Interplay Between CR Geometry and Algebraic Geometry

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1 Introduction

CR manifolds are abstract models of real hypersurfaces in complex spaces. The abstract definition of the boundary as a *CR* structure on a complex manifold is essentially in Cartan [7]. For more detail, see [24, 25]. Strongly pseudoconvex *CR* manifolds have rich geometric and analytic structures. Namely, there is an intrinsic pseudo conformed geometry for which complete local invariants have been obtained, see for example [8, 14, 41], as well as a deep analysis of the $\overline{\partial}_b$ complex, see for example [12, 22, 23, 46]. The harmonic theory for the $\overline{\partial}_b$ complex on compact strongly pseudoconvex *CR* manifolds was developed by Kohn [21]. Using this theory, Boutet de Monvel [4] proved that if X is a compact strongly pseudoconvex *CR* manifold of dimension $2n-1$, $n \geq 3$, then there exist C^∞

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functions $f_1, \ldots, f_N$ on X such that each $\overline{\partial}_b f_j = 0$ and $f = (f_1, \ldots, f_N)$ defines an embedding of X in $\mathbb{C}^N$. Thus, any compact strongly pseudoconvex *CR* manifold of dimension ≥ 5 can be *CR* embedded in some complex Euclidean space. On the other hand, 3-dimensional strongly pseudoconvex compact orientable *CR*-manifolds are not necessarily embeddable. The first example is due to Andreotti according to [37]. This example also appeared in the list of homogeneous structures of Cartan although the embeddability question was not addressed. Nirenberg [35] first proved that 3-dimensional *CR* manifolds might not be locally embeddable. Jacobowitz and Treves [19, 20] showed that in fact non-embeddable *CR* structures are, in some sense, dense in the space of *CR*-structures over a 3-dimensional manifold. The theory of harmonic integrals on strongly pseudoconvex *CR* structures over small balls was due to Kuranishi [23]. Using this theory, Kuranishi [23] proved that any strongly pseudoconvex *CR* manifold of dimension $2n-1$ with $n \geq 5$ can be locally *CR* embedded as a real hypersurface in $\mathbb{C}^n$. For $n = 4$, Akahori [1] proved that Kuranishi's local embedding theorem is also true. However, the 5-dimensional case of local embeddability of *CR* manifolds remains open.

Throughout this paper, our *CR* manifolds are always assumed to be compact orientable and embeddable in some $\mathbb{C}^N$. By a beautiful theorem of Harvey and Lawson [16, 17], these *CR* manifolds are the boundaries of subvarieties in $\mathbb{C}^N$. This allowed the first author [46] to relate *CR* geometry and algebraic geometry of singularities for the first time. The purpose of this paper is to discuss the interplay between *CR* geometry and algebraic geometry. Our paper is organized as follows. In Sect. 2, we shall recall the basic notion of *CR* geometry. In Sect. 3, we show how to use the Bergman function of the first author to give canonical construction of moduli space for complete Reinhardt domains. In Sect. 4, we use algebraic geometry to study the complex Plateau problem. In Sect. 5, we study the minimal embedding dimension of compact *CR* manifolds in complex Euclidean space. Finally in Sect. 6, we study invariants of compact strongly pseudoconvex *CR* manifolds arising from geometry of singularities.

2 Preliminary

Definition 2.1 Let X be a connected orientable manifold of real dimension $2n-1$. A *CR* structure on X is an $(n-1)$-dimensional subbundle S of the complexified tangent bundle $\mathbb{C}TX$ such that

(1) $S \cap \overline{S} = \{0\}$
(2) If L, L' are local sections of S, then so is $[L, L']$.

A manifold with a *CR* structure is called a *CR* manifold. There is a unique subbundle $\mathcal{H}$ of the tangent bundle $T(X)$ such that $\mathbb{C}\mathcal{H} = S \oplus \overline{S}$. Furthermore, there is a unique homomorphism $J : \mathcal{H} \longrightarrow \mathcal{H}$ such that $J^2 = -1$ and $S = \{v - iJv : v \in \mathcal{H}\}$. The pair $(\mathcal{H}, J)$ is called the real expression of the *CR* structure.

Definition 2.2 Let $L_1, \ldots, L_{n-1}$ be a local frame of S. Then $\overline{L}_1, \ldots, \overline{L}_{n-1}$ is a local frame of $\overline{S}$ and one may choose a local section N of TX such that $L_1, \ldots, L_{n-1}$, $\overline{L}_1, \ldots, \overline{L}_{n-1}, N$ is a local frame of $\mathbb{C}TX$. The matrix (c_{ij}) defined by

$$[L_i, \overline{L}_j] = \Sigma a_{ij}^k L_k + \Sigma b_{ij}^k \overline{L}_k + \sqrt{-1} c_{ij} N$$

is Hermitian and is called the Levi form of X.

Proposition 2.1 *The number of non-zero eigenvalues and the absolute value of the signature of the Levi form* (c_{ij}) *at each point are independent of the choice of* $L_1, \ldots, L_{n-1}, N$.

Definition 2.3 The CR manifold X is called strongly pseudoconvex if the Levi form is definite at each point of X.

Theorem 2.2 (Boutet de Monvel [4]) *If X is a compact strongly pseudoconvex CR manifold of dimension* $(2n-1)$ *and* $n \geq 3$*, then X is CR embeddable in* $\mathbb{C}^N$.

Although there are non-embeddable compact 3-dimensionable *CR* manifolds, in this paper all *CR* manifolds are assumed to be embeddable in complex Euclidean space. The following theorem links *CR* geometry and algebraic geometry together.

Theorem 2.3 (Harvey-Lawson [16, 17]) *For any compact connected embeddable CR manifold X, there is a unique complex variety V in $\mathbb{C}^N$ for some N such that the boundary of V is X and V has only normal isolated singularities.*

3 Bergman Function and Moduli Space of Complete Reinhardt Domains

Recall that a complex manifold M is called strictly pseudoconvex if there is a compact set B in M, and a continuous real valued function ϕ on M, which is strictly plurisubharmonic outside B and such that for each $c \in \mathbb{R}$, the set $M_c = \{x \in M\colon \phi(x) < c\}$ is relatively compact in M. Note that a strictly pseudoconvex complex manifold is a modification of a Stein space at a finite many points.

Let V be a Stein variety of dimension $n \geqslant 2$ in $\mathbb{C}^N$ with only irreducible isolated singularities. We assume that ∂V is a smooth *CR* manifold. Let $\pi\colon M \to V$ be a resolution of singularity with E as an exceptional set. We shall define the k-th order Bergman function $B_M^{(k)}(z)$ on M which is a biholomorphic invariant of M.

Definition 3.1 Let F (respectively, F_k) be the set of all L^2 integrable holomorphic n-forms Ψ on M (respectively, vanishing at least the k-th order on the exception set E of M). Let $\{w_j\}$ (respectively, $\{w_j^{(k)}\}$) be a complete orthonormal basis of F (respectively, F_k). The Bergman kernel (respectively Bergman kernel vanishing on the exceptional set of k-th order) is defined to be $K(z) = \sum w_j(z) \wedge \overline{w_j(z)}$ (respectively, $K^{(k)}(z) = \sum w_j^{(k)}(z) \wedge \overline{w_j^{(k)}(z)}$).

Lemma 3.1 *F/F_k is a finite dimensional vector space.*

Lemma 3.2 *Bergman kernel vanishing on the exceptional set of k-th order $K^{(k)}(z)$ is independent of the choice of the complete orthonormal basis of F_k and $K^{(k)}(z)$ is invariant under biholomorphic maps.*

Definition 3.2 Let M be a resolution of a Stein variety V of dimension $n \geqslant 2$ in $\mathbb{C}^N$ with only irreducible isolated singularity at the origin. The k-th order Bergman function $B_M^{(k)}$ on M is defined to be $K_M^{(k)}/K_M$.

Theorem 3.1 *$B_M^{(k)}$ is a global function defined on M which is invariant under biholomorphic maps. Moreover, $B_M^{(k)}$ is nowhere vanishing outside the exceptional set of M. If the canonical bundle is generated by its global sections in a neighborhood of the exceptional set, then the zero set of the k-th order Bergman function $B_M^{(k)}$ is precisely the exceptional set of M.*

Theorem 3.2 *Let M be a strictly pseudoconvex complex manifold of dimension $n \geqslant 2$ with exceptional set E. Let A be a compact submanifold contained in E. Let $\pi: M_1 \to M$ be the blow up of M along A. Then we have $K_{M_1}^{(k)}(z) = \pi^* K_M^{(k)}(z)$ and $K_{M_1}(z) = \pi^* K_M(z)$. Consequently $B_{M_1}^{(k)}(z) = \pi^* B_M^{(k)}(z)$.*

Let $\pi_i: M_i \to V$, $i = 1, 2$, be two resolutions of singularities of V. By Hironaka's theorem [18], there exists a resolution $\tilde{\pi}: \tilde{M} \to V$ of singularities of V such that $\tilde{M}$ can be obtained from M_i, $i = 1, 2$, by successive blowing up along submanifolds in exceptional set. In view of Theorems 3.1 and 3.2, the following definition is well defined if the canonical bundle is generated by its global sections in a neighborhood of the exceptional set.

Definition 3.3 Let V be a Stein variety in $\mathbb{C}^N$ with only irreducible isolated singularities. Let $\pi: M \to V$ be a resolution of singularities of V such that the canonical bundle is generated by its global sections in a neighborhood of the exceptional set. Define the k-th order Bergman function $B_V^{(k)}$ on V to be the push forward of the k-th order Bergman function $B_M^{(k)}$ by the map π.

Theorem 3.3 *Let V be a Stein variety in $\mathbb{C}^N$ with only irreducible isolated singularities. Assume that there exists a resolution M of singularities of V such that the canonical bundle is generated by its global sections in a neighborhood of the exceptional set. Then the k-th order Bergman function $B_V^{(k)}$ on V is invariant under biholomorphic maps and $B_V^{(k)}$ vanishes precisely on the singular set of V.*

For the convenience of the readers, we recall the following two important theorems.

Theorem 3.4 ([13]) *A biholomorphic mapping between two strictly pseudoconvex domains is smooth up to boundary and the induced boundary mapping gives a CR-equivalence between the boundaries.*

Theorem 3.5 ([40]) *Two n-dimensional bounded Reinhardt domains D_1 and D_2 are mutually equivalent if and only if there exists a transformation $\phi : \mathbb{C}^n \to \mathbb{C}^n$ given by $z_i \mapsto r_i z_{\sigma(i)} (r_i > 0, i = 1, \cdots, n$ and σ being a permutation of the indices i) such that $\phi(D_1) = D_2$.*

The following Proposition 3.1 tells us how to use singularity structures to distinguish *CR* structures.

