**COMPLETE CHARACTERIZATION OF ISOLATED
HOMOGENEOUS HYPERSURFACE SINGULARITIES**

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Dedicated to Professor Michael Artin on the occasion of his 80th birthday

Let X be a nonsingular projective variety in $\mathbb{CP}^{n-1}$. Then the cone over X in $\mathbb{C}^n$ is an affine variety V with an isolated singularity at the origin. It is a very natural and important question to ask when an affine variety with an isolated singularity at the origin is a cone over nonsingular projective variety.

This problem is very hard in general. In this paper we shall treat the hypersurface case. Given a function f with an isolated singularity at the origin, we can ask whether f is a weighted homogeneous polynomial or a homogeneous polynomial after a biholomorphic change of coordinates. The former question was answered in a celebrated 1971 paper by Saito. However, the latter question had remained open for 40 years until Xu and Yau solved it for f with three variables. Recently, Yau and Zuo solved it for f with up to six variables. However, the methods they used are hard to generalize. In this paper, we solve the latter question for general n completely; i.e., we show that f is a homogeneous polynomial after a biholomorphic change of coordinates if and only if $\mu = \tau = (\nu - 1)^n$, where μ, τ and ν are the Milnor number, Tjurina number and multiplicity of the singularity respectively. We also prove that there are at most $\mu^{1/n} + 1$ multiplicities within the same topological type of the isolated hypersurface singularity, while the famous Zariski multiplicity problem asserts that there is only one multiplicity.

1. Introduction

Let $w = (w_1, \ldots, w_n)$ be an n-tuple of positive rational numbers. A polynomial $f(z_1, \ldots, z_n)$ is said to be a weighted homogeneous polynomial with weight w if each monomial $\alpha z_1^{a_1} z_2^{a_2} \cdots z_n^{a_n}$ of f satisfies $a_1 w_1 + \cdots + a_n w_n = 1$. It has an isolated critical point at $0 \in \mathbb{C}^n$ if grad $f = (\partial f/\partial z_1, \ldots, \partial f/\partial z_n)$ is zero at 0 but grad $f(z) \neq 0$ for all z in a neighborhood of 0.

Recall that a polynomial $f(z_1, \ldots, z_n)$ is called quasihomogeneous if f is in the Jacobian ideal of f, i.e., $f \in (\partial f/\partial z_1, \ldots, \partial f/\partial z_n)$.

By a theorem of Saito (see Theorem 2.7), if f is quasihomogeneous with an isolated critical point at 0, then after a biholomorphic change of coordinates, f becomes a weighted homogeneous polynomial.

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be the germ of a complex analytic function with an isolated critical point at the origin. Let $V = \{z \in \mathbb{C}^n : f(z) = 0\}$. It is a natural question to ask when V is defined by a weighted homogeneous polynomial up to biholomorphic change of coordinates. Saito [1971] solved this question. He gave a necessary and sufficient condition for V to be defined by a weighted homogeneous polynomial. It is a natural and important question to characterize homogeneous polynomial with an isolated critical point at the origin. This question has remained open for 40 years. In fact it is the first important case of the following interesting problem. Let X be a nonsingular projective variety in $\mathbb{CP}^{n-1}$. Then the cone over X in $\mathbb{C}^n$ is an affine variety V with an isolated singularity at the origin. It is then natural to ask when an affine variety with an isolated singularity at the origin is a cone over nonsingular projective variety.

