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ZERO-DIMENSIONAL GRADIENT SINGULARITIES*

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Dedicated to Henry Laufer on the occasion of his 70th birthday

Abstract. We discuss an approach to the problem of classifying zero-dimensional gradient quasihomogeneous singularities using simple properties of deformation theory. As an example, we enumerate all such singularities with modularity $\wp = 0$ and with Milnor number not greater than 12. We also compute normal forms and monomial vector-bases of the first cotangent homology and cohomology modules, the corresponding Poincaré polynomials, inner modality, inner modularity, primitive ideals, etc.

Key words. Gradient singularities, multiple points, deformations, cotangent homology and cohomology, primitive ideals, complete intersections, inner modality and modularity.

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Introduction. Lists of normal forms of quasihomogeneous functions with inner modality 0 and 1 are given in [10]. The next step in classifying singularities with respect to this invariant seems to be the case of one-dimensional complete intersections. This case has been investigated in detail in the article [2], where a list of one-dimensional unimodal quasihomogeneous complete intersections under contact $\mathscr{K}$-equivalence was also obtained. The description of the corresponding lists of *zero*-dimensional complete intersections can become a big leap forward in the general theory of classification. It is appropriate at this point to recall also that a list of $\mathscr{K}$*-simple* zero-dimensional complete intersections has been obtained in [16], and all $\mathscr{K}$-unimodal germs (not necessarily quasihomogeneous) are enumerated in [14].

In this connection it should be noted that the theory of zero-dimensional singularities differs essentially from the theory of singularities of *positive* dimension even in the case of complete intersections. For instance, gradings of zero-dimensional singularities are not defined uniquely, sometimes there exist several derivations of weighted degree zero, including the Euler vector fields. Next, the cotangent homology and cohomology, as well as the Milnor number, are not determined uniquely by the type of homogeneity, and so on.

The aim of the present note is to study zero-dimensional *gradient* singularities, which are determined by the Jacobian ideals of quasihomogeneous functions with isolated critical points. We develop an approach to the problem of classifying such singularities, which is mainly based on closed relations between deformation theories of hypersurfaces and the associated gradient singularities.

In the first two sections we recall some standard notions and results which will be used in the article. Then we discuss some useful properties of deformation theory of zero-dimensional singularities following the paper [6]. In the fifth section we introduce notions of the inner modality and inner modularity, taking into account the case of complete intersections of *positive* dimension studied in [2] and [3]. Then, as an example, we enumerate all zero-dimensional gradient singularities with modularity

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$\wp = 0$ and with Milnor number not greater than 12. We also compute normal forms and monomial vector-bases of the first cotangent homology and cohomology modules, the corresponding Poincaré polynomials, inner modality, inner modularity, primitive ideals, etc.

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1. Kähler differentials and derivations. In what follows, we denote a field of characteristic 0 by k. A unitary commutative Noetherian ring is called a ring, a commutative finitely generated k-algebra with unity is called a k-algebra and all k-homomorphisms of such k-algebras are assumed to be unitary.

Let k be the ground field and let A be a k-algebra with multiplication

$$\varrho\colon A \otimes_{\mathsf{k}} A \longrightarrow A, \qquad \varrho(a_1 \otimes a_2) = a_1 a_2.$$

Let $\Omega^1_{A/\mathsf{k}}$ denote the A-module of the Kähler differentials (or k-differentials of the first order) of the algebra A (for brevity, we shall often write Ω^1_A), i. e., $\Omega^1_{A/\mathsf{k}} = I_A/I_A^2$, where $I_A := \mathrm{Ker}(\varrho)$.

Next, given any A-module M, we use $\mathrm{Der}_{\mathsf{k}}(A, M)$ to denote an A-module of k-derivations (or, equivalently, k-differentiations of the first order) of A with values in M. It is well-known that, on the category of A-modules, there exists a fundamental isomorphism

$$\mathrm{Der}_{\mathsf{k}}(A, M) \cong \mathrm{Hom}_{\mathsf{k}}(\Omega^1_{A/\mathsf{k}}, M),$$

so that $\mathrm{Der}_{\mathsf{k}}(A) = \mathrm{Der}_{\mathsf{k}}(A, A)$ is the module of k-derivations of the algebra A. Let d denote the *universal* derivation $A \to \Omega^1_{A/\mathsf{k}}$ given by the rule $d(a) \equiv (a \otimes 1 - 1 \otimes a) \pmod{I_A^2}$.

ASSERTION 1. *Suppose that a k-algebra A is generated by its subset $\{a_1, \dots, a_m\}$. Then the A-module $\Omega^1_{A/\mathsf{k}}$ is generated by the elements $da_1, \dots, da_m$.*

Proof. From definition it follows that $\Omega^1_{A/\mathsf{k}}$, as a k-module, is generated by the elements $b\,d(a)$ for $a, b \in A$. □

EXAMPLE 1. In the *algebroid* (algebraic local) case, $P = \mathsf{k}\langle z_1, \dots, z_m\rangle$ is the localization of the ring $\mathsf{k}[z_1, \dots, z_m]$ of polynomials in m variables in its maximal ideal $(z_1, \dots, z_m)$, and the P-module $\Omega^1_{P/\mathsf{k}}$ is generated by the differentials $dz_1, \dots, dz_m$. If I is an ideal of P and $A = P/I$ is the corresponding k-algebra, then the A-module $\Omega^1_{A/\mathsf{k}}$ of Kähler differentials of k-algebra A is determined by the exact sequence

$$I/I^2 \xrightarrow{D} \Omega^1_{P/\mathsf{k}} \otimes_P A \longrightarrow \Omega^1_{A/\mathsf{k}} \longrightarrow 0, \tag{$\star$}$$

where the homomorphism D is given by the rule $D(\overline{f}) = d(f) = d(f) \otimes 1$ for an element $f \in I$ and its class $\overline{f}$ in the *conormal* module I/I^2, and where $d\colon P \to \Omega^1_{P/\mathsf{k}}$ is the universal derivation. Then $\Omega^0_{A/\mathsf{k}} = A$, $\Omega^p_{A/\mathsf{k}} = \bigwedge^p \Omega^1_{A/\mathsf{k}}$ for all $p \geqslant 1$, and $\Omega^p_{A/\mathsf{k}} = 0$ for $p < 0$ and $p > m$. The derivation d, extended to the family of modules

$\Omega^p_{A/\mathsf{k}}$, $p \geqslant 0$, determines the increasing *de Rham* complex $(\Omega^\bullet_A, d)$, its homology groups are denoted by $H^q_{\mathrm{DR}}(A)$, $q \geqslant 0$, etc.

