{R}p_{1,\mathcal{Y}*}(p_{2,\mathcal{Y}}^* \mathbf{G} \otimes^L \Delta_{\mathcal{Y}*} \rho_{\mathcal{M}}^* Q) \\ &\cong i_{\mathcal{Y}*}(\mathbf{G} \otimes^L \rho_{\mathcal{M}}^* Q) \end{aligned}$$

for any object  $Q$  of  $\mathcal{D}^b(\mathcal{M})$ . Then using diagrams (54) and (55) one obtains the commutative diagram

$$(58) \quad \begin{array}{ccc} \mathbb{F} & \xrightarrow{\alpha_{\mathcal{X}/\mathcal{X}}(\mathbb{F})} & \mathbb{F} \otimes^L \pi_{\mathcal{M}}^* \mathbb{L}_{\mathcal{M}}^{\bullet}[1] \\ \downarrow \mathbf{1} & & \downarrow \wr \\ i_{\mathcal{Y}*} \mathbf{G} & \xrightarrow{i_{\mathcal{Y}*}(\alpha_{\mathcal{Y}/\mathcal{Y}}(\mathbf{G}))} & i_{\mathcal{Y}*}(\mathbf{G} \otimes^L \rho_{\mathcal{M}}^* \mathbb{L}_{\mathcal{M}}^{\bullet}[1]) \end{array}$$

where the left vertical arrow is the canonical isomorphism given by the projection formula. This implies equation (43).

q.e.d.

**3.3. Comparison of obstruction theories.** The next step is to translate Proposition 3.2 into a statement on obstruction theories using [27, Theorem 4.1].

Using the canonical isomorphism

$$(59) \quad \mathbf{R}\mathrm{Hom}_{\mathcal{D}^b(\mathcal{X})}(\mathbb{F}, \mathbb{F} \otimes^L \pi_{\mathcal{M}}^* \mathbb{L}_{\mathcal{M}}^{\bullet}[1]) \cong \mathbf{R}\mathrm{Hom}_{\mathcal{D}^b(\mathcal{X})}(\mathbf{R}\mathcal{H}om_{\mathcal{D}^b(\mathcal{X})}(\mathbb{F}, \mathbb{F}), \pi_{\mathcal{M}}^* \mathbb{L}_{\mathcal{M}}^{\bullet}[1]).$$

the relative Atiyah class  $\alpha_{\mathcal{X}/\mathcal{X}}(\mathbb{F})$  is identified with a map

$$(60) \quad \beta_{\mathcal{X}} : \mathbf{R}\mathcal{H}om_{\mathcal{D}^b(\mathcal{X})}(\mathbb{F}, \mathbb{F}) \rightarrow \pi_{\mathcal{M}}^* \mathbb{L}_{\mathcal{M}}^{\bullet}[1].$$

In complete analogy, the relative Atiyah class  $\alpha_{\mathcal{Y}/\mathcal{Y}}$  is also identified with a map

$$(61) \quad \beta_{\mathcal{Y}} : \mathbf{R}\mathcal{H}om_{\mathcal{D}^b(\mathcal{Y})}(\mathbf{G}, \mathbf{G}) \rightarrow \rho_{\mathcal{M}}^* \mathbb{L}_{\mathcal{M}}^{\bullet}[1].$$

Since  $X$  is  $K$ -trivial, as in [27, Sect 4.2], Grothendieck-Verdier duality yields an isomorphism

$$(62) \quad \begin{aligned} & \mathbf{R}\mathrm{Hom}_{\mathcal{G}^b(\mathcal{X})}(\mathbf{R}\mathcal{H}om_{\mathcal{G}^b(\mathcal{X})}(\mathbb{F}, \mathbb{F}), \pi_{\mathcal{M}}^* \mathbb{L}_{\mathcal{M}}^\bullet[1]) \\ & \cong \mathbf{R}\mathrm{Hom}_{\mathcal{G}^b(\mathcal{M})}(\mathbf{R}\mathcal{H}om_{\pi_{\mathcal{M}}}(\mathbb{F}, \mathbb{F}), \mathbb{L}_{\mathcal{M}}^\bullet)[1-d]. \end{aligned}$$

where  $d = \dim(X)$ . Therefore the relative Atiyah class is further identified with a map

$$a_X : \mathbf{R}\mathcal{H}om_{\pi_{\mathcal{M}}}(\mathbb{F}, \mathbb{F})[d-1] \rightarrow \mathbb{L}_{\mathcal{M}}^\bullet.$$

As shown in [27, Thm. 4.1], this map is an obstruction theory for  $\mathcal{M}$ , as defined in [2, Definition 4.4].

Again, in complete analogy, the relative Atiyah class  $\alpha_{\mathcal{Y}/\mathcal{Y}}(\mathbb{G})$  yields a second obstruction theory

$$a_Y : \mathbf{R}\mathcal{H}om_{\rho_{\mathcal{M}}}(\mathbb{G}, \mathbb{G} \otimes \mathcal{K})[d-2] \rightarrow \mathbb{L}_{\mathcal{M}}^\bullet,$$

where  $\rho_{\mathcal{M}} : \mathcal{Y} \rightarrow \mathcal{M}$  is the canonical projection, and  $\mathcal{K}$  is the pull-back  $\mathcal{K} = \rho_Y^* K_Y$ . Moreover, as shown in Lemma 3.1 there is a natural isomorphism

$$(63) \quad \begin{aligned} & \mathbf{R}\mathcal{H}om_{\mathcal{G}^b(\mathcal{X})}(\mathbb{F}, \mathbb{F}) \cong \\ & i_{\mathcal{Y}*} \mathbf{R}\mathcal{H}om_{\mathcal{G}^b(\mathcal{Y})}(\mathbb{G} \otimes \mathcal{K}^{-1}, \mathbb{G})[-1] \oplus i_{\mathcal{Y}*} \mathbf{R}\mathcal{H}om_{\mathcal{G}^b(\mathcal{Y})}(\mathbb{G}, \mathbb{G}). \end{aligned}$$

This yields a decomposition

$$(64) \quad \begin{aligned} & \mathbf{R}\mathcal{H}om_{\pi_{\mathcal{M}}}(\mathbb{F}, \mathbb{F})[d-1] \cong \\ & \mathbf{R}\mathcal{H}om_{\rho_{\mathcal{M}}}(\mathbb{G}, \mathbb{G} \otimes \mathcal{K})[d-2] \oplus \mathbf{R}\mathcal{H}om_{\rho_{\mathcal{M}}}(\mathbb{G}, \mathbb{G})[d-1]. \end{aligned}$$

The goal of this section is to prove:

**Proposition 3.6.** *The map  $a_X$  has components  $a_X = (a_Y, 0)$  with respect to the decomposition (64).*

*Proof.* Proposition 3.6 follows from Proposition 3.2 and identities similar to (59). In addition to (64), Lemma A.1 also yields an isomorphism

$$(65) \quad \begin{aligned} & \mathbf{R}\mathcal{H}om_{\mathcal{G}^b(\mathcal{X})}(\mathbb{F}, \mathbb{F} \otimes^L \pi_{\mathcal{M}}^* \mathbb{L}_{\mathcal{M}}^\bullet[1]) \cong \\ & i_{\mathcal{Y}*} \mathbf{R}\mathcal{H}om_{\mathcal{G}^b(\mathcal{Y})}(\mathbb{G}, \mathbb{G} \otimes^L \rho_{\mathcal{M}}^* \mathbb{L}_{\mathcal{M}}^\bullet[1]) \\ & \oplus i_{\mathcal{Y}*} \mathbf{R}\mathcal{H}om_{\mathcal{G}^b(\mathcal{Y})}(\mathbb{G} \otimes \mathcal{K}^{-1}, \mathbb{G} \otimes^L \rho_{\mathcal{M}}^* \mathbb{L}_{\mathcal{M}}^\bullet). \end{aligned}$$

This identity is canonically equivalent to

$$(66) \quad \begin{aligned} & \mathbf{R}\mathcal{H}om_{\mathcal{G}^b(\mathcal{X})}(\mathbf{R}\mathcal{H}om_{\mathcal{G}^b(\mathcal{X})}(\mathbb{F}, \mathbb{F}), \pi_{\mathcal{M}}^* \mathbb{L}_{\mathcal{M}}^\bullet[1]) \cong \\ & i_{\mathcal{Y}*} \mathbf{R}\mathcal{H}om_{\mathcal{G}^b(\mathcal{Y})}(\mathbf{R}\mathcal{H}om_{\mathcal{G}^b(\mathcal{Y})}(\mathbb{G}, \mathbb{G}), \rho_{\mathcal{M}}^* \mathbb{L}_{\mathcal{M}}^\bullet[1]) \oplus \\ & i_{\mathcal{Y}*} \mathbf{R}\mathcal{H}om_{\mathcal{G}^b(\mathcal{Y})}(\mathbf{R}\mathcal{H}om_{\mathcal{G}^b(\mathcal{Y})}(\mathbb{G}, \mathbb{G} \otimes \mathcal{K}), \rho_{\mathcal{M}}^* \mathbb{L}_{\mathcal{M}}^\bullet) \end{aligned}$$

Next, applying the global derived functor  $\mathbf{R}\Gamma_{\mathcal{D}^b(\mathcal{X})}$  to both sides of equation (66) yields the isomorphism

$$\begin{aligned} \mathbf{R}\mathrm{Hom}_{\mathcal{D}^b(\mathcal{X})}(\mathbf{R}\mathcal{H}om_{\mathcal{D}^b(\mathcal{X})}(\mathbb{F}, \mathbb{F}), \pi_{\mathcal{M}}^* \mathbb{L}_{\mathcal{M}}^{\bullet}[1]) &\cong \\ \mathbf{R}\mathrm{Hom}_{\mathcal{D}^b(\mathcal{Y})}(\mathbf{R}\mathcal{H}om_{\mathcal{D}^b(\mathcal{Y})}(\mathbb{G}, \mathbb{G}), \rho_{\mathcal{M}}^* \mathbb{L}_{\mathcal{M}}^{\bullet}[1]) &\oplus \\ \mathbf{R}\mathrm{Hom}_{\mathcal{D}^b(\mathcal{Y})}(\mathbf{R}\mathcal{H}om_{\mathcal{D}^b(\mathcal{Y})}(\mathbb{G}, \mathbb{G} \otimes \mathcal{K}), \rho_{\mathcal{M}}^* \mathbb{L}_{\mathcal{M}}^{\bullet}) & \end{aligned}$$

Then Proposition (3.2) implies that

$$\beta_{\mathcal{X}/\mathcal{X}}(\mathbb{F}) : \mathbf{R}\mathcal{H}om_{\mathcal{D}^b(\mathcal{X})}(\mathbb{F}, \mathbb{F}) \rightarrow \pi_{\mathcal{M}}^* \mathbb{L}_{\mathcal{M}}^{\bullet}[1]$$

has components

$$(67) \quad \beta_{\mathcal{X}} = (\beta_{\mathcal{Y}}, 0)$$

with respect to the above decomposition, where  $\beta_{\mathcal{X}}, \beta_{\mathcal{Y}}$  are the maps obtained in equations (60), (61) respectively.

Now note that  $Li_{\mathcal{Y}}^* \pi_{\mathcal{M}}^* \mathbb{L}_{\mathcal{M}}^{\bullet} \cong \rho_{\mathcal{M}}^* \mathbb{L}_{\mathcal{M}}^{\bullet}$ , as shown in equation (41). This yields the isomorphism

$$i_{\mathcal{Y}}^! \pi_{\mathcal{M}}^* \mathbb{L}_{\mathcal{M}}^{\bullet} \cong \mathcal{K} \otimes \rho_{\mathcal{M}}^* \mathbb{L}_{\mathcal{M}}^{\bullet}[-1].$$

Then using Grothendieck-Verdier duality for the closed embedding  $i_{\mathcal{Y}} : \mathcal{Y} \rightarrow \mathcal{X}$ , identity (66) is further equivalent to

$$(68) \quad \begin{aligned} \mathbf{R}\mathcal{H}om_{\mathcal{D}^b(\mathcal{X})}(\mathbf{R}\mathcal{H}om_{\mathcal{D}^b(\mathcal{X})}(\mathbb{F}, \mathbb{F}), \pi_{\mathcal{M}}^* \mathbb{L}_{\mathcal{M}}^{\bullet}[1]) &\cong \\ \mathbf{R}\mathcal{H}om_{\mathcal{D}^b(\mathcal{X})}(i_{\mathcal{Y}*}(\mathbf{R}\mathcal{H}om_{\mathcal{D}^b(\mathcal{Y})}(\mathbb{G}, \mathbb{G} \otimes \mathcal{K}), \pi_{\mathcal{M}}^* \mathbb{L}_{\mathcal{M}}^{\bullet}[2]) &\oplus \\ \mathbf{R}\mathcal{H}om_{\mathcal{D}^b(\mathcal{X})}(i_{\mathcal{Y}*} \mathbf{R}\mathcal{H}om_{\mathcal{D}^b(\mathcal{Y})}(\mathbb{G}, \mathbb{G}), \pi_{\mathcal{M}}^* \mathbb{L}_{\mathcal{M}}^{\bullet}[1]). & \end{aligned}$$

Finally, using Grothendieck-Verdier duality for the projection  $\pi_{\mathcal{M}} : \mathcal{X} \rightarrow \mathcal{M}$  in (68), one obtains the decomposition

$$(69) \quad \begin{aligned} \mathbf{R}\mathrm{Hom}_{\mathcal{D}^b(\mathcal{M})}(\mathbf{R}\mathcal{H}om_{\pi_{\mathcal{M}}}(\mathbb{F}, \mathbb{F}), \mathbb{L}_{\mathcal{M}}^{\bullet})[1-d] &\cong \\ \mathbf{R}\mathrm{Hom}_{\mathcal{D}^b(\mathcal{M})}(\mathbf{R}\mathcal{H}om_{\rho_{\mathcal{M}}}(\mathbb{G}, \mathbb{G} \otimes \mathcal{K}), \mathbb{L}_{\mathcal{M}}^{\bullet})[2-d] &\oplus \\ \mathbf{R}\mathrm{Hom}_{\mathcal{D}^b(\mathcal{M})}(\mathbf{R}\mathcal{H}om_{\rho_{\mathcal{M}}}(\mathbb{G}, \mathbb{G}), \mathbb{L}_{\mathcal{M}}^{\bullet})[1-d]. & \end{aligned}$$

This is in fact the global version of (64). Then equation (67) implies that  $a_{\mathcal{X}}$  has indeed components  $(a_{\mathcal{Y}}, 0)$  with respect to decomposition (64).  $\square$

#### 4. Donaldson-Thomas invariants for sheaves on local fourfolds

In this paper, the main application of Theorem 1.1 is the construction of virtual counting invariants of sheaves on a local Calabi-Yau fourfolds, which is presented below.