For a two-dimensional isolated hypersurface singularity V, Xu and Yau [1992; 1993] found a coordinate-free characterization when V is defined by a homogeneous polynomial. Recently, necessary and sufficient conditions were given for three-dimensional isolated hypersurface singularities with $p_g \geq 0$ [Lin and Yau 2004; Lin et al. 2006a; Xu and Yau 1996] and four-dimensional isolated hypersurface singularities with $p_g > 0$ [Chen et al. 2011], where p_g is the geometric genus of the singularity. Based on the classification of weighted homogeneous singularities, Yau and Zuo [2012] solved the problem for f with up to six variables. However, it is quite difficult to generalize their methods to characterize the homogeneous polynomials for general n. Ten years ago, Yau formulated the Yau homogeneous characterization conjecture: (1) Let μ and ν be the Milnor number and multiplicity of $(V, 0)$ respectively. Then $\mu \geq (\nu - 1)^n$, and equality holds if and only if f is a semihomogeneous function (i.e., $f = f_\nu + g$, where f_ν is a homogeneous polynomial of degree ν defining an isolated singularity at the origin and g consists of terms of degree at least $\nu + 1$) after a biholomorphic change of coordinates. (2) Moreover, if f is a weighted homogeneous function, then $\mu = (\nu - 1)^n$ if and only if f is a homogeneous polynomial after a biholomorphic change of coordinates. In this paper we verify the Yau homogeneous characterization conjecture affirmatively. As a result, we have solved the characterization problem of homogeneous polynomials with an isolated critical point at the origin, i.e., we have shown that f is a homogeneous polynomial after a biholomorphic change of coordinates if and only if $\mu = \tau = (\nu - 1)^n$.

Recall that the multiplicity of the singularity V is defined to be the order of the lowest nonvanishing term in the power series expansion of f at 0. The Milnor number μ and the Tjurina number τ of the singularity $(V, 0)$ are defined by

$$\mu = \dim \mathbb{C}\{z_1, z_2, \ldots, z_n\}/(f_{z_1}, \ldots, f_{z_n}),$$

$$\tau = \dim \mathbb{C}\{z_1, z_2, \ldots, z_n\}/(f, f_{z_1}, \ldots, f_{z_n}).$$

They are numerical invariants of $(V, 0)$.

Let $\pi : (M, A) \to (V, 0)$ be a resolution of singularity of dimension n with exceptional set $A = \pi^{-1}(0)$. The geometric genus p_g of the singularity $(V, 0)$ is the dimension of $H^{n-1}(M, \mathbb{O})$ and is independent of the resolution M.

Using p_g, μ and ν, Yau made another conjecture in 1995 (see [Lin and Yau 2004; Chen et al. 2011]) describing when a weighted homogeneous singularity is a homogeneous singularity. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a weighted homogeneous polynomial with an isolated singularity at the origin. Let μ, p_g and ν be the Milnor number, geometric genus and multiplicity of the singularity $V = \{z : f(z) = 0\}$; then

$$\mu - p(\nu) \geq n! p_g,$$

where $p(\nu) = (\nu - 1)^n - \nu(\nu - 1) \cdots (\nu - n + 1)$, and equality holds if and only if f is a homogeneous polynomial after a biholomorphic change of coordinates.

In fact, we shall prove in this paper that if $p_g = 0$, then the Yau homogeneous characterization conjecture implies the 1995 Yau conjecture.

These conjectures are sharp estimates and have some important applications in geometry. However, they were proved only for low-dimensional singularities. For the Yau homogeneous characterization conjecture, Lin, Wu, Yau and Luk [Lin et al. 2006b] proved the following two theorems.

Theorem 1.1. *Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be the germ of a holomorphic function defining an isolated plane curve singularity $V = \{z \in \mathbb{C}^2 : f(z) = 0\}$ at the origin. Let μ and ν be the Milnor number and multiplicity of $(V, 0)$, respectively. Then*

$$(1\text{-}1) \qquad\qquad \mu \geq (\nu - 1)^2.$$

Furthermore, if V has at most two irreducible branches at the origin, or if f is a quasihomogeneous function, then equality holds in (1-1) if and only if f is a homogeneous polynomial (after a biholomorphic change of coordinates).