Example 2. Analogously, in the *analytic* case $\mathsf{k} = \mathbb{C}$, $H = \mathsf{k}\langle\langle z_1, \ldots, z_m\rangle\rangle$ is the ring of convergent power series in m variables and the H-module $\Omega^1_{H/k}$ is generated by the set of differentials $\{dz_1, \ldots, dz_m\}$. Next, let $(X, \mathfrak{o})$ be a germ of complex space. We choose one of its representatives embedded in an open neighborhood U of the point $\mathfrak{o}$ in $\mathbb{C}^m$ with coordinates $z_1, \ldots, z_m$, and denote it by X. Then X is determined by the ideal I of the structure algebra $\mathscr{O}_U$ in the neighborhood U, generated by a sequence of holomorphic functions $f_1, \ldots, f_k \in \mathscr{O}_U$, so that the localization in the distinguished point of the quotient algebra $A = H/I$ is the *dual* local analytic $\mathbb{C}$-algebra of the germ X. By definition, the coherent sheaves of regular *holomorphic* p-forms Ω^p_X, $p \geqslant 1$, on X are determined by the restriction to X of the corresponding quotient modules:

$$\Omega^p_X = \Omega^p_U/((f_1, \ldots, f_k)\Omega^p_U + df_1 \wedge \Omega^{p-1}_U + \cdots + df_k \wedge \Omega^{p-1}_U)\big|_X,$$

so that $\Omega^p_X \cong \bigwedge^p \Omega^1_X$, and $\Omega^0_X = \mathscr{O}_X$. As well as in the algebraic setting, the de Rham complex $(\Omega^\bullet_X, d)$ is defined; it is often called the *Poincaré complex*.

Assertion 2. *Let X be a germ with an isolated singularity of dimension $n \geqslant 1$, then the modules Ω^p_X are finite-dimensional vector spaces for all $p > n$.*

Proof. For every point $x \in X \setminus \{\mathfrak{o}\}$ we have $\Omega^p_{X,x} = 0$ for the indicated values of p. The coherence implies the statement. □

For simplicity of notations, in what follows, we shall call any germ $(X, \mathfrak{o})$ of complex space or its suitable representative (even the distinguished point $\mathfrak{o}$ is *nonsingular*) the *singularity* and denote it by X. Similarly, the term "singularity A" is often used in reference to any singularity with dual analytic k-algebra A as well.

2. Cotangent functors. First, we recall some basic properties of the cotangent functors which will be used below.

In the notation of Section 1, let $A = P/I$, and let $\mathrm{T}_i(A/\mathsf{k}, -)$ and $\mathrm{T}^i(A/\mathsf{k}, -)$, $i \geqslant 0$, be the i-th *lower* and the i-th *upper* cotangent functors of k-algebra A, respectively. Given an A-module M, the A-modules $\mathrm{T}_i(A/\mathsf{k}, M)$ and $\mathrm{T}^i(A/\mathsf{k}, M)$ are often called the i-th cotangent *homology* and *cohomology* of the k-algebra A with coefficients in M. By definition, there are canonical isomorphisms

$$\mathrm{T}_0(A/\mathsf{k}, M) \cong \Omega^1_{A/\mathsf{k}} \otimes_A M, \qquad \mathrm{T}^0(A/\mathsf{k}, M) \cong \mathrm{Der}_{\mathsf{k}}(A, M),$$

and the following exact sequences of A-modules

$$0 \to \mathrm{T}_1(A/\mathsf{k}, M) \to I/I^2 \otimes_A M \xrightarrow{D} \Omega^1_{P/\mathsf{k}} \otimes_P M \to \mathrm{T}_0(A/\mathsf{k}, M) \to 0,$$

$$0 \to \mathrm{T}^0(A/\mathsf{k}, M) \to \mathrm{Hom}_A(\Omega^1_{P/\mathsf{k}} \otimes_P A, M) \xrightarrow{D^*} \mathrm{Hom}_A(I/I^2, M) \to \mathrm{T}^1(A/\mathsf{k}, M) \to 0.$$

The first of these exact sequences is the tensor product of sequence $(\star)$ by M over A, the second is the result of application of the functor $\mathrm{Hom}_A(-, M)$ to the same sequence. The two morphisms $D \otimes 1_M$ and $D^* \otimes 1^*_M$ we simply denote by D and D^*, respectively.

ASSERTION 3. *Let $A = P/I$ be a local* k*-algebra. Then*

$$\dim_{\mathsf{k}} \mathrm{T}_1(A/\mathsf{k}, \mathsf{k}) = \#(I), \quad \dim_{\mathsf{k}} \mathrm{T}_0(A/\mathsf{k}, \mathsf{k}) = e(A),$$

where the minimal number of generators of the ideal I is denoted by $\#(I)$, and $e(A) = \dim_{\mathsf{k}} \mathfrak{m}_A/\mathfrak{m}_A^2$ is the embedded dimension of the germ.

Hereafter we shall often denote the A-modules $\mathrm{T}_i(A/\mathsf{k}, A)$ and $\mathrm{T}^i(A/\mathsf{k}, A)$ by $\mathrm{T}_i(A)$ and $\mathrm{T}^i(A)$, respectively. Consequently

$$\mathrm{T}_0(A) \cong \Omega^1_{A/\mathsf{k}}, \quad \mathrm{T}^0(A) \cong \mathrm{Der}_{\mathsf{k}}(A, A), \quad \mathrm{T}_1(A) \cong \mathrm{Ker}(D), \quad \mathrm{T}^1(A) \cong \mathrm{Coker}(D^*).$$

Recall also some well-known properties of cotangent homology of *complete intersections*. In this case, $A \cong P/I$, where $P = \mathsf{k}\langle z_1, \dots, z_m\rangle$, the ideal I is generated by a *regular* P-sequence $f_1, \dots, f_k$, and $\mathrm{T}_i(A) = \mathrm{T}^i(A) = 0$ for all $i \geqslant 2$. Next, if A is zero-dimensional then $m = k$, and the nontrivial cotangent homology and cohomology spaces have the same dimension over the ground field (see, e.g., [4, Theorem 1] or [5, Corollary 2.2]):

$$\dim_{\mathsf{k}} \mathrm{T}_i(A) = \dim_{\mathsf{k}} \mathrm{T}^j(A), \quad i, j = 0, 1.$$

The dimension of $\mathrm{T}^1(A)$ is usually called the Tjurina number of the singularity A and is denoted by $\tau(A)$. Hence,

$$\dim_{\mathsf{k}} \Omega^1_{A/\mathsf{k}} = \dim_{\mathsf{k}} \mathrm{T}_1(A) = \dim_{\mathsf{k}} \mathrm{Der}_{\mathsf{k}}(A) = \dim_{\mathsf{k}} \mathrm{T}^1(A) = \tau(A).$$

ASSERTION 4. *Let $A = P/I$ be a zero-dimensional singularity and $e(A) = 1$. Then*

$$\tau(A) = \mu(A).$$

where $\mu(A) = \dim_{\mathsf{k}}(A) - 1$ is the Milnor number of the singularity.