Proposition 3.1 ([50]) *Let X_1, X_2 be two strictly pseudoconvex CR manifolds of dimension $2n-1$ which bound varieties V_1, V_2 respectively in $\mathbb{C}^N$ with only isolated normal singularities. If $\Phi: X_1 \to X_2$ is a CR-isomorphism, then Φ can be extended to a biholomorphic map from V_1 to V_2.*

In view of the above Proposition 3.1, if X_1 and X_2 are two strictly pseudoconvex *CR* manifolds which bound varieties V_1 and V_2 respectively with non-isomorphic singularities, then X_1 and X_2 are not *CR* equivalent. Therefore to study the *CR* equivalence of two strictly pseudoconvex *CR* manifolds X_1 and X_2, it remains to consider the case when X_1 and X_2 are lying on the same variety V. It is known that the global invariant Bergman function of k-th order can be used to study the *CR* equivalence problem of smooth *CR* manifolds lying on the same variety. As an example, we shall show explicitly how *CR* manifolds varies in the A_n-variety $\tilde{V}_n = \{(x, y, z) \in \mathbb{C}^3 : f(x, y, z) = xy - z^{n+1} = 0\}$. An explicit resolution $\tilde{\pi}: \tilde{M}_n \to \tilde{V}_n$ can be given in terms of coordinate charts and transition functions as follows:

$$\text{Coordinate charts:} \quad \tilde{W}_k = \mathbb{C}^2 = \{(u_k, v_k)\},\ k = 0, 1, \cdots, n.$$

$$\text{Transition functions:} \quad \begin{cases} u_{k+1} = \dfrac{1}{v_k} \\ v_{k+1} = u_k {v_k}^2 \end{cases} \quad \text{or} \quad \begin{cases} u_k = {u_{k+1}}^2 v_{k+1} \\ v_k = \dfrac{1}{u_{k+1}} \end{cases}$$

$$\text{Resolution map:} \quad \tilde{\pi}(u_k, v_k) = \left(u_k^{k+1} v_k^k, u_k^{n-k} v_k^{n+1-k}, u_k v_k\right) \quad \text{or}$$

$$(x, y, z) = (u_0, u_0^n v_0^{n+1}, u_0 v_0) = \cdots = ({u_n}^{n+1} v_n^n, v_n, u_n v_n)$$

$$\text{Exceptional set:} \quad E = \tilde{\pi}^{-1}(0) = C_k = \{u_{k-1} = 0\} \cup \{v_k = 0\},$$
$$k = 1, \cdots, n.$$

From now on, we suppose V to be a bounded complete Reinhardt domain in $\tilde{V}_n$ (cf. Definition 3.5). Then let $M = \tilde{\pi}^{-1}(V) = \cup_{k=0}^n W_k$, where $W_k = \tilde{\pi}^{-1}(V) \cap \tilde{W}_k$, $k = 0, 1, \cdots, n$. Observe that under $\pi := \tilde{\pi}|_M : M \to V$, $W_0 \backslash C_1$ is mapped biholomorphically onto $V \backslash y$-axis. In particular $M \backslash W_0$ is of measure zero in the obvious sense. Hence, we may compute integrals on M using the (u_0, v_0) coordinate on the chart W_0 alone.

The following proposition is a general consequence of the proof of Proposition 3.2 of [50].

Proposition 3.2 ([10]) *In the above notations, let* $\phi_{\alpha\beta} = u_0^\alpha v_0^\beta \, du_0 \wedge dv_0$, $\alpha, \beta = 0, 1, 2, \ldots$. *Then* $\left\{ \frac{\phi_{\alpha\beta}}{\|\phi_{\alpha\beta}\|_M} : \alpha \geqslant \frac{n}{n+1}\beta \right\}$ *is a complete orthonormal base of* F *and* $\left\{ \frac{\phi_{\alpha\beta}}{\|\phi_{\alpha\beta}\|_M} : \alpha \geqslant \frac{n}{n+1}\beta \text{ and } \alpha \geqslant k \right\}$ *is a complete orthonormal base of* F_k. *Therefore the Bergman kernel vanishing on the exceptional set of k-th order* $K_M^{(k)}$ *and the Bergman kernel* K_M *are given respectively by:*

$$K_M^{(k)}(u_0, v_0) = \Theta_M^{(k)} \, du_0 \wedge dv_0 \wedge d\overline{u_0} \wedge d\overline{v_0}$$

where

$$\Theta_M^{(k)} = \sum_{\substack{\alpha \geqslant \frac{n}{n+1}\beta \\ \alpha \geqslant k}} \frac{|u_0|^{2\alpha} |v_0|^{2\beta}}{\|\phi_{\alpha\beta}\|_M^2},$$

and

$$K_M(u_0, v_0) = \left(\frac{1}{\|\phi_{00}\|_M^2} + \sum_{\substack{\alpha \geqslant \frac{n}{n+1}\beta \\ 1 \leqslant \alpha \leqslant k-1}} \frac{|u_0|^{2\alpha} |v_0|^{2\beta}}{\|\phi_{\alpha\beta}\|_M^2} + \Theta_M^{(k)} \right) du_0 \wedge dv_0 \wedge d\overline{u_0} \wedge d\overline{v_0}.$$

The following results generalize Theorem 3.3 in [50].

Theorem 3.6 ([10]) *In the above notations, the k-th order Bergman function for the strongly pseudoconvex complex manifold M is given by*

$$B_M^{(k)}(u_0, v_0) = \frac{\Theta_M^{(k)}}{\left(\frac{1}{\|\phi_{00}\|_M^2} + \sum_{\substack{\alpha \geqslant \frac{n}{n+1}\beta \\ \alpha \geqslant 1}} \frac{|u_0|^{2\alpha} |v_0|^{2\beta}}{\|\phi_{\alpha\beta}\|_M^2} \right)}$$

The k-th order Bergman function for the variety is given by

$$B_V^{(k)}(x, y) = \frac{\Theta_V^{(k)}}{\left(\frac{1}{\|\phi_{00}\|_M^2} + \Theta_V^{(1)} \right)},$$

where

$$\Theta_V^{(k)} = \sum_{\substack{\alpha \geqslant \frac{n}{n+1}\beta \\ \alpha \geqslant k}} \frac{|x|^{2\alpha - \frac{2n\beta}{n+1}} |y|^{\frac{2\beta}{n+1}}}{\|\phi_{\alpha\beta}\|_M^2}$$

Definition 3.4 An open subset $D \subseteq \mathbb{C}^n$ is a complete Reinhardt domain if, whenever $(z_1, \cdots, z_n) \in D$ then $(\xi_1 z_1, \cdots, \xi_n z_n) \in D$ for all complex numbers ξ_j with $|\xi_j| \leqslant 1$.

It is well known that $\tilde{V}_n = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$ is the quotient of $\mathbb{C}^2$ by a cyclic group of order $n+1$, i.e. $\delta.(z_1, z_2) = (\delta z_1, \delta^n z_2)$, where δ is a primitive $(n+1)$-th root of unit. The quotient map $\pi : \mathbb{C}^2 \to \tilde{V}$ is given by $\pi(z_1, z_2) = (z_1^{n+1}, z_2^{n+1}, z_1 z_2)$.

Definition 3.5 An open set V in the A_n-variety $\tilde{V}_n = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$ is called a complete Reinhardt domain if $\pi^{-1}(V)$ is a complete Reinhardt domain in $\mathbb{C}^2$.

Theorem 3.7 ([10]) *Let V_i, $i = 1, 2$, be two bounded complete Reinhardt domains in A_n-variety $\tilde{V}_n = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$. Let*

$$g^{(\alpha,\beta)} = \frac{\|\phi_{10}\|^{\alpha - \frac{n}{n+1}\beta} \|\phi_{n,n+1}\|^{\frac{\beta}{n+1}}}{\|\phi_{\alpha\beta}\| \, \|\phi_{00}\|^{\alpha - \frac{n-1}{n+1}\beta - 1}}.$$

If V_1 is biholomorphic to V_2, then

$$\xi^{(\alpha,\beta)} := g^{(\alpha,\beta)} \cdot g^{(n\alpha - (n-1)\beta, (n+1)\alpha - n\beta)},$$

$$\zeta^{(\alpha,\beta)} := g^{(\alpha,\beta)} + g^{(n\alpha - (n-1)\beta, (n+1)\alpha - n\beta)},$$

$$\eta^{(\alpha,p,q)} := (g^{(\alpha,p)} - g^{(n\alpha - (n-1)p, (n+1)\alpha - np)}) \cdot (g^{(\alpha,q)} - g^{(n\alpha - (n-1)q, (n+1)\alpha - nq)})$$

and

$$\omega^{(\alpha_1,\alpha_2,p_1,p_2)} := (g^{(\alpha_1,p_1)} - g^{(n\alpha_1 - (n-1)p_1, (n+1)\alpha_1 - np_1)}) \cdot (g^{(\alpha_2,p_2)} - g^{(n\alpha_2 - (n-1)p_2, (n+1)\alpha_2 - np_2)}),$$

where

$$\alpha \geqslant 1, \alpha \geqslant \frac{n}{n+1}\beta, 0 \leqslant p, q \leqslant \left[\frac{n+1}{n}\alpha\right], p \neq q,$$

$$0 \leqslant p_i \leqslant \left[\frac{n+1}{n}\alpha_i\right], \alpha_i \geqslant 1, \alpha_1 \neq \alpha_2, i = 1, 2,$$

are all invariants, i.e.

$$\xi_{V_1}^{(\alpha,\beta)} = \xi_{V_2}^{(\alpha,\beta)}, \zeta_{V_1}^{(\alpha,\beta)} = \zeta_{V_2}^{(\alpha,\beta)}, \eta_{V_1}^{(\alpha,p,q)} = \eta_{V_2}^{(\alpha,p,q)},$$

$$\omega_{V_1}^{(\alpha_1,\alpha_2,p_1,p_2)} = \omega_{V_2}^{(\alpha_1,\alpha_2,p_1,p_2)}.$$

The following Theorem says that these invariants in Theorem 3.7 determine completely the Bergman function up to automorphisms of A_n-variety.

Theorem 3.8 ([10]) *Let V_i, $i = 1, 2$, be two bounded complete Reinhardt strictly pseudoconvex (respectively C^ω-smooth pseudoconvex) domains in $\tilde{V}_n = \{(x, y, z) \in \mathbb{C}^3 \colon xy = z^{n+1}\}$. If*

$$\xi_{V_1}^{(\alpha,\beta)} = \xi_{V_2}^{(\alpha,\beta)}, \zeta_{V_1}^{(\alpha,\beta)} = \zeta_{V_2}^{(\alpha,\beta)}, \eta_{V_1}^{(\alpha,p,q)} = \eta_{V_2}^{(\alpha,p,q)},$$

$$\omega_{V_1}^{(\alpha_1,\alpha_2,p_1,p_2)} = \omega_{V_2}^{(\alpha_1,\alpha_2,p_1,p_2)},$$

where

$$\alpha \geqslant 1, \alpha \geqslant \frac{n}{n+1}\beta, 0 \leqslant p,\ q \leqslant \left[\frac{n+1}{n}\alpha\right], p \neq q,$$

$$0 \leqslant p_i \leqslant \left[\frac{n+1}{n}\alpha_i\right], \alpha_i \geqslant 1, \alpha_1 \neq \alpha_2, i = 1, 2,$$

then there exists an automorphism $\Psi = (\psi_1, \psi_2, \psi_3)$ of A_n-variety $\tilde{V}_n = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$ given by either

$$(\psi_1, \psi_2, \psi_3) =$$

$$\left(\frac{\|\phi_{10}\|_{M_2}}{\|\phi_{00}\|_{M_2}}\frac{\|\phi_{00}\|_{M_1}}{\|\phi_{10}\|_{M_1}}x,\ \frac{\|\phi_{n,n+1}\|_{M_2}}{\|\phi_{00}\|_{M_2}}\frac{\|\phi_{00}\|_{M_1}}{\|\phi_{n,n+1}\|_{M_1}}y,\ \frac{\|\phi_{11}\|_{M_2}}{\|\phi_{00}\|_{M_2}}\frac{\|\phi_{00}\|_{M_1}}{\|\phi_{11}\|_{M_1}}z\right),$$

or

$$(\psi_1, \psi_2, \psi_3) =$$

$$\left(\frac{\|\phi_{10}\|_{M_2}}{\|\phi_{00}\|_{M_2}}\frac{\|\phi_{00}\|_{M_1}}{\|\phi_{n,n+1}\|_{M_1}}y,\ \frac{\|\phi_{n,n+1}\|_{M_2}}{\|\phi_{00}\|_{M_2}}\frac{\|\phi_{00}\|_{M_1}}{\|\phi_{10}\|_{M_1}}x,\ \frac{\|\phi_{11}\|_{M_2}}{\|\phi_{00}\|_{M_2}}\frac{\|\phi_{00}\|_{M_1}}{\|\phi_{11}\|_{M_1}}z\right).$$

such that Ψ sends V_1 to V_2.

As an immediate corollary of Theorem 3.8 above, we have the following theorem.

Theorem 3.9 ([10]) *The moduli space of bounded complete Reinhardt strictly pseudoconvex (respectively C^ω-smooth pseudoconvex) domains in A_n-variety $\tilde{V}_n = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$ is given by the image of the map $\Phi : \{V : V$ a bounded complete Reinhardt strictly pseudoconvex (respectively C^ω-smooth pseudoconvex) domain in $\tilde{V}_n\} \to \mathbb{R}^\infty$, where the component function of Φ are the invariant functions*

$$\xi^{(\alpha,\beta)}, \zeta^{(\alpha,\beta)}, \eta^{(\alpha,p,q)}, \omega^{(\alpha_1,\alpha_2,p_1,p_2)},$$

$$\alpha \geqslant 1, \alpha \geqslant \frac{n}{n+1}\beta, 0 \leqslant p,\ q \leqslant \left[\frac{n+1}{n}\alpha\right], p \neq q,$$

$$0 \leqslant p_i \leqslant \left[\frac{n+1}{n}\alpha_i\right], \alpha_i \geqslant 1, \alpha_1 \neq \alpha_2, i = 1, 2.$$

defined in Theorem 3.7.