Theorem 1.2. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be the germ of a holomorphic function defining an isolated hypersurface singularity $V = \{z \in \mathbb{C}^n : f(z) = 0\}$ at the origin. Let μ, ν and $\tau = \dim \mathbb{C}\{z_1, \ldots, z_n\}/(f, \partial f/\partial z_1, \ldots, \partial f/\partial z_n)$ be the Milnor number, multiplicity and Tjurina number of $(V, 0)$, respectively. Suppose $\mu = \tau$ and n is either 3 or 4. Then*

$$(1\text{-}2) \qquad\qquad \mu \geq (\nu - 1)^n,$$

and equality holds in (1-2) *if and only if f is a homogeneous polynomial* (*after a biholomorphic change of coordinates*).

For the 1995 Yau conjecture, Lin, Tu and Yau [Lin and Yau 2004; Lin et al. 2006a] have the following theorem:

Theorem 1.3. *Let $(V, 0)$ be a three-dimensional isolated hypersurface singularity defined by a weighted homogeneous polynomial $f(x, y, z, w) = 0$. Let μ, ν and p_g be the Milnor number, multiplicity and geometric genus of the singularity, respectively. Then*

$$(1\text{-}3) \qquad \mu - (2\nu^3 - 5\nu^2 + 2\nu + 1) \geq 4! p_g$$

and equality holds in (1-3) *if and only if f is a homogeneous polynomial.*

Remark. The above theorem is proved in [Lin and Yau 2004] with $p_g > 0$. For $p_g = 0$, the theorem is proved in [Lin et al. 2006a].

An immediate corollary of Theorem 1.3 is the following:

Corollary 1.4 [Lin et al. 2006a]. *Let $(V, 0)$ be a three-dimensional isolated hypersurface singularity defined by a polynomial $f(x, y, z, w) = 0$. Let μ, ν, p_g and τ be the Milnor number, multiplicity, geometric genus and Tjurina number of the singularity, respectively. Then f is a homogeneous polynomial after a biholomorphic change of coordinates if and only if $\mu = \tau$ and $\mu - (2\nu^3 - 5\nu^2 + 2\nu + 1) = 4! p_g$.*

Chen, Lin, Yau and Zuo generalized the above theorem to four-dimensional isolated hypersurface singularities with the additional assumption $p_g > 0$.

Theorem 1.5 [Chen et al. 2011]. *Let $(V, 0)$ be a four-dimensional isolated hypersurface singularity defined by a weighted homogeneous polynomial $f(x, y, z, w, t) = 0$. Let μ, ν and p_g be the Milnor number, multiplicity and geometric genus of the singularity, respectively. If $p_g > 0$ then*

$$(1\text{-}4) \qquad \mu - [(\nu - 1)^5 + \nu(\nu - 1)(\nu - 2)(\nu - 3)(\nu - 4)] \geq 5! p_g,$$

and equality holds in (1-4) *if and only if f is a homogeneous polynomial after a biholomorphic change of coordinates.*

Corollary 1.6 [Chen et al. 2011]. *Let $(V, 0)$ be a four-dimensional isolated hypersurface singularity defined by a polynomial $f(x, y, z, w, t) = 0$. Let μ, ν, p_g and τ be the Milnor number, multiplicity, geometric genus and Tjurina number of the singularity, respectively. Moreover, if $p_g > 0$ then f is a homogeneous polynomial after a biholomorphic change of coordinates if and only if $\mu = \tau$ and*

$$\mu - [(\nu - 1)^5 + \nu(\nu - 1)(\nu - 2)(\nu - 3)(\nu - 4)] = 5! p_g.$$

Yau and Zuo [2012] proved the following theorem:

Theorem 1.7. *Let* $f : (\mathbb{C}^k, 0) \to (\mathbb{C}, 0)$, *where k is either 5 or 6, be a polynomial with an isolated singularity at the origin. Let μ, ν and τ be the Milnor number, multiplicity and Tjurina number of the singularity $V = \{z \colon f(z) = 0\}$ respectively. Then f is a homogeneous polynomial after a biholomorphic change of coordinates if and only if $\mu = \tau = (\nu - 1)^k$.*

The purpose of this paper is to prove the following results:

Theorem A. *Let* $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ *be a weighted homogeneous polynomial with an isolated singularity at the origin. Let μ and ν be the Milnor number and multiplicity of the singularity $V = \{z \colon f(z) = 0\}$ respectively. Then*

$$(1\text{-}5) \qquad\qquad \mu \geq (\nu - 1)^n,$$

and equality holds in (1-5) if and only if f is a homogeneous polynomial after a biholomorphic change of coordinates.