Proof. In the notation of Example 1, $A \cong P/(z_1^{k+1})P$ for some $k \geqslant 0$. Hence, it is an A_k-singularity and $\tau(A) = \mu(A) = k$. □

There is a useful representation of the first cotangent homology module $\mathrm{T}_1(A)$ in terms of elements of the ideal I. By definition, the *primitive* ideal $\int I \subset I$ consists of all the elements $g \in I$, such that $V(g) \in I$ for all $V \in \mathrm{Der}_{\mathsf{k}}(P)$. Some properties of the primitive ideal are given below (see [19, §2]):

(*i*) $I^2 \subseteq \int I \subseteq I$;
(*ii*) $\int(I_1 \cap I_2) = \int I_1 \cap \int I_2$ for any ideals $I_1, I_2 \subset P$;
(*iii*) if an ideal I is primary then $\int I$ is primary also.

PROPOSITION 1. *There exists an exact sequence of A-modules*

$$0 \to \textstyle\int I/I^2 \to I/I^2 \xrightarrow{D} \Omega^1_{P/\mathsf{k}} \otimes_P A \to \Omega^1_{A/\mathsf{k}} \to 0,$$

which implies the natural isomorphisms

$$\mathrm{Ker}(D) = \textstyle\int I/I^2 \cong \mathrm{T}_1(A).$$

Thus, the sequence $(\star)$ is left exact if and only if $\int I = I^2$; this is equivalent to the triviality of the first cotangent homology, that is, $\mathrm{T}_1(A) = 0$.

Proof. It follows from the definitions. □

EXAMPLE 3. Suppose that $I = (f)$, $f = \prod_{i=1}^{h} g_i^{\ell_i}$, $\ell_i \geq 1$, and $(g_i)P$ are *prime* ideals, g_i are mutually prime, $i = 1, \dots, h$. Then X is a hypersurface germ and the quotient module $\textstyle\int I/I^2$ is generated over $A = P/I$ by the function $F = \prod_{i=1}^{h} g_i^{\ell_i+1}$. That is, $\mathrm{T}_1(A)$ is a *cyclic* A-module. Moreover, in this case $\mathrm{T}_1(A) \neq 0$ if and only if there is at least one exponent ℓ_i greater than or equal to 2. In other words, if and only if the germ X is nonreduced and contains at least one *multiple* irreducible component.

PROPOSITION 2. *Suppose that the ideal I is generated by the regular sequence $z_1^{\ell_1}, \dots, z_m^{\ell_m}$, $\ell_i \geqslant 2$, $i = 1, \dots, m$. Then*

$$\dim_{\mathsf{k}} \mathrm{T}_1(A) = m\Pi(\ell) - \textstyle\sum_{i=1}^{m} \Pi(\ell)/\ell_i,$$

where $\Pi(\ell) = \ell_1 \cdots \ell_m$.

Proof. By definition, $\textstyle\int(x^a) = \langle x^{a+1}, x^{a+2}, \dots \rangle$. Hence, $\dim_{\mathsf{k}} \textstyle\int(x^a)/(x^{2a}) = a - 1$. Next,

$$\textstyle\int(x^a, y^b)/(x^a, y^b)^2 = \langle x^i y^j, x^k y^\ell \rangle_{ijk\ell}\,,$$

where $a + 1 \leqslant i \leqslant 2a - 1$, $0 \leqslant j \leqslant b - 1$, $0 \leqslant k \leqslant a - 1$, $b + 1 \leqslant \ell \leqslant 2b - 1$. Hence, $\dim_{\mathsf{k}} \textstyle\int(x^a, y^b)/(x^a, y^b)^2 = 2ab - a - b$, and so on. □

THEOREM 1 ([19, (4.1)]). *The first cotangent homology of a local algebra A is trivial if and only if it is reduced, that is, $A = A_{\mathrm{red}}$, and the primary decomposition of the ideal I^2 has no associated embedded prime components.*

It should be noted that a local algebra A, with $\textstyle\int I = I^2$ or $\mathrm{T}_1(A) = 0$, is often called an L-algebra (see [19, (1.1), (2.2)]). In particular, if A is a *reduced* complete intersection, then $\mathrm{T}_1(A) = 0$. By Proposition 1, this is equivalent to the relation $\textstyle\int I = I^2$ and the left exactness of sequence $(\star)$.

COROLLARY 1. *Let A be an Artinian* k*-algebra of dimension greater than one. Then* $\mathrm{T}_1(A) \neq 0$.

3. Quasihomogeneous singularities. Suppose that $A = P/I$ is a local k-algebra and the ideal $I \subset \mathfrak{m}_P$ is generated by a sequence $f_1, \dots, f_k$ of functions in variables $z_1, \dots, z_m$, which are *quasihomogeneous* of weighted degrees $d_1, \dots, d_k$ with respect to weights $w_1, \dots, w_m$. Then $\pi(A) = (d_1, \dots, d_k; w_1, \dots, w_m) \in \mathbb{Z}^k \times \mathbb{Z}^m$ is the *type* (of homogeneity) of the singularity.

In this case the modules Ω_A^p for all $p \geq 0$, as well as $\mathrm{T}_i(A)$ and $\mathrm{T}^i(A)$ for $i \geq 0$, are equipped with a *natural* $\mathbb{Z}$-grading in which $\deg(df_j) = d_j$, $j = 1, \dots, k$, and $\deg(dz_i) = w_i$, $i = 1, \dots, m$. The elements of the homogeneous component $\mathrm{T}^0(A)_\nu \cong \mathrm{Der}_{\mathsf{k}}(A)_\nu$ are called vector fields of *weight* ν. In particular, the weight of $\partial/\partial z_i$ is equal to $-w_i$, $i = 1, \dots, m$. The element $\mathcal{V}_0 = \sum_{i=1}^{m} w_i z_i \partial/\partial z_i \in \mathrm{Der}_{\mathsf{k}}(A)_0$ of *zero* weight is usually called the Euler vector field.

If the local k-algebra A is endowed with a *positive* grading, that is, $\pi(A) \in \mathbb{Z}_+^k \times \mathbb{Z}_+^m$, then A is often called a singularity with *effective* $\mathbb{C}^*$-action or a *weighted homogeneous* singularity.

Given a $\mathbb{Z}$-graded A-module $M = \bigoplus_{\nu \in \mathbb{Z}} M_\nu$ with *finite-dimensional* homogeneous components, the formal Laurent series

$$\mathscr{P}(M; t) = \sum_{\nu \in \mathbb{Z}} \dim_{\mathsf{k}}(M_\nu)\, t^\nu$$

is well defined; it is called the Poincaré series of M. If the sum is finite, then $\mathscr{P}(M;t)$ is usually called the Poincaré *polynomial*; in this case $\mathscr{P}(M;1) = \dim_{\mathsf{k}}(M)$.

