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A Note on Failure of Energy Reversal for Classical Fractional Singular Integrals

Eric T. Sawyer[1], Chun-Yen Shen[2], and Ignacio Uriarte-Tuero[3]

[1]Department of Mathematics & Statistics, McMaster University, 1280 Main Street West, Hamilton, ON, Canada L8S 4K1, [2]Department of Mathematics, National Central University, Chungli 32054, Taiwan, and [3]Department of Mathematics, Michigan State University, East Lansing, MI, USA

Correspondence to be sent to: ignacio@math.msu.edu

For $0 \leq \alpha < n$, we demonstrate the failure of energy reversal for the vector of α-fractional Riesz transforms, and more generally for the vector of all α-fractional convolution singular integrals having a kernel with vanishing integral on every great circle of the sphere.

1 Introduction

In the recent two-part solution to the Nazarov–Treil–Volberg conjecture for the two weight Hilbert transform inequality (Lacey et al. Part I [2], Lacey Part II [1]), crucial use was made of the following point-wise equivalence for the difference quotients of the Hilbert transform $H\mu$ associated with x, x' in an interval J and a positive measure μ

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supported outside the double $2J$:

$$\begin{aligned}\frac{H\mu(x)-H\mu(x')}{x-x'} &= \frac{1}{x-x'}\int_{\mathbb{R}\setminus 2J}\left\{\frac{1}{y-x}-\frac{1}{y-x'}\right\}\mathrm{d}\mu(y)\\ &= \frac{1}{|J|}\int_{\mathbb{R}\setminus 2J}\frac{|J|}{(y-x)(y-x')}\,\mathrm{d}\mu(y)\\ &\approx \frac{\mathrm{P}(J,\mathbf{1}_{\mathbb{R}\setminus 2J}\mu)}{|J|}. \end{aligned}\tag{1.1}$$

Here, $H\mu(x)\equiv\int_{\mathbb{R}}\frac{1}{y-x}\,\mathrm{d}\mu(y)$ is the Hilbert transform of the measure μ on J, and $\mathrm{P}(J,\nu)\equiv\int_{\mathbb{R}}\frac{|J|}{|J|^2+|y-c_J|^2}\,\mathrm{d}\nu(y)$ is the Poisson integral of ν associated with the interval J, and where c_J is the center of J. The important consequence of this point-wise equivalence is that a difference quotient of the singular integral $H\mu$ is comparable with a positive quantity $\frac{\mathrm{P}(J,\mu)}{|J|}$, that is in turn monotone increasing in the measure μ. A striking difference between the one and higher dimensional settings for singular integrals is that this equivalence fails in both directions $\lesssim$ and $\gtrsim$ for singular integrals in $\mathbb{R}^n$ with $n>1$. In fact, the forward direction $\lesssim$ requires an additional energy term derived by Lacey and Wick [3], and with this addition the forward inequality provides a suitable substitute for use in higher dimensions. However, there is a particularly spectacular failure of the reverse inequality $\gtrsim$, which we refer to as energy reversal, a terminology explained below. In fact, this failure of energy reversal in higher dimensions underlies the restrictive nature of higher dimensional analogs of the two weight theorem for the Hilbert transform in both [3, 5], where in earlier versions of each of these papers, errors were made in assuming some version of energy reversal.

To set direction for this paper, we first recall a special case of Theorem 1 from our paper [5], using notation from that paper, which we now review here, albeit very briefly. Let $0\le\alpha<n$. Consider a kernel function $K^\alpha(x,y)$ defined on $\mathbb{R}^n\times\mathbb{R}^n$ satisfying the following fractional size and smoothness conditions of order $1+\delta$ for some $\delta>0$,

$$\begin{aligned} |K^\alpha(x,y)| &\le C_{\mathrm{CZ}}|x-y|^{\alpha-n},\\ |\nabla K^\alpha(x,y)| &\le C_{\mathrm{CZ}}|x-y|^{\alpha-n-1},\\ |\nabla K^\alpha(x,y)-\nabla K^\alpha(x',y)| &\le C_{\mathrm{CZ}}\left(\frac{|x-x'|}{|x-y|}\right)^{\delta}|x-y|^{\alpha-n-1},\quad \frac{|x-x'|}{|x-y|}\le\frac{1}{2},\\ |\nabla K^\alpha(x,y)-\nabla K^\alpha(x,y')| &\le C_{\mathrm{CZ}}\left(\frac{|y-y'|}{|x-y|}\right)^{\delta}|x-y|^{\alpha-n-1},\quad \frac{|y-y'|}{|x-y|}\le\frac{1}{2}. \end{aligned}\tag{1.2}$$

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We define a standard α-fractional Calderón–Zygmund operator associated with such a kernel as follows.

Definition 1. We say that T^α is a *standard α-fractional integral operator with kernel* K^α if T^α is a bounded linear operator from some $L^p(\mathbb{R}^n)$ to some $L^q(\mathbb{R}^n)$ for some fixed $1 < p \le q < \infty$, that is

$$\|T^\alpha f\|_{L^q(\mathbb{R}^n)} \le C \|f\|_{L^p(\mathbb{R}^n)}, \qquad f \in L^p(\mathbb{R}^n),$$

if $K^\alpha(x, y)$ is defined on $\mathbb{R}^n \times \mathbb{R}^n$ and satisfies (1.2), and if T^α and K^α are related by

$$T^\alpha f(x) = \int K^\alpha(x, y) f(y)\, \mathrm{d}y, \quad \text{a.e.-}x \notin \operatorname{supp} f,$$

whenever $f \in L^p(\mathbb{R}^n)$ has compact support in $\mathbb{R}^n$. We say $K^\alpha(x, y)$ is a *standard α-fractional kernel* if it satisfies (1.2). □

In higher dimensions, there are the two α-fractional Poisson integrals of μ on a cube Q are given by

$$\begin{aligned} \mathrm{P}^\alpha(Q, \mu) &\equiv \int_{\mathbb{R}^n} \frac{|Q|^{\frac{1}{n}}}{(|Q|^{\frac{1}{n}} + |x - x_Q|)^{n+1-\alpha}}\, \mathrm{d}\mu(x), \\ \mathcal{P}^\alpha(Q, \mu) &\equiv \int_{\mathbb{R}^n} \left(\frac{|Q|^{\frac{1}{n}}}{(|Q|^{\frac{1}{n}} + |x - x_Q|)^2} \right)^{n-\alpha} \mathrm{d}\mu(x). \end{aligned} \tag{1.3}$$

We refer to P^α as the *standard* Poisson integral and to $\mathcal{P}^\alpha$ as the *reproducing* Poisson integral. We refer the reader to [5] for the more technical definitions of elliptic and strongly elliptic operators T^α, and various notions of energy.

Finally, we define the *α-energy condition* constant to be

$$\begin{aligned} (\mathcal{E}_\alpha)^2 \equiv \sup_{\substack{Q = \dot\cup Q_r \\ Q, Q_r \in \mathcal{D}^n}} \frac{1}{|I|_\sigma} \sum_{r=1}^\infty \sum_{J \in \mathcal{M}_{\mathbf{r}\text{-deep}}(Q_r)} \left(\frac{\mathrm{P}^\alpha(J, \mathbf{1}_Q \sigma)}{|J|^{\frac{1}{n}}} \right)^2 \|\mathsf{P}_J^{\mathrm{subgood},\omega} \mathbf{x}\|_{L^2(\omega)}^2 \\ + \sup_{\ell \ge 0} \sup_Q \frac{1}{|I|_\sigma} \sum_{J \in \mathcal{M}^\ell_{\mathbf{r}\text{-deep}}(Q)} \left(\frac{\mathrm{P}^\alpha(J, \mathbf{1}_Q \sigma)}{|J|^{\frac{1}{n}}} \right)^2 \|\mathsf{P}_J^{\mathrm{subgood},\omega} \mathbf{x}\|_{L^2(\omega)}^2, \end{aligned}$$

where the supremum is taken over all dyadic grids $\mathcal{D}^n$ and all dyadic cubes in $\mathcal{D}^n$, and where the goodness parameters $\mathbf{r}$ and ε implicit in the definition of $\mathcal{M}_{\mathbf{r}\text{-deep}}(K)$ and $\mathcal{M}^\ell_{\mathbf{r}\text{-deep}}(K)$ are fixed sufficiently large and small, respectively, depending on n and α. Here the collection $\mathcal{M}_{\mathbf{r}\text{-deep}}(K)$ consists of the *maximal* $\mathbf{r}$-deeply embedded dyadic subcubes J of a cube K (a subcube J of K is a *dyadic* subcube of K if $J \in \mathcal{D}$ when $\mathcal{D}$ is a

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dyadic grid containing K), and for each $\ell \geq 0$, the collection $\mathcal{M}^{\ell}_{\mathbf{r}\text{-deep}}(K)$ consists of those $J \in \mathcal{M}_{\mathbf{r}\text{-deep}}(\pi^{\ell} K)$ (where $\pi^{\ell} K$ denotes the ℓ-fold dyadic parent of K) that are $\mathbf{r}$-deeply embedded in K, that is,

$$\mathcal{M}^{\ell}_{\mathbf{r}\text{-deep}}(K) \equiv \{J \in \mathcal{M}_{\mathbf{r}\text{-deep}}(\pi^{\ell} K) : J \subset L \text{for some } L \in \mathcal{M}_{\mathbf{r}\text{-deep}}(K)\}.$$

Of course $\mathcal{M}^{0}_{\mathbf{r}\text{-deep}}(K) = \mathcal{M}_{\mathbf{r}\text{-deep}}(K)$, but $\mathcal{M}^{\ell}_{\mathbf{r}\text{-deep}}(K)$ is in general a finer subdecomposition of K the larger ℓ is, and may in fact be empty. There is a similar definition of the dual α-energy condition constant $\mathcal{E}^{*}_{\alpha}$. We refer the reader to [5], or almost any other paper on the subject, for the detailed definition and properties of *good* dyadic cubes.

Theorem 1. Suppose that σ and ω are locally finite positive Borel measures in $\mathbb{R}^n$ with no common point masses, and assume the finiteness of the α-*energy condition* constant $\mathcal{E}_{\alpha}$ and its dual constant $\mathcal{E}^{*}_{\alpha}$. Let $\mathbf{T}^{\alpha}$ be a standard strongly elliptic α-fractional Calderón–Zygmund operator in Euclidean space $\mathbb{R}^n$. Then $\mathbf{T}^{\alpha}$ is bounded from $L^2(\sigma)$ to $L^2(\omega)$ *if and only if* the $\mathcal{A}^{\alpha}_{2}$ condition

$$\mathcal{A}^{\alpha}_{2} \equiv \sup_{Q \in \mathcal{Q}^n} \mathcal{P}^{\alpha}(Q, \sigma) \frac{|Q|_{\omega}}{|Q|^{1-\frac{\alpha}{n}}} < \infty \tag{1.4}$$

and its dual hold, the cube testing conditions

$$\int_Q |\mathbf{T}^{\alpha}(\mathbf{1}_Q \sigma)|^2 \omega \leq \mathfrak{T}^2_{T^{\alpha}} \int_Q \mathrm{d}\sigma \quad \text{and} \quad \int_Q |(\mathbf{T}^{\alpha})^{*}(\mathbf{1}_Q \omega)|^2 \sigma \leq \mathfrak{T}^2_{T^{\alpha}} \int_Q \mathrm{d}\omega, \tag{1.5}$$

hold for all cubes Q in $\mathbb{R}^n$, and the weak boundedness property for $\mathbf{T}^{\alpha}$ holds:

$$\left| \int_Q \mathbf{T}^{\alpha}(1_{Q'} \sigma)\, \mathrm{d}\omega \right| \leq \mathcal{WBP}_{\mathbf{T}^{\alpha}} \sqrt{|Q|_{\omega} |Q'|_{\sigma}},$$

$$\text{for all cubes } Q, Q' \text{ with } \frac{1}{C} \leq \frac{|Q|^{\frac{1}{n}}}{|Q'|^{\frac{1}{n}}} \leq C,$$

$$\text{and either } Q \subset 3Q' \setminus Q' \text{ or } Q' \subset 3Q \setminus Q. \qquad \square$$

In [7], we used Theorem 1 to prove the $T1$ theorem for the vector of Riesz transforms in $\mathbb{R}^n$ in the special case when one of the measures σ, ω is supported on a line in $\mathbb{R}^n$. The key to that proof was proving control of the above energy constants $\mathcal{E}_{\alpha}$ and $\mathcal{E}^{*}_{\alpha}$ in terms of the constants in the hypotheses (1.4) and (1.5), and this in turn used an energy reversal that exploited the 1D support of one of the measures. As mentioned above, a number of attempts have been made to prove such control of various different energy conditions by invoking an *energy reversal* for the Riesz transforms and similar operators—see (2.4)—but all of these attempts have been met with failure. The purpose

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of this paper is to show first that *energy reversal* is false, not only for the vector of α-fractional Riesz transforms in the plane when $0 \le \alpha < 2$, but also for the vectors of classical α-fractional singular integrals in the plane,

$$\mathbf{T}_M^\alpha \equiv \{T_\Omega^\alpha : \Omega \in \mathcal{P}_M\},$$

$$\mathcal{P}_M \equiv \{\cos n\theta, \sin n\theta\}_{n=1}^M,$$

where T_Ω^α has convolution kernel $\frac{\Omega(\frac{x}{|x|})}{|x|^{2-\alpha}} = \frac{\Omega(\theta)}{|x|^{2-\alpha}}$ and $0 \le \alpha < 2$. The linear space $\mathcal{L}_M$ of trigonometric polynomials with vanishing mean and degree at most M is spanned by the monomials $\mathcal{P}_M$, and so we also obtain the failure of energy reversal for the infinite vector $\mathbf{T}_M^\alpha \equiv \{T_\Omega : \Omega \in \mathcal{L}_M\}$. A standard limiting argument applied to the proof below extends this failure to all sufficiently smooth $\Omega(\theta)$ with vanishing mean on the circle. Finally, we embed an analog of the planar measure constructed below into Euclidean space $\mathbb{R}^n$ to obtain the failure of energy reversal for *any* vector of classical convolution Calderón–Zygmund operators with odd kernel in $\mathbb{R}^n$—and more generally for kernels $\frac{\Omega(x')}{|x|^{n-\alpha}}$ where Ω has vanishing integral on every great circle in the sphere $\mathbb{S}^{n-1}$. A key to our proof is the positivity of the determinants $\det[\frac{\Gamma(z)^2}{\Gamma(z-|i-j|)\Gamma(z+|i-j|)}]_{i,j=1}^n$ for all $n \ge 1$, where $\Gamma(z)$ is the gamma function. See also [3] for related results regarding fractional Riesz transforms in higher dimensions.