Corollary B. *Let* $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ *be a polynomial with an isolated singularity at the origin. Let μ, ν and τ be the Milnor number, multiplicity and Tjurina number of the singularity $V = \{z \colon f(z) = 0\}$ respectively. Then f is a homogeneous polynomial after a biholomorphic change of coordinates if and only if $\mu = \tau = (\nu - 1)^n$.*

Proposition 3.1 and Theorem A answer the Yau homogeneous characterization conjecture affirmatively, and Corollary B gives a complete characterization of isolated homogeneous hypersurface singularities. Let $(V, 0)$ and $(W, 0)$ be two isolated hypersurface singularities in $\mathbb{C}^n$. We say that $(V, 0)$ and $(W, 0)$ have the same topological type if $(\mathbb{C}^n, V, 0)$ is homeomorphically equivalent to $(\mathbb{C}^n, W, 0)$. The famous Zariski multiplicity question asks whether $(V, 0)$ and $(W, 0)$ have the same multiplicity if they have the same topological type, i.e., whether there is only one multiplicity within the same topological type. For two-dimensional isolated quasihomogeneous singularities, the Zariski multiplicity question was solved [Yau 1988; Xu and Yau 1989]. Proposition 3.1 says that there are at most $\mu^{1/n} + 1$ multiplicities within the same topological type. On the other hand, Theorem C below confirms that the 1995 Yau conjecture is true for the case of $p_g = 0$.

Theorem C. *Let* $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ *be a weighted homogeneous polynomial with an isolated singularity at the origin. Let μ, p_g and ν be the Milnor number, geometric genus and multiplicity of the singularity $V = \{z \colon f(z) = 0\}$. If $p_g = 0$, then*

$$\mu - p(\nu) \geq n! p_g,$$

where $p(\nu) = (\nu - 1)^n - \nu(\nu - 1) \cdots (\nu - n + 1) \ (= 0)$, and equality holds if and only if f is a homogeneous polynomial after a biholomorphic change of coordinates.

In Section 2, we recall the material which is necessary to prove the main theorems. In Section 3, we prove the main theorems.

2. Preliminaries

In this section, we recall some known results which are needed to prove the main theorems. Let $f(z_1, \ldots, z_n)$ be a germ of an analytic function at the origin such that $f(0) = 0$. Suppose f has an isolated critical point at the origin, and suppose f can be developed in a convergent Taylor series $f(z_1, \ldots, z_n) = \sum a_\lambda z^\lambda$, where $z^\lambda = z_1^{\lambda_1} \cdots z_n^{\lambda_n}$. Recall that the Newton boundary $\Gamma(f)$ is the union of compact faces of $\Gamma_+(f)$, where $\Gamma_+(f)$ is the convex hull of the union of subsets $\{\lambda + \mathbb{R}_+^n\}$ for λ such that $a_\lambda \neq 0$. Let $\Gamma_-(f)$, the Newton polyhedron of f, be the cone over $\Gamma(f)$ with cone point at 0. For any closed face Δ of $\Gamma(f)$, we associate the polynomial $f_\Delta(z) = \sum_{\lambda \in \Delta} a_\lambda z^\lambda$. We say that f is nondegenerate if f_Δ has no critical point in $(\mathbb{C}^*)^n$ for any $\Delta \in \Gamma(f)$, where $\mathbb{C}^* = \mathbb{C} - \{0\}$. We say that a point p of the integral lattice $\mathbb{Z}^n$ in $\mathbb{R}^n$ is positive if all coordinates of p are positive.