The following theorem says that the biholomorphic equivalence problem for bounded complete Reinhardt domains in A_n-variety $\tilde{V}_n$ is the same as the biholomorphic equivalence problem for the corresponding bounded complete Reinhardt domains in $\mathbb{C}^2$.

Theorem 3.10 ([10]) *Let $\pi : \mathbb{C}^2 \to \tilde{V}_n = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$ be the quotient map given by $\pi(z_1, z_2) = (z_1^{n+1}, z_2^{n+1}, z_1 z_2)$. Let V_i, $i = 1, 2$, be bounded complete Reinhardt domains in $\tilde{V}_n$ such that $W_i := \pi^{-1}(V_i), i = 1, 2$, are bounded complete Reinhardt domain in $\mathbb{C}^2$. Then V_1 is biholomorphic to V_2 if and only if W_1 is biholomorphic to W_2. In particular, V_1 is biholomorphic to V_2 if and only if there exists a biholomorphism $\Phi : V_1 \to V_2$ given by $\Phi(x, y, z) = (a^{n+1}x, b^{n+1}y, abz)$ or $\Phi(x, y, z) = (a^{n+1}y, b^{n+1}x, abz)$ where $a, b > 0$.*

As a corollary of Theorems 3.10 and 3.9, we have the following theorem.

Theorem 3.11 ([10])

(1) Let $\mathcal{W} = \{W : W = \pi^{-1}(V)$ where V is a bounded complete Reinhardt domain in A_n-variety$\}$ be the space of bounded complete Reinhardt domains in $\mathbb{C}^2$ which are invariant under the action of the cyclic group of order $n + 1$ on $\mathbb{C}^2$. Then

$$\xi^{(\alpha,\beta)}, \zeta^{(\alpha,\beta)}, \eta^{(\alpha,p,q)}, \omega^{(\alpha_1,\alpha_2,p_1,p_2)},$$

$$\alpha \geqslant 1, \alpha \geqslant \frac{n}{n+1}\beta, 0 \leqslant p,\ q \leqslant \left[\frac{n+1}{n}\alpha\right], p \neq q,$$

$$0 \leqslant p_i \leqslant \left[\frac{n+1}{n}\alpha_i\right], \alpha_i \geqslant 1, \alpha_1 \neq \alpha_2, i = 1, 2,$$

defined in Theorem 3.7 are invariants of $\mathcal{W}$.

(2) *Let $\mathcal{W}_P = \{W : W = \pi^{-1}(V)$ where V is a complete Reinhardt pseudoconvex C^ω-smooth domain in A_n-variety$\}$ and $\mathcal{W}_{SP} = \{W : W = \pi^{-1}(V)$ where V is a complete Reinhardt strictly pseudoconvex domain in A_n-variety$\}$. Then the moduli space of $\mathcal{W}_{\mathcal{P}}$ (respectively $\mathcal{W}_{SP}$) is given by the image of the map $\tilde{\Phi}_P : \mathcal{W}_P \to \mathbb{R}^\infty$ (respectively $\tilde{\Phi}_{SP} : \mathcal{W}_{SP} \to \mathbb{R}^\infty$), where the component functions of $\tilde{\Phi}_P$ (respectively $\tilde{\Phi}_{SP}$) are the invariant functions*

$$\xi^{(\alpha,\beta)}, \zeta^{(\alpha,\beta)}, \eta^{(\alpha,p,q)}, \omega^{(\alpha_1,\alpha_2,p_1,p_2)},$$

$$\alpha \geqslant 1, \alpha \geqslant \frac{n}{n+1}\beta, 0 \leqslant p,\ q \leqslant \left[\frac{n+1}{n}\alpha\right], p \neq q,$$

$$0 \leqslant p_i \leqslant \left[\frac{n+1}{n}\alpha_i\right], \alpha_i \geqslant 1, \alpha_1 \neq \alpha_2, i = 1, 2,$$

defined in Theorem 3.7. In particular, the moduli space of $\mathcal{W}_P$ (respectively $\mathcal{W}_{SP}$) is the same as the moduli space of bounded complete Reinhardt pseudoconvex C^ω-smooth domains (respectively bounded complete Reinhardt strictly pseudoconvex domains) in A_n-variety $\tilde{V}_n = \{(x,y,z) \in \mathbb{C}^3 : xy = z^{n+1}\}$.

It is an interesting question to study the geometry of the moduli space of bounded complete Reinhardt domains in A_n-variety. As an example, we look at two families of domains in A_n-variety and construct the moduli space of these families explicitly. More specifically, consider

$$V^{(d)}_{(a,b,c)} = \{(x,y,z) : xy = z^2, a|x|^{2d} + b|y|^{2d} + c|z|^{2d} < \varepsilon_0\}.$$

Here we assume that a, b, c are strictly greater than zero, and d is a fixed integer greater than or equal to one. This is a 3 parameters family of pseudoconvex domains in A_1-variety $\tilde{V}_1 = \{(x,y,z) \in \mathbb{C}^3 : xy = z^2\}$. Using our Bergman function theory, we can write down the explicit moduli space of this family as shown in the following theorem by means of the invariant

$$\nu^{(\alpha,\beta)} = \frac{(\xi^{(\alpha,\beta)})^{\frac{1}{2}} \cdot (\xi^{(n\alpha-(n-1)\beta,(n+1)\alpha-n\beta)})^{\frac{1}{2}}}{(\xi^{(\alpha,\alpha)})^{\frac{1}{2}}}, \quad \text{for} \quad n = 1.$$

Theorem 3.12 ([10]) *Let*

$$V^{(d)}_{(a,b,c)} = \{(x,y,z) \in \mathbb{C}^3 : xy = z^2, a|x|^{2d} + b|y|^{2d} + c|z|^{2d} < \varepsilon_0\}.$$

Let $\sim$ denote the biholomorphic equivalence. Then the map

$$\varphi : \{V^{(d)}_{(a,b,c)}\} \to \mathbb{R}_+,\ V^{(d)}_{(a,b,c)} \mapsto \nu^{(2d-1,d-1)}$$

is injective up to a biholomorphism equivalence. More precisely, the induced map

$$\tilde{\varphi}\colon \{V^{(d)}_{(a,b,c)}\}/\sim \to \mathbb{R}_+$$

is one-to-one map from $\{V^{(d)}_{(a,b,c)}\}/\sim$ *onto* $\left(0, \dfrac{2}{\pi}\right)$. *So the moduli space of* $\{V^{(d)}_{(a,b,c)}\}$ *is an open interval* $\left(0, \dfrac{2}{\pi}\right)$.

The biholomorphically equivalent problem of domains in A_1-variety is not only interesting in its own right, but also has application to the classical biholomorphically equivalent problem of domains in $\mathbb{C}^2$. In fact, let

$$W^{(d)}_{(a,b,c)} = \{(x, y)\colon a|x|^{2d} + b|y|^{2d} + c|xy|^d < \varepsilon_0\}$$

Corollary 3.1 ([10]) *The moduli space of* $W^{(d)}_{(a,b,c)}$ *is the same as the moduli space of* $V^{(d)}_{(a,b,c)}$, *which is* $(0, \frac{2}{\pi})$.

As an application to the above theory, it is easy to compute explicitly the invariant $\nu^{(3,1)}$ for two domains $V^{(1)}_{(1,1,1)}$ and $V^{(2)}_{(1,1,1)}$ in A_1-variety. As a consequence, we see that $V^{(1)}_{(1,1,1)}$ is not biholomorphic to $V^{(2)}_{(1,1,1)}$ and the domain $W^{(1)}_{(1,1,1)}$ in $\mathbb{C}^2$ is not biholomorphic to the domain $W^{(2)}_{(1,1,1)}$ in $\mathbb{C}^2$.

One of the basic problems in complex geometry is to find a reasonable object which parametrizes all non-isomorphic complex manifolds. This is the well known moduli problem. Let D_1 and D_2 be two domains in $\mathbb{C}^n$. One of the most fundamental problems in complex geometry is to find necessary and sufficient conditions which will imply that D_1 and D_2 are biholomorphically equivalent. For $n = 1$, the celebrated Riemann mapping theorem states that any simply connected domains in $\mathbb{C}$ are biholomorphically equivalent. For $n \geq 2$, there are many domains which are topologically equivalent to the ball but not biholomorphically equivalent to the ball [36]. Poincaré studied the invariance properties of the *CR* manifolds, which are real hypersurfaces in $\mathbb{C}^n$, with respect to biholomorphic transformations. The systematic study of such properties for real hypersurface was made by Cartan [7] and later by Chern and Moser [8]. A main result of the theory is the existence of a complete system of local differential invariants for *CR*-structures on real hypersurface. In 1974, Fefferman [13] proved that a biholomorphic mapping between two strongly pseudoconvex domains is smooth up to the boundaries and the induced boundary mapping is a *CR*-equivalence on the boundary. Thus, one can use Chern-Moser invariants to study the biholomorphically equivalent problem of two strongly pseudoconvex domains. Using the Chern-Moser theory, Webster [44] gave a complete characterization when two ellipsoids in $\mathbb{C}^n$ are biholomorphically equivalent. In 1978, Burns Shnider and Wells [6] showed that the number of moduli of a moduli space of a strongly pseudoconvex bounded domain has to be infinite. Thus the moduli problem of open manifolds is really a very difficult one.

Lempert [27] made significant progress in the subject. He was able to construct the moduli space of bounded strictly convex domains of $\mathbb{C}^n$ with marking at the origin. Although the theory established by Lempert is beautiful, the computation of his invariants is a hard problem.

In [10], Du and Yau studied the moduli problem of complete Reinhardt domains in $\mathbb{C}^2$. The main tool to solve this moduli problem with geometry information is the new biholomorphic invariant Bergman function defined by Yau [50]. In fact Yau's Bergman function theory can also solve the biholomorphic equivalence problem or moduli problem for complete Reinhardt pseudoconvex domains in $\mathbb{C}^n$ for all $n \geq 2$. In order to describe the complete biholomorphic invariants of bounded complete Reinhardt domains in $\mathbb{C}^n$, we introduce the following notations. Let S_n be the symmetric group of degree n. Recall that group ring $\mathbb{R}[S_n]$ is a ring of the form $\mathbb{R}[\tau_1, \tau_2, \dots, \tau_{n!}]$ with $\tau_i \in S_n$ for $1 \leqslant i \leqslant n!$. Let $\sum\limits_i x_i\tau_i$ and $\sum\limits_j y_j\tau_j$, where x_i, y_j are in $\mathbb{R}$, be two elements in $\mathbb{R}[S_n]$. Then

$$(\sum_i x_i\tau_i)(\sum_j y_j\tau_j) := \sum_{i,j} x_iy_j(\tau_i \cdot \tau_j),$$

where $\tau_i \cdot \tau_j$ is the product in the group S_n. We shall consider $\mathbb{R}[S_n] \times \cdots \times \mathbb{R}[S_n]$ the product of the group ring with itself. Such a product has a natural S_n-module structure in the following manner. Let $\sigma \in S_n$ and $(\sum\limits_i x_i\tau_i, \cdots, \sum\limits_i y_i\tau_i) \in (\mathbb{R}[S_n] \times \cdots \times \mathbb{R}[S_n])$. Then

$$\sigma(\sum_i x_i\tau_i, \cdots, \sum_i y_i\tau_i) = (\sum_i x_i(\tau_i\sigma), \cdots, \sum_i y_i(\tau_i\sigma)).$$

Definition 3.6 Two elements f, g in $\mathbb{R}[S_n] \times \cdots \times \mathbb{R}[S_n]$ are said to be equivalent and denoted by $f \sim g$ if there exists a $\sigma \in S_n$ such that $\sigma(f) = g$.