ASSERTION 5. *Let A be a zero-dimensional singularity with effective $\mathbb{C}^*$-action and $I \subset \mathfrak{m}_P^2$. Then*

$$\tau(A) \geqslant \mu(A) + e(A) - 1.$$

Proof. The required statement is a direct consequence of the following general inequality valid for arbitrary zero-dimensional singularities:

$$\dim_{\mathsf{k}} \Omega^1_{A/\mathsf{k}} \geqslant \dim_{\mathsf{k}} A - \dim_{\mathsf{k}} \operatorname{Ker}(d) + e(A) - 1,$$

where $d\colon A \to \Omega^1_A$ is the universal derivation (see [4, Theorem 3], [6, 4.2] or [5, §2]). Indeed, A is a weighted homogeneous singularity and, consequently, $H^0_{\mathrm{DR}}(A) \cong \operatorname{Ker}(d) \cong \mathsf{k}$. □

COROLLARY 2. *Under the assumptions of Assertion 5, suppose that $e(A) \geqslant 2$. Then*

$$\tau(A) \geqslant \mu(A) + 1.$$

COROLLARY 3. *Under the same assumptions, the equality*

$$\tau(A) = \mu(A) + 1$$

holds if and only if $e(A) = 2$ and $A \cong P/(xy, x^p + y^q)$.

Proof. If $e(A) = 1$, then $\tau = \mu$ by Assertion 4. If $e(A) \geqslant 3$, then $\tau(A) \geqslant \mu(A)+2$. Suppose now that $e(A) = 2$, $A = P/I$ and the ideal I is generated by two functions $f(x,y)$ and $g(x,y)$. Under our assumptions, $H^0_{\mathrm{DR}}(A) \cong \mathsf{k}$ and the Poincaré (de Rham) complex, completed from the left:

$$0 \longrightarrow \mathsf{k} \longrightarrow A \xrightarrow{d} \Omega^1_A \xrightarrow{d} \Omega^2_A \longrightarrow 0,$$

is exact in view of [18]. Therefore,

$$1 - \dim_{\mathsf{k}} A + \dim_{\mathsf{k}} \Omega^1_A - \dim_{\mathsf{k}} \Omega^2_A = 0.$$

As a result, $\dim_{\mathsf{k}} \Omega^1_A = \mu(A) + \dim_{\mathsf{k}} \Omega^2_A$. On the other hand, $\Omega^2_A \cong A/(f'_x, f'_y, g'_x, g'_y)A$, so that $\dim_{\mathsf{k}} \Omega^2_A \geqslant 1$, and $\Omega^2_A \cong \mathsf{k}$ if and only if $(f'_x, f'_y, g'_x, g'_y) \cong \mathfrak{m}_A = (x,y)$. In order to complete the proof, it remains to examine the following two possibilities (see, e.g., [5, Theorem 4.3]):

1) $xy \,|\, f$ and $g(x,0) \neq 0,\ g(0,y) \neq 0$,
2) $x \,|\, f,\ y \,|\, g$ and $f(x,0) \neq 0,\ g(0,y) \neq 0$. □

REMARK 1. In the notation of [16, V, Tableau 4], the plane germs $(xy,\ x^p + y^q)$, denoted by $F^{p,q}_{p+q-1}$, are *simple* for all $p, q \geqslant 2$.

REMARK 2. It is well-known that for complete intersections with isolated singularities of *positive* dimension one has $\tau \leqslant \mu$, with equality $\tau = \mu$ if and only if the corresponding singularity is quasihomogeneous (see, e.g., [20]).

REMARK 3. In the course of the latter statements one may ask how to enumerate zero-dimensional weighted homogeneous complete intersections with $\tau = \mu + 2$, $\tau = \mu + 3$, and so on. A similar question seems to be more complicated for zero-dimensional complete intersections which are not weighted homogeneous. It is very probably that the difference $\varrho = \tau - \mu$ is, in fact, a highly suitable and useful invariant in the classification theory of singularities. Thus, for a bouquet of curves, which are *noncomplete* intersections, the invariant ϱ is called the *defect* of the corresponding one-dimensional singularity X; it is equal to the number of *nonvanishing* 1-cocycles of the de Rham complex of X, i.e., $\varrho = \dim_{\mathsf{k}} H^1_{\mathrm{DR}}(X)$ (see [1]).

4. Gradient singularities. Let F be an analytic function and X the corresponding hypersurface germ. Then the partial derivatives $f_i = \partial F/\partial z_i$, $i = 1, \dots, m$, generate the *Jacobian* ideal of F denoted by $\mathrm{Jac}(F)$. This ideal determines the singular locus $\mathrm{Sing}\, X \subset X$. The corresponding germ with a natural structure given by the Jacobian ideal, as well as its dual local (analytic) $\mathbb{C}$-algebra $A = P/\mathrm{Jac}(F)$ (if there is no ambiguity), is called the *gradient* singularity associated with function F and is denoted by $\mathrm{grad}(F)$ (cf. [10, 4.5, Example], [6, §5]).

If F has an *isolated* critical point at the origin then the partial derivatives $f_1, \dots, f_m$ form a *regular* P-sequence. In this case, the hypersurface germ X has an isolated singularity and $\mathrm{Sing}\, X$ is a *multiple* point (in other terms, a *fat* or *thick* point). In our terminology, it is a zero-dimensional gradient singularity. The dual local algebra A of $\mathrm{Sing}\, X$ is an Artinian complete intersection of finite dimension over the ground field and $\dim_{\mathsf{k}} A = \mu(A) + 1$.

Proposition 3. *Let F be a quasihomogeneous function with an isolated critical point. Then*

$$\tau(\mathrm{grad}(F)) \geqslant \mu(\mathrm{grad}(F)) + e(\mathrm{grad}(F)) - 1.$$

Proof. As seen in Section 3, Saito's normalized grading of F is positive. Hence, the singularity $\mathrm{grad}(F)$ is endowed with a positive grading as well. It remains to use the inequality from Assertion 5. □

Corollary 4. *Under the same assumptions, there are the following relations:*

$$\mu(\mathrm{grad}(F)) = \mu(F) - 1, \quad \tau(\mathrm{grad}(F)) \geqslant \tau(F) + e(\mathrm{grad}(F)) - 2.$$

Proof. The first equality follows from the definition. Next, in view of Proposition 3, one has $\tau(A) \geqslant \mu(A) + e(A) - 1 = \mu(F) + e(A) - 2 \geqslant \tau(F) + e(A) - 2$, where A is the local analytic algebra of the corresponding zero-dimensional gradient singularity. □

Proposition 4. *Let F be a function with an isolated critical point of type A_k, $k \geqslant 2$. Then $\mathrm{T}_1(\mathrm{grad}(F))$ is a cyclic module of length $k-1$ and $\tau(\mathrm{grad}(A_k)) = \mu(\mathrm{grad}(A_k)) = k-1$.*

Proof. Under our assumptions, we have $F = x^{k+1}$, $e(F) = 1$ and the local algebra of $\mathrm{grad}(F)$ is isomorphic to $\mathsf{k}\langle x\rangle/(x^k)\mathsf{k}\langle x\rangle$. Similarly to the proof of Proposition 2 and Example 3, it is easily to verify that

$$\mathrm{T}_1(\mathrm{grad}(F)) \cong \textstyle\int (x^k)/(x^{2k}) \cong \mathsf{k}\langle x^{k+1}, x^{k+2}, \cdots, x^{2k-1}\rangle = \langle 1, x, x^2, \cdots, x^{k-2}\rangle\, F.$$

Consequently, $\mathrm{T}_1(\mathrm{grad}(F))$ is a cyclic module of length $k-1$ generated by the function F. In addition, $\dim_{\mathsf{k}} \mathrm{T}_1(\mathrm{grad}(F)) = \mu(\mathrm{grad}(F)) = \mu(F) - 1 = \mu(A_k) - 1 = k-1$ (cf. Assertion 4 and Corollary 4). □

Corollary 5. *The module of derivations of a* $\mathrm{grad}(A_k)$*-singularity is cyclic and is generated by the Euler vector field.*

REMARK 4. It is well-known that the module of derivations of any quasihomogeneous complete intersection with an isolated singularity of *positive* dimension is generated by the Euler vector field modulo Hamiltonian (equivalently, trivial) derivations (see [3, §6]). In the zero-dimensional case there are no Hamiltonian derivations and Corollary 5 asserts that the module of derivations of a gradient A_k-singularity satisfies the same condition.