Theorem 2.1 [Merle and Teissier 1980]. *Let $(V, 0)$ be an isolated hypersurface singularity defined by a nondegenerate holomorphic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. Then the geometric genus $p_g = \#\{p \in \mathbb{Z}^n \cap \Gamma_-(f) : p$ is positive$\}$.*

Recall that a polynomial $f(z_1, \ldots, z_n)$ is a weighted homogeneous polynomial of type $(w_1, \ldots, w_n)$, where $w_1, \ldots, w_n$ are fixed positive rational numbers, if it can be expressed as a linear combination of monomials $z_1^{i_1} \cdots z_n^{i_n}$ for which $i_1 w_1 + \cdots + i_n w_n = 1$. As a consequence of Theorem 2.1, for an isolated singularity defined by a weighted homogeneous polynomial, computing the geometric genus is equivalent to counting the number of positive integral points in the tetrahedron defined by $x_1 w_1 + \cdots + x_n w_n \leq 1$, $x_1 \geq 0, \ldots, x_n \geq 0$. We also need the following result:

Theorem 2.2 [Milnor and Orlik 1970]. *Let $f(z_1, \ldots, z_n)$ be a weighted homogeneous polynomial of type $(w_1, \ldots, w_n)$ with an isolated singularity at the origin. Then the Milnor number μ is equal to $(1/w_1 - 1) \cdots (1/w_n - 1)$.*

Yau [1977] gave a lower bound for p_g of a hypersurface singularity.

Theorem 2.3 [Yau 1977]. *Let*

$$f(z_1, \ldots, z_{n-1}, z_n) = z_n^m + a_1(z_1, \ldots, z_{n-1})z_n^{m-1} + \cdots + a_m(z_1, \ldots, z_{n-1})$$

be holomorphic near $(0, \ldots, 0)$. Let d_i be the order of the zero of $a_i(z_1, \ldots, z_{n-1})$ at $(0, \ldots, 0)$, with $d_i \geq i$. Let $d = \min_{1 \leq i \leq m}(d_i / i)$. Suppose that

$$V = \{(z_1, \ldots, z_n) : f(z_1, \ldots, z_n) = 0\},$$

defined in a suitably small polydisc, has $p = (0, \ldots, 0)$ as its only singularity. Let $\pi : M \to V$ be a resolution of V. Then $\dim H^{n-2}(M, \mathbb{O}) > (m-1)d - (n-1)$.

Remark. Here, the singularity is $(n - 1)$-dimensional, so $\dim H^{n-2}(M, \mathbb{O}) = p_g$.

Let $f \in \mathbb{C}\{z_1, \ldots, z_n\}$ define an isolated singularity at the origin. Let $w = (w_1, \ldots, w_n)$ be a weight on the coordinates $(z_1, \ldots, z_n)$ for positive integers w_i, $i = 1, \ldots, n$. We have the weighted Taylor expansion $f = f_\rho + f_{\rho+1} + \cdots$ with respect to w, where $f_\rho \neq 0$ and f_k is a weighted homogeneous polynomial of type $(w_1, \ldots, w_n; k)$ for $k \geq \rho$, i.e., f_k is a linear combination of monomials $z_1^{i_1} \cdots z_n^{i_n}$ for which $i_1 w_1 + \cdots + i_n w_n = k$.

Theorem 2.4 [Furuya and Tomari 2004]. *Let $f \in \mathbb{C}\{z_1, \ldots, z_n\}$ define an isolated singularity at the origin. With the above situation, then*:

(1) *The following inequality holds*:

$$\mu(f) \geq \left(\frac{\rho}{w_1} - 1\right) \cdots \left(\frac{\rho}{w_n} - 1\right).$$

(2) *Equality holds in* (1) *if and only if f_ρ defines an isolated singularity at the origin.*

Here we recall that f is called a semiquasihomogeneous function if the initial term f_ρ defines an isolated singularity at the origin.