Let $\vec{\alpha} = (\alpha_1, \dots, \alpha_n)$ be an n-tuple of nonnegative integers. Denote $\phi_{\vec{\alpha}} = \left(\prod_{\vec{\alpha}=1}^n z_i^{\alpha_i}\right) dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n$. For a domain D in $\mathbb{C}^n$, we shall use notation $\|\phi_{\vec{\alpha}}\|_D^2 := \int_D \phi_{\vec{\alpha}} \wedge \overline{\phi}_{\vec{\alpha}}$. In [9], the authors showed that all biholomorphic invariants of a bounded complete Reinhardt domains are contained in $(\mathbb{R}[S_n] \times \cdots \times \mathbb{R}[S_n])/\sim$ where there are $n!$ copies of $\mathbb{R}[S_n]$ and $\sim$ is the equivalence relation defined in Definition 3.6.

Theorem 3.13 ([9]) *Let D be a bounded complete Reinhardt domain in $\mathbb{C}^n$. Let $\vec{\alpha} = (\alpha_1, \dots, \alpha_n)$ be a n-tuple of nonnegative integers and $\tau \in S_n$. Denote*

$$g_D^\tau(\vec{\alpha}) = \frac{\|\phi_{\vec{0}}\|_D^{\Sigma\alpha_i - 1} \|\phi_{\tau(\vec{\alpha})}\|_D}{\prod\limits_{i=1}^n \|\phi_{\vec{e}_i}\|_D^{\alpha_{\tau(i)}}}$$

where $\tau(\vec{\alpha}) = (\alpha_{\tau(1)}, \ldots, \alpha_{\tau(n)})$ *and* $\vec{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ *with 1 in the ith component. Then for all n-tuple of nonnegative integers*

$$\vec{\beta}_1, \cdots, \vec{\beta}_{n!},\ \xi_D^{(\vec{\beta}_1, \cdots, \vec{\beta}_{n!})} = (\sum_{\tau \in S_n} g_D^{\tau}(\vec{\beta}_1)\tau, \cdots, \sum_{\tau \in S_n} g_D^{\tau}(\vec{\beta}_{n!})\tau)$$

as an element in $(\mathbb{R}[S_n] \times \cdots \times \mathbb{R}[S_n])/\sim$ *is a biholomorphic invariant. In fact, if* D_1 *and* D_2 *are two such domains which are biholomorphically equivalent, then there exists a* $\sigma \in S_n$ *such that*

$$g_D^{\tau}(\vec{\alpha}) = g_{D_2}^{\tau \cdot \sigma}(\vec{\alpha}) \qquad \forall \tau \in S_n \text{and } \forall\, \vec{\alpha} \text{ n-tuple of nonnegative integers.}$$

The invariants in Theorem 3.13 are complete invariants for bounded complete Reinhardt pseudoconvex domains with C^1 boundaries.

Theorem 3.14 ([9]) *Let* D_i, $i = 1, 2$, *be two bounded complete Reinhardt pseudoconvex domains in* $\mathbb{C}^n$ *with* C^1 *boundaries. If for all* $\vec{\alpha_1}, \cdots, \vec{\alpha_{n!}}$ *n-tuples of non-negative integers,* $\xi_{D_1}^{(\vec{\alpha_1}, \cdots, \vec{\alpha_{n!}})} = \xi_{D_2}^{(\vec{\alpha_1}, \cdots, \vec{\alpha_{n!}})}$ *in* $(\mathbb{R}[S_n] \times \cdots \times \mathbb{R}[S_n])/\sim$, *where*

$$\xi_D^{(\vec{\alpha_1}, \cdots, \vec{\alpha_{n!}})} = (\sum_{\tau \in S_n} g_D^{\tau}(\vec{\alpha_1})\tau, \cdots, \sum_{\tau \in S_n} g_D^{\tau}(\vec{\alpha_{n!}})\tau),$$

then there exists $\sigma \in S_n$ *and a biholomorphic map*

$$\Psi_\sigma(z_1, \ldots, z_n) = (a_1 z_{\sigma(1)}, \ldots, a_n z_{\sigma(n)}),$$

where $a_i = \frac{\|\phi_{\vec{0}}\|_{D_1} \|\phi_{\vec{e_i}}\|_{D_2}}{\|\phi_{\vec{e}_{\sigma(i)}}\|_{D_1} \|\phi_{\vec{0}}\|_{D_2}}$, *such that* Ψ_σ *sends* D_1 *onto* D_2.

Theorems 3.13 and 3.14 above give a complete characterization of two bounded complete Reinhardt domains with real analytic boundaries in $\mathbb{C}^n$ to be biholomorphically equivalent in terms of the group ring $(\mathbb{R}[S_n] \times \cdots \times \mathbb{R}[S_n])/\sim$. In case $n = 2$, we can actually write down the complete numerical invariants for two bounded complete Reinhardt domains with real analytic boundaries in $\mathbb{C}^2$ to be biholomorphically equivalent.

Theorem 3.15 ([9]) *Let* D_1, D_2 *be two bounded complete Reinhardt pseudoconvex domains in* $\mathbb{C}^2$ *with* C^1 *boundaries. Then* D_1 *is biholomorphic to* D_2 *if and only if*

(1) $g_{D_1}(\alpha_1, \alpha_2) + g_{D_1}(\alpha_2, \alpha_1) = g_{D_2}(\alpha_1, \alpha_2) + g_{D_2}(\alpha_2, \alpha_1)$

(2) $g_{D_1}(\alpha_1, \alpha_2) g_{D_1}(\alpha_2, \alpha_1) = g_{D_2}(\alpha_1, \alpha_2) g_{D_2}(\alpha_2, \alpha_1)$

(3) $(g_{D_1}(\alpha_1, \alpha_2) - g_{D_1}(\alpha_2, \alpha_1))(g_{D_1}(\beta_1, \beta_2) - g_{D_1}(\beta_2, \beta_1))$
$= (g_{D_2}(\alpha_1, \alpha_2) - g_{D_2}(\alpha_2, \alpha_1))(g_{D_2}(\beta_1, \beta_2) - g_{D_2}(\beta_2, \beta_1))$

for all non-negative integers α_i, β_i, *where*

$$g_{D_i}(\alpha_1, \alpha_2) = \frac{\|\phi_{\vec{0}}\|_{D_i}^{\alpha_1+\alpha_2-1} \|\phi_{(\alpha_1,\alpha_2)}\|_{D_i}}{\prod_{j=1}^{2} \|\phi_{\vec{e_j}}\|_{D_i}^{\alpha_j}}$$

Corollary 3.2 ([9]) *The moduli space of bounded complete Reinhardt domains with* C^1 *boundaries in* $\mathbb{C}^2$ *can be constructed explicitly as the image of the complete family of numerical invariants:* $g_D(\alpha_1, \alpha_2) + g_D(\alpha_2, \alpha_1)$, $g_D(\alpha_1, \alpha_2)g_D(\alpha_2, \alpha_1)$ *and*

$$(g_D(\alpha_1, \alpha_2) - g_D(\alpha_2, \alpha_1))(g_D(\beta_1, \beta_2) - g_D(\beta_2, \beta_1))$$

$\forall\ \alpha_i, \beta_i$ *non-negative integers.*

In order to find the complete numerical biholomorphic invariants of bounded complete Reinhardt domain in $\mathbb{C}^n$ for $n \geqslant 3$, we need to consider the finite symmetric group $S_n = \{\sigma_1, \sigma_2, \ldots, \sigma_{n!}\}$ of degree n acting on the affine space $\mathbb{C}^{n!n!} = \mathbb{C}^{n!} \times \cdots \times \mathbb{C}^{n!}$, which is the product of $n!$ copies of $\mathbb{C}^{n!}$, in the following manner. Let $\tau \in S_n$ and

$$(x_{\sigma_1}, \ldots, x_{\sigma_{n!}}; \cdots ; y_{\sigma_1}, \ldots, y_{\sigma_{n!}}) \in \mathbb{C}^{n!} \times \cdots \times \mathbb{C}^{n!} = \mathbb{C}^{n!n!}.$$

Then

$$\tau \cdot (x_{\sigma_1}, \ldots, x_{\sigma_{n!}}; \cdots ; y_{\sigma_1}, \ldots, y_{\sigma_{n!}}) = (x_{\sigma_1\tau}, \ldots, x_{\sigma_{n!}\tau}; \cdots ; y_{\sigma_1\tau}, \ldots, y_{\sigma_{n!}\tau}).$$

Since S_n is linearly reductive, by Hilbert Theorem, the ring of invariants

$$\mathbb{C}[x_{\sigma_1}, \ldots, x_{\sigma_{n!}}; \cdots ; y_{\sigma_1}, \ldots, y_{\sigma_{n!}}]^{S_n}$$

is finitely generated. Moreover, the generators can be listed explicitly by Göbel's theorem [15]. Before we give the statement of Göbel's theorem, we shall introduce some definitions first.

Definition 3.7 Suppose that a finite group G acts as permutations on a finite set X. We then refer to X together with the G-action as a finite G-set. A subset $B \subset X$ is called an orbit if G permutes the elements of B among themselves and the induced permutation action of G on B is transitive.

Definition 3.8 If $K = (k_1, \cdots, k_n)$ is an n-tuple of non-negative integers, then K is called an exponent sequence. The associated partition of K is the ordered set consisting of the n numbers $k_1, \cdots, k_n$ rearranged in weakly decreasing order. We denote by $\lambda(K)$ the partition associated to K, so

$$\lambda(K) = (\lambda_1(K) \geq \lambda_2(K) \geq \cdots \geq \lambda_n(K))$$

and the n-tuple $(\lambda_1(K), \lambda_2(K), \cdots, \lambda_n(K))$ is a permutation of $k_1, \cdots, k_n$. The monomial x^K is called special if the associated partition $\lambda(K)$ of the exponent sequence K satisfies

(1) $\lambda_i(K) - \lambda_{i+1}(K) \leq 1$ for all $i = 1, \cdots, n-1$ and

(2) $\lambda_n(K) = 0$.

Notice that if two exponent sequences A and B are permutations of each other, then $\lambda(A) = \lambda(B)$.

Theorem 3.16 *Let G be a finite group, X a finite G-set, and R a commutative ring. Then the ring of invariants $R[X]^G$ is generated as an algebra by $e_{|X|} = \prod_{x \in X} x$, the top degree elementary symmetric polynomial in the elements of X, and the orbit sums of special monomials.*

Theorem 3.17 ([9]) *Let $f_1, \ldots, f_N \in \mathbb{C}[x_{\sigma_1}, \ldots, x_{\sigma_{n!}}; \ldots; y_{\sigma_1}, \ldots, y_{\sigma_{n!}}]^{S_n}$ be the generators of the ring of invariant polynomials computed by Theorem 3.16. Let D be a bounded complete Reinhardt domain in $\mathbb{C}^n$. Then, for $\vec{\alpha}_1, \vec{\alpha}_2, \ldots, \vec{\alpha}_{n!}$ n-tuples of non-negative integers,*

$$f_1(g_D^\sigma(\vec{\alpha}_1), \ldots, g_D^\sigma(\vec{\alpha}_{n!}))_{\sigma \in S_n}, \ldots, f_N(g_D^\sigma(\vec{\alpha}_1), \ldots, g_D^\sigma(\vec{\alpha}_{n!}))_{\sigma \in S_n}$$

are biholomorphic invariants, where

$$g_D^\sigma(\vec{\beta}) = \frac{\|\phi_{\vec{0}}\|_D^{\sum \beta_i - 1} \|\phi_{\sigma(\vec{\beta})}\|_D}{\prod_{i=1}^n \|\phi_{\vec{e}_i}\|_D^{\sigma(i)}}, \quad \vec{\beta} = (\beta_1, \beta_2, \ldots, \beta_n)$$

The following theorem says that the above invariants are actually complete in case the domain D is pseudoconvex.