PROPOSITION 5. *Let F be a function with an isolated critical point of type D_k, $k \geqslant 4$. Then the first cotangent homology module* $\mathrm{T}_1(\mathrm{grad}(F))$ *has length k. In particular,* $\tau(\mathrm{grad}(D_k)) = \mu(\mathrm{grad}(D_k)) + 1 = k$.

Proof. Let us consider the normal form $F = x^{k-1} + xy^2$ of a D_k-singularity. An easy computation shows that the primitive ideal of $\mathrm{Jac}(F)$ is generated by the two functions, F and $y^3+3(k-1)x^{k-2}y$. As a result, the length of the module $\mathrm{T}_1(\mathrm{grad}(F))$ is equal to k. Hence, $\tau(\mathrm{grad}(F)) = \mu(\mathrm{grad}(F)) + 1 = \mu(F) = \mu(D_k) = \tau(D_k) = k$. □

EXERCISE 1. For the plane germs $(xy,\ x^p + y^q)$ from Corollary 3 the primitive ideal of $\mathrm{Jac}(F)$ is generated by the two polynomials $\varphi_1 = x(x^p + (p + 1)y^q)$ and $\varphi_2 = y(y^q + (q+1)x^p)$ for $p, q \geqslant 2$; the length of the first cotangent homology module is equal to $p + q$.

There is a series of natural dualities in the cotangent cohomology of zero-dimensional singularities which are very useful in computations. For convenience, the corresponding statements, adapted to the case of *gradient* quasihomogeneous singularities, are reproduced below.

THEOREM 2 ([4, Theorem 2]). *Let F be a quasihomogeneous function of weighted degree d with respect to weights $w_1, \dots, w_m$, which determines a zero-dimensional gradient singularity $A = \mathrm{grad}(F)$. Then there exist natural nondegenerate pairings*

$$\begin{array}{rcl}
\mathrm{T}_0(A) \times \mathrm{T}^0(A) & \longrightarrow & \mathsf{k}[-c], \\
\mathrm{T}_1(A) \times \mathrm{T}^1(A) & \longrightarrow & \mathsf{k}[-c], \\
\mathrm{T}_0(A) \times \mathrm{T}_1(A) & \longrightarrow & \mathsf{k}[-c-d], \\
\mathrm{T}^0(A) \times \mathrm{T}^1(A) & \longrightarrow & \mathsf{k}[-c+d],
\end{array}$$

where $c = md - 2\sum w_i$ is the weighted degree of the Hessian of the function F.

REMARK 5. It should be noted that Theorem 2 has been proved in [6, §7]. However, both exponents in the last two pairings were printed out in the cited work with misprints, which have been corrected somewhat later (see [8, Corollary 7.15]).

COROLLARY 6. *Under the same assumptions, there are the following relations:*

$$\begin{array}{rcl}
\mathscr{P}(\mathrm{T}_i(A);t) & = & t^c \mathscr{P}(\mathrm{T}^i(A);t^{-1}), \quad i = 0,1, \\
\mathscr{P}(\mathrm{T}_0(A);t) & = & t^{c+d} \mathscr{P}(\mathrm{T}_1(A);t^{-1}), \\
\mathscr{P}(\mathrm{T}^0(A);t) & = & t^{c-d} \mathscr{P}(\mathrm{T}^1(A);t^{-1}).
\end{array}$$

As a consequence, one obtains a series of useful relations for the Poincaré polynomials:

$$\begin{array}{rcl}
\mathscr{P}(\mathrm{Der}_{\mathsf{k}}(A);t) & = & t^{-d} \mathscr{P}(\mathrm{T}_1(A);t) \quad = \quad t^{c-d} \mathscr{P}(\mathrm{T}^1(A);t^{-1}) = t^c \mathscr{P}(\Omega^1_{A/\mathsf{k}};t^{-1}), \\
\mathscr{P}(\mathrm{T}^1(A);t) & = & t^{-d} \mathscr{P}(\mathrm{T}_0(A);t) \quad = \quad t^{-d}\, \mathscr{P}(\Omega^1_A;t), \\
\mathscr{P}(\mathrm{T}_1(A);t) & = & t^{d}\, \mathscr{P}(\mathrm{Der}_{\mathsf{k}}(A);t) \quad = \quad t^{c+d} \mathscr{P}(\Omega^1_A;t^{-1}),
\end{array}$$

and so on. Thus, if one knows the Poincaré polynomial of any cotangent module, then all others are obtained immediately from this one.

REMARK 6. In fact, natural perfect pairings between $\mathrm{T}_i(A)$ and $\mathrm{T}^i(A)$, $i \geqslant 0$, exist in the case of arbitrary zero-dimensional complete intersections of type $(d_1, \dots, d_m; w_1, \dots, w_m)$ with $c = \sum d_j - \sum w_i$. Moreover, there are similar pairings in the case of zero-dimensional Gorenstein singularities *without grading* (see [6, §3, Remark]). Moreover, in general, analogous pairings can be constructed in the case of compact complex spaces of *any dimension* (see [11]).

REMARK 7. For completeness, it should be mentioned also that one can develop in a similar manner the theory of duality for "almost" gradient zero-dimensional singularities. By definition (cf. [7, §4]), a singularity of such kind is defined by a regular sequence of holomorphic functions $f_1, \dots, f_m$, which satisfy the relations $\partial F/\partial z_i = \psi f_i$, $i = 1, \dots, m$, for some *meromorphic* (or rational) functions F and ψ (the so-called "integrating factor").

5. The inner modality and the inner modularity. We shall follow the exposition of the article [2], where basic notions, definitions, properties and useful references are discussed in full detail.