Definition 2.1. Let $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be germs of holomorphic functions defining isolated hypersurface singularities $V_f = \{z : f(z) = 0\}$ and $V_g = \{z : g(z) = 0\}$. Let $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be the germ of a biholomorphic map.

(1) f is contact-equivalent to g if $\phi(V_f) = V_g$.

(2) f is right-equivalent to g if $g = f \circ \phi$.

The Milnor number is an invariant under right-equivalence and the Tjurina number is an invariant under contact equivalence. It is a nontrivial theorem that the Milnor number is indeed an invariant under contact equivalence:

Theorem 2.5 [Greuel 1975]. *Let $\mathbb{K}$ be an algebraically closed field of characteristic zero and $f, g \in \mathbb{K}\{z_1, z_2, \ldots, z_n\}$. If f is contact-equivalent to g, then $\mu(f) = \mu(g)$.*

Theorem 2.6 [Shoshitaishvili 1976; Benson and Yau 1990]. *If f and g are germs of isolated weighted homogeneous singularities at the origin in $\mathbb{C}^n$, then f and g are right-equivalent if and only if f and g are contact-equivalent.*

Theorem 2.7 [Saito 1971]. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be the germ of a complex analytic function with an isolated critical point at the origin.*

(a) *f is right-equivalent to a weighted homogeneous polynomial if and only if $\mu = \tau$ or*

$$f \in J_f := \left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right).$$

(b) *If f is weighted homogeneous with normalized weight system $(w_1, \ldots, w_n, 1)$ with $0 < w_1 \leq \cdots \leq w_n < 1$ and if $f \in \boldsymbol{m}_{\mathbb{C}^n,0}^3$, then the weight system is unique and $0 < w_1 \leq \cdots \leq w_n < \frac{1}{2}$.*

(c) *If $f \in J_f$, then f is right-equivalent to a weighted homogeneous polynomial $g(z_1, \ldots, z_k) + z_{k+1}^2 + \cdots + z_n^2$ with $g \in \boldsymbol{m}_{\mathbb{C}^n,0}^3$. Specifically, its normalized weight system satisfies $0 < w_1 \leq \cdots \leq w_k < w_{k+1} = \cdots = w_n = \frac{1}{2}$.*

(d) *If f and $\bar{f} \in \mathbb{O}_{\mathbb{C}^n,0}$ are right-equivalent and weighted homogeneous with normalized weight systems $(w_1, \ldots, w_n, 1)$ and $(\bar{w}_1, \ldots, \bar{w}_n, 1)$ with $w_1 \leq \cdots \leq w_n \leq \frac{1}{2}$ and $\bar{w}_1 \leq \cdots \leq \bar{w}_n \leq \frac{1}{2}$, then $w_i = \bar{w}_i$.*

3. Proof of the main theorems

The following statement is well known.

Proposition 3.1 [Teissier 1973]. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic germ defining an isolated hypersurface singularity $V = \{z : f(z) = 0\}$ at the origin. Let μ and ν be the Milnor number and multiplicity of $(V, 0)$ respectively. Then*

$$(3\text{-}1) \qquad \mu \geq (\nu - 1)^n,$$

and equality holds in (3-1) if and only if f is a semihomogeneous function (i.e., $f = f_\nu + g$, where f_ν is a nondegenerate homogeneous polynomial of degree ν and g consists of terms of degree at least $\nu + 1$) after a biholomorphic change of coordinates.