Let  $S$  be a smooth projective surface, let  $\mathcal{O}_S(1)$  be a very ample line bundle on  $S$  and let  $h = c_1(\mathcal{O}_S(1))$ . Throughout this paper it will be further assumed that  $H^1(\mathcal{O}_S) = 0$  and that the integral cohomology of  $S$  is torsion free. In the framework of Section 3 let  $Y$  be the total space of  $K_S(D)$ , where  $D \in \text{Pic}(S)$  be an effective non-zero divisor on  $S$ . Then  $K_Y \cong q^*\mathcal{O}_S(-D)$ , where  $\pi_S : Y \rightarrow S$  denotes the canonical projection. Hence  $X$  is isomorphic to the total space of the rank two bundle  $V = K_S(D) \oplus \mathcal{O}_S(-D)$  on  $S$ . Let  $g = \pi_S \circ q : X \rightarrow S$  denote the canonical projection.

**4.1. Stable two dimensional sheaves.** Let  $\text{Coh}_c(X)$  denote the abelian category of coherent sheaves on  $X$  with proper support. Since the projection  $g : X \rightarrow S$  is affine, for any such sheaf  $\mathcal{F}$  the derived pushforward  $\mathbf{R}_{g_*}\mathcal{F}$  is isomorphic to the ordinary pushforward,  $g_*\mathcal{F}$ . Therefore using the Grothendieck-Riemann-Roch theorem, the topological invariants of  $\mathcal{F}$  are completely determined by the total Chern class  $c(g_*\mathcal{F}) \in H^{\text{even}}(S, \mathbb{Z})$ . Using Poincaré duality on  $S$ , the latter is canonically identified with an element of  $H_{\text{even}}(S, \mathbb{Z})$ , which can be written as

$$(r(\mathcal{F})[S], \beta(F), n(F)[pt]) \in H_4(S, \mathbb{Z}) \oplus H_2(S, \mathbb{Z}) \oplus H_0(S, \mathbb{Z}).$$

Since the generators  $[S], [pt]$  are canonical, this is further identified with the triple

$$\gamma(F) = (r(\mathcal{F}), \beta(F), n(F)) \in \mathbb{Z} \oplus H_2(S, \mathbb{Z}) \oplus \mathbb{Z}.$$

In particular  $r(\mathcal{F}) = \text{rk}(g_*\mathcal{F})$  will be informally referred to as the rank of  $\mathcal{F}$  over  $S$  in the following.

Next note that the polarization  $h$  determines naturally a stability condition on  $\text{Coh}_c(X)$  using the Hilbert polynomial

$$P_h(\mathcal{F}; m) = \chi(\mathcal{F} \otimes g^*\mathcal{O}_S(m)).$$

As in [26, Definition 1.2.4],  $\mathcal{F}$  is  $h$ -(semi)stable if and only if  $\mathcal{F}$  is pure, and

$$p_h(\mathcal{F}'; m) (\leq) p_h(\mathcal{F}; m) \quad m \gg 0$$

for any proper nonzero subsheaf  $\mathcal{F}' \subset \mathcal{F}$ . Here  $p_h(\mathcal{F}; m)$  denotes the reduced Hilbert polynomial of  $\mathcal{F}$ .

In terms of  $\gamma(\mathcal{F}) = (r(\mathcal{F}), \beta(\mathcal{F}), n(\mathcal{F}))$  the Hilbert polynomial reads

$$P_h(\mathcal{F}; m) = \frac{r(\mathcal{F})h^2}{2} m^2 + h \cdot \left( \beta(\mathcal{F}) + \frac{r(\mathcal{F})c_1(S)}{2} \right) m + \frac{\beta \cdot (\beta + c_1(S))}{2} - n(\mathcal{F}) + r(\mathcal{F})\chi(\mathcal{O}_S).$$

Hence, for sheaves with  $r(\mathcal{F}) > 0$ ,

$$p_h(\mathcal{F}, m) = m^2 + \left( \frac{2h \cdot \beta(\mathcal{F})}{r(\mathcal{F})h^2} + \frac{h \cdot c_1(S)}{h^2} \right) m + \frac{2}{r(\mathcal{F})h^2} \left( \frac{\beta \cdot (\beta + c_1(S))}{2} - n(\mathcal{F}) \right) + \frac{2\chi(\mathcal{O}_S)}{h^2}.$$

For any such sheaf  $\mathcal{F}$  let

$$(70) \quad \mu_h(\mathcal{F}) = \frac{h \cdot \beta(\mathcal{F})}{r(\mathcal{F})}, \quad v_h(\mathcal{F}) = \frac{1}{r(\mathcal{F})} \left( \frac{\beta \cdot (\beta + c_1(S))}{2} - n(F) \right).$$

Then  $\mathcal{F}$  is  $h$ -(semi)stable if and only if it is pure, and any nonzero proper subsheaf  $\mathcal{F}' \subset \mathcal{F}$  satisfies

$$(71) \quad \mu_h(\mathcal{F}') (\leq) \mu_h(\mathcal{F})$$

while in case of equality,

$$(72) \quad v_h(\mathcal{F}') (\leq) v_h(\mathcal{F}).$$

Furthermore, as in [26, Definition 2.1.12], a pure sheaf  $\mathcal{F}$  with  $r(\mathcal{F}) > 0$  is said to be  $\mu_h$ -(semi)stable if and only if any nonzero proper subsheaf  $\mathcal{F}' \subset \mathcal{F}$  with  $0 < r(\mathcal{F}') < r(\mathcal{F})$  satisfies inequality (71). As usual,  $h$ -semistability implies  $\mu_h$ -semistability for any such sheaf.

**Lemma 4.1.** *Let  $\mathcal{F}$  be a pure compactly supported sheaf on  $X$  with  $r(\mathcal{F}) > 0$ . Then  $\mathcal{F}$  is  $\mu_h$ -(semi)stable if and only if  $\mathcal{F} \otimes g^* \mathcal{O}_S(-D)$  is  $\mu_h$ -(semi)stable.*

*Proof.* Note that for any compactly supported sheaf  $\mathcal{E}$  on  $X$  with  $r(\mathcal{E}) > 0$  one has

$$(73) \quad \mu_h(\mathcal{E} \otimes g^* \mathcal{O}_S(-D)) = \mu_h(\mathcal{E}) - h \cdot D,$$

Then the claim follows by a straightforward verification of the  $\mu_h$ -(semi)stability condition. □

**Corollary 4.2.** *Let  $\mathcal{F}$  be a pure  $\mu_h$ -semistable compactly supported sheaf on  $X$  with  $r(\mathcal{F}) > 0$ . Then the following vanishing result holds*

$$(74) \quad \text{Ext}_X^0(\mathcal{F}, \mathcal{F} \otimes g^* \mathcal{O}_S(-D)) = 0.$$

*In particular this holds for all  $h$ -semistable sheaves  $\mathcal{F}$  with  $r(\mathcal{F}) > 0$ .*

*Proof.* Since  $\mathcal{F}$  is  $\mu_h$ -semistable, as shown in Lemma 4.1 above,  $\mathcal{F} \otimes g^* \mathcal{O}_S(-D)$  is also  $\mu_h$ -semistable. Moreover, since  $D$  is effective and non-zero,

$$\mu_h(\mathcal{F} \otimes g^* \mathcal{O}_S(-D)) = \mu_h(\mathcal{F}) - h \cdot D < \mu_h(\mathcal{F}).$$

Then the claim follows from Theorem 1.6.6 and Proposition 1.2.7 in [26]. □

**Corollary 4.3.** *Let  $\mathcal{F}$  be a pure compactly supported  $\mu_h$ -semistable sheaf on  $X$  such that  $r(\mathcal{F}) > 0$ . Then  $\mathcal{F}$  is scheme theoretically supported on  $Y$ , and as a sheaf on  $Y$ , the following holds:*

$$(75) \quad \text{Ext}_Y^3(\mathcal{F}, \mathcal{F}) = 0.$$

In particular this holds for all  $h$ -semistable sheaves  $\mathcal{F}$  with  $r(\mathcal{F}) > 0$ .

*Proof.* Note that  $Y \subset X$  is the zero locus of the tautological section

$$\xi \in H^0(X, g^* \mathcal{O}_S(-D)).$$

In order to prove the first claim it suffices to show that the morphism

$$\mathbf{1}_{\mathcal{F}} \otimes \xi : \mathcal{F} \rightarrow \mathcal{F} \otimes g^* \mathcal{O}_S(-D)$$

is identically zero for any sheaf  $\mathcal{F}$  as in Lemma 4.3. This follows from the first vanishing result in (74).

The vanishing result (75) follows by Serre duality for compactly supported sheaves, noting that  $K_Y \cong \pi_S^* \mathcal{O}_S(-D) \simeq g^* \mathcal{O}_S(-D)|_Y$ .  $\square$

**4.2. Reduced obstruction theory.** As explained in Section 3.3, the moduli space  $\mathcal{M}$  has two natural obstruction theories, associated to deformations of sheaves on  $X$  and  $Y \subset X$  respectively. Using the Atiyah class formalism of [27, Thm 4.1], these are given by the maps

$$(76) \quad a_X : \mathbf{R}\mathcal{H}om_{\pi_{\mathcal{M}}}(\mathbb{F}, \mathbb{F})[3] \rightarrow \mathbb{L}_{\mathcal{M}}^{\bullet}$$

and

$$(77) \quad a_Y : \mathbf{R}\mathcal{H}om_{\rho_{\mathcal{M}}}(\mathbb{G}, \mathbb{G} \otimes \rho_Y^* K_Y)[2] \rightarrow \mathbb{L}_{\mathcal{M}}^{\bullet}$$

respectively. As in Section 3, here  $\pi_{\mathcal{M}}, \rho_{\mathcal{M}}$  denote the canonical projections  $X \times \mathcal{M} \rightarrow \mathcal{M}$  and  $Y \times \mathcal{M} \rightarrow \mathcal{M}$  respectively. The projections onto the  $X$  and  $Y$  factors will be respectively denoted by  $p_X$  and  $\rho_Y$ . Proposition 3.6 shows that the first obstruction theory splits as a direct sum  $a_X = (a_Y, 0)$  with respect to the decomposition (64), which in the present case reads

$$(78) \quad \begin{aligned} \mathbf{R}\mathcal{H}om_{\pi_{\mathcal{M}}}(\mathbb{F}, \mathbb{F})[3] &\cong \\ \mathbf{R}\mathcal{H}om_{\rho_{\mathcal{M}}}(\mathbb{G}, \mathbb{G} \otimes \rho_Y^* K_Y)[2] &\oplus \mathbf{R}\mathcal{H}om_{\rho_{\mathcal{M}}}(\mathbb{G}, \mathbb{G})[3]. \end{aligned}$$

In particular the obstruction theory (76) of  $\mathcal{M}$  as a moduli space of sheaves on  $X$  can be naturally truncated to the (77) without any loss of information. This result is similar to [4, Lemma 6.4], except that here it is proven for global obstruction theories as opposed to local complex analytic Kuranishi maps.