Theorem 3.18 ([9]) *Let D_i, $i = 1, 2$, be two bounded complete Reinhardt pseudoconvex domains in $\mathbb{C}^n$ with C^1 boundaries. Let $f_1, \ldots, f_N \in \mathbb{C}[x_{\sigma_1}, \ldots, x_{\sigma_{n!}}; \ldots; y_{\sigma_i}, \ldots, y_{\sigma_{n!}}]^{S_n}$ be the generators of the ring of invariant polynomials computed by Theorem 3.16. If for all $\vec{\alpha}_1, \ldots, \vec{\alpha}_{n!}$ n-tuples of non-negative integers*

$$f_i(g_{D_1}^\sigma(\vec{\alpha}_1), \ldots, g_{D_1}^\sigma(\vec{\alpha}_{n!}))_{\sigma \in S_n} = f_i(g_{D_2}^\sigma(\vec{\alpha}_1), \ldots, g_{D_2}^\sigma(\vec{\alpha}_{n!}))_{\sigma \in S_n},$$
$$i = 1, 2, \ldots, N,$$

then there exists $\tau \in S_n$ and a biholomorphic map

$$\Psi_\tau : \mathbb{C}^n \to \mathbb{C}^n, \Psi_\tau(z_1, \ldots, z_n) = (a_1 z_{\tau(1)}, \ldots, a_n z_{\tau(n)}),$$

where

$$a_i = \frac{\|\phi_{\vec{o}}\|_{D_1}\|\phi_{\vec{e_i}}\|_{D_2}}{\|\phi_{e_{\sigma(i)}}\|_{D_1}\|\phi_{\vec{o}}\|_{D_2}},$$

such that Ψ_τ *sends* D_1 *onto* D_2.

Corollary 3.3 ([9]) *The moduli space of bounded complete Reinhardt pseudoconvex domains with* C^1 *boundaries in* $\mathbb{C}^n$ *can be constructed explicitly as the image of the complete family of numerical invariants* $f_i(g_D^\sigma(\vec{\alpha_1}), \ldots, g_D^\sigma(\vec{\alpha_{n!}}))_{\sigma \in S_n}$, $1 \leqslant i \leqslant N$, *where* $\vec{\alpha_1}, \ldots, \vec{\alpha_{n!}}$ *are all possible n-tuples of nonnegative integers.*

Remark 3.1 One can compute explicitly the relation of the generators

$$f_1, \ldots, f_N \in \mathbb{C}[x_{\sigma_1}, \ldots, x_{\sigma_{n!}}; \ldots; y_{\sigma_i}, \ldots, y_{\sigma_{n!}}]^{S_n}.$$

These relations define an algebraic variety in R^∞ where the moduli space lies.

For complete Reinhardt pseudoconvex domains with real analytic boundaries, we can use fewer numerical invariants to characterize these domains. More precisely, we have the following theorems.

Theorem 3.19 ([9]) *Let* D_i, $i = 1, 2$, *be two bounded complete Reinhardt pseudoconvex domains in* $\mathbb{C}^n$ *with real analytic boundaries. Then* D_1 *is biholomorphically equivalent to* D_2 *If and only if for all* $\vec{\alpha}$ *n-tuple of non-negative integers,* $\xi_{D_1}^{\vec{\alpha}} = \xi_{D_2}^{\vec{\alpha}}$ *in* $\mathbb{R}[S_n]/\sim$ *where* $\xi_{D_i}^{\vec{\alpha}} = \sum_{\tau \in S_n} g_{D_i}^\tau(\vec{\alpha})\tau$. *In this case, there exists* $\sigma \in S_n$ *and a biholomorphic map*

$$\Psi_\sigma(z_1, \ldots, z_n) = (a_1 z_{\sigma(1)}, \ldots, a_n z_{\sigma(n)}),$$

where $a_i = \frac{\|\phi_{\vec{0}}\|_{D_1}\|\phi_{\vec{e_i}}\|_{D_2}}{\|\phi_{\vec{e}_{\sigma(i)}}\|_{D_1}\|\phi_{\vec{0}}\|_{D_2}}$, *such that* Ψ_σ *sends* D_1 *onto* D_2.

Theorem 3.20 ([9]) *Let* D_1, D_2 *be two bounded complete Reinhardt pseudoconvex domains in* $\mathbb{C}^2$ *with real analytic boundaries. Then* D_1 *is biholomorphic to* D_2 *if and only if*

$$g_{D_1}(\alpha_1, \alpha_2) + g_{D_1}(\alpha_2, \alpha_1) = g_{D_2}(\alpha_1, \alpha_2) + g_{D_2}(\alpha_2, \alpha_1)$$
$$g_{D_1}(\alpha_1, \alpha_2) g_{D_1}(\alpha_2, \alpha_1) = g_{D_2}(\alpha_1, \alpha_2) g_{D_2}(\alpha_2, \alpha_1)$$

for all non-negative integers α_1, α_2, *where*

$$g_{D_i}(\alpha_1, \alpha_2) = \frac{\|\phi_{\vec{0}}\|_{D_i}^{\alpha_1+\alpha_2-1}\|\phi_{(\alpha_1,\alpha_2)}\|_{D_i}}{\prod_{j=1}^{2}\|\phi_{\vec{e_j}}\|_{D_i}^{\alpha_j}}$$

Theorem 3.21 ([9]) *Let D_i, $i = 1, 2$, be two bounded complete Reinhardt pseudoconvex domains in $\mathbb{C}^n$ with real analytic boundaries. Let*

$$f_1, \ldots, f_N \in \mathbb{C}[x_{\sigma_1}, \ldots, x_{\sigma_{n!}}]^{S_n}$$

be the generators of the ring of invariant polynomials computed by Theorem 3.16. Then D_1 is biholomorphically equivalent to D_2 if and only if for all $\vec{\alpha}$ n-tuples of nonnegative integers

$$f_i(g^{\sigma}_{D_1}(\vec{\alpha}))_{\sigma \in S_n} = f_i(g^{\sigma}_{D_2}(\vec{\alpha}))_{\sigma \in S_n}, \quad i = 1, \ldots, N.$$

In this case, there exists $\tau \in S_n$ and a biholomorphic map $\Psi_\tau : \mathbb{C}^n \to \mathbb{C}^n$, $\Psi_\tau(z_1, \ldots, z_n) = (a_1 z_{\tau(1)}, \ldots, a_n z_{\tau(n)})$, where

$$a_i = \frac{\|\phi_{\vec{0}}\|_{D_1} \|\phi_{\vec{e_i}}\|_{D_2}}{\|\phi_{e_{\sigma(i)}}\|_{D_1} \|\phi_{\vec{0}}\|_{D_2}}$$

such that Ψ_τ sends D_1 onto D_2.

4 Complex Plateau Problem

Let X be a compact connected CR manifold of dimension $2n-1$ in $\mathbb{C}^N$. The famous complex Plateau problem asks under what conditions on X, X will be a boundary of a complex submanifold in $\mathbb{C}^N$. By a theorem of Harvey and Lawson [16], X is a boundary of a unique complex variety V in $\mathbb{C}^N$. Therefore we need to understand under what conditions on X, V will be a complex submanifold.

In 1963, J.J. Kohn solved the famous $\overline{\partial}$-Neumann problem. Based on this work, Kohn and Rossi [22] in 1965 introduced the fundamental CR invariants, the Kohn-Rossi cohomology groups $H^{p,q}_{KR}(X)$. They proved the finite dimensionality of their cohomology groups for $1 \leq q \leq n-2$ if X is strongly pseudoconvex. Following Tanaka [42], we shall recall the definition of Kohn-Rossi cohomology groups as follows.

Let $\{\mathcal{A}^k(X), d\}$ be the De-Rham complex of X with complex coefficients, and let $H^k(X)$ be the De-Rham cohomology groups. There is a natural filtration of the De-Rham complex as follows. For any integer p and k, put $A^k(X) = \Lambda^k(\mathbb{C}T(X)^*)$ and denoted by $F^p(A^k(X))$ the subbundle of $A^k(X)$ consisting of all $\phi \in A^k(X)$ which satisfy the equality

$$\phi(Y_1, \ldots, Y_{p-1}, \overline{Z}_1, \ldots, \overline{Z}_{k-p+1}) = 0$$

for all $Y_1, \ldots, Y_{p-1} \in \mathbb{C}T(X)_x$ and $Z_1, \ldots, Z_{k-p+1} \in S_x, x$ being the origin of ϕ. Then

$$A^k(X) = F^0(A^k(X)) \supseteq F^1(A^k(X)) \supseteq \cdots \supseteq F^k(A^k(X)) \supseteq F^{k+1}(A^k(X)) = 0$$

setting $F^p(\mathcal{A}^k(X)) = \Gamma(F^p(A^k(X)))$, we have

$$\mathcal{A}^k(X) = F^0(\mathcal{A}^k(X)) \supseteq F^1(\mathcal{A}^k(X)) \supseteq \cdots \supseteq F^k(\mathcal{A}^k(X)) \supseteq F^{k+1}(\mathcal{A}^k(X)) = 0.$$

Since clearly $dF^p(\mathcal{A}^k(X)) \subseteq F^p(\mathcal{A}^{k+1}(X))$, the collection $\{F^p(\mathcal{A}^k(X))\}$ gives a filtration of the De-Rham complex.

Definition 4.1 $H^{p,q}_{KR}(X)$, the Kohn-Rossi cohomology group of type (p, q), is defined to be the group $E_1^{p,q}(X)$ of the spectral sequence $\{E_r^{p,q}(X)\}$ associated with the filtration $\{F^p(\mathcal{A}^k(X))\}$.

More explicitly, let

$$A^{p,q}(X) = F^p(A^{p+q}(X)), \qquad \mathcal{A}^{p,q}(X) = \Gamma(A^{p,q}(X))$$
$$C^{p,q}(X) = A^{p,q}(X)\big/A^{p+1,q-1}(X), \ \mathcal{C}^{p,q}(X) = \Gamma(C^{p,q}(X)).$$

Since $d : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q+1}(x)$ maps $\mathcal{A}^{p+1,q-1}(X)$ into $\mathcal{A}^{p+1,q}(X)$, it induces an operator $\overline{\partial}_b : \mathcal{C}^{p,q}(X) \to \mathcal{C}^{p,q+1}(X)$. $H^{p,q}_{KR}(X)$ are then the cohomology groups of the complex $\{\mathcal{C}^{p,q}(X), \overline{\partial}_b\}$.

Definition 4.2 $H^k_h(X)$, the holomorphic De-Rham cohomology group of degree k, is defined to be the group $E_2^{k,0}(X)$ of the spectral sequence $\{E_r^{p,q}(X)\}$ associated with the filtration $\{F^p(\mathcal{A}^k(X))\}$.

More explicitly, recall $E_0^{p,q}(X) = \mathcal{C}^{p,q}(X)$ and $d_0 : \mathcal{C}^{p,q}(X) \longrightarrow \mathcal{C}^{p,q+1}(X)$ is the map $\overline{\partial}_b$ above. Note that $E_0^{k,0}(X) = \mathcal{C}^{k,0}(X) = \mathcal{A}^{k,0}(X) \subseteq \mathcal{A}^k(X)$. Next,

$$E_1^{p,q}(X) = \frac{\operatorname{Ker}(d_0 : \mathcal{C}^{p,q}(X) \longrightarrow \mathcal{C}^{p,q+1}(X))}{\operatorname{Im}(d_0 : \mathcal{C}^{p,q-1}(X) \longrightarrow \mathcal{C}^{p,q}(X))}$$

and $d_1 : E_1^{p,q}(X) \longrightarrow E_1^{p+1,q}(X)$ is the naturally induced map. In particular,

$$\begin{aligned} E_1^{k,0}(X) &= \ker(d_0 : \mathcal{C}^{k,0}(X) \longrightarrow \mathcal{C}^{k,1}(X)) \\ &= \{\phi \in \mathcal{A}^{k,0}(X) : d\phi \in \mathcal{A}^{k+1,0}(X)\} \end{aligned}$$

and d_1 is just d on $E_1^{k,0}(X) \subseteq \mathcal{A}^k(X)$. $E_1^{k,0}(X)$ is called the space of holomorphic k-forms on X. Denoting $E_1^{k,0}(X)$ by $\mathcal{S}^k(X)$, we have the holomorphic De Rham complex $\{\mathcal{S}^k(X), d\}$. Then

$$E_2^{k,0}(X) = \frac{\mathrm{Ker}(d : \mathcal{S}^k(X) \longrightarrow \mathcal{S}^{k+1}(X))}{\mathrm{Im}(d : \mathcal{S}^{k-1}(X) \longrightarrow \mathcal{S}^k(x))}$$
$$= \frac{\{\text{closed holomorphic } k\text{-forms on } X\}}{\{\text{exact holomorphic } k\text{-forms on } X\}},$$

is the holomorphic De Rham cohomology $H_h^k(X)$.