Proof. Let $f(z_1, \ldots, z_n) : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic function with an isolated singularity at the origin. Let μ and ν be the Milnor number and multiplicity of the singularity $V = \{z : f(z) = 0\}$. By an analytic change of coordinates, one can assume that the z_n-axis is not contained in the tangent cones of V, so that $f(0, \ldots, 0, z_n) \neq 0$. By the Weierstrass preparation theorem, near 0, the germ f can be represented as a product $f(z_1, \ldots, z_n) = u(z_1, \ldots, z_n) g(z_1, \ldots, z_n)$, where $u(0, \ldots, 0) \neq 0$ and

$$g(z_1, \ldots, z_{n-1}, z_n) = z_n^\nu + a_1(z_1, \ldots z_{n-1}) z_n^{\nu-1} + \cdots + a_\nu(z_1, \ldots, z_{n-1}),$$

where ν is the multiplicity of $f(z_1, \ldots, z_n)$ and $a_i \in (z_1, \ldots, z_{n-1})^i$ for $i = 1, \ldots, \nu$. Therefore $f(z_1, \ldots, z_n)$ is contact-equivalent to $g(z_1, \ldots, z_n)$.

Let d_i be the order of the zero of $a_i(z_1, \ldots, z_{n-1})$ at $(0, \ldots, 0)$, with $d_i \geq i$. Let $d = \min_{1 \leq i \leq \nu}(d_i/i)$; then $d \geq 1$. We define a new weight w on the coordinate system: $w(z_n) = d$, $w(z_i) = 1$ for $1 \leq i \leq n - 1$. With respect to the new weights, z_n^ν has degree $d\nu$ and $a_i(z_1, \ldots, z_{n-1}) z_n^{\nu-i}$ has degree at least $d(\nu - i) + d_i \geq d\nu - di + di = d\nu$. Thus the initial term of $f(z_1, \ldots, z_n)$ has degree $\rho = d\nu$. By

Theorem 2.5, the Milnor number is an invariant under the contact equivalence. By Theorem 2.4(1), we have

$$\mu = \mu(g) \geq \left(\frac{dv}{d} - 1\right)\left(\frac{dv}{1} - 1\right) \cdots \left(\frac{dv}{1} - 1\right) = (v-1)(dv-1)^{n-1} \geq (v-1)^n.$$

Thus, we proved the inequality (3-1).

We need to show that equality in (3-1) holds if and only if f is a semihomogeneous function after a biholomorphic change of coordinates.

$\Rightarrow$: If $\mu = (v-1)^n$, then by the fact that $\mu \geq (v-1)(dv-1)^{n-1} \geq (v-1)^n$, we have $d = 1$, and by Theorem 2.4(2), $g_{dv}(z_1, \ldots, z_n) = g_v(z_1, \ldots, z_n)$ is a homogeneous polynomial of degree v defining an isolated singularity. Hence $f(z_1, \ldots, z_n)$ is contact-equivalent to a semihomogeneous singularity g; i.e., f is a semihomogeneous function after a biholomorphic change of coordinates.

$\Leftarrow$: Suppose f is a semihomogeneous polynomial after a biholomorphic change of coordinates. Since the Milnor number of f is the same as the Milnor number of its initial part (see [Arnold 1974]) which is a homogeneous polynomial with degree v, so $\mu = (v-1)^n$ is obvious. $\qquad \square$

Proof of Theorem A. By Proposition 3.1, it is sufficient to show that if f is a weighted homogeneous singularity, then $\mu = (v-1)^n$ if and only if f is equivalent to a homogeneous singularity.

The "if" part is trivial. We only need to consider the "only if" part. By Saito's theorem (see Theorem 2.7(c)), we can choose normalized weights for f, which means that these weights satisfy $0 < w_i \leq \frac{1}{2}$, $1 \leq i \leq n$. By what we have proved above, we know that there exists a $g(z_1, \ldots, z_n)$ so that f is contact equivalent to g; moreover the initial part of g is g_v, a homogeneous polynomial with degree v, and g_v also defines an isolated singularity at the origin. We can rewrite f and g as

$$f(z_1, \ldots, z_n) = f_v(z_1, \ldots, z_n) + f_{v+1}(z_1, \ldots, z_n) + \cdots,$$
$$g(z_1, \ldots, z_n) = g_v(z_1, \ldots, z_n) + g_{v+1}(z_1, \ldots, z_n) + \cdots,$$