Let $f = (f_1, \dots, f_k)\colon \mathbb{C}^m \to \mathbb{C}^k$ be a flat map and let $X = f^{-1}(0)$ be a complete intersection with an isolated singularity. It is well-known that for quasihomogeneous hypersurface singularities (i. e., $f = f_1$) there is an isomorphism of degree $d = \deg(f)$ between the finite-dimensional vector spaces $\mathrm{T}^1(X) \cong \mathscr{O}_f$, where $\mathscr{O}_f = P/\mathrm{Jac}(f)$ is the Milnor algebra (equivalently, the moduli algebra) of the function germ $f = f_1$ at its critical point. More precisely, $\mathrm{T}^1(X)_{\nu-d} \cong (\mathscr{O}_f)_\nu$ for any $\nu \in \mathbb{Z}$ or, equivalently, $\mathrm{T}^1(X)(-d) \cong \mathscr{O}_f$ in the standard notation for shifting the grading. In general, the vector space $\mathrm{T}^1(X)$ can be generated by vector-columns of height $k \geqslant 1$ containing the only nonzero monomial entry. The corresponding basis of $\mathrm{T}^1(X)$ is called a *monomial vector-basis* or, shortly, a monomial basis. The natural grading of $\mathrm{T}^1(X)$ is defined in such a way that the vector-column $(1, 0, \dots, 0)^{tr}$ has weight $-d_1$, the vector-column $(0, 1, 0, \dots, 0)^{tr}$ has weight $-d_2$, and so on. Similarly to [10, §7], an element from $\mathrm{T}^1(X)$ is called a lower, diagonal or upper vector-monomial if it belongs to $\sum_{\nu<0} \mathrm{T}^1(X)_\nu$, $\mathrm{T}^1(X)_0$ or $\sum_{\nu>0} \mathrm{T}^1(X)_\nu$, respectively (see [2, 2.9]). Of course, the number of elements of each type does not depend on the choice of monomial basis of the module $\mathrm{T}^1(X)$.

DEFINITION 1. The *inner* (or intrinsic) *modality* of a quasihomogeneous isolated complete intersection singularity of positive dimension is defined to be $m_0(X) = \sum_{\nu \geqslant 0} \dim_{\mathsf{k}} \mathrm{T}^1(X)_\nu$; it is equal to the total number of diagonal and upper vector-monomials in some monomial basis of $\mathrm{T}^1(X)$. The number of *diagonal* vector-monomials is denoted by $\wp_0(X)$, that is, $\wp_0(X) = \dim_{\mathsf{k}} \mathrm{T}^1(X)_0$. By analogy, we shall call $\wp_0$ the *inner modularity.* Evidently, $m_0(X) \geqslant \wp_0(X)$.

Recall that the grading of complete intersections with isolated singularities of *positive* dimension is defined *uniquely*, except for the case of hypersurfaces of multiplicity two (see [3, (6.4)]). Moreover, such germs are *positively* graded (in the exceptional case one should use Saito's *normalized* grading), the dimensions of graded components $\mathrm{T}^1(X)_\nu$, $\nu \in \mathbb{Z}$, and, consequently, the numbers $m_0(X)$ and $\wp_0(X)$, are integer analytic invariants of X, etc.

In the *zero-dimensional* case we have a completely different situation because there may exist an infinite number of various gradings.

The maximal weight of *nontrivial* weighted components of the first cotangent cohomology module is denoted by

$$\mathfrak{a}_\pi = \sup\{\nu \in \mathbb{Z} : \mathrm{T}^1(X)_\nu \neq 0\}$$

(cf. [16, II]). Let $\#(\pi)$ be the total number of elements of a monomial basis whose weights are *non-negative* in the grading π, and let σ_π be the sum of the corresponding non-negative weights. For convenience of notation, we set $\sigma_\pi = -1$, if $\mathrm{T}^1(X)_\nu = 0$ for all $\nu \geqslant 0$.

Definition 2. A grading π of a zero-dimensional singularity is called *minimal* if the couple of integers $(\#(\pi),\ \sigma_\pi)$ are *minimal* possible (in the standard lexicographic order); it is denoted by $\pi_{\min}$. In such grading the inner modality and inner modularity are minimal possible as well.

Example 4. In general, a minimal grading is *not unique.* For instance, let us assume that X is determined by the ideal (x^2, y^3); it is a G_5-singularity from the list [16, V, Tableau 4] with $\mu = 5$ and $\tau = 7$. In this case, there is an infinite number of distinct gradings $\pi = (2a, 3b; a, b)$, where $a, b \in \mathbb{Z}$. The vector space $\mathrm{T}^1(X)$ is generated by the seven vector-monomials:

$$\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}y\\0\end{pmatrix}, \begin{pmatrix}y^2\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}0\\x\end{pmatrix}, \begin{pmatrix}0\\y\end{pmatrix}, \begin{pmatrix}0\\xy\end{pmatrix},$$

with weights $-2a$, $b-2a$, $2b-2a$, $-3b$, $a-3b$, $-2b$, $a-2b$, respectively. In particular, taking into account the third and the seventh vector-monomials, we see that $\mathfrak{a}_\pi$ is *negative* if one takes $b < a < 2b$. Thus, for $a = 3$ and $b = 2$, the monomial basis consists of elements with weights

$$-6,\ -4,\ -2,\ -6,\ -3,\ -4,\ -1.$$

That is, $\pi_{\min} = (6, 6; 3, 2)$, $\mathfrak{a}_{\pi_{\min}} = -1$ and $(\#(\pi_{\min}),\ \sigma_{\pi_{\min}}) = (0, -1)$, hence $m_0(X) = 0$. Similarly, one can choose another minimal grading: set $a = 7$ and $b = 5$, then the monomial basis consists of elements with the following weights

$$-14,\ -9,\ -4,\ -15,\ -8,\ -10,\ -3,$$

that is, $\pi = (14, 15; 7, 5)$, $\mathfrak{a}_\pi = -3$ and $(\#(\pi),\ \sigma_\pi) = (0, -1)$ as before. Anyway, by definition, the inner modality is equal to zero. On the other hand, X has contact modality 0; it is a *simple* zero-dimensional singularity.

Remark 8. If X has inner modality 0, then the space $\mathrm{T}^1(X)$ can be endowed with a *negative* (minimal) grading. It is well-known that singularities of such kind have a nice deformation theory. Next, isolated complete intersection singularities of *positive* dimension with inner modality 0 are *simple.* However, the zero-dimensional case is different from that. For instance, the *nongradient* singularity X, defined by the ideal $I = (x^2 + yz, z^2 + y^3, xy)$, has type $\pi(X) = (10, 12, 9; 5, 4, 6)$, $\mu = 8$ and $\tau = 11$. The vector space $\mathrm{T}^1(X)$ is generated by the eleven vector-monomials:

$$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}x\\0\\0\end{pmatrix}, \begin{pmatrix}z\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\x\\0\end{pmatrix}, \begin{pmatrix}0\\y\\0\end{pmatrix}, \begin{pmatrix}0\\y^2\\0\end{pmatrix}, \begin{pmatrix}0\\yz\\0\end{pmatrix}, \begin{pmatrix}0\\z\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\z\end{pmatrix},$$

whose weights are $-10, -5, -4, -12, -7, -8, -4, -2, -6, -9, -3$, respectively. In fact, it is a *minimal* grading and the inner modality of X is equal to 0. Nevertheless, the *contact modality* of X is equal to 1 in view of [14, Table 3]. In particular, the singularity X is not simple.