The next observation is that the obstruction theory (77) can be further truncated as shown in [27, Section 4.4]. Namely, Grothendieck duality for the projection  $\rho_{\mathcal{M}} : \mathcal{Y} \rightarrow \mathcal{M}$  yields an isomorphism

$$(79) \quad \mathbf{R}\mathcal{H}om_{\rho_{\mathcal{M}}}(\mathbb{G}, \mathbb{G} \otimes \rho_Y^* K_Y)[2] \cong \mathbf{R}\mathcal{H}om_{\rho_{\mathcal{M}}}(\mathbb{G}, \mathbb{G})^{\vee}[-1].$$

There is also a natural map of complexes

$$\mathcal{O}_{\mathcal{M}} \rightarrow \mathbf{R}\mathcal{H}om_{\rho_{\mathcal{M}}}(\mathbb{G}, \mathbb{G})$$

which restricts to the identity map on each fiber  $Y_m$ ,  $m \in \mathcal{M}$ . The cone of this map defines the truncation  $\tau^{\geq 1} \mathbf{R}\mathcal{H}om_{\rho_{\mathcal{M}}}(\mathbf{G}, \mathbf{G})$ . Moreover, using the vanishing result (75), the global version of Nakayama's lemma [27, Lemma 4.2] yields a further truncation  $\tau^{[1,2]} \mathbf{R}\mathcal{H}om_{\rho_{\mathcal{M}}}(\mathbf{G}, \mathbf{G})$  of amplitude  $[1, 2]$ . As shown in [27, Sect. 4.4] the map (77) also admits a truncation

$$(80) \quad \tau^{[-1,0]} a_Y : (\tau^{[1,2]} \mathbf{R}\mathcal{H}om_{\rho_{\mathcal{M}}}(\mathbf{G}, \mathbf{G}))^\vee[-1] \rightarrow \mathbb{L}_{\mathcal{M}}^\bullet$$

which is still an obstruction theory for  $\mathcal{M}$ . Moreover, this obstruction theory is perfect i.e. it admits a two term locally free resolution. However, as observed [15], [31], the resulting virtual cycle invariants would be identically zero since the obstruction sheaf induced by the above obstruction theory contains a trivial direct summand. This problem is solved in loc. cit. by constructing a reduced perfect obstruction theory.

More precisely, it is shown in [15, Proposition 2.4], and [31, Theorem 6.1, Theorem 6.5] that there is a further splitting

$$(81) \quad \tau^{[-1,0]} a_Y = (a_Y^{\text{red}}, 0) : \begin{array}{c} (\tau^{[1,2]} \mathbf{R}\mathcal{H}om_{\rho_{\mathcal{M}}}(\mathbf{G}, \mathbf{G}))^\vee[-1]_\perp \\ \oplus \\ \rho_{\mathcal{M}*}(\pi_S^* K_S)[2] \end{array} \rightarrow \mathbb{L}_{\mathcal{M}}^\bullet$$

where  $\pi_S : S \times \mathcal{M} \rightarrow \mathcal{M}$  is the natural projection. Furthermore, it is also proven in loc. cit. that the map

$$(82) \quad a_Y^{\text{red}} : (\tau^{[1,2]} \mathbf{R}\mathcal{H}om_{\rho_{\mathcal{M}}}(\mathbf{G}, \mathbf{G}))^\vee[-1]_\perp \rightarrow \mathbb{L}_{\mathcal{M}}^\bullet$$

is still a perfect obstruction theory for  $\mathcal{M}$ , hence it yields a reduced virtual cycle  $[\mathcal{M}]_{\text{red}}^{\text{vir}}$ .

As stated in Corollary 1.2, it is now clear that the fourfold obstruction theory (76) admits a reduction to (82), which is moreover perfect.

**4.3. Reduced Donaldson-Thomas invariants.** The Donaldson-Thomas invariants for sheaves on  $X$  can be naturally defined by integration against the reduced virtual cycle associated to (82), provided that compactness issues are properly addressed. Again, this is similar to [4, Theorem 6.5], although the logic employed here is slightly different.

In the present situation, compactness issues are easily addressed using the natural torus action  $\mathbf{T} \times Y \rightarrow Y$  which scales the fibers of the line bundle  $K_S(D)$  with weight  $+1$ . Clearly, this admits a lift to a torus action on  $X$  scaling the fibers of  $K_S(D) \oplus \mathcal{O}_S(-D)$  with weights  $(+1, -1)$ . The fixed point set coincides with the zero section,  $S \subset Y$ , which is obviously proper. This implies that the induced action  $\mathbf{T} \times \mathcal{M} \rightarrow \mathcal{M}$  on the moduli space has proper fixed locus  $\mathcal{M}^{\mathbf{T}}$ . Furthermore,

by the virtual localization theorem [21], the fixed locus has an induced reduced virtual cycle,  $[\mathcal{M}^{\mathbf{T}}]_{\text{red}}^{\text{vir}}$ , and an equivariant virtual normal bundle

$$(83) \quad \mathcal{N}^{\text{vir}} = \left( \tau^{[1,2]} \mathbf{R}\mathcal{H}om_{\rho_{\mathcal{M}}}(\mathbb{G}, \mathbb{G})|_{\mathcal{M}^{\mathbf{T}}} \right)^m.$$

Here the superscript  $m$  denotes the moving part with respect to the torus action. Then the equivariant residual Donaldson-Thomas invariants of  $X$  are defined by

$$(84) \quad DT_X(\gamma) = \int_{[\mathcal{M}^{\mathbf{T}}]_{\text{red}}^{\text{vir}}} \frac{1}{e_{\mathbf{T}}(\mathcal{N}^{\text{vir}})}.$$

## 5. Rank two invariants and universality

The goal of this section is to prove certain universality properties of the equivariant residual invariants (84) for rank  $r = 2$ . The surface  $S$  will be subject to the same conditions as listed in the beginning of Section 4.

**5.1. Rank two fixed loci.** Let  $S$  be a smooth projective surface as in Section 4 and  $h = c_1(\mathcal{O}_S(1))$  a polarization on  $S$ . Note that under the current assumptions the natural map  $\text{Pic}(S) \rightarrow H_2(S, \mathbb{Z})$  is an isomorphism, hence the lattices will be implicitly identified in the following. Moreover, since  $H^1(\mathcal{O}_S) = 0$  and element  $\beta \in H_2(S, \mathbb{Z})$  determines a unique line bundle on  $S$  up to isomorphism.

Now let  $\gamma = (2, \beta, n)$  with  $\beta \in H_2(S, \mathbb{Z})$  and  $n \in \mathbb{Z}$  and let  $\mathcal{M} = \mathcal{M}_h(Y, \gamma)$ . Let  $\mathcal{M}_h(S, \gamma)$  be the moduli space of  $h$ -stable torsion free sheaves on  $S$  with topological invariants  $\gamma$ . For simplicity, in this section set  $L = K_S(D)$ .

The rank two torus fixed loci in the moduli space  $\mathcal{M}_h(Y, \gamma)$  were determined in [15, Section 3] and [31, Section 7]. This section will review the main points using the approach of [15, Section 3]. As shown in loc. cit., for arbitrary rank  $r \geq 1$  the torus fixed loci in the moduli space  $\mathcal{M}$  are classified by partitions  $\lambda$  of  $r$ . For a given partition  $\lambda = (\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_{\ell(\lambda)})$  the corresponding fixed locus consists of sheaves  $\mathcal{F}$  such that the torsion free sheaf  $\mathcal{E} = q_* \mathcal{F}$  has a character decomposition

$$\mathcal{E} \cong \bigoplus_{i=0}^{\ell(\lambda)-1} \mathcal{E}_{-i} \otimes \mathbf{t}^{-i},$$

where  $\mathcal{E}_{-i}$  is a rank  $\lambda_{i+1}$  torsion free sheaf on  $S$  for each  $0 \leq i \leq \ell(\lambda)$ . Here  $\mathbf{t}$  denotes the trivial equivariant line bundle on  $S$  equipped with the weight one action of  $\mathbf{T} = \mathbb{C}^\times$  on each fiber. Moreover, since

$$q_* \mathcal{O}_Y \cong \bigoplus_{i=0}^{\infty} L^{-i} \otimes \mathbf{t}^{-i}$$



the  $\mathcal{O}_Y$ -module structure of  $\mathcal{F}$  is specified by a collection of morphisms of  $\mathcal{O}_S$ -modules

$$\psi_i : \mathcal{E}_{-i} \rightarrow \mathcal{E}_{-i-1} \otimes L.$$

In particular, for  $r = 2$  one obtains two types of components corresponding to  $\lambda = (2)$  and  $\lambda = (1, 1)$  which will be called *type I* and *type II* respectively. The union of type I fixed loci will be denoted by  $\mathcal{M}_{(2)}^{\mathbf{T}}$  while the union of type II fixed loci will be denoted by  $\mathcal{M}_{(1,1)}^{\mathbf{T}}$ . As shown in [15, Proposition 3.2] there is a natural isomorphism of schemes

$$(85) \quad \mathcal{M}_{(2)}^{\mathbf{T}} \cong \mathcal{M}_h(S, \gamma).$$

Moreover, as shown in [15, Proposition 3.8], the type II fixed locus is isomorphic to a union of nested Hilbert schemes on  $S$ . Using the notation of loc. cit., given a pair of curve classes  $(\alpha_1, \alpha_2) \in H^2(S, \mathbb{Z})^{\oplus 2}$  and a pair of integers  $(n_1, n_2) \in \mathbb{Z}^2$ ,  $n_1, n_2 \geq 0$ , let  $S_{(\alpha_1, \alpha_2)}^{[n_1, n_2]}$  be the nested Hilbert scheme parametrizing flags of twisted ideal sheaves

$$\mathcal{I}_1 \otimes L_1 \subset \mathcal{I}_2 \otimes L_2$$

where  $\mathcal{I}_1, \mathcal{I}_2$  are ideal sheaves of zero dimensional subschemes of  $S$  of length  $n_1, n_2$  respectively, and  $L_1, L_2$  are line bundles on  $S$  with  $c_1(L_i) = -\alpha_i$ ,  $1 \leq i \leq 2$ . Note in particular that the curve class  $\alpha_2 - \alpha_1$  must be effective or zero, which will be indicated by  $\alpha_2 - \alpha_1 \succeq 0$ . Then the type II fixed locus is isomorphic to

$$(86) \quad \mathcal{M}_{(1,1)}^{\mathbf{T}} \cong \bigcup'_{\substack{\beta_1 + \beta_2 = \beta \\ \beta_2 + c_1(L) - \beta_1 \succeq 0}} \bigcup'_{\substack{n_1, n_2 \geq 0 \\ n_1 + n_2 + \beta_1 \cdot \beta_2 = n}} S_{(\beta_1, \beta_2 + c_1(L))}^{[n_1, n_2]}$$

where the  $'$  superscript indicates that the union in the right hand side is subject to the following additional constraints:

- (II.a) If  $\beta_2 + c_1(L) - \beta_1 = 0$  then  $n_1 \geq n_2$ , and
- (II.b)  $h \cdot \beta_2 < h \cdot \beta_1$ .

Using the virtual localization theorem, the residual Donaldson-Thomas invariant  $DT_h(X, \gamma)$  then splits as a sum

$$DT_h(X, \gamma) = DT_h(X, \gamma)_I + DT_h(X, \gamma)_{II}.$$

The type I contributions are determined in [15, Proposition 3.2 and Corollary 3.3] while the type II contributions are determined in [15, Proposition 3.11 and Corollary 3.13]. In order to write down the resulting formulas, it will be convenient to adopt the following notation conventions for any component  $\mathcal{T}$  of the fixed locus, of type I or II.

**Notation.** Let  $\pi : S \times \mathcal{T} \rightarrow \mathcal{T}$  denote the canonical projection and let  $\mathbf{R}\mathcal{H}om_\pi$  denote the derived functor  $\mathbf{R}_{\pi*}\mathbf{R}\mathcal{H}om$ . Moreover, the pull-back of any sheaf  $M$  on  $S$  to  $S \times \mathcal{T}$  will be denoted again by  $M$  for simplicity. The distinction will be clear from the context.

For type I components, let  $\mathbb{E}$  denote the universal sheaf on  $S \times \mathcal{M}_h(S, \gamma)$ . Note also that  $\mathcal{M}_h(S, \gamma)$  has a natural trace-free virtual cycle  $[\mathcal{M}_h(S, \gamma)]_0^{\text{vir}}$ , as shown in [27, Theorem 4.1]. Then Proposition 3.2 and Corollary 3.3 of [15] prove:

$$(87) \quad DT_h(X, \gamma)_I = \int_{[\mathcal{M}_h(S, \gamma)]_0^{\text{vir}}} \frac{1}{e(\mathbf{R}\mathcal{H}om_\pi(\mathbb{E}, \mathbb{E} \otimes L \cdot \mathbf{t}))},$$

where  $L \cdot \mathbf{t}$  denotes  $L \otimes \mathbf{t}$  for simplicity.