2 Failure of Reversal of Energy

Recall the energy $\mathsf{E}(J,\omega)$ of a measure ω on a cube J,

$$\mathsf{E}(J,\omega)^2 \equiv \frac{1}{|J|_\omega}\frac{1}{|J|_\omega}\int_J\int_J \left|\frac{x-z}{|J|^{\frac{1}{n}}}\right|^2 \mathrm{d}\omega(x)\,\mathrm{d}\omega(z) = 2\frac{1}{|J|_\omega}\int_J \left|\frac{x-\mathbb{E}_J^\omega x}{|J|^{\frac{1}{n}}}\right|^2 \mathrm{d}\omega(x).$$

Define its associated *coordinate* energies $\mathsf{E}^j(J,\omega)$ by

$$\mathsf{E}^j(J,\omega)^2 \equiv \frac{1}{|J|_\omega}\frac{1}{|J|_\omega}\int_J\int_J \left|\frac{x^j-z^j}{|J|^{\frac{1}{n}}}\right|^2 \mathrm{d}\omega(x)\,\mathrm{d}\omega(z), \quad j=1,2,\ldots,n,$$

and the rotations $\mathsf{E}_{\mathcal{R}}^j(J,\omega)$ of the coordinate energies by a rotation $\mathcal{R} \in \mathrm{SO}(n)$, which we refer to as *partial* energies,

$$\mathsf{E}_{\mathcal{R}}^j(J,\omega)^2 \equiv \frac{1}{|J|_\omega}\frac{1}{|J|_\omega}\int_J\int_J \left|\frac{x_{\mathcal{R}}^j-z_{\mathcal{R}}^j}{|J|^{\frac{1}{n}}}\right|^2 \mathrm{d}\omega(x)\,\mathrm{d}\omega(z), \quad j=1,2,\ldots,n,$$

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where for $\mathcal{R} \in \mathrm{SO}(n)$, $x_{\mathcal{R}} = (x^j_{\mathcal{R}})_{j=1}^n = \mathcal{R}(x^j)_{j=1}^n = \mathcal{R}x$. Set $\mathsf{E}_{\mathcal{R}}(J,\omega)^2 \equiv \mathsf{E}^1_{\mathcal{R}}(J,\omega)^2 + \cdots + \mathsf{E}^n_{\mathcal{R}}(J,\omega)^2$. We have the following elementary computations.

Lemma 1. For $\mathcal{R} \in \mathrm{SO}(n)$ we have

$$\mathsf{E}_{\mathcal{R}}(J,\omega)^2 = \mathsf{E}^1_{\mathcal{R}}(J,\omega)^2 + \cdots + \mathsf{E}^n_{\mathcal{R}}(J,\omega)^2 = \mathsf{E}(J,\omega)^2. \tag{2.1}$$

More generally, if $\mathfrak{R} = \{\mathcal{R}_j\}_{j=1}^n \subset \mathrm{SO}(n)$ is a collection of rotations such that the matrix $M_{\mathfrak{R}} = \begin{bmatrix} \mathcal{R}_1\mathbf{e}^1 \\ \vdots \\ \mathcal{R}_n\mathbf{e}^1 \end{bmatrix}$ with rows $\mathcal{R}_\ell \mathbf{e}^1$ is nonsingular, then

$$\mathsf{E}(J,\omega)^2 \le \frac{1}{\epsilon_{\mathfrak{R}}} \sum_{\ell=1}^n \mathsf{E}^1_{\mathcal{R}_\ell}(J,\omega)^2, \tag{2.2}$$

where $\epsilon_{\mathfrak{R}}$ is the least eigenvalue of $M^*_{\mathfrak{R}} M_{\mathfrak{R}}$. □

Proof. We have

$$\begin{aligned} |x^1_{\mathcal{R}} - z^1_{\mathcal{R}}|^2 + \cdots + |x^n_{\mathcal{R}} - z^n_{\mathcal{R}}|^2 &= |\mathcal{R}(x-z)|^2 \\ &= |x-z|^2 = |x^1 - z^1|^2 + \cdots + |x^n - z^n|^2, \end{aligned}$$

so that

$$\begin{aligned} \mathsf{E}_{\mathcal{R}}(J,\omega)^2 &\equiv \mathsf{E}^1_{\mathcal{R}}(J,\omega)^2 + \cdots + \mathsf{E}^n_{\mathcal{R}}(J,\omega)^2 \\ &= \mathsf{E}^1(J,\omega)^2 + \cdots + \mathsf{E}^n(J,\omega)^2 = \mathsf{E}(J,\omega)^2. \end{aligned}$$

More generally, if $M^\ell_{\mathfrak{R}}$ denotes the ℓth row of the matrix $M_{\mathfrak{R}}$, we have

$$\begin{aligned} \epsilon_{\mathfrak{R}} |x-z|^2 &\le (x-z)^{\mathrm{tr}} M^*_{\mathfrak{R}} M_{\mathfrak{R}} (x-z) \\ &= \sum_{\ell=1}^n |\mathcal{R}_\ell \mathbf{e}^1 \cdot (x-z)|^2, \end{aligned}$$

so that

$$\begin{aligned} \epsilon_{\mathfrak{R}} \mathsf{E}(J,\omega)^2 &= \left(\frac{1}{|J|_\omega |J|^{\frac{1}{n}}}\right)^2 \int_J \int_J \epsilon_{\mathfrak{R}} |x-z|^2 \, d\omega(x)\, d\omega(z) \\ &\le \left(\frac{1}{|J|_\omega |J|^{\frac{1}{n}}}\right)^2 \int_J \int_J \left\{ \sum_{\ell=1}^n |\mathcal{R}_\ell \mathbf{e}^1 \cdot (x-z)|^2 \right\} d\omega(x)\, d\omega(z) \\ &= \sum_{\ell=1}^n \mathsf{E}^1_{\mathcal{R}_\ell}(J,\omega)^2. \end{aligned}$$

■

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The point of the estimate (2.2) is that it could hopefully be used to help obtain a reversal of energy for a vector transform $\mathbf{T}^{n,\alpha} = \{T_\ell^{n,\alpha}\}_{\ell=1}^n$, where the convolution kernel $K_\ell^{n,\alpha}(w)$ of the operator $T_\ell^{n,\alpha}$ has the form

$$K_\ell^{n,\alpha}(w) = \frac{\Omega_\ell^n\left(\frac{w}{|w|}\right)}{|w|^{n-\alpha}}, \tag{2.3}$$

and where Ω_ℓ^n is smooth on the sphere $\mathbb{S}^{n-1}$. We refer to the operator $T_\ell^{n,\alpha}$ as an α-fractional *convolution* Calderón–Zygmund operator. If in addition we require that Ω_ℓ^n has vanishing integral on the sphere $\mathbb{S}^{n-1}$, we refer to $T_\ell^{n,\alpha}$ as a *classical* α-fractional Calderón–Zygmund operator.

However, we now dash this hope, at least for the most familiar singular operators in the plane, in a spectacular way. Here is the key definition we work with.

Definition 2. A vector $\mathbf{T}^\alpha = \{T_\ell^\alpha\}_{\ell=1}^N$ of α-fractional transforms in Euclidean space $\mathbb{R}^n$ satisfies a *strong* reversal of ω-energy on a cube J if there is a positive constant C_0 such that for all $\gamma \geq 2$ sufficiently large and for all positive measures μ supported outside γJ, we have the inequality

$$\mathsf{E}(J,\omega)^2 \mathrm{P}^\alpha(J,\mu)^2 \leq C_0 \mathbb{E}_J^{d\omega(x)} \mathbb{E}_J^{d\omega(z)} |\mathbf{T}^\alpha \mu(x) - \mathbf{T}^\alpha \mu(z)|^2. \tag{2.4}$$

□

Note that in dimension $n = 1$, (2.4) is an immediate consequence of (1.1)—simply square $H\mu(x) - H\mu(x') \approx \frac{x-x'}{|J|}\mathrm{P}(J,\mu)$ and take ω-expectations over J in both x and x'. We show that (2.4) is false in higher dimensions by stating and proving a variant of Lemma 9 in an earlier paper [6] that has since been withdrawn.

Lemma 2 (Failure of Reverse Energy)**.** Suppose that J is a square in the plane $\mathbb{R}^2$, $0 \leq \alpha < 2$, $\gamma > 2$ and that $\mathbf{R}^\alpha = \{R_\ell^\alpha\}_{\ell=1}^2$ is the vector of α-fractional Riesz transforms in the plane $\mathbb{R}^2$ with kernels $K_\ell^\alpha(w) = \frac{\Omega_\ell(\frac{w}{|w|})}{|w|^{2-\alpha}}$ and $\Omega_\ell(\frac{w}{|w|}) = \frac{w_\ell}{|w|}$. Finally, suppose that $C_0 > 0$ is given. For γ sufficiently large, there exists a positive measure μ on $\mathbb{R}^2$ supported outside γJ and depending only on α and γ, such that the strong reversal of ω-energy inequality (2.4) *fails* for a certain class of measures ω. Moreover, we can choose μ as above so that in addition, for any $M \geq 1$, the strong reversal of ω-energy inequality (2.4) fails for the vector $\mathbf{T}_M^\alpha$. □

As a corollary of the proof of this lemma we easily obtain an extension to higher dimensions by simply embedding an appropriate planar measure into Euclidean space $\mathbb{R}^n$.

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Corollary 1 (of the proof of Lemma 2)**.** Suppose that J is a cube in $\mathbb{R}^n$, $0 \le \alpha < n$, $\gamma > 2$ and suppose that $C_0 > 0$ is given. For γ sufficiently large, there exists a positive measure μ on $\mathbb{R}^n$ supported outside γJ and depending only on n, α, and γ, such that for a certain class of measures ω the strong reversal of ω-energy inequality (2.4) *fails* for any vector $\mathbf{T}^\alpha = \{T_\ell^\alpha\}_{\ell=1}^N$ of α-fractional smooth Calderón–Zygmund operators in $\mathbb{R}^n$ with kernels $K_\ell^\alpha(w) = \frac{\Omega_\ell(\frac{w}{|w|})}{|w|^{n-\alpha}}$, where Ω_ℓ has vanishing integral on every great circle in the sphere $\mathbb{S}^{n-1}$—in particular this holds if each K_ℓ^α is odd. □

2.1 Failure of weak reversal of energy

The right-hand side of (2.4) is clearly dominated by $C_0 \mathbb{E}_J^{d\omega} |\mathbf{T}^\alpha \mu|^2$, and so we say that $\mathbf{T}^\alpha = \{T_\ell^\alpha\}_{\ell=1}^2$ satisfies a *weak* reversal of ω-energy on a cube J if for γ and μ as above, we have the weaker inequality

$$\mathsf{E}(J,\omega)^2 \mathrm{P}^\alpha(J,\mu)^2 \le C_0 \mathbb{E}_J^{d\omega} |\mathbf{T}^\alpha \mu|^2. \tag{2.5}$$

Now we show that even this weaker form of energy reversal fails, although not as spectacularly as the strong energy reversal. We content ourselves with the following special case for the Riesz transform vector $\mathbf{R}^\alpha$.

Lemma 3 (Failure of Weak Reverse Energy)**.** Suppose that J is a square in the plane $\mathbb{R}^2$, $0 \le \alpha < 2$, $\gamma > 2$ and that $\mathbf{R}^\alpha = \{R_\ell^\alpha\}_{\ell=1}^2$ is the vector of α-fractional Riesz transforms with kernels $K_\ell^\alpha(w) = \frac{\Omega_\ell(\frac{w}{|w|})}{|w|^{2-\alpha}}$. Finally suppose that $C_0 > 0$ is given. For $\gamma > 2$ sufficiently large, there is a positive measure $\mu = \mu_{\alpha,\gamma,T^\alpha}$ on $\mathbb{R}^2$ supported outside γJ and depending only on α, γ, and $\mathbf{R}^\alpha$, such that for a certain class of measures ω the *weak* reversal of ω-energy inequality (2.5) *fails*. □

3 The Proofs

Proof of Lemma 2 for the Riesz transform vector. Let $\varepsilon > 0$. We let $\frac{\Omega_\ell(\frac{w}{|w|})}{|w|^{2-\alpha}}$ be an arbitrary standard kernel for the moment with smoothness index δ in (1.2). With $K_\ell^\alpha(x,y) = K_\ell^\alpha(x-y)$ and $x, z \in J$, there is a point $\xi_{x,z}$ on the line joining x and z such that

$$\begin{aligned} T_\ell^\alpha \mu(x) - T_\ell^\alpha \mu(z) &= \int \{K_\ell^\alpha(x-y) - K_\ell^\alpha(z-y)\}\, d\mu(y) \\ &= \int \{(x-z) \cdot \nabla K_\ell^\alpha(\xi_{x,z} - y)\}\, d\mu(y) \end{aligned}$$

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$$
\begin{aligned}
&= \int \{(x-z)\cdot \nabla K_\ell^\alpha(c_J - y)\}\,\mathrm{d}\mu(y) \\
&\quad + \int \{(x-z)\cdot [\nabla K_\ell^\alpha(\xi_{x,z} - y) - \nabla K_\ell^\alpha(c_J - y)]\}\,\mathrm{d}\mu(y) \\
&\equiv \Lambda_\ell^\alpha + E_{\ell,x,z}^\alpha.
\end{aligned}
$$

If $\gamma > 2$ is sufficiently large, (1.2) gives

$$
\begin{aligned}
|E_{\ell,x,z}^\alpha| &\le \int |x-z|\,|\nabla K_\ell^\alpha(\xi_{x,z} - y) - \nabla K_\ell^\alpha(c_J - y)|\,\mathrm{d}\mu(y) \\
&\le \int_{(\gamma J)^c} |x-z| C_{\mathrm{CZ}} \left(\frac{|\xi_{x,z} - c_J|}{|c_J - y|}\right)^\delta |c_J - y|^{\alpha-3}\,\mathrm{d}\mu(y) \\
&\le C_{\mathrm{CZ}} |x-z| \int \left(\frac{|\xi_{x,z} - c_J|}{|c_J - y|}\right)^\delta |c_J - y|^{\alpha-3}\,\mathrm{d}\mu(y),
\end{aligned}
$$

and so from (1.3) with $n=2$, we obtain

$$
|E_{\ell,x,z}^\alpha| \le C \frac{1}{\gamma^\delta} \frac{\mathrm{P}^\alpha(J,\mu)}{|J|^{\frac{1}{2}}} |x-z| \le \varepsilon \frac{\mathrm{P}^\alpha(J,\mu)}{|J|^{\frac{1}{2}}} |x-z|. \tag{3.1}
$$

The point of this inequality (3.1) is that it permits the replacement of the difference $T_\ell^\alpha \mu(x) - T_\ell^\alpha \mu(z)$ in (2.4) by the linear part Λ_ℓ^α of the Taylor expansion of the kernel K_ℓ^α.