A strongly pseudoconvex complex manifold M is a modification of a Stein space V with isolated singularities. In 1965, Kohn and Rossi [22] conjectured that in general, either there is no boundary cohomology of the boundary $X = \partial V$ in degree (p, q) for $q \neq 0, n-1$, or it must result from the interior singularities of V. Yau [46] solved the Kohn-Rossi conjecture affirmatively in 1981.

Theorem 4.1 (Yau [46]) *Let X be a compact strongly pseudoconvex CR manifold of dimension $2n-1$, $n \geq 3$, which is the boundary of a Stein space V with isolated singularities $x_1, \ldots, x_m$. Then for $1 \leq q \leq n-2$,*

$$H_{KR}^{p,q}(X) \simeq \bigoplus_{i=1}^{m} H_{\{x_i\}}^{q+1}(V, \Omega_V^p),$$

where Ω_V^p is the sheaf of germs of holomorphic p-forms on V. If $x_1, \ldots, x_m$ are hypersurface singularities, then

$$\dim H_{KR}^{p,q}(X) = \begin{cases} 0 & p+q \leq n-2, \quad 1 \leq q \leq n-2 \\ \tau_1 + \cdots + \tau_m & p+q = n-1, n, \ 1 \leq q \leq n-2 \\ 0 & p+q = n+1, \quad 1 \leq q \leq n-2 \end{cases}$$

where τ_i is the number of moduli of V at x_i.

Remark 4.1 Let $f : (\mathbb{C}^{n+1}, 0) \longrightarrow (\mathbb{C}, 0)$ be a holomorphic function. Suppose that $V = \{z \in \mathbb{C}^{n+1} : f(z) = 0\}$ has isolated singularity of the origin. Then the local moduli of V is the dimension of the parameter space of the semi universal deformation space of $(V, 0)$. This number is $\tau = \dim \mathbb{C}\{z_0, \ldots, z_n\}\big/(f, f_{z_0}, \ldots, f_{z_n})$.

As a result of the above theorem, Yau answers the classical complex Plateau problem for real codimension 3 *CR* in $\mathbb{C}^{n+1}$ satisfactory.

Theorem 4.2 (Yau [46]) *Let X be a compact connected strongly pseudoconvex CR-manifold of real dimension $2n-1, n \geq 3$, in the boundary of a bounded strongly pseudoconvex domain D in $\mathbb{C}^{n+1}$. Then X is a boundary of the complex submanifold $V \subseteq D - X$ if and only if Kohn-Rossi cohomology groups $H_{KR}^{p,q}(X)$ are zero for $1 \leq q \leq n-2$.*

For $n = 2$ in Theorem 4.2, X is a 3-dimensional *CR* manifold. The classical complex Plateau problem remains unsolved for over a quarter of a century. The main difficulty is that the Kohn-Rossi cohomology groups are infinite dimensional in this case. Let V be the complex variety with X as its boundary. Then the singularities of V are surface singularities. In order to solve the classical complex Plateau problem for $n = 2$, one would like to ask under what kind of condition on X, V will have only very mild singularities. Our basic observation is the following. Although Kohn-Rossi cohomology groups are infinite dimensional, we can derive from them the holomorphic De Rham cohomology. Let M be a complex manifold. The k-th holomorphic De Rham cohomology $H_h^k(M)$ of M is defined to be the d-closed holomorphic k-forms quotient by the d-exact holomorphic k-forms. It is well known that if M is a Stein manifold, then the holomorphic De Rham cohomology coincides with the ordinary De Rham cohomology.

Definition 4.3 Let (V, x) be an isolated singularity of dimension n. Let $\pi : (M, A) \longrightarrow (V, x)$ be a resolution of singularity with A as exceptional set. Let

$$s = \dim \Gamma(M - A, \Omega^n) \big/ [d\Gamma(M - A, \Omega^{n-1}) + \Gamma(M, \Omega^n)].$$

s is an invariant of the singularity (V, x). It turns out that the s-invariant plays an important role in the relationship between $H_h^n(M - A)$ and $H_h^n(M)$.

Theorem 4.3 (Luk-Yau [31]) *Let X be a compact connected $(2n-1)$-dimensional $(n \geq 2)$ strongly pseudoconvex CR manifold. Suppose that X is the boundary of a n-dimensional strongly pseudoconvex complex manifold M which is a modification of a Stein space V with only isolated singularities $\{x_1, \ldots, x_m\}$. Let A be the maximal compact analytic set in M which can be blown down to $\{x_1, \ldots, x_m\}$. Then*

(1) $H_h^q(X) \cong H_h^q(M - A) \cong H_h^q(M)$ *for* $1 \leq q \leq n - 1$.
(2) $H_h^n(X) \cong H_h^n(M - A)$, $\dim H_h^n(M - A) = \dim H_h^n(M) + s$

where $s = s_1 + \cdots + s_m$ and s_i is the s-invariant of the singularity (V, x_i).

Remark 4.2 The above theorem in particular asserts that up to degree $n - 1$, the holomorphic De Rham cohomology can extend across the maximal compact analytic set.

Definition 4.4 A normal surface singularity $(V, 0)$ is Gorenstein if these exists a nowhere vanishing holomorphic 2-form on $V - \{0\}$.

Recall that isolated hypersurface or complete intersection singularities are Gorenstein. It is a natural question to ask for a characterization of Gorenstein surface singularities with vanishing s-invariant.

Theorem 4.4 (Luk-Yau [31]) *Let $(V, 0)$ be a Gorenstein surface singularity. Let $\pi : M \longrightarrow V$ be a good resolution with $A = \pi^{-1}(0)$ as exceptional set. Assume that M is contractible to A. If $s = 0$, then $(V, 0)$ is a quasi-homogeneous singularity, $H^1(A, \mathbb{C}) = 0$, $\dim H^1(M, \Omega^1) = \dim H^2(A, \mathbb{C}) + \dim H^1(M, \mathcal{O})$, and $H_h^1(M) = H_h^2(M) = 0$. Conversely, if $(V, 0)$ is a two dimensional quasi-homogeneous Gorenstein singularity and $H^1(A, \mathbb{C}) = 0$, then the s-invariant vanishes.*

Let X be a compact CR manifold with CR-structure S. For any C^∞ functions u, there is a section $\overline{\partial}_b u \in \Gamma(\overline{S}^*)$ defined by $(\overline{\partial}_b u)(\overline{L}) = \overline{L}u$ for any $L \in \Gamma(S)$. This can be generalized as follows:

Definition 4.5 A complex vector bundle E over X is said to be holomorphic if there is a differential operator $\overline{\partial}_E : \Gamma(E) \longrightarrow \Gamma(E \otimes \overline{S}^*)$ such that if $\overline{L}u$ denotes $(\overline{\partial}_E u)(\overline{L})$ for $u \in \Gamma(E)$ and $L \in \Gamma(S)$, then for any $L_1, L_2 \in \Gamma(S)$ and any C^∞ function f on X:

(1) $\overline{L}(fu) = (\overline{L}f)u + f(\overline{L}u)$
(2) $[\overline{L}_1, \overline{L}_2]u = \overline{L}_1\overline{L}_2 u - \overline{L}_2\overline{L}_1 u$.

A solution u of the equation $\overline{\partial}_E u = 0$ is called a holomorphic section.

The vector bundle $\hat{T}(X) = \mathbb{C}T(X)\big/\overline{S}$ is holomorphic with respect to the following $\overline{\partial} = \overline{\partial}_{\hat{T}(X)}$. Let ω be the projection from $\mathbb{C}T(X)$ to $\hat{T}(X)$. Take any $u \in \Gamma(\hat{T}(X))$ and express it as $u = \omega(Z)$, $Z \in \Gamma(\mathbb{C}T(X))$. For any $L \in \Gamma(S)$, define $(\overline{\partial}u)(\overline{L}) = \omega([\overline{L}, Z])$. The section $(\overline{\partial}u)(\overline{L})$ of $\hat{T}(X)$ does not depend on the choice of Z and $\overline{\partial}u$ gives a section of $\hat{T}(X) \otimes \overline{S}^*$. Further the operator $\overline{\partial}$ satisfies the conditions in Definition 4.5. The resulting holomorphic vector bundle $\hat{T}(X)$ is called the holomorphic tangent bundle of X.

Lemma 4.1 *If X is a real hypersurface in a complex manifold M, then the holomorphic tangent bundle $\hat{T}(X)$ is naturally isomorphic to the restriction of X of the bundle $T^{1,0}(M)$ of all $(1,0)$ tangent vectors to M.*

Definition 4.6 Let X be a compact CR manifold of real dimension $2n-1$. X is said to be a Calabi-Yau CR manifold if there exists a nowhere vanishing holomorphic section in $\Gamma(\Lambda^n\hat{T}(X)^*)$ where $\hat{T}(X) = \mathbb{C}T(X)\big/\overline{S}$ is the holomorphic tangent bundle of X.

Remark 4.3 (a) Let X be a compact CR manifold of real dimension $2n-1$ in $\mathbb{C}^n$. Then X is a Calabi-Yau CR manifold. (b) let X be a strongly pseudoconvex CR manifold of real dimension $2n-1$ contained in the boundary of a bounded strongly pseudoconvex domain in $\mathbb{C}^{n+1}$. Then X is a Calabi-Yau manifold.

The following theorem is a fundamental theorem toward the complete solution of the classical complex Plateau problem for 3-dimensional strongly pseudoconvex Calabi-Yau CR manifold in $\mathbb{C}^n$. The theorem is interesting in its own right.

Theorem 4.5 (Luk-Yau [31]) *Let X be a strongly pseudoconvex compact Calabi-Yau CR manifold of dimension 3. Suppose that X is contained in the boundary of a strongly pseudoconvex bounded domain D in $\mathbb{C}^n$. If the holomorphic De Rham cohomology $H_h^2(X) = 0$, then X is a boundary of a complex variety V in D with boundary regularity and V has only isolated singularities in the interior and the normalizations of these singularities are Gorenstein surface singularities with vanishing s-invariant.*

Corollary 4.1 (Luk-Yau [31]) *Let X be a strongly pseudoconvex compact CR manifold of dimension 3. Suppose that X is contained in the boundary of a strongly pseudoconvex bounded domain D in* $\mathbb{C}^3$*. If the holomorphic De Rham cohomology* $H_h^2(X) = 0$*, then X is a boundary of a complex variety V in D with boundary regularity and V has only isolated quasi-homogeneous singularities such that the dual graphs of the exceptional sets in the resolution are star shaped and all the curves are rational.*

Before we proceed further, we need to introduce some invariants of singularities as well as *CR*-invariants.

Let V be a n-dimensional complex analytic subvariety in $\mathbb{C}^N$ with only isolated singularities. In [47], Yau considered four kinds of sheaves of germs of holomorphic p-forms

1. $\bar{\Omega}_V^p := \pi_* \Omega_M^p$, where $\pi : M \longrightarrow V$ is a resolution of singularities of V.
2. $\bar{\bar{\Omega}}_V^p := \theta_* \Omega^p_{V \backslash V_{sing}}$ where $\theta : V \backslash V_{sing} \longrightarrow V$ is the inclusion map and V_{sing} is the singular set of V.
3. $\Omega_V^p := \Omega^p_{\mathbb{C}^N} / \mathscr{K}^p$, where $\mathscr{K}^p = \{f\alpha + dg \wedge \beta : \alpha \in \Omega^p_{\mathbb{C}^N}; \beta \in \Omega^{p-1}_{\mathbb{C}^N}; f, g \in \mathscr{I}\}$ and $\mathscr{I}$ is the ideal sheaf of V in $\mathbb{C}^N$.
4. $\tilde{\Omega}_V^p := \Omega^p_{\mathbb{C}^N} / \mathscr{H}^p$, where $\mathscr{H}^p = \{\omega \in \Omega^p_{\mathbb{C}^N} : \omega|_{V \backslash V_{sing}} = 0\}$.

Clearly Ω_V^p, $\tilde{\Omega}_V^p$ are coherent. $\bar{\Omega}_V^p$ is a coherent sheaf because π is a proper map. $\bar{\bar{\Omega}}_V^p$ is also a coherent sheaf by a theorem of Siu (cf. Theorem A of [38]). If V is a normal variety, the dualizing sheaf ω_V of Grothendieck is actually the sheaf $\bar{\bar{\Omega}}_V^n$.