In order to write down the type II contributions, note that for any pair  $(\alpha_1, \alpha_2)$  and for any  $(n_1, n_2)$  the nested Hilbert scheme  $S_{(\alpha_1, \alpha_2)}^{[n_1, n_2]}$  is naturally isomorphic to the nested Hilbert scheme  $S_{(0, \alpha_2 - \alpha_1)}^{[n_1, n_2]}$  since  $H^1(\mathcal{O}_S) = 0$ . The latter will be denoted below by  $S_\alpha^{[n_1, n_2]}$ , where  $\alpha = \alpha_2 - \alpha_1$ . For  $n_1 = 0$  this reduces to the Hilbert scheme of one dimensional subschemes  $Z \subset S$  with

$$c_1(\mathcal{O}_Z) = \alpha, \quad c_2(\mathcal{O}_Z) = n_2.$$

For  $n_2 = 0$  and  $\alpha = 0$ , one obtains the usual Hilbert scheme  $S^{[n_1]}$  of  $n_1$  points on  $S$ . Moreover note that there are natural projections

$$S_\alpha^{[n_1, n_2]} \rightarrow S^{[n_1]}, \quad S_\alpha^{[n_1, n_2]} \rightarrow S_\alpha^{[n_2]}.$$

Therefore any equivariant sheaf or complex of sheaves  $\mathbb{F}$  on  $S^{[n_i]}$  or  $S \times S^{[n_i]}$ ,  $1 \leq i \leq n$  has a canonical pull-back to  $S_\alpha^{[n_1, n_2]}$  or  $S \times S_\alpha^{[n_1, n_2]}$  respectively. For simplicity the pull-back will be denoted again by  $\mathbb{F}$ . In particular, using these conventions, the universal flag on  $S \times S_\alpha^{[n_1, n_2]}$  will be given by  $\mathcal{I}^{[n_1]} \subset \mathcal{I}_\alpha^{[n_2]}$  where  $\mathcal{I}^{[n_1]}$ ,  $\mathcal{I}_\alpha^{[n_2]}$  are obtained by pulling back the universal objects from  $S \times S^{[n_1]}$  and  $S \times S_\alpha^{[n_2]}$  respectively.

Employing the above conventions, for any line bundle  $M$  on  $S$  and any non-zero integer  $a \in \mathbb{Z}$ ,  $a \neq 0$ , let

$$Q_\alpha^{[n_1, n_2]}(M, a) = \frac{(\mathbf{a}\mathbf{s})^{\chi(M)}}{e(\mathbf{R}\mathcal{H}om_\pi(\mathcal{I}^{[n_1]}, \mathcal{I}_\alpha^{[n_2]} \otimes M \cdot \mathbf{t}^{-a}))}.$$

Again,  $M \cdot \mathbf{t}^{-a}$  denotes  $M \otimes \mathbf{t}^{-a}$  while  $\mathbf{s} = c_1(\mathbf{t})$  is the equivariant first Chern class of  $\mathbf{T}$ , commonly referred to as the equivariant parameter. For any integer  $n \geq 0$  let also  $\mathbf{T}_{S^{[n]}}$  denote the tangent bundle to the Hilbert scheme  $S^{[n]}$ . Moreover, as shown in [16, Theorem 1], note that the nested Hilbert scheme has an intrinsic

virtual cycle  $[S_\alpha^{[n_1, n_2]}]^{vir}$ . Then the formula derived in Proposition 3.11 and Corollary 3.13 in [15] for the residual contribution of a component of the fixed locus  $\mathcal{T} \cong S_\alpha^{[n_1, n_2]}$  reads

$$(88) \quad DT_\alpha^{[n_1, n_2]} = \frac{(-1)^{-c_1(L) \cdot \alpha + c_1(L) \cdot c_1(S)/2 + 3c_1(L)^2/2}}{2\chi(L^2)(-\mathbf{s})\chi(L^2) + \chi(L) - \chi(L^{-1})} \\ \int_{[S_\alpha^{[n_1, n_2]}]^{vir}} (e(\mathbb{T}_{S^{[n_1]}} \otimes L \cdot \mathbf{t}) e(\mathbb{T}_{S^{[n_2]}} \otimes L \cdot \mathbf{t}) \mathcal{Q}_\alpha^{[n_1, n_2]}(K_S \otimes L^{-1}, -1) \\ \mathcal{Q}_\alpha^{[n_1, n_2]}(K_S \otimes L^{-1}, -1) \mathcal{Q}_\alpha^{[n_1, n_2]}(L^{-1}, -1) \mathcal{Q}_\alpha^{[n_1, n_2]}(K_2 \otimes L^{-2}, -2)^{-1}).$$

**5.2. Universality results.** As shown in [15, Proposition 4.4], using Muchizuki's wallcrossing formula [29, Theorem 1.4.6] the type I contributions in (87) are expressed in terms of Seiberg-Witten invariants coupled with certain combinatorial coefficients. The resulting formula is reviewed below.

Mochizuki's combinatorial coefficients are written in terms of equivariant integrals on products of Hilbert schemes of points on  $S$ . The equivariant structure is defined with respect to a torus  $\mathbf{T}' = \mathbb{C}^\times$  which acts trivially on  $S$  and its Hilbert schemes. One should make a clear distinction between  $\mathbf{T}'$  and the torus  $\mathbf{T}$  used above in virtual localization computations, which are completely unrelated. In fact the  $\mathbf{T}'$  action has its origin in a torus action on the master space used in the proof of [29, Theorem 1.4.6], which is not manifestly used in this paper. Let  $\mathbf{t}'$  denote the trivial line bundle on  $S$  with the  $\mathbb{C}^\times$ -action of weight 1 on the fibers and let  $\mathbf{s}' := c_1(\mathbf{t}')$ . Below let  $\pi'$  denote the projection

$$S \times S^{[n_1]} \times S^{[n_2]} \rightarrow S^{[n_1]} \times S^{[n_2]}.$$

Then for any  $\alpha \in H_2(S, \mathbb{Z})$  let  $L_\alpha$  be a line bundle on  $S$  with  $c_1(L_\alpha) = \alpha$  and let

$$(89) \quad V_{L_\alpha}^{[n_i]} := \pi'_* (\mathcal{O}_{\mathbb{Z}^{[n_i]}} \otimes L_\alpha).$$

Note that  $L_\alpha$  and hence  $V_{L_\alpha}^{[n_i]}$  are uniquely determined by  $\alpha$  up to isomorphism since  $H^1(\mathcal{O}_S) = 0$ . Moreover, for any equivariant sheaf  $\mathcal{E}$  on  $S \times S^{[n_1]} \times S^{[n_2]}$  let

$$P(\mathcal{E}) = e(-\mathbf{R}\mathcal{H}om_{\pi'}(\mathcal{E}, \mathcal{E} \otimes L \cdot \mathbf{t}))$$

Then, following Mochizuki, for any pair of effective curves classes  $(\beta_1, \beta_2) \in H^2(S, \mathbb{Z})^{\oplus 2}$  such that  $\beta = \beta_1 + \beta_2$  let the coefficients  $A(\gamma, \beta_1, \beta_2)$  be defined by

$$A(\gamma, \beta_1, \beta_2) := \sum_{\substack{n_1+n_2=\beta \\ n-\beta_1 \cdot \beta_2}} \int_{S^{[n_1]} \times S^{[n_2]}} \text{Res}_{s'=0} \left( \frac{e(V_{L\beta_1}^{[n_1]}) \cdot P(\mathcal{I}_{L\beta_1}^{[n_1]} \cdot \mathbf{t}'^{-1} \oplus \mathcal{I}_{L\beta_2}^{[n_2]} \cdot \mathbf{t}') \cdot e(V_{L\beta_2}^{[n_2]} \cdot \mathbf{t}'^2)}{(2s')^{n_1+n_2-p_g} \cdot Q(\mathcal{I}_{L\beta_1}^{[n_1]} \cdot \mathbf{t}'^{-1}, \mathcal{I}_{L\beta_2}^{[n_2]} \cdot \mathbf{t}')} \right).$$

where

$$Q(\mathcal{I}_{L\beta_1}^{[n_1]} \cdot \mathbf{t}'^{-1}, \mathcal{I}_{L\beta_2}^{[n_2]} \cdot \mathbf{t}') = e\left(-\mathbf{R}\mathcal{H}om_{\pi'}(\mathcal{I}_{L\beta_1}^{[n_1]} \cdot \mathbf{t}'^{-1}, \mathcal{I}_{L\beta_2}^{[n_2]} \cdot \mathbf{t}') - \mathbf{R}\mathcal{H}om_{\pi'}(\mathcal{I}_{L\beta_2}^{[n_2]} \cdot \mathbf{t}', \mathcal{I}_{L\beta_1}^{[n_1]} \cdot \mathbf{t}'^{-1})\right).$$

Then the following formula is proven in [15, Proposition 4.4]:

$$(90) \quad \text{DT}_h(X, \gamma)_I = - \sum_{\substack{\beta_1+\beta_2=\beta \\ \beta_1 \cdot h < \beta_2 \cdot h}} \text{SW}(\gamma_1) \cdot 2^{2-\chi(v)} \cdot A(\gamma_1, \gamma_2, v; P_1 \cup \alpha).$$

where the sum in the right hand side is over all decompositions  $\beta = \beta_1 + \beta_2$  with  $\beta_1, \beta_2$  effective curve classes.

By analogy with a result of Göttsche and Kool [19, Proposition 3.3] using [20, Lemma 5.5] and an adaptation of [9] one has the following universality statement.

**Proposition 5.1.** *The expression (89) is a universal polynomial in the topological invariants*

$$\beta_i^2, \beta_i \cdot c_1(S), \beta_i \cdot D, \beta_1 \cdot \beta_2, D^2, D \cdot c_1(S), c_1(S)^2, c_2(S).$$

with  $1 \leq i \leq 2$ .

An analogous universality result for the type II contributions (88) is not readily available, except for type II components associated to nested Hilbert scheme of points with no divisorial twists, i.e.  $S^{[n_1 \geq n_2]} := S_\alpha^{[n_1, n_2]}$  with  $\alpha = 0$ . In this case, using [16, Theorem 6] the right hand side of (88) simplifies as follows.

For any pair  $(n_1, n_2) \in \mathbb{Z}^2$ ,  $n_1, n_2 \geq 0$ , note that there is a natural closed embedding  $\iota : S^{[n_1, n_2]} \rightarrow S^{[n_1]} \times S^{[n_2]}$  and a cartesian square

$$(91) \quad \begin{array}{ccc} S \times S^{[n_1, n_2]} & \xrightarrow{\iota'} & S \times S^{[n_1]} \times S^{[n_2]} \\ \downarrow \pi & & \downarrow \pi' \\ S^{[n_1, n_2]} & \xrightarrow{\iota} & S^{[n_1]} \times S^{[n_2]}, \end{array}$$

where  $\pi, \pi'$  are the natural projections. Then, as in [15, Definition 3.6] for any line bundle  $M$  on  $S$ , let

$$E_M^{n_1, n_2} = [\mathbf{R}\pi'_* M] - [\mathbf{R}\mathcal{H}om_{\pi'}(\mathcal{I}^{[n_1]}, \mathcal{I}^{[n_2]} \otimes M)].$$

Then the following holds

**Lemma 5.2.** *The equivariant residual contribution of a Type II component  $\mathcal{T} \cong S^{[n_1, n_2]}$  is given by*

$$(92) \quad DT_0^{[n_1, n_2]} = \frac{(-1)^{c_1(S) \cdot D/2 + 3D^2/2}}{2\chi(L^2)(-\mathbf{s})\chi(L^2) + \chi(L) - \chi(L^{-1})} \int_{S^{[n_1]} \times S^{[n_2]}} \frac{c_{n_1+n_2}(E^{n_1, n_2}) \cup e(\mathbb{T}_{S^{[n_1]}} \otimes L \cdot \mathbf{t}) e(\mathbb{T}_{S^{[n_2]}} \otimes L \cdot \mathbf{t}) e(E_{K_S \otimes L^{-2}}^{n_1, n_2} \cdot \mathbf{t}^{-2})}{e(E_{K_S \otimes L^{-1}}^{n_1, n_2} \cdot \mathbf{t}^{-1}) \cdot e(E_{L^{-1}}^{n_1, n_2} \cdot \mathbf{t}^{-1})},$$

*Proof.* The  $K$ -theory class  $E_{M \cdot \mathbf{t}^a}^{n_1, n_2}$  satisfies the pull-back property

$$(93) \quad \iota^* E_{M \cdot \mathbf{t}^a}^{n_1, n_2} \cong [\mathbf{R}\pi_* M] - [\mathbf{R}\pi_*(\mathcal{I}^{[n_1]}, \mathcal{I}^{[n_2]} \otimes M \cdot \mathbf{t}^a)]$$

since the universal sheaves  $\mathcal{I}^{[n_i]}$  on  $S^{[n_1, n_2]}$  are naturally isomorphic to the restriction of the analogous objects from  $S^{[n_1]} \times S^{[n_2]}$ . This implies that

$$\iota^* e(E_{M \cdot \mathbf{t}^a}^{n_1, n_2}) = e(Q_0^{[n_1, n_2]}(M, a))$$

for any  $M$  and any  $a \in \mathbb{Z}$ . Furthermore, as shown in [16, Lemma 5.4], the equivariant class  $Q_0^{[n_1, n_2]}(M, a)$  is well behaved under good degenerations on  $S$ . For completeness, note that the right hand side of (93) was denoted by  $K_{M \cdot \mathbf{t}^a}^{[n_1 \geq n_2]}$  in [16, Lemma 5.4], assuming without loss of generality that  $n_1 \geq n_2$ . Moreover, the good degeneration property is explicitly stated in [16, Remark 5.10].