Now we make the choice

$$
\begin{aligned}
&\Omega_\ell(w) = \Omega(\theta_\ell(w)); \\
&\theta_\ell(w) \equiv \tan^{-1} \frac{(-1)^{\ell'} w^{\ell'}}{w^\ell}, \qquad 1 \le \ell \le 2,
\end{aligned}
$$

where $w^{\ell'}$ denotes the coordinate variable other than w^ℓ, that is, $\ell + \ell' = 3$. Thus, θ_1 is the usual angular coordinate on the circle and $\theta_2 = \theta_1 + \frac{\pi}{2}$. We now use

$$
\begin{aligned}
\nabla |w|^{\alpha-2} &= \left(\frac{\partial}{\partial w^1}((w^1)^2 + (w^2)^2)^{\frac{\alpha-2}{2}}, \frac{\partial}{\partial w^2}((w^1)^2 + (w^2)^2)^{\frac{\alpha-2}{2}}\right) \\
&= \frac{\alpha-2}{2}((w^1)^2 + (w^2)^2)^{\frac{\alpha-2}{2}-1} 2w \\
&= (\alpha-2)|w|^{\alpha-4} w.
\end{aligned}
$$

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and

$$\frac{\partial}{\partial w^\ell}\tan^{-1}\frac{w^{\ell'}}{w^\ell}=\frac{1}{1+(\frac{w^{\ell'}}{w^\ell})^2}\frac{-w^{\ell'}}{(w^\ell)^2}=\frac{-w^{\ell'}}{|w|^2},$$

$$\frac{\partial}{\partial w^{\ell'}}\tan^{-1}\frac{w^{\ell'}}{w^\ell}=\frac{1}{1+(\frac{w^{\ell'}}{w^\ell})^2}\frac{1}{w^\ell}=\frac{w^\ell}{|w|^2}.$$

to calculate that the gradient of the convolution kernel

$$K_\ell^\alpha(w)=\frac{\Omega_\ell(w)}{|w|^{2-\alpha}}=\frac{\Omega(\theta_\ell(w))}{|w|^{2-\alpha}}=\frac{\Omega(\tan^{-1}\frac{w^{\ell'}}{w^\ell})}{|w|^{2-\alpha}},$$

is given by,

$$\begin{aligned}\nabla K_\ell^\alpha(w)&=\nabla\left(\frac{\Omega_\ell(w)}{|w|^{2-\alpha}}\right)=\Omega(\theta_\ell(w))\nabla|w|^{\alpha-2}+|w|^{\alpha-2}\Omega'(\theta_\ell(w))\nabla\theta_\ell\\&=\frac{(\alpha-2)\Omega(\theta_\ell(w))w+\Omega'(\theta_\ell(w))w^\perp}{|w|^{4-\alpha}}.\end{aligned}$$

Thus, the linear part Λ_ℓ^α in the Taylor expansion of $T_\ell^\alpha\mu$ is given by

$$\Lambda_\ell^\alpha=(x-z)\cdot\int\nabla K_\ell^\alpha(c_J-y)\,\mathrm{d}\mu(y)\equiv(x-z)\cdot\mathbf{Z}_{\Omega_\ell}^\alpha(c_J;\mu),$$

where

$$\begin{aligned}\mathbf{Z}_{\Omega_\ell}^\alpha(c_J;\mu)&=\int_{\mathbb{R}^2}\frac{(\alpha-2)\Omega(\theta_\ell(c_J-y))(c_J-y)+\Omega'(\theta_\ell(c_J-y))(c_J-y)^\perp}{|c_J-y|^{4-\alpha}}\,\mathrm{d}\mu(y)\\&=\int_{w\in\mathbb{S}^1}\{(\alpha-2)\Omega(\theta_\ell(w))w^1-\Omega'(\theta_\ell(w))w^2\}\mathbf{e}^1\,\mathrm{d}\Psi_\mu(w)\\&\quad+\int_{w\in\mathbb{S}^1}\{(\alpha-2)\Omega(\theta_\ell(w))w^2+\Omega'(\theta_\ell(w))w^1\}\mathbf{e}^2\,\mathrm{d}\Psi_\mu(w),\end{aligned}$$

and $\mathbf{e}^\ell$ is the coordinate vector with a 1 in the ℓth position. Here the measure Ψ_μ is an *arbitrary* positive finite measure on the circle $\mathbb{S}^1$ given formally by

$$\mathrm{d}\Psi_\mu(w)=\mathrm{d}\Psi_\mu^J(w)=\int_0^\infty r^{\alpha-3}\,\mathrm{d}\mu_w^J(r)=\int_0^\infty r^{\alpha-3}\,\mathrm{d}\mu(c_J+rw),\qquad w\in\mathbb{S}^1.$$

Note that conversely, if we are given a positive finite measure Ψ on the circle, we can simply translate and dilate Ψ, preserving its mass, to be supported in a circle centered at c_J with radius R. Then, the resulting measure μ will satisfy $\mathrm{d}\Psi_\mu^J=\mathrm{d}\Psi$. Below we will typically apply this converse observation with R chosen to be $\gamma|J|^{\frac{1}{2}}$ to obtain our counterexamples.

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We use,

$$\tan\theta_\ell(w) = \frac{(-1)^{\ell'} w^{\ell'}}{w^\ell},$$

$$\csc\theta_\ell(w) = (-1)^{\ell'}\sqrt{1+\cot^2\theta_\ell(w)} = (-1)^{\ell'}\sqrt{1+\left(\frac{w^\ell}{w^{\ell'}}\right)^2} = \frac{|w|}{(-1)^{\ell'} w^{\ell'}},$$

$$\sin\theta_\ell(w) = \frac{(-1)^{\ell'} w^{\ell'}}{|w|} \quad \text{and} \quad \cos\theta_\ell(w) = \frac{w^\ell}{|w|},$$

for $w \neq 0$, to obtain

$$\begin{aligned}
\mathbf{Z}^\alpha_{\Omega_1}(c_J;\mu) &= \int_{\mathbb{S}^1} \{(\alpha-2)\Omega(\theta_1(w))\cos\theta_1(w) - \Omega'(\theta_1(w))\sin\theta_1(w)\}\mathbf{e}^1\,\mathrm{d}\Psi_\mu \\
&\quad + \int_{\mathbb{S}^1} \{(\alpha-2)\Omega(\theta_1(w))\sin\theta_1(w) + \Omega'(\theta_1(w))\cos\theta_1(w)\}\mathbf{e}^2\,\mathrm{d}\Psi_\mu \\
&\equiv \int_{\mathbb{S}^1} \{A^1_\alpha(\theta_1(w))\mathbf{e}^1 + B^1_\alpha(\theta_1(w))\mathbf{e}^2\}\,\mathrm{d}\Psi_\mu,
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{Z}^\alpha_{\Omega_2}(c_J;\mu) &= \int_{\mathbb{S}^1} \{-(\alpha-2)\Omega(\theta_2(w))\sin\theta_2(w) - \Omega'(\theta_2(w))\cos\theta_2(w)\}\mathbf{e}^1\,\mathrm{d}\Psi_\mu \\
&\quad + \int_{\mathbb{S}^1} \{(\alpha-2)\Omega(\theta_2(w))\cos\theta_2(w) - \Omega'(\theta_2(w))\sin\theta_2(w)\}\mathbf{e}^2\,\mathrm{d}\Psi_\mu \\
&\equiv \int_{\mathbb{S}^1} \{A^2_\alpha(\theta_2(w))\mathbf{e}^1 + B^2_\alpha(\theta_2(w))\mathbf{e}^2\}\,\mathrm{d}\Psi_\mu,
\end{aligned}$$

with

$$\begin{aligned}
A^1_\alpha(t) &= (\alpha-2)\Omega(t)\cos t - \Omega'(t)\sin t = B^2_\alpha(t), \\
B^1_\alpha(t) &= (\alpha-2)\Omega(t)\sin t + \Omega'(t)\cos t = -A^2_\alpha(t).
\end{aligned} \tag{3.2}$$

Now we show below in (3.7) that a necessary condition for reversal of energy on J is that the span of the pair of vectors $\{\mathbf{Z}^\alpha_{\Omega_\ell}(c_J;\mu)\}_{\ell=1}^2$ is all of $\mathbb{R}^2$:

$$\mathrm{Span}\{\mathbf{Z}^\alpha_{\Omega_\ell}(c_J;\mu)\}_{\ell=1}^2 = \mathbb{R}^2. \tag{3.3}$$

So it suffices to show the failure of (3.3), that is, that $\mathbf{Z}^\alpha_{\Omega_1}(c_J;\mu)$ and $\mathbf{Z}^\alpha_{\Omega_2}(c_J;\mu)$ are parallel, or at least one of them is the zero vector.

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At this point, we take $\ell = 1$ and set $\theta = \theta_1(w)$ so that we obtain

$$A_\alpha(\theta) \equiv A^1_\alpha(\theta_1(w)) = (\alpha - 2)\Omega(\theta)\cos\theta - \Omega'(\theta)\sin\theta,$$
$$B_\alpha(\theta) \equiv B^1_\alpha(\theta_1(w)) = (\alpha - 2)\Omega(\theta)\sin\theta + \Omega'(\theta)\cos\theta. \tag{3.4}$$

In the case $\alpha = 1$, these coefficients are perfect derivatives:

$$A_1(\theta) = -\Omega(\theta)\cos\theta - \Omega'(\theta)\sin\theta = -[\Omega(\theta)\sin\theta]',$$
$$B_1(\theta) = -\Omega(\theta)\sin\theta + \Omega'(\theta)\cos\theta = -[\Omega(\theta)\cos\theta]',$$

and so have vanishing integral on the circle. Thus with the choice $d\Psi_\mu(\theta) = d\theta$ we have

$$\mathbf{Z}_\Omega(c_J;\mu) = \int_{\mathbb{S}^1}\{A_1(\theta)\mathbf{e}^1 + B_1(\theta)\mathbf{e}^2\}\,d\theta = \mathbf{0}$$

the zero vector, for **every** choice of differentiable Ω on the circle.

In the case $0 \le \alpha < 2$ with $\alpha \ne 1$, it is no longer possible to find a nontrivial measure μ so that $\mathbf{Z}^\alpha_\Omega(c_J;\mu)$ vanishes for all differentiable Ω, but we will see that we can always find a positive measure μ such that the vectors $\mathbf{Z}^\alpha_{\Omega_1}(c_J;\mu)$ and $\mathbf{Z}^\alpha_{\Omega_2}(c_J;\mu)$ are parallel for the choice $\Omega(\theta) = \cos\theta$ that corresponds to the vector of Riesz transforms.