The *modularity* $\wp(X)$ is defined to be the dimension of the *maximal modular* germ M; it is contained in the base space of a minimal versal deformation of X

(see [17]). For isolated singularities M_{red} coincides with the support of the stratum $\{\tau_{\max} = \mathrm{const}\}$. In the quasihomogeneous case there is a canonical embedding of the tangent space at the distinguished point $\mathrm{T}_{\{\mathfrak{o}\}}(M_{\mathrm{red}})$ into $\mathrm{T}^1(X)_0$. In particular, $\wp(X) \leqslant \wp_0(X)$ (see [1, Proposition 3] or [9, Proposition 2.2]). Moreover, for isolated quasihomogeneous complete intersections of *positive* dimension the maximal modular germ is *reduced* and *smooth*, $M = M_{\mathrm{red}}$, and $\wp(X) = \wp_0(X)$ (see [2], [3]).

PROPOSITION 6. *Let F be a quasihomogeneous function with an isolated critical point. Then there are the following relations:*

$$m_0(\mathrm{grad}(F)) \geqslant m_0(F),\ \wp_0(\mathrm{grad}(F)) \geqslant \wp_0(F) = \wp(F),\ \wp_0(\mathrm{grad}(F)) \geqslant \wp(\mathrm{grad}(F)).$$

Proof. Let $\mathbf{m}$ be a diagonal or an upper monomial which determines a nontrivial deformation of the hypersurface germ given by F. Then at least one partial derivative of $\mathbf{m}$ does not vanish. Taking vector-columns whose nonzero entries are equal to these derivatives, one gets at least one nontrivial deformation of the singularity $\mathrm{grad}(F)$, and so on (cf. [2, 2.6]). □

In contrast with graded complete intersections of *positive* dimension, in the zero-dimensional case the structure of the maximal modular germ M may be, in general, very complicated. In particular, for a quasihomogeneous zero-dimensional complete intersection X the modularity $\wp(X)$ is usually not equal to $\wp_0(X)$. Moreover, M may be either smooth, or singular, or nonreduced. In fact, similar properties are typical in the case of quasihomogeneous *noncomplete* intersections of positive dimension (cf. [1]), *semi*-quasihomogeneous isolated hypersurface singularities (see [17]), etc.

This phenomenon can be explained by a highly nontrivial structure of the module of vector fields of zero-dimensional germs. More precisely, for a quasihomogeneous isolated complete intersection of positive dimension the module of derivations of its structure algebra is generated by the Euler vector and Hamiltonians (the so-called *trivial* vector fields [3]). However, in the zero-dimensional case there are no trivial vector fields (see [5, Proposition 4.2]) and the module of derivations is generated by the Euler vector field and *noncollinear* (nontrivial) ones (excluding the case of $\mathrm{grad}(A_k)$-singularities in view of Corollary 5). The latter may produce nontrivial obstructions for their prolongation to the total space of a minimal versal deformation of X (cf. [1], [9]).

6. Gradient singularities with modularity zero. In the following table the first two columns contain the standard notation of classes (or genotypes) of hypersurface singularities, their equations and types of homogeneity. For families of germs we also write down conditions under which such a family is nondegenerate in the usual sense. The equations and types of the corresponding *gradient* singularities are given in the third column. In the fourth column of the table we write down monomial vector-bases of the first cotangent space $\mathrm{T}^1(\mathrm{grad}(F))$, nonzero entries of the corresponding monomial vectors, and weights of all generators in a minimal grading. The symbol $[i]$ refers to the i-th nonzero entry of monomial vectors, all other entries are zero.

In the last three columns of our table we collect information on some useful invariants of gradient singularities. The column $\tau|\mu$ contains couples of Tjurina and Milnor numbers. The notation $m_0|\wp_0$ has a similar meaning for the inner modality and the inner modularity. If $m_0 = \wp_0$ then the only integer is pointed out, etc.

THEOREM 3. *Zero-dimensional quasihomogeneous gradient singularities with modularity $\wp = 0$ and with Milnor number not exceeded 12, $\mu \leqslant 12$, are exhausted up to $\mathscr{K}$-equivalence by the following list:*