In conclusion Theorem 6 in [16] applies to the present situation, leading to equation (92). □

To conclude, applying [20, Lemma 5.5] to our situation, [16, Corollary 5.9, Remark 5.10, Proposition 5.11, Appendix 5.A.] yield the following

**Proposition 5.3.** *Each type II contribution  $DT_0^{[n_1, n_2]}$  is a universal polynomial in the topological invariants*

$$\beta_i^2, \quad \beta_i \cdot c_1(S), \quad \beta_i \cdot D, \quad \beta_1 \cdot \beta_2, \quad D^2, \quad D \cdot c_1(S), \quad c_1(S)^2, \quad c_2(S).$$

with  $1 \leq i \leq 2$ .

## 6. Rank two calculations for local elliptic K3 surfaces

The goal of this section is to carry out explicit computations and formulate a modularity conjecture for rank two invariants in a specific class of examples. Namely, the surface  $S$  in Section 4 will be a smooth generic elliptic K3 surface in Weierstrass form and the divisor  $D$  will be a positive multiple of the elliptic fiber class.

**6.1. Chamber structure for local elliptic surfaces.** In this section  $S$  will be a smooth generic elliptic surface in Weierstrass form over the complex projective line  $C = \mathbb{P}^1$ . It will be assumed that all singular fibers of  $S$  over  $C$  are nodal elliptic curves, and there is at least one such fiber. In particular all fibers are irreducible and there are no multiple fibers. Under these assumptions [10, Thm. 24, Ch. 7] shows that  $S$  has trivial fundamental group, while Proposition 27, Ch. 7 in loc. cit. shows that  $H^2(X, \mathbb{Z})$  is torsion free. In particular  $H^1(S, \mathbb{Z}) = 0$ , hence  $H^1(\mathcal{O}_S) = 0$  as well.

Furthermore it will be assumed throughout this section that the Mordell-Weil group of  $S$  is trivial. This implies that the Picard group of  $S$  is freely generated by  $(f, \sigma)$ , where  $\mathcal{F}$  is the elliptic fiber class. Moreover, there is a natural isomorphism  $\text{Pic}(S) \cong H_2(S, \mathbb{Z})$  which will be implicitly used below. Finally, [10, Thm. 15, Ch. 7] shows that the canonical line bundle of such a surface is isomorphic to  $K_S \cong \mathcal{O}_S(kf)$  for some  $k \in \mathbb{Z}$ ,  $k \geq -1$ . In this section it will be assumed that  $k \geq 0$ . Since the canonical section is isomorphic to the projective line, a simple computation yields the intersection numbers:

$$(94) \quad \sigma^2 = -k - 2, \quad f^2 = 0, \quad \sigma \cdot f = 1.$$

Moreover, note the following.

**Lemma 6.1.** *Under the above assumptions,*

- (i) *The cone effective divisors in  $\text{Pic}(S)$  is generated by  $(\sigma, f)$ , and*
- (ii) *A real divisor class  $h = t\sigma + uf$  in  $\text{Pic}_{\mathbb{R}}(S)$ , with  $t, u \in \mathbb{R}$  is ample if and only if  $0 < (k+2)t < u$ .*

*Proof.* Suppose  $\Sigma \subset S$  is an nonempty irreducible curve on  $S$  and let  $[\Sigma] \in H_2(S, \mathbb{Z})$  denote the associated curve class. Under the current assumptions,  $\text{Pic}(S) \cong H_2(S, \mathbb{Z})$  is freely generated by  $(\sigma, f)$ . Therefore  $[\Sigma] = a\sigma + bf$  with  $a, b \in \mathbb{Z}$ . Since  $\Sigma$  is assumed irreducible, it can be either a multiple of a fiber of the projection  $\pi : S \rightarrow C$ , or an irreducible finite-to-one cover of  $C$ . In both cases, note that  $[\Sigma] \cdot f \geq 0$ , hence  $a \geq 0$ . If  $a = 0$ , it follows that  $b > 0$  since  $\Sigma$  is assumed effective, non-empty.

Suppose  $a > 0$ . Since  $\Sigma$  is irreducible, and the Mordell-Weil group of  $S$  is trivial by assumption, one of the following cases must hold:

- (a)  $\Sigma$  is set theoretically supported on the canonical section, in which case  $b = 0$ ,  
 or  
 (b)  $\Sigma$  is not set theoretically supported on the canonical section, in which case  
 $[\Sigma] \cdot \sigma > 0$ . Using (94) this implies  $b > a(k + 2) > 0$ .

In conclusion, any effective irreducible curve class must be of the form  $a\sigma + bf$  with  $a, b \geq 0$ . Hence the same holds for an arbitrary effective curve class, proving (i). The second claim follows immediately from the intersection table (94).  $\square$

Now, in the framework of Section 4, let  $S$  be as above, and let  $D = mf$  with  $m \geq 1$ . Hence the local fourfold  $X$  is the total space of the direct sum of line bundles  $K_S(mf) \oplus \mathcal{O}_S(-mf) \cong \mathcal{O}_S((k+m)f) \oplus \mathcal{O}_S(-mf)$ . Since  $m \geq 1$  by assumption, Lemma 4.3 implies that any  $h$ -stable compactly supported pure dimension two sheaf on  $X$  is scheme theoretically supported on  $Y$ . The main goal of the section is to prove a global boundedness result for such sheaves with fixed topological invariants. This will require a Bogomolov type inequality, which will be derived below from the following categorical equivalence proven in [31, Proposition 2.2].

Let  $\text{Higgs}_L(S)$  denote the abelian category of coherent Higgs sheaves on  $(\mathcal{E}, \phi)$  on  $S$ , where  $\mathcal{E}$  is a coherent sheaf on  $S$  and  $\phi : \mathcal{E} \rightarrow \mathcal{E} \otimes L$  a morphism of  $\mathcal{O}_S$ -modules. Let  $\text{Coh}_c(Y)$  denote the abelian category of compactly supported coherent sheaves on  $Y$ . Recall that  $\pi_S : Y \rightarrow S$  denotes the natural projection and  $\zeta : H^0(Y, \pi_S^*L)$  is the tautological section. Then note that any object  $\mathcal{F}$  in  $\text{Coh}_c(Y)$  determines a Higgs sheaf  $(\mathcal{E}, \phi)$  on  $S$ , where

- $\mathcal{E} = \pi_{S*}\mathcal{F}$ , and
- $\phi : \mathcal{E} \rightarrow \mathcal{E} \otimes L$  is the pushforward of the natural map

$$\zeta \otimes \mathbf{1}_{\mathcal{F}} : \mathcal{F} \otimes \pi_S^*L^{-1} \rightarrow \mathcal{F}.$$

Then the following holds ([31, Proposition 2.2]).

**Proposition 6.2.** *There is an equivalence of categories  $\text{Coh}_c(Y) \xrightarrow{\sim} \text{Higgs}_L(S)$  which maps  $\mathcal{F}$  to the pair  $(\mathcal{E}, \phi)$  constructed above.*

Recall that the topological invariants of a compactly supported sheaf  $\mathcal{F}$  on  $X$  where defined in Section 4 as the Chen classes of  $\mathcal{E} = g_*\mathcal{F}$  encoded in the triple

$$\gamma(F) = (r(\mathcal{F}), \beta(\mathcal{F}), n(\mathcal{F})) \in \mathbb{Z} \oplus H_2(S, \mathbb{Z}) \oplus \mathbb{Z}.$$

Then note the following consequence of Proposition 6.2, proven in [31, Lemma 2.9]:

**Corollary 6.3.** *Let  $h \in \text{Pic}_{\mathbb{R}}(S)$  be an ample real divisor class on  $S$ . Let  $\mathcal{F}$  be a compactly supported sheaf on  $Y$  with  $r(\mathcal{F}) > 0$ . Then  $\mathcal{F}$  is  $h$ -(semi)stable if and only if the associated Higgs sheaf  $(\mathcal{E}, \phi)$  is  $h$ -(semi)stable.*

Next note that the Bogomolov inequality holds for slope semistable Higgs sheaves on  $S$ . As shown below, this follows from [1, Thm. 4.3], which is proven analytically. In order to show that the conditions of loc. cit are satisfied, one first needs the following.

**Lemma 6.4.** *Let  $h = t\sigma + uf$  a real ample class on  $S$  with  $0 < (k+2)t < u$ . Let  $\omega_C$  be the standard Fubiny-Study Kähler form on  $C = \mathbb{P}^1$ . Then there exists a Kähler form  $\omega_h$  such that the Hodge class  $[\omega_h] \in H^{1,1}(S, \mathbb{R})$  is Poincaré dual to  $h$ , and  $\omega_h|_C = (u - (k+2)t)\omega_C$ .*

*Proof.* Since  $S$  is assumed to be in Weierstrass form, it admits a presentation as a hypersurface of bi-degree  $(6(k+2), 3)$  in the toric variety

$$Z = \left( \mathbb{C}^2 \times \mathbb{C}^3 \setminus (\mathbb{C}^2 \times \{0\} \cup \{0\} \times \mathbb{C}^3) \right) / \mathbb{C}^\times \times \mathbb{C}^\times$$

where the  $\mathbb{C}^\times \times \mathbb{C}^\times$  action is given by

$$\begin{array}{ccccc} u & v & x & y & z \\ 1 & 1 & 2(k+2) & 3(k+2) & 0 \\ 0 & 0 & 1 & 1 & 1 \end{array}$$

The divisor classes  $\sigma, f \in \text{Pic}(S)$  are obtained by restriction of toric divisors,

$$\sigma = Z|_S, \quad f = U|_S$$

where  $U, Z$  are the toric divisors defined by  $u = 0$  and  $z = 0$  respectively. These are the same as the standard generators of the Picard group of  $Z$  associated to the two  $\mathbb{C}^\times$  factors. Then the claim follows from the correspondence between toric and symplectic Kähler quotients. q.e.d.

The next result needed in the following is the existence of a suitable hermitian structure on  $L$ .

**Lemma 6.5.** *Let  $\omega_h$  be a Kähler form on  $S$  as in Lemma 6.4. Then there exists a hermitian structure on  $L$  and a hermitian connection with curvature  $\gamma$  such that the contraction  $\Lambda_{\omega_h} \gamma$  is pointwise positive semidefinite.*

*Proof.* Note that  $L$  is the pull-back of the line bundle  $\mathcal{O}_C(m+k)$ . The latter admits a hermitian structure and a hermitian connection with curvature  $(m+k)\omega_C$ . The required structure on  $L$  is obtained by pull-back. q.e.d.

As usual for any rank  $r > 0$  torsion free Higgs sheaf  $(\mathcal{E}, \phi)$  on  $S$  let

$$\Delta(\mathcal{E}) = c_2(\mathcal{E}) - \frac{r-1}{2r} c_1(\mathcal{E})^2.$$

Then the following holds.



**Lemma 6.6.** *Let  $(\mathcal{E}, \phi)$  be a  $\mu_h$ -semistable rank  $r > 0$  torsion free Higgs sheaf on  $S$ . Then*

$$\Delta(\mathcal{E}) \geq 0.$$

*Proof.* First suppose  $E$  is locally free. Then Lemmas 6.4 and 6.5 show that the conditions of [1, Thm. 4.3] are satisfied in the present situation. This implies the result in this case. If  $E$  is torsion free but not locally free, the double dual Higgs sheaf  $(\mathcal{E}^{\vee\vee}, \phi^{\vee\vee})$  is  $\mu_h$ -semistable and locally free. Moreover,  $c_2(\mathcal{E}) - c_2(\mathcal{E}^{\vee\vee}) = \chi(\mathcal{E}^{\vee\vee}/\mathcal{E}) > 0$  while  $c_1(\mathcal{E}) = c_1(\mathcal{E}^{\vee\vee})$ . q.e.d.

Using Lemma 6.6, one proves the following global boundedness result. Let  $\mathcal{F}$  be a  $\mu_h$ -stable compactly supported sheaf on  $Y$  with invariants

$$(95) \quad \gamma(F) = (r, \beta, n) \in \mathbb{Z} \oplus H_2(S, \mathbb{Z}) \oplus \mathbb{Z}.$$

Suppose  $r > 0$ . According to Corollary 6.3, the associated Higgs bundle on  $S$  is also  $\mu_h$ -stable, hence  $n \geq 0$  by Lemma 6.6. Then one has:

**Lemma 6.7.** *Suppose there exists  $t' \in \mathbb{R}$ ,  $0 < t' < t$ , such that  $E$  is not  $\mu_{h'}$ -semistable for  $h' = t'\sigma + uf$ . Then*

$$\frac{t}{u} > \frac{2}{k+2+2r^3\Delta(E)}$$

*Proof.* Completely analogous to the proof of [41, Proposition 6.2] and [8, Lemma 4.7]. Uses the Bogomolov inequality and the algebraic Hodge index theorem on  $S$ . q.e.d.