Indeed, in the special case that $\Omega(t) = \cos t$, and recalling that $\theta_2(w) = \theta_1(w) + \frac{\pi}{2} = \theta + \frac{\pi}{2}$, we have

$$A^1_\alpha(\theta_1(w)) = A^1_\alpha(\theta) = (\alpha - 2)\cos^2\theta + \sin^2\theta;$$
$$B^1_\alpha(\theta_1(w)) = B^1_\alpha(\theta) = (\alpha - 3)\cos\theta\sin\theta;$$
$$A^2_\alpha(\theta_2(w)) = -B^1_\alpha\left(\theta + \frac{\pi}{2}\right) = -(\alpha - 3)\cos\left(\theta + \frac{\pi}{2}\right)\sin\left(\theta + \frac{\pi}{2}\right)$$
$$= (\alpha - 3)\cos\theta\sin\theta;$$
$$B^2_\alpha(\theta_2(w)) = A^1_\alpha\left(\theta + \frac{\pi}{2}\right) = (\alpha - 2)\cos^2\left(\theta + \frac{\pi}{2}\right) + \sin^2\left(\theta + \frac{\pi}{2}\right)$$
$$= (\alpha - 2)\sin^2\theta + \cos^2\theta.$$

Thus we also have

$$\mathbf{Z}^\alpha_{\Omega_1}(c_J;\mu) = \int_{\mathbb{S}^1}\{A^1_\alpha(\theta_1(w))\mathbf{e}^1 + B^1_\alpha(\theta_1(w))\mathbf{e}^2\}\,d\Psi_\mu$$
$$= \int_{\mathbb{S}^1}\{[(\alpha - 2)\cos^2\theta + \sin^2\theta]\mathbf{e}^1 + [(\alpha - 3)\cos\theta\sin\theta]\mathbf{e}^2\}\,d\Psi_\mu$$
$$= \left\{\int_{\mathbb{S}^1}[(\alpha - 2)\cos^2\theta + \sin^2\theta]\,d\Psi_\mu\right\}\mathbf{e}^1 + \left\{\int_{\mathbb{S}^1}[(\alpha - 3)\cos\theta\sin\theta]\,d\Psi_\mu\right\}\mathbf{e}^2$$

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and

$$\begin{aligned}
\mathbf{Z}^{\alpha}_{\Omega_2}(c_J;\mu) &= \int_{\mathbb{S}^1} \{A^2_{\alpha}(\theta_2(w))\mathbf{e}^1 + B^2_{\alpha}(\theta_2(w))\mathbf{e}^2\}\,\mathrm{d}\Psi_\mu \\
&= \int_{\mathbb{S}^1} \{[(\alpha-3)\cos\theta\sin\theta]\mathbf{e}^1 + [(\alpha-2)\sin^2\theta+\cos^2\theta]\mathbf{e}^2\}\,\mathrm{d}\Psi_\mu \\
&= \left\{\int_{\mathbb{S}^1}[(\alpha-3)\cos\theta\sin\theta]\,\mathrm{d}\Psi_\mu\right\}\mathbf{e}^1 + \left\{\int_{\mathbb{S}^1}[(\alpha-2)\sin^2\theta+\cos^2\theta]\,\mathrm{d}\Psi_\mu\right\}\mathbf{e}^2.
\end{aligned}$$

Using

$$\begin{aligned}
&(\alpha-2)\cos^2\theta+\sin^2\theta = (\alpha-3)\cos^2\theta+1, \\
&(\alpha-2)\sin^2\theta+\cos^2\theta = (\alpha-3)\sin^2\theta+1, \\
&\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta, \quad \cos^2\theta = \frac{1+\cos 2\theta}{2}, \quad \sin^2\theta = \frac{1-\cos 2\theta}{2},
\end{aligned} \tag{3.5}$$

we see that

$$\begin{aligned}
(\alpha-2)\cos^2\theta+\sin^2\theta &= (\alpha-3)\frac{1+\cos 2\theta}{2}+1 = \frac{\alpha-3}{2}\cos 2\theta + \frac{\alpha-1}{2}, \\
(\alpha-2)\sin^2\theta+\cos^2\theta &= (\alpha-3)\frac{1-\cos 2\theta}{2}+1 = -\frac{\alpha-3}{2}\cos 2\theta + \frac{\alpha-1}{2}, \\
(\alpha-3)\cos\theta\sin\theta &= \frac{\alpha-3}{2}\sin 2\theta.
\end{aligned}$$

Plugging these formulas into those for $\mathbf{Z}^{\alpha}_{\Omega_1}(c_J;\mu)$ and $\mathbf{Z}^{\alpha}_{\Omega_2}(c_J;\mu)$, we obtain

$$\begin{aligned}
&\det\begin{bmatrix}\mathbf{Z}^{\alpha}_{\Omega_1}(c_J;\mu)\\ \mathbf{Z}^{\alpha}_{\Omega_2}(c_J;\mu)\end{bmatrix} \\
&= \det\begin{bmatrix}\int_{\mathbb{S}^1}\left[\frac{\alpha-3}{2}\cos 2\theta+\frac{\alpha-1}{2}\right]\mathrm{d}\Psi_\mu & \int_{\mathbb{S}^1}\left[\frac{\alpha-3}{2}\sin 2\theta\right]\mathrm{d}\Psi_\mu \\ \int_{\mathbb{S}^1}\left[\frac{\alpha-3}{2}\sin 2\theta\right]\mathrm{d}\Psi_\mu & \int_{\mathbb{S}^1}\left[-\frac{\alpha-3}{2}\cos 2\theta+\frac{\alpha-1}{2}\right]\mathrm{d}\Psi_\mu\end{bmatrix} \\
&= \left(\frac{\alpha-3}{2}\int_{\mathbb{S}^1}\cos 2\theta\,\mathrm{d}\Psi_\mu + \frac{\alpha-1}{2}\|\Psi_\mu\|\right)\left(-\frac{\alpha-3}{2}\int_{\mathbb{S}^1}\cos 2\theta\,\mathrm{d}\Psi_\mu + \frac{\alpha-1}{2}\|\Psi_\mu\|\right) \\
&\quad - \left(\frac{\alpha-3}{2}\int_{\mathbb{S}^1}\sin 2\theta\,\mathrm{d}\Psi_\mu\right)^2 \\
&= \left(\frac{\alpha-1}{2}\|\Psi_\mu\|\right)^2 - \left\{\left(\frac{\alpha-3}{2}\int_{\mathbb{S}^1}\cos 2\theta\,\mathrm{d}\Psi_\mu\right)^2 + \left(\frac{\alpha-3}{2}\int_{\mathbb{S}^1}\sin 2\theta\,\mathrm{d}\Psi_\mu\right)^2\right\}.
\end{aligned}$$

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Thus $\det\begin{bmatrix}\mathbf{Z}^{\alpha}_{\Omega_1}(c_J;\mu)\\ \mathbf{Z}^{\alpha}_{\Omega_2}(c_J;\mu)\end{bmatrix}=0$ if and only if the length of the vector

$$\frac{\alpha-3}{2}\begin{pmatrix}\int_{\mathbb{S}^1}\cos 2\theta\,\mathrm{d}\Psi_\mu\\ \int_{\mathbb{S}^1}\sin 2\theta\,\mathrm{d}\Psi_\mu\end{pmatrix}$$

equals $\frac{|\alpha-1|}{2}\|\Psi_\mu\|$, that is,

$$\left\|\begin{pmatrix}\int_{\mathbb{S}^1}\cos 2\theta\,\mathrm{d}\Psi_\mu\\ \int_{\mathbb{S}^1}\sin 2\theta\,\mathrm{d}\Psi_\mu\end{pmatrix}\right\|=\frac{|\alpha-1|}{|\alpha-3|}\|\Psi_\mu\|. \tag{3.6}$$

To construct a positive probability measure $\mathrm{d}\Psi_\mu$ on the circle that satisfies (3.6), we first observe that if $\mathrm{d}\Psi_\mu=\delta_{(1,0)}$ is the unit point mass at $(1,0)$, then

$$\left\|\begin{pmatrix}\int_{\mathbb{S}^1}\cos 2\theta\,\mathrm{d}\Psi_\mu\\ \int_{\mathbb{S}^1}\sin 2\theta\,\mathrm{d}\Psi_\mu\end{pmatrix}\right\|=\left\|\begin{pmatrix}\int_{\mathbb{S}^1}\mathrm{d}\Psi_\mu\\ 0\end{pmatrix}\right\|=\|\Psi_\mu\|,$$

and since $|\alpha-1|<|\alpha-3|$ for all $0\le\alpha<2$, we have

$$\left\|\begin{pmatrix}\int_{\mathbb{S}^1}\cos 2\theta\,\mathrm{d}\Psi_\mu\\ \int_{\mathbb{S}^1}\sin 2\theta\,\mathrm{d}\Psi_\mu\end{pmatrix}\right\|>\frac{|\alpha-1|}{|\alpha-3|}\|\Psi_\mu\|,$$

in this case. On the other hand, if $\mathrm{d}\Psi_\mu(\theta)=\frac{1}{2\pi}\,\mathrm{d}\theta$ is normalized Lebesgue measure on the circle, we have

$$\left\|\begin{pmatrix}\int_{\mathbb{S}^1}\cos 2\theta\,\mathrm{d}\Psi_\mu\\ \int_{\mathbb{S}^1}\sin 2\theta\,\mathrm{d}\Psi_\mu\end{pmatrix}\right\|=\left\|\begin{pmatrix}0\\ 0\end{pmatrix}\right\|=0<\frac{|\alpha-1|}{|\alpha-3|}\|\Psi_\mu\|.$$

It is now easy to see that there is a convex combination $\mathrm{d}\Psi_\mu=(1-\lambda)\delta_{(1,0)}+\lambda\frac{1}{2\pi}\,\mathrm{d}\theta$ such that (3.6) holds. Thus (3.3) fails, and we now show that energy reversal fails.

In fact, we may assume that both $\mathbf{Z}^{\alpha}_{\Omega_1}(c_J;\mu)$ and $\mathbf{Z}^{\alpha}_{\Omega_2}(c_J;\mu)$ are parallel to the coordinate vector $\mathbf{e}_2$, and in this case we will see that we can reverse at most the coordinate energy $\mathsf{E}^2(J,\omega)$, defined above by

$$\mathsf{E}^2(J,\omega)^2\equiv\frac{1}{|J|_\omega}\frac{1}{|J|_\omega}\int_J\int_J\left|\frac{x^2-z^2}{|J|^{\frac{1}{n}}}\right|^2\mathrm{d}\omega(x)\,\mathrm{d}\omega(z),$$

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and not the full energy $\mathsf{E}(J,\omega)$. More precisely, we claim that there is a measure ω such that for γ so large that $\varepsilon \ll C_0$, the strong reversal of ω-energy inequality (2.4) fails. Indeed, using that $\mathbf{Z}^{\alpha}_{\Omega_\ell}(c_J;\mu)$ is parallel to $\mathbf{e}^2$, we have that

$$\begin{aligned}
&\int_J\int_J |\mathbf{R}^{\alpha}\mu(x)-\mathbf{R}^{\alpha}\mu(z)|^2\,\mathrm{d}\omega(x)\,\mathrm{d}\omega(z)\\
&=\sum_{\ell=1}^{2}\int_J\int_J |(x-z)\cdot\mathbf{Z}^{\alpha}_{\Omega_\ell}(c_J;\mu)+[E^{\alpha}_{\ell,x}-E^{\alpha}_{\ell,z}]|^2\,\mathrm{d}\omega(x)\,\mathrm{d}\omega(z)\\
&\le\sum_{\ell=1}^{2}\int_J\int_J \left|\frac{\mathrm{P}^{\alpha}(J,\mu)}{|J|^{\frac{1}{2}}}(x-z)\cdot\frac{\mathbf{Z}^{\alpha}_{\Omega_\ell}(c_J;\mu)}{|\mathbf{Z}^{\alpha}_{\Omega_\ell}(c_J;\mu)|}\right|^2\mathrm{d}\omega(x)\,\mathrm{d}\omega(z)\\
&\quad+C\sum_{\ell=1}^{2}\int_J\int_J\left|\varepsilon\frac{\mathrm{P}^{\alpha}(J,\mu)}{|J|^{\frac{1}{2}}}|x-z|\right|^2\mathrm{d}\omega(x)\,\mathrm{d}\omega(z)\\
&\le\mathsf{E}^2(J,\omega)^2\mathrm{P}^{\alpha}(J,\mu)^2+C\varepsilon^2\mathsf{E}(J,\omega)^2\mathrm{P}^{\alpha}(J,\mu)^2\\
&\le\frac{1}{10}C_0\mathsf{E}(J,\omega)^2\mathrm{P}^{\alpha}(J,\mu)^2,
\end{aligned}\tag{3.7}$$

provided we choose γ so large that $C\varepsilon^2\le\frac{1}{10}C_0$ and provided we choose ω so that $\mathsf{E}^2(J,\omega)=0$ but $\mathsf{E}(J,\omega)>0$. This completes the proof of the first assertion in Lemma 2. ■

Remark 1. The above proof shows that for each $t\in\mathbb{R}$, the convolution kernel

$$\Phi_{\alpha,t}(x,y)=\frac{x\cos t+y\sin t}{(x^2+y^2)^{\frac{3-\alpha}{2}}},$$

in the plane with coordinates (x,y), $x,y\in\mathbb{R}$, and the probability measure $\mathrm{d}\mu_\alpha$ supported on the circle $\mathbb{S}^1=[0,2\pi)$ given by

$$\mathrm{d}\mu_\alpha(\theta)=\frac{|\alpha-1|}{|\alpha-3|}\delta_0(\theta)+\frac{|\alpha-3|-|\alpha-1|}{|\alpha-3|}\frac{\mathrm{d}\theta}{2\pi},$$

satisfy the somewhat surprising property that $\nabla(\Phi_{\alpha,t}*\mu_\alpha)(0,0)$ points in the same direction for all t. A direct calculation, which we leave for the interested reader, shows that

$$\nabla(\Phi_{\alpha,t}*\mu_\alpha)(0,0)=(\alpha-1)\begin{cases}[\cos t,0] & \text{for } 0\le\alpha<1\\ [0,\sin t] & \text{for } 1<\alpha<2\end{cases}.$$

□

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Proof of Lemma 2 for the vector of trig polynomials. Recall that with $\theta = \theta_1(w)$ we obtain

$$A_\alpha(\theta) = (\alpha - 2)\Omega(\theta)\cos\theta - \Omega'(\theta)\sin\theta$$

$$B_\alpha(\theta) = (\alpha - 2)\Omega(\theta)\sin\theta + \Omega'(\theta)\cos\theta.$$

Thus, we have

$$\begin{aligned} A_\alpha(\theta) &= \{(\alpha-2)\Omega(\theta) + \mathrm{i}\Omega'(\theta)\}\{\cos\theta + \mathrm{i}\sin\theta\} - \mathrm{i}\{(\alpha-2)\Omega(\theta)\sin\theta + \Omega'(\theta)\cos\theta\} \\ &= \{(\alpha-2)\Omega(\theta) + \mathrm{i}\Omega'(\theta)\}\{\cos\theta + \mathrm{i}\sin\theta\} - \mathrm{i}B_\alpha(\theta), \end{aligned}$$

and so

$$\{(\alpha-2)\Omega(\theta) + \mathrm{i}\Omega'(\theta)\}\{\cos\theta + \mathrm{i}\sin\theta\} = A_\alpha(\theta) + \mathrm{i}B_\alpha(\theta).$$