Definition 4.7 The Siu complex is a complex of coherent sheaves $J^\bullet$ supported on the singular points of V which is defined by the following exact sequence

$$0 \longrightarrow \bar{\Omega}^\bullet \longrightarrow \bar{\bar{\Omega}}^\bullet \longrightarrow J^\bullet \longrightarrow 0. \tag{1}$$

Definition 4.8 Let V be a n-dimensional Stein space with 0 as its only singular point. Let $\pi : (M, A) \rightarrow (V, 0)$ be a resolution of the singularity with A as exceptional set. The geometric genus p_g, the irregularity q and $g^{(p)}$ invariant of the singularity are defined as follows (cf. [39, 47]):

$$p_g := dim \Gamma(M \backslash A, \Omega^n) / \Gamma(M, \Omega^n), \tag{2}$$

$$q := dim \Gamma(M \backslash A, \Omega^{n-1}) / \Gamma(M, \Omega^{n-1}), \tag{3}$$

$$g^{(p)} := dim \Gamma(M, \Omega_M^p) / \pi^* \Gamma(V, \Omega_V^p). \tag{4}$$

And recall that the s-invariant of the singularity is defined (cf. Definition 4.3) as follows

$$s := dim \Gamma(M \backslash A, \Omega^n) / [\Gamma(M, \Omega^n) + d\Gamma(M \backslash A, \Omega^{n-1})]. \tag{5}$$

Lemma 4.2 ([31]) *Let V be a n-dimensional Stein space with* 0 *as its only singular point. Let* $\pi : (M, A) \to (V, 0)$ *be a resolution of the singularity with A as exceptional set. Let* $J^\bullet$ *be the Siu complex of coherent sheaves supported on* 0. *Then:*

1. $dim J^n = p_g$,
2. $dim J^{n-1} = q$,
3. $dim J^i = 0$, *for* $1 \leq i \leq n-2$.

Proposition 4.1 ([31]) *Let V be a n-dimensional Stein space with* 0 *as its only singular point. Let* $\pi : (M, A) \to (V, 0)$ *be a resolution of the singularity with A as exceptional set. Let* $J^\bullet$ *be the Siu complex of coherent sheaves supported on* 0. *Then the s-invariant is given by*

$$s = dim H^n(J^\bullet) = p_g - q \tag{6}$$

and

$$dim H^{n-1}(J^\bullet) = 0 \tag{7}$$

Let X be a compact connected strongly pseudoconvex *CR* manifold of real dimension 3, in the boundary of a bounded strongly pseudoconvex domain D in $\mathbb{C}^N$. By Harvey and Lawson [16], there is a unique complex variety V in $\mathbb{C}^N$ such that the boundary of V is X. Let $\pi : (M, A_1, \cdots, A_k) \to (V, 0_1, \cdots, 0_k)$ be a resolution of the singularities with $A_i = \pi^{-1}(0_i), 1 \leq i \leq k$, as exceptional sets. Then the s-invariant defined in Definition 4.8 is *CR* invariant, which is also called $s(X)$.

In order to solve the classical complex Plateau problem, we need to find some *CR*-invariant which can be calculated directly from the boundary X and the vanishing of this invariant will give the regularity of Harvey-Lawson solution to the complex Plateau problem. For this purpose, we define a new sheaf $\bar{\bar{\Omega}}_V^{1,1}$.

Definition 4.9 Let $(V, 0)$ be a Stein germ of a 2-dimensional analytic space with an isolated singularity at 0. Define a sheaf of germs $\bar{\bar{\Omega}}_V^{1,1}$ by the sheaf associated to the presheaf

$$U \mapsto < \Gamma(U, \bar{\bar{\Omega}}_V^1) \wedge \Gamma(U, \bar{\bar{\Omega}}_V^1) >,$$

where U is an open set of V.

Lemma 4.3 ([11]) *Let V be a* 2*-dimensional Stein space with* 0 *as its only singular point in* $\mathbb{C}^N$. *Let* $\pi : (M, A) \to (V, 0)$ *be a resolution of the singularity with A as exceptional set. Then* $\bar{\bar{\Omega}}_V^{1,1}$ *is coherent and there is a short exact sequence*

$$0 \longrightarrow \bar{\bar{\Omega}}_V^{1,1} \longrightarrow \bar{\bar{\Omega}}_V^2 \longrightarrow \mathscr{G}^{(1,1)} \longrightarrow 0 \tag{8}$$

where $\mathscr{G}^{(1,1)}$ *is a sheaf supported on the singular point of* V. *Let*

$$G^{(1,1)}(M\backslash A) := \Gamma(M\backslash A, \Omega_M^2)/ < \Gamma(M\backslash A, \Omega_M^1) \wedge \Gamma(M\backslash A, \Omega_M^1) >, \tag{9}$$

then $dim\mathscr{G}_0^{(1,1)} = dimG^{(1,1)}(M\backslash A)$.

Thus, from Lemma 4.3, we can define a local invariant of a singularity which is independent of resolution.

Definition 4.10 Let V be a 2-dimensional Stein space with 0 as its only singular point. Let $\pi : (M, A) \to (V, 0)$ be a resolution of the singularity with A as exceptional set. Let

$$g^{(1,1)}(0) := dim\mathscr{G}_0^{(1,1)} = dimG^{(1,1)}(M\backslash A). \tag{10}$$

We will omit 0 in $g^{(1,1)}(0)$ if there is no confusion from the context.

Let $\pi : (M, A_1, \cdots, A_k) \to (V, 0_1, \cdots, 0_k)$ be a resolution of the singularities with $A_i = \pi^{-1}(0_i), 1 \leq i \leq k$, as exceptional sets, and $A = \cup_i A_i$. In this case, we still let

$$G^{(1,1)}(M\backslash A) := \Gamma(M\backslash A, \Omega_M^2)/ < \Gamma(M\backslash A, \Omega_M^1) \wedge \Gamma(M\backslash A, \Omega_M^1) > .$$

Definition 4.11 If X is a compact connected strongly pseudoconvex CR manifold of real dimension 3 which is the boundary of a bounded strongly pseudoconvex domain D in $\mathbb{D}^N$. Suppose V in $\mathbb{C}^N$ such that the boundary of V is X. Let $\pi : (M, A = \cup_i A_i,) \to (V, 0_1, \cdots, 0_k)$ be a resolution of the singularities with $A_i = \pi^{-1}(0_i)$, $1 \leq i \leq k$, as exceptional sets. Let

$$G^{(1,1)}(M\backslash A) := \Gamma(M\backslash A, \Omega_M^2)/ < \Gamma(M\backslash A, \Omega_M^1) \wedge \Gamma(M\backslash A, \Omega_M^1) > \tag{11}$$

and

$$G^{(1,1)}(X) := \mathscr{S}^2(X)/ < \mathscr{S}^1(X) \wedge \mathscr{S}^1(X) > \tag{12}$$

where $\mathscr{S}^p$ are holomorphic cross sections of $\wedge^p(\hat{T}(X)^*)$. Then we set

$$g^{(1,1)}(M \setminus A) := dimG^{(1,1)}(M \setminus A) \tag{13}$$

$$g^{(1,1)}(X) := dimG^{(1,1)}(X). \tag{14}$$

Lemma 4.4 ([11]) *Let* X *be a compact connected strongly pseudoconvex CR manifold of real dimension* 3 *which bounds a bounded strongly pseudoconvex variety* V *with only isolated singularities* $\{0_1, \cdots, 0_k\}$ *in* $\mathbb{C}^N$. *Let* $\pi : (M, A_1, \cdots, A_k) \to (V, 0_1, \cdots, 0_k)$ *be a resolution of the singularities with* $A_i = \pi^{-1}(0_i)$, $1 \leq i \leq k$, *as exceptional sets. Then* $g^{(1,1)}(X) = g^{(1,1)}(M\backslash A)$, *where* $A = \cup A_i$, $1 \leq i \leq k$.

By Lemma 4.4 and the proof of Lemma 4.3, we can get the following lemma easily.

Lemma 4.5 *Let X be a compact connected strongly pseudoconvex CR manifold of real dimension* 3*, which bounds a bounded strongly pseudoconvex variety V with only isolated singularities* $\{0_1, \cdots, 0_k\}$ *in* $\mathbb{C}^N$*. Then* $g^{(1,1)}(X) = \sum_i g^{(1,1)}(0_i) = \sum_i dim \mathscr{G}^{(1,1)}_{0_i}$.

The following proposition is to show that $g^{(1,1)}$ is bounded above.

Proposition 4.2 ([11]) *Let V be a* 2*-dimensional Stein space with* 0 *as its only singular point. Then* $g^{(1,1)} \leq p_g + g^{(2)}$.

The following theorem is the crucial part for the classical complex Plateau problem.

Theorem 4.6 ([11]) *Let V be a* 2*-dimensional Stein space with* 0 *as its only normal singular point with* $\mathbb{C}^*$*-action. Let* $\pi : (M, A) \to (V, 0)$ *be a minimal good resolution of the singularity with A as exceptional set, then* $g^{(1,1)} \geq 1$.

In the paper [31], Luk and Yau gave a sufficient condition $H_h^2(X) = 0$ to determine when X can bound some special singularities. However, even if both $H_h^2(X)$ and $H_h^1(X)$ vanish, V still can be singular.

The *CR* invariants in Definition 4.11 (formula 14) can be used to give sufficient and necessary conditions for the variety bounded by X being smooth after normalization.

Theorem 4.7 ([11]) *Let X be a strongly pseudoconvex compact Calabi-Yau CR manifold of dimension* 3*. Suppose that X is contained in the boundary of a strongly pseudoconvex bounded domain D in* $\mathbb{C}^N$*. Then X is a boundary of the complex variety* $V \subseteq D - X$ *with boundary regularity and the variety is smooth after normalization if and only if s-invariant and* $g^{(1,1)}(X)$ *vanish.*

Corollary 4.2 ([11]) *Let X be a strongly pseudoconvex compact CR manifold of dimension* 3*. Suppose that X is contained in the boundary of a strongly pseudoconvex bounded domain D in* $\mathbb{C}^3$*. Then X is a boundary of the complex sub-manifold* $V \subset D - X$ *if and only if s-invariant and* $g^{(1,1)}(X)$ *vanish.*

Corollary 4.3 ([11]) *Let X be a strongly pseudoconvex compact Calabi-Yau CR manifold of dimension* 3*. Suppose that X is contained in the boundary of a strongly pseudoconvex bounded domain D in* $\mathbb{C}^N$ *with* $H_h^2(X) = 0$*. Then X is a boundary of the complex sub-manifold up to normalization* $V \subset D - X$ *with boundary regularity if and only if* $g^{(1,1)}(X) = 0$.

Corollary 4.4 ([11]) *Let X be a strongly pseudoconvex compact CR manifold of dimension* 3*. Suppose that X is contained in the boundary of a strongly pseudoconvex bounded domain D in* $\mathbb{C}^3$ *with* $H_h^2(X) = 0$*. Then X is a boundary of the complex sub-manifold* $V \subset D - X$ *if and only if* $g^{(1,1)}(X) = 0$.

5 Minimal Embedding Dimension of Compact CR Manifold

Let us first consider a compact strongly pseudoconvex manifold X of dimension $2n-1$ where $n \geq 3$. As mentioned above, X can be CR embedded in some $\mathbb{C}^N$. It is therefore of interest to study the minimal dimensional complex Euclidean space in which X CR embeds. Our starting point is the Theorem 4.1 which provides us with obstruction to CR embedding:

Theorem 5.1 *Let X be a compact strongly pseudoconvex CR manifold of dimension $2n-1, n \geq 3$. Then X cannot be CR embedded in $\mathbb{C}^n$ unless all $H_{KR}^{p,q}(X) = 0$, $1 \leq q \leq n-2$. Further, X cannot be CR embedded in $\mathbb{C}^{n+1}$ if one of the following does not hold:*

(1) $H_{KR}^{p,q}(X) = 0$ *for* $p+q \leq n-2$ *and* $1 \leq q \leq n-2$

(2) $\dim H_{KR}^{p,q}(X) = \dim H_{KR}^{p',q'}(X)$ *for* $\left.\begin{matrix} p+q \\ p'+q' \end{matrix}\right\} = n-1, n$ *and* $1 \leq q, q' \leq n-2$

(3) $H_{KR}^{p,q}(X) = 0$ *for* $p+q \geq n+1$ *and* $1 \leq q \leq n-2$.