The main consequence of Lemma 6.7 is the absence of marginal stability walls for sheaves  $\mathcal{F}$  with fixed topological invariants (95) in the open chamber

$$\frac{t}{u} < \frac{2}{k+2+2r^3\Delta(E)}.$$

By analogy with [43, Lemma 1.2] and also [8, Proposition 1.2] this implies a generic stability result for such sheaves. For completeness, this will be formulated and proved below in detail. Several intermediate steps are necessary.

Let  $\delta \subset C$  be the discriminant of the elliptic fibration, which, under the current assumptions, consists of  $12(k+2)$  pairwise distinct closed points.

**Lemma 6.8.** *Let  $\mathcal{F}$  be a compactly supported sheaf on  $Y$  with topological invariants (95), where  $r > 0$ . Suppose there exists an open subset  $V \subset C \setminus \delta$  such that  $\mathcal{F}_V = \mathcal{F} \otimes q^* \mathcal{O}_V$  has pure dimension two. Then for any point  $p \in V$  there is an exact sequence*

$$0 \rightarrow \mathcal{F} \otimes \mathcal{O}_Y(-Y_p) \rightarrow \mathcal{F} \rightarrow \mathcal{F}_p \rightarrow 0$$

where  $Y_p \subset Y$  is the fiber  $Y_p = \pi_S^{-1}(p)$ , and  $\mathcal{F}_p = \mathcal{F} \otimes \mathcal{O}_{Y_p}$ . Moreover,

$$\text{ch}_0(\mathcal{F}_p) = 0, \quad \text{ch}_1(\mathcal{F}_p) = rf, \quad \text{ch}_2(\mathcal{F}_p) = \beta \cdot f.$$

*Proof.* Under the given assumptions, the scheme theoretic support of  $\mathcal{F}$  is a proper horizontal divisor in  $Y$  i.e. a compact finite-to-one cover of  $S$  which intersects  $Y_p$  along a curve. In particular the local tor sheaf  $\mathcal{T}or_1^Y(\mathcal{O}_{Y_p}, \mathcal{F})$  is set theoretically supported in dimension one, hence the natural map  $\mathcal{T}or_1^Y(\mathcal{O}_{Y_p}, \mathcal{F}) \rightarrow \mathcal{F}$  must be identically zero for any  $p \in V$ , by the purity assumption for  $\mathcal{F}$ . This proves exactness. The Chern character follows by a straightforward computation in the intersection ring of  $Y$ .  $\square$

**Remark 6.9.** *In the framework of Lemma 6.8 note that  $Y_p \cong S_p \times \mathbb{A}^1$  is an elliptic surface, and  $\mathcal{F}_p$  is set theoretically supported on a finite set of elliptic fibers  $Y_p$  over  $\mathbb{A}^1$ . In particular it has compact support. In the following a compactly supported sheaf  $\mathcal{G}$  on  $Y_p$  with invariants*

$$\text{ch}_0(\mathcal{G}) = 0, \quad \text{ch}_1(\mathcal{G}) = r_{\mathcal{G}}f$$

*will be called (semi)stable if and only if*

$$\text{ch}_2(\mathcal{G}')/r_{\mathcal{G}'} (\leq) \text{ch}_2(\mathcal{G})/r_{\mathcal{G}}$$

*for any proper nontrivial subsheaf  $0 \subset \mathcal{G}' \subset \mathcal{G}$  with invariants*

$$\text{ch}_0(\mathcal{G}') = 0, \quad \text{ch}_1(\mathcal{G}') = r_{\mathcal{G}'}f, \quad r_{\mathcal{G}'} > 0.$$

Next note the following lemma, which follows from Theorems 2.1.5 and 2.3.2 in [26].

**Lemma 6.10.** *Let  $\mathcal{F}$  be a compactly supported pure dimension two sheaf on  $Y$  with invariants  $(r, \beta, n)$ ,  $r > 0$ . Then there exist an open subset  $V \subset C \setminus \delta$  and a relative Harder-Narasimhan filtration*

$$(96) \quad 0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_l = \mathcal{F}_V$$

*such that all successive quotients are flat over  $V$ .*

Using Lemma 6.10, one next proves:

**Lemma 6.11.** *Let  $\mathcal{F}$  be a compactly supported sheaf on  $Y$  of pure dimension two with invariants  $(r, \ell f, n)$ ,  $r > 0$ ,  $\ell \in \mathbb{Z}$ . Suppose  $\mathcal{F}$  is  $\mu_h$ -semistable and*

$$(97) \quad \frac{t}{u} < \frac{2}{k + 2 + 2nr^3}.$$

*Then there exists an open subset  $V \subset C \setminus \delta$  such that the restriction  $\mathcal{F}_p = \mathcal{F}|_{Y_p}$  is semistable for any closed point  $p \in V$ .*

*Proof.* This statement is analogous to [43, Lemma 1.2] and also [8, Proposition 1.2]. The proof is given below for completeness.

Let  $\mathcal{F}$  be a sheaf as in Lemma 6.11 and let  $V \subset C \setminus \delta$  be the open subset of Lemma 6.10. By construction the successive quotients of the filtration (96) are flat over  $V$  and the filtration restricts to a Harder-Narasimhan filtration for  $\mathcal{F}_p$  on each fiber  $Y_p$ ,  $p \in V$ . Using [25, Ex. 5.15(d)], this filtration extends to a filtration of sheaves

$$0 \subset \overline{\mathcal{F}}_1 \subset \cdots \subset \overline{\mathcal{F}}_1 = \mathcal{F}$$

on  $Y$ . Let

$$\gamma(\overline{\mathcal{F}}_1) = (r_1, \beta_1, n_1) \in \mathbb{Z} \oplus H_2(S, \mathbb{Z}) \oplus \mathbb{Z}.$$

be the invariants of  $\overline{\mathcal{F}}_1$ . By construction,  $r_1 > 0$ . Using Lemma 6.8, one finds

$$\text{ch}_0(\overline{\mathcal{F}}_{1,p}) = 0, \quad \text{ch}_1(\overline{\mathcal{F}}_{1,p}) = r_1 f, \quad \text{ch}_2(\overline{\mathcal{F}}_{1,p}) = \beta_1 \cdot f.$$

At the same time

$$\text{ch}_0(\overline{\mathcal{F}}_p) = 0, \quad \text{ch}_1(\overline{\mathcal{F}}_p) = r f, \quad \text{ch}_2(\overline{\mathcal{F}}_p) = 0.$$

Therefore one must have

$$\beta_1 \cdot f \geq 0.$$

However, by assumption,  $\mathcal{F}$  is  $\mu_h$ -slope semistable for any  $t > 0$  satisfying inequality (100). This implies that

$$\beta_1 \cdot f \leq 0$$

Therefore  $\beta_1 \cdot f = 0$ , which implies that  $\mathcal{F}_p$  is semistable. □

**6.2. Type II fixed loci for elliptic K3 surfaces.** Lemma 6.11 has important consequences for the structure of type II fixed loci, assuming  $S$  to be an elliptic K3 surface. In this section the rank will be fixed,  $r = 2$ . Recall that each connected component of the type II fixed locus is isomorphic to a nested Hilbert scheme  $S_{(\beta_1, \beta_2 + c_1(L))}^{[n_1, n_2]}$  parameterizing flags of twisted ideal sheaves  $\mathcal{I}_1 \otimes L_1 \subset \mathcal{I}_2 \otimes L_2 \otimes L$ . Here  $\mathcal{I}_i$ ,  $1 \leq i \leq 2$  are ideal sheaves of zero dimensional subschemes of  $S$  and  $L_1, L_2$  are line bundles on  $S$  such that

$$c_2(\mathcal{I}_i) = n_i, \quad c_1(L_i) = \beta_i, \quad 1 \leq i \leq 2.$$

Moreover, the topological invariants  $(n_1, n_2)$ ,  $\beta_1, \beta_2$  are subject to the conditions specified in equation (86) as well as (II.a) and (II.b) below (86). In particular the curve class  $\alpha = \beta_2 + c_1(L)_\beta - 1$  must be effective. As shown in [15, Section 3.1], the associated stable sheaf  $\mathcal{F}$  on  $Y$  is then an extension of  $\mathcal{O}_Y$ -modules

$$(98) \quad 0 \rightarrow \mathcal{I}_2 \otimes L_2 \otimes L \rightarrow \mathcal{F} \rightarrow \mathcal{I}_1 \otimes L_1 \rightarrow 0.$$

For completeness note that this also follows from Proposition 2.2 in [31]. Denoting the injection  $\mathcal{I}_1 \otimes L_1 \subset \mathcal{I}_2 \otimes L_2 \otimes L$  by  $\varphi$ , each such triple determines a Higgs sheaf  $(\mathcal{E}, \phi)$  on  $S$  with

$$\mathcal{E} = \mathcal{I}_1 \otimes L_1 \oplus \mathcal{I}_2 \otimes L_2, \quad \phi = \begin{pmatrix} 0 & 0 \\ \varphi & 0 \end{pmatrix}.$$

Then [31, Proposition 2.2] implies that the associated sheaf on  $\mathcal{F}$  must be an extension of the form (98).

For an elliptic surface as above, each class  $\beta_i$  is of the form  $\beta_i = a_i\sigma + b_if$ ,  $1 \leq i \leq 2$ . Then Lemma 6.11 yields the following.

**Lemma 6.12.** *Let  $\mathcal{F}$  be an extension of the form (98) corresponding to a flag of twisted ideal sheaves  $\mathcal{I}_1 \otimes L_1 \subset \mathcal{I}_2 \otimes L_2 \otimes L$  with  $\alpha = \beta_2 + c_1(L) - \beta_1$  effective. Suppose  $\mathcal{F}$  is  $\mu_h$ -semistable,  $a_1 + a_2 = 0$ , and*

$$(99) \quad \frac{t}{u} < \frac{2}{k+2+16n}.$$

Then  $a_1 = a_2 = 0$ .

*Proof.* As shown in Lemma 6.11, under the given conditions,  $\mathcal{F}$  has to be  $\mu_h$ -semistable for all polarizations  $h' = t'\sigma + uf$ , with  $0 < t' < t$ . This implies  $a_2 \leq 0$ . Since  $\alpha$  is assumed effective, Lemma 6.1 implies that  $a_1 = a_2 = 0$ . □

**Lemma 6.13.** *Let  $S$  be an elliptic K3 surface in Lemma 6.12. Let  $\mathcal{F}$  be a  $\mu_h$ -semistable extension of the form (98) with  $\beta_1 = bf$ ,  $\beta_2 = (1-b)f$ . Moreover, suppose that*

$$(100) \quad \frac{t}{u} < \frac{1}{8n+1}.$$

Then the following inequalities hold

$$(101) \quad 1 \leq b \leq (m+1)/2.$$

*Proof.* In this case  $\alpha = (m+1-2b)f$ . Since this must be effective by assumption,  $2b \leq m+1$ . Moreover, condition (II.b) below equation (86) implies  $b \geq 1$ . □

In conclusion, using equation (86), Lemma 6.13 yields:

**Lemma 6.14.** *Let  $S$  be a smooth generic elliptic K3 surface in Weierstrass form and let  $\gamma = (2, f, n)$  be a triple of topological invariants with  $n \geq 0$ . Suppose inequality (100) is satisfied. Then the type II fixed locus is a union*

$$(102) \quad \mathcal{M}_h(X, \gamma)_{II}^{\mathbf{T}} = \bigcup_{1 \leq b \leq (m+1)/2} \bigcup_{\substack{n_1, n_2 \geq 0 \\ n_1 + n_2 = n \\ n_1 \geq n_2 \text{ if } b = (m+1)/2}} S_{(m+1-2b)f}^{[n_1, n_2]}.$$

**6.3. Partition functions and modularity conjectures.** In this section  $S$  will be a smooth generic elliptic K3 surface in Weierstrass form. Recall that  $X$  is the total space  $K_S(D)$ , which in this case is isomorphic to  $\mathcal{O}_S(mf)$ ,  $m \geq 1$ . For any triple  $\gamma = (2, f, n)$ , with  $n \geq 0$ , let

$$DT_\infty(X, \gamma)_I = DT_h(X, \gamma)_I, \quad DT_\infty(X, \gamma)_{II} = DT_h(X, \gamma)_{II},$$

for any polarization  $h = t\sigma + uf$  satisfying inequality (100). Lemmas 6.7 and 6.14 show that these invariants are independent of the choice of  $h$  as long as inequality (100) is satisfied. Then let

$$Z_\infty(X, 2, f; q)_I = \sum_{n \geq 0} DT_\infty(X, 2, f, n)_I q^{n-2}$$

and

$$Z_\infty(X, 2, f; q)_{II} = \sum_{n \geq 0} DT_\infty(X, 2, f, n)_{II} q^{n-2}$$

the generating functions of such invariants. Note that a priori the above expressions are elements of  $q^{-2}\mathbf{Q}(\mathbf{s})[[q]] \subset \mathbf{Q}(\mathbf{s})[[q^{-1}, q]]$ .