This shows that in complex notation,

$$\begin{aligned} \mathbf{Z}^\alpha_\Omega(c_J;\mu) &= \int_{\mathbb{S}^1} \{A_\alpha(\theta) + \mathrm{i}B_\alpha(\theta)\}\,\mathrm{d}\Psi_\mu \\ &= \int_{\mathbb{S}^1} \{(\alpha-2)\Omega(\theta) + \mathrm{i}\Omega'(\theta)\}\{\cos\theta + \mathrm{i}\sin\theta\}\,\mathrm{d}\Psi_\mu \\ &= \int_{\mathbb{S}^1} \Omega_\alpha(\theta)\,\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\Psi_\mu, \end{aligned}$$

where

$$\Omega_\alpha(\theta) \equiv (\alpha-2)\Omega(\theta) + \mathrm{i}\Omega'(\theta).$$

Recall the product formulas

$$2\cos A\cos B = \cos(A-B) + \cos(A+B);$$

$$2\sin A\sin B = \cos(A-B) - \cos(A+B);$$

$$2\sin A\cos B = \sin(A-B) + \sin(A+B).$$

In the special case that $\Omega_1^k(t) = \cos kt$ we thus have

$$\begin{aligned} A_\alpha(\theta) &= (\alpha-2)\cos k\theta\cos\theta + k\sin k\theta\sin\theta \\ &= (\alpha-2)\frac{1}{2}[\cos(k-1)\theta + \cos(k+1)\theta] + k\frac{1}{2}[\cos(k-1)\theta - \cos(k+1)\theta] \\ &= \left\{\frac{\alpha+k}{2} - 1\right\}\cos(k-1)\theta + \left\{\frac{\alpha-k}{2} - 1\right\}\cos(k+1)\theta; \end{aligned}$$

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$$B_\alpha(\theta) = (\alpha - 2)\cos k\theta \sin\theta - k\sin k\theta\cos\theta$$

$$= (\alpha - 2)\frac{1}{2}[-\sin(k-1)\theta + \sin(k+1)\theta] - k\frac{1}{2}[\sin(k-1)\theta + \sin(k+1)\theta]$$

$$= -\left\{\frac{\alpha+k}{2} - 1\right\}\sin(k-1)\theta + \left\{\frac{\alpha-k}{2} - 1\right\}\sin(k+1)\theta,$$

and so

$$\mathbf{Z}^\alpha_{\Omega^k_1}(c_J;\mu) = \int_{\mathbb{S}^1}\{A_\alpha(\theta) + \mathrm{i}B_\alpha(\theta)\}\,\mathrm{d}\Psi_\mu$$

$$= \int_{\mathbb{S}^1}\left[\left\{\frac{\alpha+k}{2} - 1\right\}\cos(k-1)\theta + \left\{\frac{\alpha-k}{2} - 1\right\}\cos(k+1)\theta\right]\mathrm{d}\Psi_\mu$$

$$+ \mathrm{i}\int_{\mathbb{S}^1}\left[-\left\{\frac{\alpha+k}{2} - 1\right\}\sin(k-1)\theta + \left\{\frac{\alpha-k}{2} - 1\right\}\sin(k+1)\theta\right]\mathrm{d}\Psi_\mu$$

$$= \int_{\mathbb{S}^1}\left\{\left(\frac{\alpha+k-2}{2}\right)\mathrm{e}^{-\mathrm{i}(k-1)\theta} + \left(\frac{\alpha-k-2}{2}\right)\mathrm{e}^{\mathrm{i}(k+1)\theta}\right\}\mathrm{d}\Psi_\mu$$

$$= \left(\frac{\alpha+k-2}{2}\right)\overline{\widehat{\Psi_\mu}(k-1)} + \left(\frac{\alpha-k-2}{2}\right)\widehat{\Psi_\mu}(k+1).$$

Next we take $\Omega^k_2(\theta) = \sin k\theta$ so that

$$A_\alpha(\theta) = (\alpha - 2)\sin k\theta\cos\theta - k\cos k\theta\sin\theta$$

$$= (\alpha - 2)\frac{1}{2}[\sin(k-1)\theta + \sin(k+1)\theta] - k\frac{1}{2}[-\sin(k-1)\theta + \sin(k+1)\theta]$$

$$= \left\{\frac{\alpha+k}{2} - 1\right\}\sin(k-1)\theta + \left\{\frac{\alpha-k}{2} - 1\right\}\sin(k+1)\theta;$$

$$B_\alpha(\theta) = (\alpha - 2)\sin k\theta\sin\theta + k\cos k\theta\cos\theta$$

$$= (\alpha - 2)\frac{1}{2}[\cos(k-1)\theta - \cos(k+1)\theta] + k\frac{1}{2}[\cos(k-1)\theta + \cos(k+1)\theta]$$

$$= \left\{\frac{\alpha+k}{2} - 1\right\}\cos(k-1)\theta - \left\{\frac{\alpha-k}{2} - 1\right\}\cos(k+1)\theta.$$

Thus with $\Omega^k_2(\theta) = \sin k\theta$ we obtain

$$\mathbf{Z}^\alpha_{\Omega^k_2}(c_J;\mu) = \int_{\mathbb{S}^1}\{A_\alpha(\theta) + \mathrm{i}B_\alpha(\theta)\}\,\mathrm{d}\Psi_\mu$$

$$= \int_{\mathbb{S}^1}\left[\left\{\frac{\alpha+k}{2} - 1\right\}\sin(k-1)\theta + \left\{\frac{\alpha-k}{2} - 1\right\}\sin(k+1)\theta\right]\mathrm{d}\Psi_\mu$$

$$+ \mathrm{i}\int_{\mathbb{S}^1}\left[\left\{\frac{\alpha+k}{2} - 1\right\}\cos(k-1)\theta - \left\{\frac{\alpha-k}{2} - 1\right\}\cos(k+1)\theta\right]\mathrm{d}\Psi_\mu$$

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$$= \int_{\mathbb{S}^1} \left\{ \left(\frac{\alpha + k - 2}{2} \right) \mathrm{i} e^{-\mathrm{i}(k-1)\theta} - \left(\frac{\alpha - k - 2}{2} \right) \mathrm{i} e^{\mathrm{i}(k+1)\theta} \right\} \mathrm{d}\Psi_\mu$$

$$= \mathrm{i} \left(\frac{\alpha + k - 2}{2} \right) \overline{\widehat{\Psi_\mu}(k-1)} - \mathrm{i} \left(\frac{\alpha - k - 2}{2} \right) \widehat{\Psi_\mu}(k+1).$$

Altogether we have

$$\begin{aligned} \mathbf{Z}^\alpha_{\Omega^k_1}(c_J;\mu) &= \left(\frac{\alpha + k - 2}{2} \right) \overline{\widehat{\Psi_\mu}(k-1)} + \left(\frac{\alpha - k - 2}{2} \right) \widehat{\Psi_\mu}(k+1); \\ \mathbf{Z}^\alpha_{\Omega^k_2}(c_J;\mu) &= \mathrm{i} \left[\left(\frac{\alpha + k - 2}{2} \right) \overline{\widehat{\Psi_\mu}(k-1)} - \left(\frac{\alpha - k - 2}{2} \right) \widehat{\Psi_\mu}(k+1) \right]. \end{aligned} \tag{3.8}$$

Thus $\det \begin{bmatrix} \mathbf{Z}^\alpha_{\Omega^k_1}(c_J;\mu) \\ \mathbf{Z}^\alpha_{\Omega^k_2}(c_J;\mu) \end{bmatrix}$ is the imaginary part of $\mathbf{Z}^\alpha_{\Omega^k_1}(c_J;\mu)\overline{\mathbf{Z}^\alpha_{\Omega^k_2}(c_J;\mu)}$, which is -1 times the real part of

$$\begin{aligned} &\left\{ \left(\frac{\alpha + k - 2}{2} \right) \overline{\widehat{\Psi_\mu}(k-1)} + \left(\frac{\alpha - k - 2}{2} \right) \widehat{\Psi_\mu}(k+1) \right\} \\ &\quad \times \left\{ \left(\frac{\alpha + k - 2}{2} \right) \widehat{\Psi_\mu}(k-1) - \left(\frac{\alpha - k - 2}{2} \right) \overline{\widehat{\Psi_\mu}(k+1)} \right\} \\ &\quad = \left(\frac{\alpha + k - 2}{2} \right)^2 |\widehat{\Psi_\mu}(k-1)|^2 - \left(\frac{\alpha - k - 2}{2} \right)^2 |\widehat{\Psi_\mu}(k+1)|^2 \\ &\qquad + \mathfrak{Re} \left[\left(\frac{\alpha + k - 2}{2} \right) \left(\frac{\alpha - k - 2}{2} \right) (\widehat{\Psi_\mu}(k+1)\widehat{\Psi_\mu}(k-1) - \overline{\widehat{\Psi_\mu}(k-1)\widehat{\Psi_\mu}(k+1)}) \right] \\ &\quad = \left(\frac{\alpha + k - 2}{2} \right)^2 |\widehat{\Psi_\mu}(k-1)|^2 - \left(\frac{\alpha - k - 2}{2} \right)^2 |\widehat{\Psi_\mu}(k+1)|^2, \end{aligned}$$

since $\widehat{\Psi_\mu}(k+1)\widehat{\Psi_\mu}(k-1) - \overline{\widehat{\Psi_\mu}(k-1)\widehat{\Psi_\mu}(k+1)}$ is pure imaginary. We conclude that

$$\det \begin{bmatrix} \mathbf{Z}^\alpha_{\Omega^k_1}(c_J;\mu) \\ \mathbf{Z}^\alpha_{\Omega^k_2}(c_J;\mu) \end{bmatrix} = 0 \iff |\widehat{\Psi_\mu}(k+1)| = \left| \frac{\alpha + k - 2}{\alpha - k - 2} \right| |\widehat{\Psi_\mu}(k-1)|, \quad \text{all } k. \tag{3.9}$$

We also have that $\det \begin{bmatrix} \mathbf{Z}^\alpha_{\Omega^k_1}(c_J;\mu) \\ \mathbf{Z}^\alpha_{\Omega^\ell_1}(c_J;\mu) \end{bmatrix}$ is the imaginary part of $\mathbf{Z}^\alpha_{\Omega^k_1}(c_J;\mu)\overline{\mathbf{Z}^\alpha_{\Omega^\ell_1}(c_J;\mu)}$, that is, the imaginary part of

$$\begin{aligned} &\left\{ \left(\frac{\alpha + k - 2}{2} \right) \overline{\widehat{\Psi_\mu}(k-1)} + \left(\frac{\alpha - k - 2}{2} \right) \widehat{\Psi_\mu}(k+1) \right\} \\ &\quad \times \left\{ \left(\frac{\alpha + \ell - 2}{2} \right) \widehat{\Psi_\mu}(\ell-1) + \left(\frac{\alpha - \ell - 2}{2} \right) \overline{\widehat{\Psi_\mu}(\ell+1)} \right\}. \end{aligned}$$

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If we now suppose that $\widehat{\Psi_\mu}(n)$ is real for all n, then $\mathbf{Z}^\alpha_{\Omega_1^k}(c_J;\mu)$ is real for all k, and it follows that

$$\det\begin{bmatrix}\mathbf{Z}^\alpha_{\Omega_1^k}(c_J;\mu)\\ \mathbf{Z}^\alpha_{\Omega_1^\ell}(c_J;\mu)\end{bmatrix}=\mathfrak{Im}(\mathbf{Z}^\alpha_{\Omega_1^k}(c_J;\mu)\,\overline{\mathbf{Z}^\alpha_{\Omega_1^\ell}(c_J;\mu)})=0,\quad \text{all } k,\ell. \tag{3.10}$$

We are now ready to construct the measure μ with an appropriate density Ψ_μ. In the case $1\le\alpha<2$ there is a choice of density that is easy to prove positive, and we give that first. Then we give a density for all cases $0\le\alpha<2$, but that is much harder to prove positive. Finally, we give a particularly simple proof for the case $\alpha=0$.

Construction of a density in the case $1\le\alpha<2$:

Define a density $\Psi(\theta)$ by

$$\Psi(\theta)=1+2\sum_{n=1}^\infty b_n\cos(2n\theta)=1+\sum_{n=1}^\infty b_n\{e^{i2n\theta}+e^{-i2n\theta}\},$$

where

$$b_n=\left|\frac{\alpha+(2n-3)}{\alpha-(2n+1)}\frac{\alpha+(2n-5)}{\alpha-(2n-1)}\cdots\frac{\alpha+3}{\alpha-7}\frac{\alpha+1}{\alpha-5}\frac{\alpha-1}{\alpha-3}\right|=a_na_{n-1}\cdots a_2a_1,\quad n\ge1;$$

with

$$a_n=\left|\frac{\alpha+(2n-3)}{\alpha-(2n+1)}\right|=\left|\frac{2n-1-x}{2n-1+x}\right|\quad \text{if } x=2-\alpha.$$

Then, we have

$$\widehat{\Psi}(2n)=b_n=\widehat{\Psi}(-2n),\quad n\ge1,$$

$$\widehat{\Psi}(k)=0\quad\text{if } k \text{ is odd},$$

and in particular that $|\widehat{\Psi}(k+1)|=|\frac{\alpha+k-2}{\alpha-k-2}||\widehat{\Psi}(k-1)|$ for all $k\ge1$. Now choose a measure μ giving rise to the density Ψ. In the case $1\le\alpha<2$ we have $|\frac{\alpha+k-2}{\alpha-k-2}|=-\frac{\alpha+k-2}{\alpha-k-2}$ for $k\ge1$, and so from (3.8) we actually obtain that $\mathbf{Z}^\alpha_{\Omega_1^k}(c_J;\mu)=0$ for all $k\ge1$, and that $\mathbf{Z}^\alpha_{\Omega_2^k}(c_J;\mu)$ is imaginary for all $k\ge1$. Thus, all of the vectors $\{\mathbf{Z}^\alpha_{\Omega_1^k}(c_J;\mu),\mathbf{Z}^\alpha_{\Omega_2^k}(c_J;\mu)\}_{k=1}^\infty$ are multiples of the unit vector $(0,1)$ in the plane (it is the failure of such a conclusion for the case $0<\alpha<1$ that forces a different construction below).