We next consider an interesting class of CR manifolds.

Definition 5.1 Let X be a CR manifold with structure bundle S. Let α be a smooth S^1-action on X and V be its generating vector field. The S^1-action α is called holomorphic of $\mathcal{L}_V\Gamma(S) \subseteq \Gamma(S)$ where $\mathcal{L}_V$ denotes the Lie derivative. It is called transversal if V is transversal to $S \oplus \overline{S}$ in $\mathbb{C}TX$ at every point of X.

For a CR manifold X which admits a transversal holomorphic S^1-action, the invariant Kohn-Rossi cohomology is defined as follows.

Definition 5.2 With the notation in Definition 5.1, consider first the differential operator on k forms $N : \mathcal{A}^k(X) \to \mathcal{A}^k(X)$ defined by $N\phi = \sqrt{-1}\mathcal{L}_V\phi$, $\phi \in \mathcal{A}^k(X)$. Observe that N leaves invariant the spaces $\mathcal{A}^{p,q}(X)$ and $\mathcal{C}^{p,q}(X)$, and commutes with the operators d and $\overline{\partial}_b$. Hence N acts on the cohomology groups $H_{KR}^{p,q}(X)$. Now define the invariant Kohn-Rossi cohomology by $\tilde{H}_{KR}^{p,q}(X) = \{c \in H_{KR}^{p,q}(X) : Nc = 0\}$.

For a compact strongly pseudoconvex CR manifold X of dimension $2n-1, n \geq 3$, which admits a transversal holomorphic S^1-action, the invariant Kohn-Rossi cohomology $\tilde{H}_{KR}^{p,q}(X)$, for $1 \leq p+q \leq 2n-N-1$, are obstructions to CR embedding in $\mathbb{C}^N$. This is implied by the following theorem.

Theorem 5.2 (Luk-Yau [33]) *Let X be a compact strongly pseudoconvex CR manifold of dimension $2n-1, n \geq 3$, which admits a transversal holomorphic S^1-action. Suppose that X is CR embeddable in $\mathbb{C}^N$. Then $\tilde{H}_{KR}^{p,q}(X) = 0$ for all $1 \leq p+q \leq 2n-N-1$.*

The proof of Theorem 5.2 contains two main parts. The first part depends heavily on the work of Lawson-Yau [26], which provides us with topological restrictions on X. In particular it can be shown that the De Rham cohomology groups $H^k(X) = 0$ for $1 \leq k \leq 2n-N-1$. The second part follows Tanaka's differential geometric study on the $\overline{\partial}_b$ cohomology groups [42]. The existence of the vector field V with $[V, \Gamma(S)] \subseteq \Gamma(S)$ entails a formalism analogous to Kähler geometry linking the various cohomology groups via harmonic forms. The details of the proof of Theorem 5.2 are contained in [33].

For 3 dimensional compact strongly pseudoconvex *CR* manifolds, global *CR* embedding in complex Euclidean space may fail and much work has been done on this phenomenon. See for example [3, 5, 28]. We only remark that as a consequence of the global invariants to be discussed in the next section, we find obstructions to *CR* embedding in $\mathbb{C}^3$, assuming that the 3-dimensional strongly pseudoconvex *CR* manifold is *CR* embeddable in some $\mathbb{C}^N$ to begin with. These obstructions provide us with numerous examples of such 3-dimensional *CR* manifolds not *CR* embeddable in $\mathbb{C}^3$.

Remark 5.1 It is interesting to note that there are compact strongly pseudoconvex 3 dimensional *CR* manifolds with arbitrarily large minimal embedding dimensions. For any positive integer N, take any 2-dimensional strongly pseudoconvex complex manifold with maximal compact analytic set A which is a smooth rational curve having self intersection number $-N$. The corresponding weighted dual graph is hence

$$\underset{-N}{\bullet}\ .$$

On blowing down A, one gets a 2-dimensional rational singularity (V, x). The minimal embedding dimension of (V, x) is $-A \cdot A + 1 = N + 1$. Let X be the intersection of V with a small sphere centered at x. Then the minimal embedding dimension of X is $N + 1$.

6 Global Invariants of Compact Strongly Pseudoconvex CR Manifolds

As a first step towards the difficult classification problem of compact strongly pseudoconvex *CR* manifolds [43], it would be useful to understand the following notion of equivalence which is weaker than *CR* equivalence.

Definition 6.1 Assume that X_1, X_2 are compact strongly pseudoconvex embeddable CR manifolds of dimension $2n - 1$, $n \geq 2$. By [16, 17], there are unique complex varieties $V_1 \subseteq \mathbb{C}^{N_1}$ and $V_2 \subseteq \mathbb{C}^{N_2}$ such that $\partial V_1 = X_1$, $\partial V_2 = X_2$, V_1 and V_2 have only isolated normal singularities. X_1, X_2 are called algebraically equivalent if V_1 and V_2 have isomorphic singularities Y_1, Y_2, i.e. $(V_1, Y_1) \cong (V_2, Y_1)$ as germs of varieties.

Remark 6.1 It is not difficult to show that *CR* equivalence implies algebraic equivalence. Hence all algebro-geometric invariants of the singularities of V are *CR* invariants of X.

In case a compact strongly pseudoconvex *CR* manifold X of dimension $2n-1$ embeds in $\mathbb{C}^{n+1}$, $n \geq 2$, it is the boundary of a complex hypersurface V with isolated singularities $x_1, \ldots, x_m$. In this case, an Artinian algebra can be associated to X as follows.

Definition 6.2 With the above notation, let f_i be a defining function of the germ (V, x_i), $1 \leq i \leq m$. Then the $\mathbb{C}$-algebra $A_i = \mathcal{O}_{n+1}\Big/(f_i, \frac{\partial f_i}{\partial z_0}, \ldots, \frac{\partial f_i}{\partial z_n})$ is a commutative local Artinian algebra called the moduli algebra of (V, x_i). The moduli algebra is independent of the choice of defining function. We associate to the *CR* manifold X the Artinian algebra $A(X) = \bigoplus_{i=1}^{m} A_i$.

By the work of Mather-Yau [34] on isolated hypersurface singularities, it can be shown that the associated Artinian algebras are complete algebraic *CR* invariants in the following sense.

Theorem 6.1 (Luk-Yau [30]) *Two compact strongly pseudoconvex real codimension 3 CR manifolds X_1, X_2 are algebraically equivalent if and only if the associated Artinian algebras $A(X_1)$, $A(X_2)$ are isomorphic $\mathbb{C}$-algebras.*

Definition 6.3 With the above notation, let $L(X)$ be the algebra of derivations of $A(X)$. Since $A(X)$ is finite dimensional as $\mathbb{C}$-vector space and $L(X)$ is contained in the endomorphism algebra of $A(X)$, consequently $L(X)$ is a finite dimensional Lie algebra with the obvious Lie algebra structure.

Theorem 6.2 (Yau [48, 49]) *With the above notation, $L(X)$ is a finite dimensional solvable Lie algebra.*

We remark that there are Torelli type examples in which the Lie algebras $L(X_t)$ associated to a family of compact strongly pseudoconvex real codimension 3 *CR* manifolds X_t suffice to distinguish *CR* equivalence. For example, in the family $X_t = \{(x, y, z) \in \mathbb{C}^3 : x^6 + y^3 + z^2 + tx^4y = 0 \text{ and } |x|^2 + |y|^2 + |z|^2 = \epsilon^2\}$ where $\epsilon > 0$ is a small fixed number and $t \in \mathbb{C}$ with $4t^2 + 27 \neq 0$, X_{t_1}, X_{t_2} are *CR* equivalent if and only if $L(X_{t_1}), L(X_{t_2})$ are isomorphic Lie algebras.

Question 6.1 *How can one compute $A(X)$ and $L(X)$ directly from X without going through V?*

For the rest of this section we consider embeddable 3 dimensional compact strongly pseudoconvex *CR* manifolds. By taking resolutions of the singularities of the subvariety V bounded by such a *CR* manifold X in complex Euclidean space, numerical invariants under algebraic equivalence may be defined as follows.

Definition 6.4 Let $\pi : M \to V$ be a resolution of the singularities Y of V such that the exceptional set $A = \pi^{-1}(Y)$ has normal crossing, i.e., the irreducible components A_i of A are nonsingular, they intersect transversally and no three meet at a point. According to Artin [2], there exists a unique minimal positive divisor Z, called the fundamental cycle, with support on A, such that $Z \cdot A_i \leq 0$ for all A_i. For any positive divisor $D = \Sigma d_i A_i$, let $\mathcal{O}_M(-D)$ be the sheaf of germs of holomorphic functions on M vanishing to order d_i on A_i, let $\mathcal{O}_D = \mathcal{O}_M/\mathcal{O}_M(-D)$ and let $\chi(\mathcal{O}_D) = \sum_{i=0}^{2}(-1)^i \dim H^i(M, \mathcal{O}_D)$. It can be proved that $p_f(X) := 1-\chi(\mathcal{O}_Z)$, $p_a(X) := \sup(1 - \chi(\mathcal{O}_D))$ where D ranges over all positive divisors with support on A and $p_g(X) := \dim H^1(M, \mathcal{O})$ are defined independent of the resolution π and are invariants of X under algebraic equivalence. The detailed proofs are contained in [32]. We refer to $p_f(X), p_a(X)$ and $p_g(X)$ as the fundamental genus, arithmetic genus and geometric genus of X respectively.

The following facts are known:

- $0 \leq p_f(X) \leq p_a(X) \leq p_g(X)$
- $p_f(X) = 0 \Leftrightarrow p_a(X) = 0 \Leftrightarrow p_g(X) = 0$.

Further numerical invariants under algebraic equivalence are given by $m_Z(X) := Z \cdot Z$, $q(X) := \dim H^0(M-A, \Omega^1) \big/ H^0(M, \Omega^1)$, $\chi(X) := K \cdot K + \chi_T(A)$ and $\omega(X) := K \cdot K + \dim H^1(M, \Omega^1)$, where Ω^1 is the sheaf of germs of holomorphic 1-form on M, $\chi_T(A)$ is the topological Euler characteristic of A and K is the canonical divisor on M. These invariants are defined independent of the choice of the resolution π. Since K is a divisor with rational coefficient, $\chi(X)$ and $\omega(X)$ are in general rational numbers.

Using the above invariants, one may attempt a rough algebraic classification of embeddable 3 dimensional compact strongly pseudoconvex *CR* manifolds.

Definition 6.5 An embeddable 3 dimensional compact strongly pseudoconvex *CR* manifold X is called a rational (respectively elliptic) *CR* manifold if $p_a(X) = 0$ (respectively $p_a(X) = 1$).

If X is a rational or an elliptic *CR* manifold embeddable in $\mathbb{C}^3$ and M_0 is the minimal good resolution of the subvariety V bounded by X in $\mathbb{C}^3$, then the weighted dual graph for the exceptional set of M_0 is completely classified. The same also holds for those X embeddable in $\mathbb{C}^3$ and has $p_g(X) = 1$. With the weighted dual graphs classified, the topology of the embedding of the exceptional set in M_0 is well understood.

As an application, one obtains obstructions to embedding in $\mathbb{C}^3$ for the above three classes of *CR* manifolds when their weighted dual graphs fail to have the required forms. For example, a rational *CR* manifolds whose weighted dual graph is not a direct sum of the graphs A_k, D_k, E_6, E_7, E_8 is not embeddable in $\mathbb{C}^3$.

Similarly in view of the following theorem, one obtains numerical obstructions to embedding in $\mathbb{C}^3$ for those *CR* manifolds failing the conditions in the theorem.

Theorem 6.3 ([29, 32]) *Let X be a compact strongly pseudoconvex 3-dimensional CR manifold embeddable in* $\mathbb{C}^3$. *Then*

(1) $\chi(X)$ *and* $\omega(X)$ *are integers.*
(2) $10p_g(X) + \omega(X) \geq 0$
(3) If $p_a(X) = 1$, *then* $\chi(X) \geq -3$
(4) If X admits a transversal holomorphic S^1*-action, then* $6p_g(X) + \chi(X) > 0$.

We remark that (4) depends on the Durfee conjecture which is solved by Xu and Yau [45].

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