*Proof of Proposition 1.3.* Recall that Proposition 1.3 claims that  $Z_\infty(X, 2, f; q)_I$  belongs to  $\mathbf{Q}(\mathbf{s})[[q]] \subset \mathbf{Q}(\mathbf{s})[[q^{-1}, q]]$  and that the following identity holds in  $\mathbf{Q}(\mathbf{s})[[q^{1/2}]]$ :

$$(103) \quad Z_\infty(X, 2, f; q)_I = \frac{\mathbf{s}^{-1}}{2} (\Delta^{-1}(q^{1/2}) + \Delta^{-1}(-q^{1/2}))$$

In order to prove this note that the type I contributions are given by equation (87), which is reproduced below for convenience:

$$DT_h(X, \gamma)_I = \int_{[\mathcal{M}_h(S, \gamma)]_0^{\text{vir}}} \frac{1}{e(\mathbf{R}\mathcal{H}om_\pi(\mathbb{E}, \mathbb{E} \otimes L \cdot \mathbf{t}))}.$$

Under the current assumptions Proposition 5.1 shows that the virtual integral in the right hand side is independent of  $L \cong \mathcal{O}_S(mf)$  since

$$D^2 = D \cdot c_1(S) = D \cdot \beta = 0.$$

As a result, the above integral equals

$$DT_h(X, \gamma)_I = \int_{[\mathcal{M}_h(S, \gamma)]_0^{\text{vir}}} \frac{1}{e(\mathbf{R}\mathcal{H}om_\pi(\mathbb{E}, \mathbb{E} \otimes \mathbf{t}))}.$$

As explained in [15, Remark. 2.10], these invariants are related to the Vafa-Witten invariants  $\text{VW}_h(S, \gamma)$  defined in [31] by

$$DT_h(X, \gamma)_I = \mathbf{s}^{-1} \text{VW}_h(S, \gamma).$$

Moreover, Proposition 7.4 of loc. cit. shows that under the present assumptions, the latter are given by the topological Euler character of the moduli space  $\mathcal{M}_h(S, \gamma)$ , which is in this case smooth. In conclusion, one has

$$DT_h(X, \gamma)_I = \mathbf{s}^{-1} \chi^{\text{top}}(\mathcal{M}_h(S, \gamma)).$$

Finally, note that the moduli space  $\mathcal{M}_h(S, \gamma)$  is shown in [30] and [26, Section 6] to be deformation equivalent to the Hilbert scheme  $S^{[2n-3]}$ . In particular the moduli space is empty for  $0 \leq n \leq 1$ , which proves that indeed the series  $Z_\infty(X, 2, f; q)_I$  belongs to  $\mathbb{Q}(\mathbf{s})[[q]] \subset \mathbb{Q}(\mathbf{s})[[q^{-1}, q]]$ . Moreover, identity (103) follows by straightforward computations from Goettsche's formula.  $\square$

**Proposition 6.15.** *Under the same conditions as in Lemma 6.13, the type II contributions  $DT_\infty(X, \gamma)_{II}$  vanish for  $m$  even, and reduced to*

$$(104) \quad DT_\infty(X, \gamma)_{II} = \sum_{\substack{n_1 \geq n_2 \geq 0 \\ n_1 + n_2 = m}} DT_0^{[n_1, n_2]}$$

for  $m$  odd.

*Proof.* Note that for  $1 \leq b \leq (m+1)/2$  one has  $h^0(S, \mathcal{O}_S((m+1-2b)f)) > 0$  and

$$h^2(S, \mathcal{O}_S((m+1-2b)f)) = \begin{cases} 0, & \text{if } 2b < m+1, \\ 1, & \text{if } 2b = m+1. \end{cases}$$

This shows that under to given assumptions all type II fixed loci on  $S$  satisfy the vanishing conditions of [15, Thm. 7] for  $2b < m+1$ , i.e.

$$h^0(S, \mathcal{O}_S(m+1-2b)) > 0, \quad h^2(S, \mathcal{O}_S(m+1-2b)) = 0$$

If  $2b = m+1$ , one has  $\alpha = 0$ , hence all type II components satisfying this condition are isomorphic to nested Hilbert schemes without divisorial twists. The condition  $n_1 \geq n_2$  comes from equation (102).  $\square$

In conclusion, for  $m$  even, the partition function  $Z_\infty(X, 2, f; q)$  equals the generating function of type I contributions (103). For  $m$  odd one obtains the additional type II contributions (104). Each term in this formula is an equivariant residual integral of the form (92). Again, Proposition 5.3 shows that under the present conditions, the integrals (92) are independent of  $m$  since  $D^2 = D \cdot c_1(S) = D \cdot \beta = 0$ . Therefore each integral (92) is in fact equal to its  $m = 0$  counterpart. There is no contradiction here, since the condition that  $m$  be odd only occurred as a selection rule for type II contributions in Lemma 6.14 and Proposition 6.15. Once that

is done, the resulting integrals (92) can be computed in isolation since they are certainly well defined for any value of  $m$ , in particular  $m = 0$ .

Setting  $m = 0$  in (92), as explained in [15, Remark 2.10], the resulting invariants *formally* coincide up to an equivariant factor to the type II Vafa-Witten invariants defined and computed in Sections 8.7-8.9 of [31]. It should be emphasized that while the resulting virtual integrals are formally the same, the theoretical framework for these two computations is not the same. In the framework of this paper the type II contributions studied in [31] occur in the local fourfold theory with  $L = K_S$  where  $S$  is a general type surface with  $c_1(S) \neq 0$  and primitive. In particular such contributions are ruled out by stability for local K3 surfaces with  $L = K_S \cong \mathcal{O}_S$  and for primitive invariants  $\gamma$ . The main point of the above argument is to point out that formally identical equivariant integrals also occur in the  $L = K_S(mf)$ -twisted theory with  $S$  a K3 surface, at least if  $m$  is odd. In the latter case they are allowed by the stability conditions, as shown in Lemma 6.14. In conclusion, while the context is different, the resulting contributions are formally identical up to a factor of  $\mathbf{s}^{-1}$ , which was explained in [15, Remark 2.10]. Therefore the computations in Sections 8.7-8.9 of [31] lead to Conjecture 1.4.

## Appendix A. Background results

### A.1. Adjunction.

**Lemma A.1.** *Let  $W$  be a quasi-projective scheme over  $\mathbb{C}$  and let  $V$  be the total space of a line bundle  $\mathcal{Q}$  on  $W$ . Let  $\sigma : W \rightarrow V$  denote the zero section. Let  $C^\bullet$  be an object of  $\mathcal{D}^b(W)$ . There is an isomorphism*

$$(105) \quad L\sigma^* \mathbf{R}\sigma_* C^\bullet \cong C^\bullet \oplus C^\bullet \otimes \mathcal{Q}^{-1}[1]$$

in  $\mathcal{D}^b(W)$ . Moreover, given any objects  $C_1^\bullet, C_2^\bullet$  in  $\mathcal{D}^b(W)$ , there is a splitting

$$(106) \quad \begin{aligned} \mathbf{R}\mathcal{H}om_{\mathcal{D}^b(V)}(\sigma_* C_1^\bullet, \sigma_* C_2^\bullet) &\cong \mathbf{R}\sigma_* \mathbf{R}\mathcal{H}om_{\mathcal{D}^b(W)}(C_1^\bullet, C_2^\bullet) \oplus \\ &\mathbf{R}\sigma_* \mathbf{R}\mathcal{H}om_{\mathcal{D}^b(W)}(C_1^\bullet \otimes \mathcal{Q}^{-1}, C_2^\bullet)[-1] \end{aligned}$$

in  $\mathcal{D}^b(W)$ .

*Proof.* Since  $W$  is quasi-projective,  $C^\bullet$  has a quasi-coherent, possibly infinite, locally free resolution  $F_{C^\bullet}^\bullet$ . Let  $p : V \rightarrow W$  be the canonical projection, which is affine and flat. Let  $\zeta \in H^0(V, p^* \mathcal{Q})$  denote the tautological section. Then the locally free complex

$$\text{Cone}(\mathbf{1}_{F_{C^\bullet}^\bullet} \otimes \zeta : p^*(F_{C^\bullet}^\bullet \otimes p^* \mathcal{L}^{-1}) \rightarrow p^* F_{C^\bullet}^\bullet)$$

is quasi-isomorphic to  $\mathbf{R}_{\sigma^*}C$ . Since the zero locus of  $\zeta$  coincides with the image of the zero section  $\sigma$ ,

$$L\sigma^*\mathbf{R}_{\sigma^*}C \cong \sigma^*F_{\mathbb{C}}^\bullet \oplus F_{\mathbb{C}}^\bullet \otimes \mathcal{Q}^{-1}[1].$$

This proves equation (105).

Next, [24, Prop. II.5.10] shows that for any objects  $C_1^\bullet, C_2^\bullet$  of  $\mathcal{D}^b(W)$  there is a natural isomorphism

$$\mathbf{R}_{\sigma^*}\mathbf{R}\mathcal{H}om_{\mathcal{D}^b(W)}(L\sigma^*\mathbf{R}_{\sigma^*}C_1^\bullet, \mathbf{R}_{\sigma^*}C_2^\bullet) \cong \mathbf{R}\mathcal{H}om_{\mathcal{D}^b(V)}(\mathbf{R}_{\sigma^*}C_1^\bullet, \mathbf{R}_{\sigma^*}C_2^\bullet).$$

Then equation (106) follows from (105).  $\square$

**A.2. Products and diagonals.** This section consists of some background results on product schemes needed in the proof of Theorem 2.5.

**Lemma A.2.** *Suppose  $W_1, W_2$  are quasi-projective over  $\mathbb{C}$  and let  $\mathcal{G}_1$  be a coherent sheaf on  $W_1$ . Then  $\pi_{W_1}^*\mathcal{G}_1$  is flat over  $W_2$  and the derived pull-back  $L\pi_{W_1}^*\mathcal{G}_1$  is isomorphic to the ordinary pull-back  $\pi_{W_1}^*\mathcal{G}_1$ . Moreover, for any coherent sheaves  $\mathcal{G}_i$  on  $W_i$  with  $1 \leq i \leq 2$ , one has the transversality result*

$$(107) \quad \mathcal{T}or_{W_1 \times W_2}^k(\pi_{W_1}^*\mathcal{G}_1, \pi_{W_2}^*\mathcal{G}_2) = 0$$

for  $k \geq 1$ . In particular, given any the derived tensor product  $L\pi_{W_1}^*\mathcal{G}_1 \otimes^L L\pi_{W_2}^*\mathcal{G}_2$  is isomorphic to the ordinary tensor product  $\pi_{W_1}^*\mathcal{G}_1 \otimes \pi_{W_2}^*\mathcal{G}_2$ .

*Proof.* This follows from the cartesian diagram

$$\begin{array}{ccc} W_1 \times W_2 & \xrightarrow{\pi_{W_1}} & W_1 \\ \pi_{W_2} \downarrow & & \downarrow f_1 \\ W_2 & \xrightarrow{f_2} & \text{Spec}(\mathbb{C}). \end{array}$$

Since  $f_2$  is flat, there is an isomorphism  $\pi_{W_2^*}\pi_{W_1}^*\mathcal{G}_1 \cong f_2^*f_{1*}\mathcal{G}_1$ , while  $f_{1*}\mathcal{G}_1$  is obviously locally free on  $\text{Spec}(\mathbb{C})$ . Moreover,  $\pi_{W_1}$  is flat since  $f_2$  is flat, which implies the second statement.