We must now show that the density $\Psi(\theta)$ is nonnegative. We have $\Psi(\theta)=\Phi(2\theta)$ where $\hat{\Phi}(0)=1$ and

$$\hat{\Phi}(n)=\hat{\Phi}(-n)=b_n=a_na_{n-1}\cdots a_2a_1,\quad n\ge1.$$

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We claim that the nonnegative sequence $\{1, b_1, b_2, \ldots\}$ is convex for $0 < x \le 2$, and has limit 0 as $n \to \infty$. With this established, the density Φ is a positive sum of Féjer kernels, and hence $\Phi(\theta) \ge 0$. Since $a_n = \frac{2n-1-x}{2n-1+x} = 1 - \frac{2x}{2n-1+x}$ and $\sum_{n=1}^{\infty} \frac{2x}{2n-1+x} = \infty$, we see that $\lim_{n\to\infty} b_n = \prod_{n=1}^{\infty}(1 - \frac{2x}{2n-1+x}) = 0$. To see the convexity we note that

$$b_{n+1} + b_{n-1} - 2b_n = a_{n+1}a_n[a_{n-1}\cdots a_2a_1] + [a_{n-1}\cdots a_2a_1] - 2a_n[a_{n-1}\cdots a_2a_1]$$
$$= [a_{n+1}a_n + 1 - 2a_n][a_{n-1}\cdots a_2a_1]$$

is positive if and only if $a_{n+1}a_n + 1 - 2a_n$ is positive. But for $n \ge 2$ and $0 < x \le 2$, we have $a_n = \frac{2n-1-x}{2n-1+x}$ and so

$$\begin{aligned} a_{n+1}a_n + 1 - 2a_n &= (a_{n+1} - 2)a_n + 1 \\ &= \left(\frac{2n+1-x}{2n+1+x} - 2\right)\frac{2n-1-x}{2n-1+x} + 1 \\ &= -\left(\frac{2n+1+3x}{2n+1+x}\right)\frac{2n-1-x}{2n-1+x} + 1 \\ &= \frac{(2n+1+x)(2n-1+x) - (2n+1+3x)(2n-1-x)}{(2n+1+x)(2n-1+x)} \\ &= \frac{4x^2+4x}{(2n+1+x)(2n-1+x)} > 0. \end{aligned}$$

This calculation is valid also when $n = 1$ and $0 < x \le 1$, so it remains to consider only the case $n = 1$ and $1 \le x \le 2$. But then we have $a_1 = \frac{x-1}{1+x}$ and so

$$\begin{aligned} a_2a_1 + 1 - 2a_1 &= (a_2 - 2)a_1 + 1 \\ &= \left(\frac{3-x}{3+x} - 2\right)\frac{x-1}{1+x} + 1 = \frac{6-2x}{3+x} > 0. \end{aligned}$$

Construction of a density in the general case $0 \le \alpha < 2$:

This time we modify the definition of our density to be

$$\tilde{\Psi}(\theta) = 1 + 2\sum_{n=1}^{\infty} b_n \cos(2n\theta) = 1 + \sum_{n=1}^{\infty} b_n\{e^{i2n\theta} + e^{-i2n\theta}\},$$

where

$$b_n = \frac{\alpha+(2n-3)}{\alpha-(2n+1)}\,\frac{\alpha+(2n-5)}{\alpha-(2n-1)}\cdots\frac{\alpha+3}{\alpha-7}\,\frac{\alpha+1}{\alpha-5}\,\frac{\alpha-1}{\alpha-3} = a_na_{n-1}\cdots a_2a_1, \qquad n \ge 1;$$

where

$$a_n = \frac{\alpha+(2n-3)}{\alpha-(2n+1)} = -\frac{2n-1-x}{2n-1+x} \quad \text{if } x = 2-\alpha.$$

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Then, we have

$$\hat{\tilde{\Psi}}(2n) = b_n = \hat{\tilde{\Psi}}(-2n), \quad 1 \le n \le N,$$

$$\hat{\tilde{\Psi}}(k) = 0 \quad \text{if } k \text{ is odd},$$

and in particular, if $\tilde{\mu}$ is chosen to give rise to the density $\tilde{\Psi}$, then from (3.8) we obtain that $\mathbf{Z}^{\alpha}_{\Omega_2^k}(c_J; \tilde{\mu}) = 0$ for all $k \ge 1$, and that $\mathbf{Z}^{\alpha}_{\Omega_1^k}(c_J; \tilde{\mu})$ is real for all $k \ge 1$. Thus all of the vectors $\{\mathbf{Z}^{\alpha}_{\Omega_1^k}(c_J; \tilde{\mu}), \mathbf{Z}^{\alpha}_{\Omega_2^k}(c_J; \tilde{\mu})\}_{k=1}^{\infty}$ are multiples of the unit vector $(1, 0)$ in the plane.

Finally, we must show that the density $\tilde{\Psi}(\theta)$ is positive. Now

$$\hat{\tilde{\Psi}}(2n) = b_n = a_n a_{n-1} \cdots a_2 a_1,$$

and so by Bôchner's theorem (more precisely Herglotz's theorem in this application—see, e.g., Rudin [4] for an extension to locally compact abelian groups), it suffices to check that the following matrices are positive semidefinite for $n \ge 2$:

$$\mathbf{B}_n = \begin{bmatrix} \hat{\tilde{\Psi}}(0) & \hat{\tilde{\Psi}}(2) & \hat{\tilde{\Psi}}(4) & \cdots & \hat{\tilde{\Psi}}(2n) \\ \overline{\hat{\tilde{\Psi}}(2)} & \hat{\tilde{\Psi}}(0) & \hat{\tilde{\Psi}}(2) & \cdots & \hat{\tilde{\Psi}}(2n-2) \\ \overline{\hat{\tilde{\Psi}}(4)} & \overline{\hat{\tilde{\Psi}}(2)} & \hat{\tilde{\Psi}}(0) & \cdots & \hat{\tilde{\Psi}}(2n-4) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{\hat{\tilde{\Psi}}(2n)} & \overline{\hat{\tilde{\Psi}}(2n-2)} & \overline{\hat{\tilde{\Psi}}(2n-4)} & \cdots & \hat{\tilde{\Psi}}(0) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & a_1 & a_2 a_1 & \cdots & a_n \cdots a_1 \\ a_1 & 1 & a_1 & \cdots & a_{n-1} \cdots a_1 \\ a_2 a_1 & a_1 & 1 & \cdots & a_{n-2} \cdots a_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n \cdots a_1 & a_{n-1} \cdots a_1 & a_{n-2} \cdots a_1 & \cdots & 1 \end{bmatrix}.$$

Since $a_n = -\frac{2n-1-x}{2n-1+x}$, the matrix $\mathbf{B}_n$ is

$$\mathbf{B}_n(x) = \begin{bmatrix} 1 & -\frac{1-x}{1+x} & \frac{3-x}{3+x}\frac{1-x}{1+x} & \cdots & \cdots & (-1)^{n+1}\frac{(2n-3)-x}{(2n-3)+x}\cdots\frac{3-x}{3+x}\frac{1-x}{1+x} \\ -\frac{1-x}{1+x} & 1 & -\frac{1-x}{1+x} & \cdots & \cdots & \vdots \\ \frac{3-x}{3+x}\frac{1-x}{1+x} & -\frac{1-x}{1+x} & 1 & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \\ \vdots & \vdots & & \ddots & 1 & -\frac{1-x}{1+x} \\ (-1)^{n+1}\frac{(2n-3)-x}{(2n-3)+x}\cdots\frac{3-x}{3+x}\frac{1-x}{1+x} & \cdots & \cdots & & -\frac{1-x}{1+x} & 1 \end{bmatrix}, \tag{3.11}$$

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and a standard reduction in matrix theory shows that it is enough to show that $\det \mathbf{B}_n(x) \geq 0$ for all $n \geq 2$.

In Appendix, we prove that these determinants satisfy the recursion formula

$$\frac{\det \mathbf{B}_{n+1}(x)}{\det \mathbf{B}_n(x)} = 2^{2n} \frac{n!(n-1+x)(n-2+x)\cdots(x)}{[(2n-1+x)(2n-3+x)\cdots(1+x)]^2}, \quad n \geq 1. \tag{3.12}$$

From this recursion we immediately obtain that for $x > 0$, the determinants $\det \mathbf{B}_n(x)$ and $\det \mathbf{B}_{n+1}(x)$ have the same sign. Then since $\det \mathbf{B}_1(x) = 1$, induction shows that

$$\det \mathbf{B}_n(x) > 0 \quad \text{for all } x > 0,\ n \geq 1. \tag{3.13}$$

This completes the proof that the matrices $\mathbf{B}_n$ are positive definite for all $n \geq 1$ and $x > 0$, and hence that the density $\tilde{\Psi}$ is positive. We note that this completes the proof of Lemma 2 for all $0 \leq \alpha < 2$.

Construction of the density in the case $\alpha = 0$:

The case $\alpha = 0$ corresponds to the usual singular integrals in the plane, and for this case there is an especially simple proof of the nonnegativity of the density $\tilde{\Psi}$. We simply note that the density $\tilde{\Psi}$ is nonnegative by taking absolute values inside the sum,

$$\tilde{\Psi}(\theta) = 1 + 2\sum_{n=1}^{\infty} b_n \cos(2n\theta) \geq 1 - 2\sum_{n=1}^{\infty} |b_n|,$$

and then calculating that

$$\begin{aligned}
|b_n| &= |a_n a_{n-1} \cdots a_2 a_1| \\
&= \frac{(2n-3)}{(2n+1)} \frac{(2n-5)}{(2n-1)} \cdots \frac{3}{7}\frac{1}{5}\frac{1}{3} \\
&= \frac{1}{(2n+1)(2n-1)} = \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right),
\end{aligned}$$

hence

$$\sum_{n=1}^{\infty} |b_n| = \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) = \frac{1}{2}. \qquad \blacksquare$$

Now we show how to adapt the above proof to prove Corollary 1.

Proof of Corollary 1. First we note that if Ω is sufficiently smooth with vanishing integral on the circle, then it is an absolutely convergent sum of the trigonometric functions $\cos n\theta$ and $\sin n\theta$ for $n \geq 1$. Thus a standard limiting argument extends the

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above failure of energy reversal to any finite vector of such Ω. Now embed the measure $\tilde{\mu}$ with density $\tilde{\Psi}$ constructed above into Euclidean space $\mathbb{R}^n$ via the embedding $\mathbb{R}^2 \ni (x_1, x_2) \to (x_1, x_2, x_3, \dots, x_n) \in \mathbb{R}^2 \times \mathbb{R}^{n-2}$. Here we are letting the parameter $x = n - \alpha$ lie in the interval $(0, n]$. Then the above proof shows that for cubes J with center $c_J \in \mathbb{R}^2 \times \{0\}$, the gradients $\mathbf{Z}^\alpha_\Omega(c_J; \tilde{\mu})$ of the kernels Ω have their planar projections parallel to $(1, 0)$, and hence all the gradients $\mathbf{Z}^\alpha_\Omega(c_J; \tilde{\mu})$ are perpendicular to the fixed direction $(0, 1, 0, \dots, 0)$ in $\mathbb{R}^n$. As a consequence, reversal of energy fails in J for the measure $\tilde{\mu}$, and it remains only to show that the density $\tilde{\Psi}$ is positive. But this is implied by the positivity of $\det \mathbf{B}_n(x)$ for $x \in (0, n]$, which follows from the recursion (3.12) and the fact that $\det \mathbf{B}_1(x) = 1 > 0$. ■

Before turning to the proof of the recursion (3.12) in Appendix, we give a proof of the failure of *weak* reverse energy in Lemma 3.

Proof of Lemma 3. To show the failure of the weak reversal of energy inequality (2.5) for the fractional Riesz transform vector $\mathbf{R}^\alpha$, we exploit the assumption that $\int_{\mathbb{S}^1} \Omega_\ell(\theta) = 0$ together with the following observation. Given any fractional singular integral T^α we note that

$$\begin{aligned}
\mathbb{E}^\omega_J T^\alpha \mu &= \int_{\mathbb{R}^2} \left\{ \frac{1}{|J|_\omega} \int_J \frac{\Omega(y-x)}{|y-x|^{2-\alpha}} \, d\omega(x) \right\} d\mu(y) \\
&= \int_{\xi \in \mathbb{S}^1} \Omega(\xi) \left\{ \frac{1}{|J|_\omega} \int_{\substack{(x,y) \in J \times \mathbb{R}^2 \\ \xi = \frac{y-x}{|y-x|}}} \frac{d\omega(x)\, d\mu(y)}{|\xi|^{2-\alpha}} \right\} d\theta \\
&= \int_{\mathbb{S}^1} \Omega(\theta) \, d\Phi_\mu(\theta),
\end{aligned}$$

where

$$\frac{d\Phi_\mu}{d\theta}(\xi) = \frac{1}{|J|_\omega} \int_{\substack{(x,y) \in J \times \mathbb{R}^2 \\ \xi = \frac{y-x}{|y-x|}}} \frac{d\omega(x)\, d\mu(y)}{|y-x|^{2-\alpha}}, \qquad \xi = (\cos\theta, \sin\theta) \in \mathbb{S}^1.$$

We will now apply a transformation to μ that moves its mass along rays away from c_J, but leaves the density $\frac{d\Psi_\mu}{d\theta}$ invariant. Given a measurable function $\varphi : \mathbb{S}^1 \to [1, \infty)$, define the measure $\tilde{\mu}$ in the plane by

$$d\tilde{\mu}(y) \equiv \varphi\left(\frac{y-c_J}{|y-c_J|}\right)^{3-\alpha} d\mu\left(c_J + \frac{y-c_J}{\varphi\left(\frac{y-c_J}{|y-c_J|}\right)}\right),$$