where f_i and g_i, $i \geq v$, are the homogeneous parts of f and g respectively and g_v defines an isolated singularity at the origin. Since for weighted homogeneous singularities contact equivalence is the same as right equivalence (see Theorem 2.6), there exists a biholomorphism at the origin:

$$\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0),$$
$$(z_1, \ldots, z_n) \mapsto (\phi_1(z_1, \ldots, z_n), \ldots, \phi_n(z_1, \ldots, z_n)),$$

such that $f(z_1, \ldots, z_n) = g(\phi_1(z_1, \ldots, z_n), \ldots, \phi_n(z_1, \ldots, z_n))$ and

$$\phi_1(z_1, \ldots, z_n) = a_{11}z_1 + \cdots + a_{1n}z_n + H_1^2 + H_1^3 + \cdots,$$

$$\vdots$$

$$\phi_n(z_1, \ldots, z_n) = a_{n1}z_1 + \cdots + a_{nn}z_n + H_n^2 + H_n^3 + \cdots,$$

where $H_i^j = \sum_{\alpha_1 + \cdots + \alpha_n = j} c_i(\alpha_1, \ldots, \alpha_n) z_1^{\alpha_1} \cdots z_n^{\alpha_n}$. Since ϕ is a biholomorphism at the origin, we have $|(a_{ij})| = \det(a_{ij}) \neq 0$. It follows from the equality $f(z_1, \ldots, z_n) = g(\phi_1(z_1, \ldots, z_n), \ldots, \phi_n(z_1, \ldots, z_n))$ that

$$g_\nu(\phi_1(z_1, \ldots, z_n), \ldots, \phi_n(z_1, \ldots, z_n))$$
$$+ g_{\nu+1}(\phi_1(z_1, \ldots, z_n), \ldots, \phi_n(z_1, \ldots, z_n)) + \cdots$$
$$= f_\nu(z_1, \ldots, z_n) + f_{\nu+1}(z_1, \ldots, z_n) + \cdots.$$

Comparing the degree of each side, we have

$$g_\nu(\bar{\phi}_1(z_1, \ldots, z_n), \ldots, \bar{\phi}_n(z_1, \ldots, z_n)) = f_\nu(z_1, \ldots, z_n),$$

where $\bar{\phi}_i = a_{i1}z_1 + \cdots + a_{in}z_n$, $1 \leq i \leq n$. Since $\det(a_{ij}) \neq 0$, f_ν is right-equivalent to g_ν. Therefore f_ν also defines an isolated singularity. Now we have two normalized weights for f_ν: one is $(w_1, \ldots, w_n)$, because each monomial in f_ν comes from f, and the other is $(1/\nu, \ldots, 1/\nu)$ which follows from the fact that f_ν is a homogeneous polynomial with degree ν. By Theorem 2.7, we have $w_1 = w_2 = \cdots = w_n = 1/\nu$. Therefore $f(z_1, \ldots, z_n) = f_\nu(z_1, \ldots, z_n)$ is a homogeneous polynomial. $\square$

Proof of Corollary B. This follows from Theorem A and Theorem 2.7(a). $\square$

Proof of Theorem C. Since $p_g = 0$, by Theorem 2.3, we have $0 > (\nu - 1)d - (n - 1)$, where $d = \min_{1 \leq i \leq \nu}(d_i/i)$, and d_i is the order of the zero of $a_i(x_1, \ldots, x_n)$ at $(0, \ldots, 0)$, with $d_i \geq i$. Then $\nu < (n - 1)/d + 1$. Since $d \geq 1$, ν is an integer at least 2 for an isolated hypersurface singularity, so we have $2 \leq \nu \leq n - 1$. Therefore $p(\nu) = (\nu - 1)^n - \nu(\nu - 1) \cdots (\nu - n + 1) = (\nu - 1)^n$. The theorem is reduced to proving that

$$\mu \geq (\nu - 1)^n,$$

where equality holds if and only if f is a homogeneous polynomial after a biholomorphic change of coordinates. The proof follows from Theorem A immediately. $\square$

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