Since both schemes are quasi-projective, the sheaves  $\mathcal{G}_1, \mathcal{G}_2$  admit locally free resolutions. Then the transversality result follows immediately from the first two statements.  $\square$

**Lemma A.3.** *Let  $W_1, W_2$  be quasi-projective schemes over  $\mathbb{C}$  and let  $W = W_1 \times W_2$ . Let  $C^\bullet$  be a complex of coherent sheaves on  $W_2$ . Then there is a canonical isomorphism*

$$(108) \quad \Delta_{W^*}\pi_{W_2}^*C^\bullet \cong \mathcal{O}_{\Delta_{W_1}} \boxtimes \Delta_{W_2^*}C^\bullet.$$



*Proof.* Consider the commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{\Delta_W} & W \times W \\ \downarrow \pi_{W_2} & & \downarrow \pi_{W_2 \times W_2} \\ W_2 & \xrightarrow{\Delta_{W_2}} & W_2 \times W_2. \end{array}$$

and note the canonical isomorphism

$$C^\bullet \cong \Delta_{W_2}^* \Delta_{W_2*} C^\bullet.$$

This implies

$$\pi_{W_2}^* C^\bullet \cong \pi_{W_2}^* \Delta_{W_2}^* \Delta_{W_2*} C^\bullet \cong \Delta_W^* \pi_{W_2 \times W_2}^* \Delta_{W_2*} C^\bullet.$$

Hence

$$\begin{aligned} \Delta_{W*} \pi_{W_2}^* C^\bullet &\cong \Delta_{W*} \Delta_W^* \pi_{W_2 \times W_2}^* \Delta_{W_2*} C^\bullet \cong \pi_{W_2 \times W_2}^* \Delta_{W_2*} C^\bullet / \mathcal{I}_{\Delta_W} \pi_{W_2 \times W_2}^* \Delta_{W_2*} C^\bullet \\ &\cong \mathcal{O}_{\Delta_W} \otimes \pi_{W_2 \times W_2}^* \Delta_{W_2*} C^\bullet \end{aligned}$$

However, Lemma A.2 implies that the scheme theoretic intersection of  $\pi_{W_1 \times W_1}^{-1}(\Delta_{W_1})$  and  $\pi_{W_2 \times W_2}^{-1}(\Delta_{W_2})$  is transverse and coincides with  $\Delta_W$ . Therefore

$$\begin{aligned} \mathcal{O}_{\Delta_W} \otimes \pi_{W_2 \times W_2}^* \Delta_{W_2*} C^\bullet &\cong (\mathcal{O}_{\Delta_{W_1}} \boxtimes \mathcal{O}_{\Delta_{W_2}}) \otimes \pi_{W_2 \times W_2}^* \Delta_{W_2*} C^\bullet \\ &\cong \mathcal{O}_{\Delta_{W_1}} \boxtimes \pi_{W_2 \times W_2}^* \Delta_{W_2*} C^\bullet. \end{aligned}$$

□

**Lemma A.4.** *Let  $W_1, W_2$  be quasi-projective schemes over  $\mathbb{C}$  and let  $W = W_1 \times W_2$ . Then there are canonical exact sequences of  $\mathcal{O}_{W \times W}$ -modules*

$$(109) \quad 0 \rightarrow \mathcal{I}_{\Delta_{W_1}} \boxtimes \mathcal{O}_{W_2 \times W_2} \xrightarrow{j_W} \mathcal{I}_{\Delta_W} \xrightarrow{\gamma_W} \mathcal{O}_{\Delta_{W_1}} \boxtimes \mathcal{I}_{\Delta_{W_2}} \rightarrow 0$$

and

$$(110) \quad 0 \rightarrow \mathcal{I}_{\Delta_{W_1}} \boxtimes \mathcal{I}_{\Delta_{W_2}} \xrightarrow{\delta_W} \mathcal{I}_{\Delta_{W_1}} \boxplus \mathcal{I}_{\Delta_{W_2}} \xrightarrow{\rho_W} \mathcal{I}_{\Delta_W} \rightarrow 0$$

where the  $\delta_W$  is the antidiagonal map defined by multiplication, and  $\gamma_W, \rho_W$  are the natural projections. Moreover, there is a commutative diagram

$$(111) \quad \begin{array}{ccc} \mathcal{I}_{\Delta_{W_1}} \boxplus \mathcal{I}_{\Delta_{W_2}} & \xrightarrow{\rho_W} & \mathcal{I}_{\Delta_W} \\ \downarrow (0, f_{W_1} \boxtimes \mathbf{1}) & & \downarrow \gamma_W \\ \mathcal{O}_{\Delta_{W_1}} \boxtimes \mathcal{I}_{\Delta_{W_2}} & \xrightarrow{\mathbf{1}} & \mathcal{O}_{\Delta_{W_1}} \boxtimes \mathcal{I}_{\Delta_{W_2}} \end{array}$$

where  $f_{W_1} : \mathcal{O}_{W_1 \times W_1} \rightarrow \mathcal{O}_{\Delta_{W_1}}$  is the canonical projection.

*Proof.* Note that  $\mathcal{I}_{\Delta_W} = \pi_{W_1 \times W_1}^* \mathcal{I}_{\Delta_{W_1}} + \pi_{W_2 \times W_2}^* \mathcal{I}_{\Delta_{W_2}}$ , the sum of ideal sheaves on  $W \times W$ . In particular there is a canonical exact sequence

$$0 \rightarrow \pi_{W_2 \times W_2}^* \mathcal{I}_{\Delta_{W_2}} \rightarrow \mathcal{I}_{\Delta_W} \rightarrow \pi_{W_1 \times W_1}^* \mathcal{I}_{\Delta_{W_1}} / (\pi_{W_1 \times W_1}^* \mathcal{I}_{\Delta_{W_1}} \cap \pi_{W_2 \times W_2}^* \mathcal{I}_{\Delta_{W_2}}) \rightarrow 0.$$

Moreover, Lemma A.2 shows that  $\pi_{W_1 \times W_1}^* \mathcal{I}_{\Delta_{W_1}}$  is flat over  $W_2 \times W_2$ , and the pull-back of the canonical exact sequence

$$0 \rightarrow \mathcal{I}_{\Delta_{W_2}} \rightarrow \mathcal{O}_{W_2 \times W_2} \rightarrow \mathcal{O}_{\Delta_{W_2}} \rightarrow 0$$

to  $W \times W$  remains exact. This yields an isomorphism

$$\pi_{W_1 \times W_1}^* \mathcal{I}_{\Delta_{W_1}} / (\pi_{W_1 \times W_1}^* \mathcal{I}_{\Delta_{W_1}} \cap \pi_{W_2 \times W_2}^* \mathcal{I}_{\Delta_{W_2}}) \cong \mathcal{I}_{\Delta_{W_1}} \boxtimes \mathcal{O}_{\Delta_{W_2}}.$$

For the second sequence, the relation  $\mathcal{I}_{\Delta_W} = \pi_{W_1 \times W_1}^* \mathcal{I}_{\Delta_{W_1}} + \pi_{W_2 \times W_2}^* \mathcal{I}_{\Delta_{W_2}}$  yields the canonical exact sequence

$$0 \rightarrow \pi_{W_1 \times W_1}^* \mathcal{I}_{\Delta_{W_1}} \cap \pi_{W_2 \times W_2}^* \mathcal{I}_{\Delta_{W_2}} \rightarrow \mathcal{I}_{\Delta_{W_1}} \boxplus \mathcal{I}_{\Delta_{W_2}} \rightarrow \mathcal{I}_{\Delta_W} \rightarrow 0.$$

Moreover, Lemma A.2 implies that the multiplication map

$$\mathcal{I}_{\Delta_{W_1}} \boxtimes \mathcal{I}_{\Delta_{W_2}} \rightarrow \pi_{W_1 \times W_1}^* \mathcal{I}_{\Delta_{W_1}} \cap \pi_{W_2 \times W_2}^* \mathcal{I}_{\Delta_{W_2}}$$

is an isomorphism.

Diagram (111) follows immediately by construction. □

**Corollary A.5.** *Under the same conditions as in Lemma A.4 the map  $\rho_W$  in (110) induces an isomorphism*

$$(112) \quad \bar{\rho}_W : \mathcal{I}_{\Delta_{W_1}} / \mathcal{I}_{\Delta_{W_1}}^2 \boxtimes \mathcal{O}_{\Delta_{W_2}} \oplus \mathcal{O}_{\Delta_{W_1}} \boxtimes \mathcal{I}_{\Delta_{W_2}} / \mathcal{I}_{\Delta_{W_2}}^2 \xrightarrow{\sim} \mathcal{I}_{\Delta_W} / \mathcal{I}_{\Delta_W}^2.$$

Moreover, there is a commutative diagram

$$(113) \quad \begin{array}{ccc} \mathcal{I}_{\Delta_{W_1}} / \mathcal{I}_{\Delta_{W_1}}^2 \boxtimes \mathcal{O}_{\Delta_{W_2}} \oplus \mathcal{O}_{\Delta_{W_1}} \boxtimes \mathcal{I}_{\Delta_{W_2}} / \mathcal{I}_{\Delta_{W_2}}^2 & \xrightarrow{\bar{\rho}_W} & \mathcal{I}_{\Delta_W} / \mathcal{I}_{\Delta_W}^2 \\ \downarrow (0,1) & & \downarrow \bar{\gamma}_W \\ \mathcal{O}_{\Delta_{W_1}} \boxtimes \mathcal{I}_{\Delta_{W_2}} / \mathcal{I}_{\Delta_{W_2}}^2 & \xrightarrow{\mathbf{1}} & \mathcal{O}_{\Delta_{W_1}} \boxtimes \mathcal{I}_{\Delta_{W_2}} / \mathcal{I}_{\Delta_{W_2}}^2 \end{array}$$

where  $\bar{\gamma}_W : \mathcal{I}_{\Delta_W} / \mathcal{I}_{\Delta_W}^2 \rightarrow \mathcal{O}_{\Delta_{W_1}} \boxtimes \mathcal{I}_{\Delta_{W_2}} / \mathcal{I}_{\Delta_{W_2}}^2$  is the natural projection determined by  $\gamma_W : \mathcal{I}_{\Delta_W} \rightarrow \mathcal{O}_{\Delta_{W_1}} \boxtimes \mathcal{I}_{\Delta_{W_2}}$ .

*Proof.* Note that the image of the map  $\delta_W$  in (110) is contained in the subsheaf

$$(\mathcal{I}_{\Delta_{W_1}} \boxtimes \mathcal{O}_{W_2 \times W_2}) \cap \mathcal{I}_{\Delta_W}^2 \oplus (\mathcal{O}_{W_1 \times W_1} \boxtimes \mathcal{I}_{\Delta_{W_2}}) \cap \mathcal{I}_{\Delta_W}^2 \subset \mathcal{I}_{\Delta_{W_1}} \boxtimes \mathcal{I}_{\Delta_{W_2}}.$$

Then the exact sequence (110) yields the following commutative diagram

$$(114) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{I}_{\Delta_{W_1}} \boxtimes \mathcal{I}_{\Delta_{W_2}} & \xrightarrow{\delta'_W} & (\mathcal{I}_{\Delta_{W_1}} \boxtimes \mathcal{O}_{W_2 \times W_2}) \cap \mathcal{I}_{\Delta_W}^2 & \xrightarrow{\rho'_W} & \mathcal{I}_{\Delta_W}^2 \longrightarrow 0 \\ & & \downarrow \mathbf{1} & & \oplus & & \downarrow \\ & & \mathcal{I}_{\Delta_{W_1}} \boxtimes \mathcal{I}_{\Delta_{W_2}} & \xrightarrow{\delta_W} & (\mathcal{O}_{W_1 \times W_1} \boxtimes \mathcal{I}_{\Delta_{W_2}}) \cap \mathcal{I}_{\Delta_W}^2 & \xrightarrow{\rho_W} & \mathcal{I}_{\Delta_W} \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{I}_{\Delta_{W_1}} \boxtimes \mathcal{I}_{\Delta_{W_2}} & \xrightarrow{\delta_W} & \mathcal{I}_{\Delta_{W_1}} \boxplus \mathcal{I}_{\Delta_{W_2}} & \xrightarrow{\rho_W} & \mathcal{I}_{\Delta_W} \longrightarrow 0 \end{array}$$

with exact rows, where  $\delta'_W$  is naturally determined by  $\delta_W$  and  $\rho'_W$  is the restriction of  $\rho_W$ . Moreover, note that

$$\mathcal{I}_{\Delta_W}^2 = \mathcal{I}_{\Delta_{W_1}}^2 \boxtimes \mathcal{O}_{W_2 \times W_2} + \mathcal{I}_{\Delta_{W_1}} \boxtimes \mathcal{I}_{\Delta_{W_2}} + \mathcal{O}_{W_1 \times W_1} \boxtimes \mathcal{I}_{\Delta_{W_2}}^2.$$

Therefore there are canonical isomorphisms

$$(\mathcal{I}_{\Delta_{W_1}} \boxtimes \mathcal{O}_{W_2 \times W_2}) \cap \mathcal{I}_{\Delta_W}^2 \cong \mathcal{I}_{\Delta_{W_1}}^2 \boxtimes \mathcal{O}_{W_2 \times W_2} + \mathcal{I}_{\Delta_{W_1}} \boxtimes \mathcal{I}_{\Delta_{W_2}}$$

and

$$(\mathcal{O}_{W_1 \times W_1} \boxtimes \mathcal{I}_{\Delta_{W_2}}) \cap \mathcal{I}_{\Delta_W}^2 \cong \mathcal{I}_{\Delta_{W_1}} \boxtimes \mathcal{I}_{\Delta_{W_2}} + \mathcal{O}_{W_1 \times W_1} \boxtimes \mathcal{I}_{\Delta_{W_2}}^2$$

These yield further isomorphisms

$$\mathcal{I}_{\Delta_{W_1}} / (\mathcal{I}_{\Delta_{W_1}} \boxtimes \mathcal{O}_{W_2 \times W_2}) \cap \mathcal{I}_{\Delta_W}^2 \cong \mathcal{I}_{\Delta_{W_1}} / \mathcal{I}_{\Delta_{W_1}}^2 \boxtimes \mathcal{O}_{W_2 \times W_2}$$

and

$$\mathcal{I}_{\Delta_{W_2}} / (\mathcal{I}_{\Delta_{W_1}} \boxtimes \mathcal{O}_{W_2 \times W_2}) \cap \mathcal{I}_{\Delta_W}^2 \cong \mathcal{O}_{W_1 \times W_1} \boxtimes \mathcal{I}_{\Delta_{W_2}} / \mathcal{I}_{\Delta_{W_2}}^2$$

Then isomorphism (112) follows from (114) by the snake lemma. Diagram (113) follows from (111). □

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