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so that for $\xi \in \mathbb{S}^1$,

$$\frac{d\Psi_{\tilde{\mu}}}{d\theta}(\xi) = \int_0^\infty r^{\alpha-3}\, d\tilde{\mu}(r\xi) = \int_0^\infty r^{\alpha-3}\varphi(\xi)^{3-\alpha}\, d\mu\left(\frac{r\xi}{\varphi(\xi)}\right)$$
$$= \int_0^\infty [s\varphi(\xi)]^{\alpha-3}\varphi(\xi)^{3-\alpha}\, d\mu(s\xi) = \frac{d\Psi_\mu}{d\theta}(\xi).$$

Now we compute

$$\int_{\mathbb{S}^1} \Omega(\theta)\, d\Phi_\mu(\theta) = \int_{\{\Omega(\theta)>0\}} \Omega(\theta)\, d\Phi_\mu(\theta) + \int_{\{\Omega(\theta)<0\}} \Omega(\theta)\, d\Phi_\mu(\theta)$$

and if this integral does not already vanish, then we may assume without loss of generality that it is negative. To prepare for a vanishing integral in our transformation below, we pick an arc K with γK contained in the set $P_\delta \equiv \{\theta : \Omega(\theta) > \delta\}$ for some $\delta > 0$. We then apply a transformation of the above type to μ with

$$\varphi(\theta) = \begin{cases} 1 & \text{if } \theta \notin P_\delta \\ M & \text{if } \theta \in P_\delta \end{cases},$$

and where $M \geq 1$ will be chosen below. From the definition of $\frac{d\Phi_{\tilde{\mu}}}{d\theta}$, and the change of variable

$$y' = c_J + \frac{y - c_J}{\varphi\left(\frac{y-c_J}{|y-c_J|}\right)},$$

we have

$$\frac{y' - c_J}{|y' - c_J|} = \frac{\frac{y-c_J}{\varphi(\frac{y-c_J}{|y-c_J|})}}{\left|\frac{y-c_J}{\varphi\left(\frac{y-c_J}{|y-c_J|}\right)}\right|} = \frac{y - c_J}{|y - c_J|},$$

and so

$$\frac{d\Phi_{\tilde{\mu}}}{d\theta}(\xi) = \frac{1}{|J|_\omega} \int_{\substack{(x,y)\in J\times\mathbb{R}^2 \\ \xi = \frac{y-x}{|y-x|}}} \frac{d\omega(x)\, d\tilde{\mu}(y)}{|y-x|^{2-\alpha}}$$
$$= \frac{1}{|J|_\omega} \int_{\substack{(x,y)\in J\times\mathbb{R}^2 \\ \xi = \frac{y-x}{|y-x|}}} \varphi\left(\frac{y-c_J}{|y-c_J|}\right)^{3-\alpha} \frac{d\omega(x)\, d\mu(y')}{|y-x|^{2-\alpha}}$$
$$= \frac{1}{|J|_\omega} \int_{\substack{(x,y')\in J\times\mathbb{R}^2 \\ \xi = \frac{\varphi\left(\frac{y'-c_J}{|y'-c_J|}\right)(y'-c_J)-x}{\left|\varphi\left(\frac{y'-c_J}{|y'-c_J|}\right)(y'-c_J)-x\right|}}} \varphi\left(\frac{y'-c_J}{|y'-c_J|}\right)^{3-\alpha} \frac{d\omega(x)\, d\mu(y')}{\left|\varphi\left(\frac{y'-c_J}{|y'-c_J|}\right)(y'-c_J) - x\right|^{2-\alpha}}.$$

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It is clear from this formula that for $\theta \in K$, only the values of $\frac{\mathrm{d}\Phi_{\tilde{\mu}}}{\mathrm{d}\theta}$ on γK are modified by the transformation, and since $\gamma K \subset P_\delta$ with $\delta > 0$, we conclude that for $\theta \in K$, we have $\lim_{M\to\infty} \frac{\mathrm{d}\Phi_{\tilde{\mu}}}{\mathrm{d}\theta}(\theta) = \infty$. Thus, there is a choice of M such that the integral of $\Omega(\theta)\,\mathrm{d}\Phi_\mu(\theta)$ over $\{\Omega(\theta) > 0\}$ equals $-\int_{\{\Omega(\theta)<0\}} \Omega(\theta)\,\mathrm{d}\Phi_\mu(\theta)$. Then for the resulting measure $\tilde{\mu}$, the density $\frac{\mathrm{d}\Psi_{\tilde{\mu}}}{\mathrm{d}\theta} = \frac{\mathrm{d}\Psi_\mu}{\mathrm{d}\theta}$ remains unchanged, and the density $\frac{\mathrm{d}\Phi_{\tilde{\mu}}}{\mathrm{d}\theta}$ satisfies

$$\mathbb{E}_J^\omega T^\alpha \tilde{\mu} = \int_{\mathbb{S}^1} \Omega(\theta)\,\mathrm{d}\Phi_{\tilde{\mu}}(\theta) = 0.$$

Now we construct a measure $\tilde{\mu}$ as above, but where

$$\mathbb{E}_J^\omega R_\ell^\alpha \mu = \int_{\mathbb{S}^1} \Omega_\ell(\theta)\,\mathrm{d}\Phi_{\tilde{\mu}}(\theta) = 0,$$

for *both* $\ell = 1$ and $\ell = 2$, and where now R_ℓ^α are the components of the fractional Riesz vector $\mathbf{R}^\alpha$. This can be achieved by choosing an appropriate pair of disjoint arcs K_1 and K_2 and using some simple algebra, and the details are left to the interested reader.

Then, we start with μ as in the proof above for the Riesz transform vector $\mathbf{R}^\alpha$, and we have that both $\mathbf{Z}_{\Omega_1}^\alpha(c_J;\mu)$ and $\mathbf{Z}_{\Omega_2}^\alpha(c_J;\mu)$ are parallel to the coordinate vector $\mathbf{e}_2$. We thus conclude that $\mathbf{R}^\alpha$ and the transformed measure $\tilde{\mu}$ fail the strong reversal of energy inequality (2.4), and also that $\mathbb{E}_J^\omega R_\ell^\alpha \tilde{\mu} = 0$ for $\ell = 1, 2$. Combining these two facts and taking γ sufficiently large gives the failure of the weak reversal of energy inequality (2.5) for $\tilde{\mu}$. Indeed, we have

$$\begin{aligned}\int_J\int_J |\mathbf{R}^\alpha\tilde{\mu}(x)|^2\,\mathrm{d}\omega(x)\,\mathrm{d}\omega(z) &= \int_J\int_J |\mathbf{R}^\alpha\tilde{\mu}(x) - \mathbb{E}_J^\omega \mathbf{R}^\alpha\tilde{\mu}|^2\,\mathrm{d}\omega(x)\,\mathrm{d}\omega(z)\\ &= \frac{1}{2}\int_J\int_J |\mathbf{R}^\alpha\tilde{\mu}(x) - \mathbf{R}^\alpha\tilde{\mu}(z)|^2\,\mathrm{d}\omega(x)\,\mathrm{d}\omega(z),\end{aligned}$$

where we have used the simple fact that $\frac{1}{|J|_\omega}\int_J (\mathbb{E}_J^\omega \mathbf{R}^\alpha\tilde{\mu} - \mathbf{R}^\alpha\tilde{\mu}(x))\,\mathrm{d}\omega(x) = 0$ in the last equality. We can now apply inequality (3.7) with $\tilde{\mu}$ in place of μ. This completes the proof of Lemma 3. ∎

Appendix

We can rewrite the recursion (3.12) above as

$$\frac{\det \mathbf{B}_{n+1}(x)}{\det \mathbf{B}_n(x)} = \frac{\Omega_n^n(x-1)}{[\Omega_n^n(\frac{x-1}{2})]^2}, \quad n \geq 1, \tag{A.1}$$

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where for any positive integer n and real number a we define the combinatorial coefficient

$$\Omega_n^n(a) \equiv \frac{(n+a)(n-1+a)\cdots(1+a)}{(n)(n-1)\cdots(1)}.$$

We now prove the recursion formula (A.1) using the well-known block determinant formula

$$\det\begin{bmatrix} \mathbf{B} & \mathbf{c} \\ \mathbf{r} & a \end{bmatrix} = a\det\mathbf{B} - \mathbf{r}[\mathrm{co}\,\mathbf{B}]^{\mathrm{tr}}\mathbf{c} = \det\mathbf{B}\{a - \mathbf{r}\mathbf{B}^{-1}\mathbf{c}\}, \tag{A.2}$$

where $\mathbf{B}$ is an $n\times n$ matrix and $\mathbf{r}$ and $\mathbf{c}$ are n-dimensional row and column vectors, respectively. Here $[\mathrm{co}\mathbf{B}]^{\mathrm{tr}}$ denotes the transposed cofactor matrix of $\mathbf{B}$ and the inverse of $\mathbf{B}$ is given by $\mathbf{B}^{-1} = \frac{1}{\det\mathbf{B}}[\mathrm{co}\mathbf{B}]^{\mathrm{tr}}$. If we apply this with $\mathbf{B} = \mathbf{B}_n(x)$ and $\left[\begin{smallmatrix} \mathbf{B} & \mathbf{c} \\ \mathbf{r} & a \end{smallmatrix}\right] = \mathbf{B}_{n+1}(x)$ we get

$$\begin{aligned}\det\mathbf{B}_{n+1}(x) &= \det\begin{bmatrix} \mathbf{B}_n(x) & \mathbf{c}^n(x) \\ \mathbf{r}_n(x) & 1 \end{bmatrix} \\ &= \det\mathbf{B}_n(x)\{1 - \mathbf{r}_n(x)\mathbf{B}_n(x)^{-1}\mathbf{c}^n(x)\},\end{aligned} \tag{A.3}$$

where $\mathbf{r}_n(x)$ denotes the n-dimensional row vector consisting of the first n entries of the bottom row of $\mathbf{B}_{n+1}(x)$, and similarly $\mathbf{c}^n(x)$ denotes the n-dimensional column vector consisting of the first n entries of the rightmost column of $\mathbf{B}_{n+1}(x)$. Note also that $\mathbf{r}_n(x)$ and $\mathbf{c}^n(x)$ are transposes of each other.

Motivated by computer algebra calculations, we **define** the column vector

$$\mathbf{v}^n(x) \equiv (-1)^{n-1}\left[(-1)^k \binom{n}{k} \Gamma_k^n\left(\frac{x-1}{2}\right)\right]_{k=0}^{n-1}, \tag{A.4}$$

where

$$\Gamma_k^n(a) \equiv \frac{\Gamma(k+a+1)\Gamma(n-k+a)}{\Gamma(n+a+1)\Gamma(a)} = \frac{(k+a)\cdots(a)}{(n+a)\cdots(n-k+a)}.$$

Lemma A.1. For $n \geq 1$, we have

$$\mathbf{B}_n(x)^{-1}\mathbf{c}^n(x) = \mathbf{v}^n(x).$$ □

Proof. It suffices to show the vector identity

$$\mathbf{B}_n(x)\mathbf{v}^n(x) = \mathbf{c}^n(x), \quad n \geq 1,$$

and to prove this we will use the well known fact that an nth order difference of a polynomial of degree less than n vanishes. More specifically, the polynomial in question

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will be

$$P_{n-1}(s) \equiv \frac{\Gamma(n-1+s)}{\Gamma(s)} = (n-1+s)\cdots(1+s)s.$$

Indeed,

$$\begin{aligned}
\mathbf{v}^n(x) &\equiv \left[(-1)^k \binom{n}{n-1-k} \Gamma^n_{n-1-k}\left(\frac{x-1}{2}\right)\right]_{k=0}^{n-1} \\
&= \left[(-1)^k \binom{n}{k+1} \frac{\Gamma\left(n-k+\frac{x-1}{2}\right)\Gamma\left(1+k+\frac{x-1}{2}\right)}{\Gamma\left(n+1+\frac{x-1}{2}\right)\Gamma\left(\frac{x-1}{2}\right)}\right]_{k=0}^{n-1} \\
&= \left[(-1)^{k-1} \binom{n}{k} \frac{\Gamma(n-k+z)\Gamma(k-1+z)}{\Gamma(n+z)\Gamma(-1+z)}\right]_{k=1}^{n},
\end{aligned}$$

where

$$z = \frac{x-1}{2} + 1 = \frac{x+1}{2}.$$

Now we use

$$\begin{aligned}
\frac{(x-1)(x-3)(x-5)\cdots(x-(2n-1))}{(x+1)(x+3)(x+5)\cdots(x+(2n-1))} &= \frac{\left(\frac{x-1}{2}\right)\left(\frac{x-1}{2}-1\right)\left(\frac{x-1}{2}-2\right)\cdots\left(\frac{x-1}{2}-(n-1)\right)}{\left(\frac{x-1}{2}+1\right)\left(\frac{x-1}{2}+2\right)\left(\frac{x-1}{2}+3\right)\cdots\left(\frac{x-1}{2}+n\right)} \\
&= \frac{\Gamma\left(\frac{x-1}{2}+1\right)\Gamma\left(\frac{x-1}{2}+1\right)}{\Gamma\left(\frac{x-1}{2}-(n-1)\right)\Gamma\left(\frac{x-1}{2}+n+1\right)} \\
&= \frac{\Gamma(z)^2}{\Gamma(z-n)\Gamma(z+n)},
\end{aligned}$$

to obtain that

$$\mathbf{B}_n(x) = \left[\frac{\Gamma(z)^2}{\Gamma(z-|j-i|)\Gamma(z+|j-i|)}\right]_{i,j=1}^{n}$$

Thus, the first row of $\mathbf{B}_n(x)$ is

$$\begin{aligned}
&\left(1 \quad \frac{x-1}{x+1} \quad \frac{x-3}{x+3}\frac{x-1}{x+1} \quad \cdots \quad \frac{(x-1)(x-3)(x-5)\cdots(x-(2n-1))}{(x+1)(x+3)(x+5)\cdots(x+(2n-1))}\right) \\
&= \left(\frac{\Gamma(z)^2}{\Gamma(z)\Gamma(z)} \quad \frac{\Gamma(z)^2}{\Gamma(z-1)\Gamma(z+1)} \quad \frac{\Gamma(z)^2}{\Gamma(z-2)\Gamma(z+2)} \quad \cdots \quad \frac{\Gamma(z)^2}{\Gamma(z-(n-1))\Gamma(z+(n-1))}\right) \\
&= \left[\frac{\Gamma(z)^2}{\Gamma(z-(k-1))\Gamma(z+(k-1))}\right]_{k=1}^{n}.
\end{aligned}$$

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Thus, we get

$$\left[\frac{\Gamma(z)^2}{\Gamma(z-(k-1))\Gamma(z+(k-1))}\right]_{k=1}^{n} \cdot \mathbf{v}^n(x)$$
$$= -\sum_{k=1}^{n}(-1)^k \binom{n}{k} \frac{\Gamma(n-k+zt)\Gamma(k-1+z)}{\Gamma(n+z)\Gamma(-1+z)} \frac{\Gamma(z)^2}{\Gamma(z-(k-1))\Gamma(z+(k-1))}$$
$$= -\sum_{k=1}^{n}(-1)^k \binom{n}{k} \frac{\Gamma(z)^2}{\Gamma(z+n)\Gamma(z-1)} \frac{\Gamma(z-k+n)}{\Gamma(z-k+1)}$$
$$= -\frac{\Gamma(z)^2}{\Gamma(z+n)\Gamma(z-1)} \sum_{k=1}^{n}(-1)^k \binom{n}{k} \{(z-k+n-1)\cdots(z-k+1)\}$$
$$= -\frac{\Gamma(z)^2}{\Gamma(z+n)\Gamma(z-1)} \sum_{k=1}^{n}(-1)^k \binom{n}{k} P_z^n(k)$$

where $P_z^n(w) = (z-w+n-1)\cdots(z-w+1)$ is a polynomial of degree $n-1$. Now recall that if $\triangle f \equiv f(1) - f(0)$ is the unit difference operator at 0, then

$$\triangle^n f = \sum_{k=0}^{n}(-1)^k \binom{n}{k} f(k)$$

Thus we have

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k} P_z^n(k) = \triangle^n P_z^n = 0$$

since P_z^n has degree less than n, and so

$$\left[\frac{\Gamma(z)^2}{\Gamma(z-(k-1))\Gamma(z+(k-1))}\right]_{k=1}^{n} \cdot \mathbf{v}^n(x) = \frac{\Gamma(z)^2}{\Gamma(z+n)\Gamma(z-1)}(z+n-1)\cdots(z+1)$$
$$= \frac{\Gamma(z)^2}{\Gamma(z+1)\Gamma(z-1)}$$

which is the first component of $\mathbf{c}^n(x)$ as required. A similar argument proves the equality of the remaining components, and this completes the proof of Lemma A.1. ∎

Lemma A.2. For $n \geq 1$, we have

$$1 - \mathbf{r}_n(x) \cdot \mathbf{v}_n(x) = \frac{\Omega_n^n(x-1)}{[\Omega_n^n(\frac{x-1}{2})]^2}.$$ □

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Proof. Again, this is an application of the fact that an nth order difference of a polynomial of degree less than n vanishes, but a bit more complicated. Recall that

$$\Omega_n^n(a) \equiv \frac{(n+a)(n-1+a)\cdots(1+a)}{(n)(n-1)\cdots(1)} = \frac{\Gamma(n+1+a)}{\Gamma(1+a)n!},$$

so that we have

$$\frac{\Omega_n^n(x-1)}{\left[\Omega_n^n\left(\frac{x-1}{2}\right)\right]^2} = n!\frac{(n+x-1)(n-1+x-1)\cdots(1+x-1)}{\left(n+\frac{x-1}{2}\right)^2\left(n-1+\frac{x-1}{2}\right)^2\cdots\left(1+\frac{x-1}{2}\right)^2}$$
$$= \frac{\Gamma(n+1)\Gamma(n+x)\Gamma\left(1+\frac{x-1}{2}\right)^2}{\Gamma(x)\Gamma\left(n+1+\frac{x-1}{2}\right)^2}.$$

We also have

$$\mathbf{v}^n(x) \equiv \left[(-1)^k\binom{n}{n-1-k}\Gamma_{n-1-k}^n\left(\frac{x-1}{2}\right)\right]_{k=0}^{n-1}$$
$$= \left[(-1)^k\binom{n}{k+1}\frac{\Gamma\left(n-k+\frac{x-1}{2}\right)\Gamma\left(1+k+\frac{x-1}{2}\right)}{\Gamma\left(n+1+\frac{x-1}{2}\right)\Gamma\left(\frac{x-1}{2}\right)}\right]_{k=0}^{n-1},$$

and from (3.11), we have

$$\mathbf{r}_n(x) = \left((-1)^n\frac{(2n-1)-x}{(2n-1)+x}\cdots\frac{3-x}{3+x}\frac{1-x}{1+x}\quad\cdots\quad\cdots\quad\frac{3-x}{3+x}\frac{1-x}{1+x}\quad-\frac{1-x}{1+x}\right)$$
$$= \left[(-1)^{k+1}\frac{(2k+1)-x}{(2k+1)+x}\cdots\frac{3-x}{3+x}\frac{1-x}{1+x}\right]_{k=0}^{n-1}$$
$$= \left[\frac{x-(2k+1)}{(2k+2)+x-1}\cdots\frac{x-3}{4+x-1}\frac{x-1}{2+x-1}\right]_{k=0}^{n-1}$$
$$= \left[\frac{-2k+x-1}{(2k+2)+x-1}\cdots\frac{-2+x-1}{4+x-1}\frac{x-1}{2+x-1}\right]_{k=0}^{n-1},$$

and hence dividing all factors top and bottom by 2, we get

$$\mathbf{r}_n(x) = \left[\frac{-k+\frac{x-1}{2}}{k+1+\frac{x-1}{2}}\cdots\frac{-1+\frac{x-1}{2}}{2+\frac{x-1}{2}}\frac{\frac{x-1}{2}}{1+\frac{x-1}{2}}\right]_{k=0}^{n-1}$$
$$= \left[\frac{\Gamma\left(1+\frac{x-1}{2}\right)\Gamma\left(1+\frac{x-1}{2}\right)}{\Gamma\left(-k+\frac{x-1}{2}\right)\Gamma\left(k+2+\frac{x-1}{2}\right)}\right]_{k=0}^{n-1}$$
$$= \left[\frac{\Gamma\left(1+\frac{x-1}{2}\right)^2}{\Gamma\left(-k+\frac{x-1}{2}\right)\Gamma\left(k+2+\frac{x-1}{2}\right)}\right]_{k=0}^{n-1}.$$

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Thus, our identity to be proved is

$$\sum_{k=0}^{n-1}(-1)^k\binom{n}{k+1}\frac{\Gamma\left(1+\frac{x-1}{2}\right)^2}{\Gamma\left(-k+\frac{x-1}{2}\right)\Gamma\left(k+2+\frac{x-1}{2}\right)}\times\frac{\Gamma\left(n-k+\frac{x-1}{2}\right)\Gamma\left(k+1+\frac{x-1}{2}\right)}{\Gamma\left(n+1+\frac{x-1}{2}\right)\Gamma\left(\frac{x-1}{2}\right)}$$
$$=1-\frac{\Gamma(n+1)\Gamma(n+x)\Gamma\left(1+\frac{x-1}{2}\right)^2}{\Gamma(x)\Gamma\left(n+1+\frac{x-1}{2}\right)^2}. \tag{A.5}$$

If we set $z=1+\frac{x-1}{2}$, then this identity becomes

$$\sum_{k=0}^{n-1}(-1)^k\binom{n}{k+1}\frac{\Gamma(z)^2}{\Gamma(-k-1+z)\Gamma(k+1+z)}\frac{\Gamma(n-k-1+z)\Gamma(k+z)}{\Gamma(n+z)\Gamma(-1+z)}$$
$$=1-\frac{\Gamma(n+1)\Gamma(n-1+2z)\Gamma(z)^2}{\Gamma(-1+2z)\Gamma(n+z)^2},$$

and if we replace k by $k-1$ we get

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\frac{\Gamma(z)^2}{\Gamma(-k+z)\Gamma(k+z)}\frac{\Gamma(n-k+z)\Gamma(k-1+z)}{\Gamma(n+z)\Gamma(-1+z)}$$
$$=1-\frac{\Gamma(n+1)\Gamma(n-1+2z)\Gamma(z)^2}{\Gamma(-1+2z)\Gamma(n+z)^2}.$$

Note that the term $k=0$ in the sum on the left would be -1, so that we can subtract 1 from both sides, and then multiply by -1 to get

$$\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}\frac{\Gamma(z)^2}{\Gamma(-k+z)\Gamma(k+z)}\frac{\Gamma(n-k+z)\Gamma(k-1+z)}{\Gamma(n+z)\Gamma(-1+z)}$$
$$=\frac{\Gamma(n+1)\Gamma(n-1+2z)\Gamma(z)^2}{\Gamma(-1+2z)\Gamma(n+z)^2},$$

which is equivalent to

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{\Gamma(z+n-k)\Gamma(z+k-1)}{\Gamma(z-k)\Gamma(z+k)}=\frac{\Gamma(n+1)\Gamma(z-1)\Gamma(2z+n-1)}{\Gamma(z+n)\Gamma(2z-1)}. \tag{A.6}$$

We now use

$$\frac{\Gamma(s+m+1)}{\Gamma(s)}=(s+m)(s+m-1)\cdots(s+1)s$$

to rewrite (A.6) as

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(z+n-k-1)\cdots(z-k)}{(z+k-1)}=n!\frac{(2z+n-2)\cdots(2z)(2z-1)}{(z+n-1)\cdots(z)(z-1)}. \tag{A.7}$$

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Denote the left- and right-hand sides of (A.7) by $\mathrm{LHS}_n(z)$ and $\mathrm{RHS}_n(z)$, respectively. Then the left hand side $\mathrm{LHS}_n(z)$ of (A.7) is

$$\begin{aligned}\mathrm{LHS}_n(z) &= \sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{(z+n-k-1)\cdots(z+1-k)([z+k-1]-[2k-1])}{z+k-1}\\ &= \sum_{k=0}^{n}(-1)^k \binom{n}{k} (z+n-k-1)\cdots(z+1-k)\\ &\quad + \sum_{k=0}^{n}(-1)^{k+1} \binom{n}{k} \frac{(z+n-k-1)\cdots(z+1-k)}{(z+k-1)}(2k-1),\end{aligned}$$

where the first sum on the right-hand side above vanishes since it is an nth order difference of the polynomial

$$P(w) \equiv (z+n-w-1)\cdots(z+1-w)$$

of degree $n-1$. Thus, we have

$$\begin{aligned}\mathrm{LHS}_n(z) &= \sum_{k=0}^{n}(-1)^{k+1} \binom{n}{k} \frac{(z+n-k-1)\cdots(z+2-k)([z+k-1]-[2k-2])}{(z+k-1)}(2k-1)\\ &= \sum_{k=0}^{n}(-1)^{k+1} \binom{n}{k} (z+n-k-1)\cdots(z+2-k)(2k-1)\\ &\quad + \sum_{k=0}^{n}(-1)^{k+2} \binom{n}{k} \frac{(z+n-k-1)\cdots(z+2-k)}{(z+k-1)}(2k-2)(2k-1),\end{aligned}$$

where the first sum on the right-hand side above vanishes since it is an nth order difference of the polynomial

$$P(w) \equiv (z+n-w-1)\cdots(z+2-w)(2w-1)$$

of degree $n-1$. Continuing in this way we get

$$\mathrm{LHS}_n(z) = \sum_{k=0}^{n}(-1)^{k+n} \binom{n}{k} \frac{1}{(z+k-1)}(2k-n)\cdots(2k-2)(2k-1).$$

Now the right-hand side $\mathrm{RHS}_n(z)$ of (A.7) is a quotient of a polynomial of degree n by a polynomial of degree $n+1$, and so has a partial fraction decomposition of the form

$$\mathrm{RHS}_n(z) = n!\frac{(2z+n-2)\cdots(2z)(2z-1)}{(z+n-1)\cdots(z)(z-1)} = \sum_{k=0}^{n} \frac{A_k}{z+k-1},$$

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for uniquely determined coefficients $A_0, \ldots, A_n$. So the proof of (A.7) has been reduced to proving the identity,

$$A_k = (-1)^{k+n}\binom{n}{k}(2k-n)\cdots(2k-2)(2k-1). \tag{A.8}$$

Now A_k is the residue of the meromorphic function $\mathrm{RHS}_n(z)$ at $z=-(k-1)$, hence using the notation $(\widehat{z+k-1})$ to indicate that the factor $(z+k-1)$ is *missing*, we get

$$\begin{aligned}
A_k &= \mathrm{res}(\mathrm{RHS}_n(z); -(k-1)) \\
&= n!\left.\frac{(2z+n-2)\cdots(2z)(2z-1)}{(z+n-1)\cdots(z+k)(\widehat{z+k-1})(z+k-2)\cdots(z)(z-1)}\right|_{z=-(k-1)} \\
&= n!\frac{(2[1-k]+n-2)\cdots(2[1-k])(2[1-k]-1)}{([1-k]+n-1)\cdots([1-k]+k)(\widehat{[1-k]+k}-1)([1-k]+k-2)\cdots([1-k])([1-k]-1)} \\
&= n!\frac{(-1)^n(2k-n)\cdots(2k-2)(2k-1)}{(n-k)\cdots(1)(\widehat{0})(-1)^k(1)\cdots(k-1)(k)} \\
&= (-1)^{n-k}\frac{n!}{(n-k)!k!}(2k-n)\cdots(2k-2)(2k-1),
\end{aligned}$$

which proves (A.8). This completes the proof of Lemma A.2. ■

The proof of our claimed recursion (A.1) is now completed by combining Lemmas A.1 and A.2 with (A.3).

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