class	*equation of* F $\pi(F)$	*ideal of* $\mathrm{grad}(F)$ $\pi_{\min}(\mathrm{grad}(F))$	T^1*–basis* *weights*	$\tau\|\mu$	$m_o\|\wp_o$	$\mathscr{P}(\mathrm{T}^1(\mathrm{grad}(F));t)$
A_k $k\geqslant 2$	x^{k+1} $(k+1;1)$	x^k $(k;1)$	$1,x,\dots,x^{k-2}$ $-k,-k+1,\dots,-2$	$k-1$	0	$t^{-2}+\ldots+t^{-k}$
D_k $k\geqslant 4$	$x^{k-1}+xy^2$ $(2(k-1);2,k-2)$	$(x^{k-2}+y^2,xy)$ $(2(k-2),k;2,k-2)$	$[1]{:}1,x,\dots,x^{k-3};\ [2]{:}1,x$ $-2k+4,-2k+6,\dots,-2;-k,2-k$	$k\|\,k-1$	0	$t^{-2}+t^{-4}+\ldots$ $+\,t^{-2k+4}+t^{-k}+t^{2-k}$
E_6	x^3+y^4 $(12;4,3)$	(x^2,y^3) $(8,9;4,3)$	$[1]{:}1,y,y^2;\ [2]{:}1,x,xy,y$ $-8,-5,-2;-9,-5,-2,-6$	$7\|5$	0	$2t^{-2}+2t^{-5}\ldots+t^{-9}$
E_7	x^3+xy^3 $(9;3,2)$	(x^2+y^3,xy^2) $(6,7;3,2)$	$[1]{:}1,x,y,xy,y^2;\ [2]{:}1,x,y$ $-6,-3,-4,-1,-2;\ -7,-4,-5$	$8\|6$	0	$t^{-1}+t^{-2}+\ldots+t^{-7}$
E_8	x^3+y^5 $(15;5,3)$	(x^2,y^4) $(10,12;5,3)$	$[1]{:}1,y,y^2,y^3;\ [2]{:}1,x,xy,xy^2,y,y^2$ $-10,-7,-4,-1;\ -12,-7,-4,-1,-9,-6$	$10\|7$	0	$2t^{-1}+2t^{-4}+\ldots+t^{-12}$
X_9^0	x^4+y^4 $(4;1,1)$	(x^3,y^3) $(9,12;3,4)$	$[1]{:}1,x,y,xy,y^2,xy^2;$ $[2]{:}1,x,y,x^2,xy,x^2y$ $-9,-6,-5,-2,-1,2;$ $-12,-9,-8,-6,-5,-2$	$12\|8$	$1\|0$	$t^2+t^{-1}+\ldots+t^{-12}$
J_{10}^0	x^3+y^6 $(6;2,1)$	(x^2,y^5) $(10,10;5,2)$	$[1]{:}1,y,y^2,y^3,y^4;$ $[2]{:}1,x,xy,xy^2,xy^3,y,y^2,y^3$ $-10,-8,-6,-4,-2;$ $-10,-5,-3,-1,1,-8,-6,-4$	$13\|9$	$1\|0$	$t+t^{-1}+\ldots+2t^{-10}$
Q_{10}	$x^3+y^4+yz^2$ $(24;8,6,9)$	(x^2,y^3+z^2,yz) $(16,18,15;8,6,9)$	$[1]{:}1,y,y^2,z,z^2;[2]{:}1,x,xy,$ $xy^2,xz,y,y^2,z;[3]{:}1,x$ $-16,-10,-4,-7,2;-18,-10,-4,$ $2,-1,-12,-6,-9;-15,-7$	$15\|9$	$2\|0$	$2t^2+t^{-1}+\ldots+t^{-18}$
Z_{11}	x^3y+y^5 $(15;4,3)$	(x^2y,x^3+y^4) $(11,12;4,3)$	$[1]{:}1,x,y,y^2,y^3;[2]{:}1,x,$ $x^2,xy,xy^2,xy^3,y,y^2,y^3$ $-11,-7,-8,-5,-2;-12,-8,$ $-4,-5,-2,1,-9,-6,-3$	$14\|10$	$1\|0$	$t+2t^{-2}+\ldots+t^{-12}$
Q_{11}	$x^3+yz^2+xy^3$ $(18;6,4,7)$	(x^2+y^3,z^2+xy^2,yz) $(12,14,11;6,4,7)$	$[1]{:}1,y,y^2,z;[2]{:}1,x,xy,xz,$ $y,y^2,y^3,y^4,z;[3]{:}1,x$ $-12,-8,-4,-5;-14,-8,-4,-1,$ $-10,-6,-2,2,-7;-11,-5$	$15\|10$	$1\|0$	$t^2+t^{-1}+\ldots+t^{-14}$
S_{11}	$x^2z+yz^2+y^4$ $(16;5,4,6)$	(xz,z^2+y^3,x^2+yz) $(11,12,10;5,4,6)$	$[1]{:}1,x,xy,y,y^2,z;$ $[2]{:}1,x,xy,y,y^2,y^2z,yz,z;[3]{:}1$ $-11,-6,-2,-7,-3,-5;$ $-12,-7,-3,-8,-4,2,-2,-6;-10$	$15\|10$	$1\|0$	$t^2+2t^{-2}+\ldots+t^{-12}$
E_{12}	x^3+y^7 $(21;7,3)$	(x^2,y^6) $(6,6;3,1)$	$[1]{:}1,y,y^2,y^3,y^4,y^5;\ [2]{:}1,x,$ $xy,xy^2,xy^3,xy^4,y,y^2,y^3,y^4$ $-6,-5,-4,-3,-2,-1;\ -6,-3,$ $-2,-1,0,1,-5,-4,-3,-2$	$16\|11$	$2\|1$	$t+1+2t^{-1}+\ldots+2t^{-6}$
Z_{12}	x^3y+xy^4 $(11;3,2)$	(x^2y+y^4,x^3+12xy^3) $(8,9;3,2)$	$[1]{:}1,x,y,y^2,y^3;\ [2]{:}1,x,x^2,$ $xy,xy^2,xy^3,y,y^2,y^3,y^4,y^5$ $-8,-5,-6,-4,-2;-9,-6,-3,$ $-4,-2,0,-7,-5,-3,-1,1$	$16\|11$	$2\|1$	$t+1+t^{-1}+\ldots+t^{-9}$
W_{12}	x^4+y^5 $(20;5,4)$	(x^3,y^4) $(15,16;5,4)$	$[1]{:}1,x,xy,xy^2,xy^3,y,y^2,y^3;$ $[2]{:}1,x,x^2,x^2y,x^2y^2,xy,xy^2,y,y^2$ $-15,-10,-6,-2,2,-11,-7,-3;$ $-16,-11,-6,-2,2,-7,-3,-12,-8$	$17\|11$	$2\|0$	$2t^2+2t^{-2}+\ldots+t^{-16}$
S_{12}	$x^2z+yz^2+xy^3$ $(13;4,3,5)$	$(xz+y^3,z^2+12xy^2,x^2+yz)$ $(9,10,8;4,3,5)$	$[1]{:}1,x,xy,y,y^2,z;\ [2]{:}1,x,xy,$ $xyz,xz,y,y^2,yz,z,z^2;\ [3]{:}1$ $-9,-5,-2,-6,-3,-4;-10,-6,-3,$ $2,-1,-7,-4,-2,-5,0;\ -8$	$17\|11$	$2\|1$	$t^2+1+t^{-1}+\ldots+t^{-10}$
Q_{12}	$x^3+yz^2+y^5$ $(15;5,3,6)$	(x^2,y^4+z^2,yz) $(10,12,9;5,3,6)$	$[1]{:}1,y,y^2,y^3,z,z^2;\ [2]{:}1,x,xy,$ $xy^2,xy^3,xz,y,y^2,y^3,z;\ [3]{:}1,x$ $-10,-7,-4,-1,-4,2;\ -12,-7,-4,$ $-1,2,-1,-9,-6,-3,-6;-9,-4$	$18\|11$	$2\|0$	$2t^2+3t^{-1}+\ldots+t^{-12}$
U_{12}	$x^3+y^3+z^4$ $(12;4,4,3)$	(x^2,y^2,z^3) $(8,8,9;4,4,3)$	$[1]{:}1,y,yz,yz^2,z,z^2;\ [2]{:}1,x,xz,xz^2,$ $z,z^2;\ [3]{:}1,x,y,z,xy,xz,yz,xyz$ $-8,-4,-1,2,-5,-2;\ -8,-4,-1,2,$ $-5,-2;-9,-5,-5,-6,-1,-2,-2,2$	$20\|11$	$3\|0$	$3t^2+3t^{-1}+\ldots+t^{-9}$
E_{13}	x^3+xy^5 $(15;5,2)$	(x^2+y^5,xy^4) $(10,13;5,2)$	$[1]{:}1,y,y^2,y^3;[2]1,x,xy,xy^2,$ $xy^3,y,y^2,y^3,y^4,y^5,y^6,y^7$ $-10,-8,-6,-4;-13,-8,-6,-4,$ $-2,-11,-9,-7,-5,-3,-1,1$	$16\|12$	$1\|0$	$t+t^{-1}+\ldots+t^{-13}$
Z_{13}	x^3y+y^6 $(18;5,3)$	(x^2y,x^3+y^5) $(13,15;5,3)$	$[1]{:}1,x,y,y^2,y^3,y^4;[2]{:}1,x,$ $x^2,xy,xy^2,xy^3,xy^4,y,y^2,y^3,y^4$ $-13,-8,-10,-7,-4,-1;-15,-10,$ $-5,-7,-4,-1,2,-12,-9,-6,-3$	$17\|12$	$1\|0$	$t^2+2t^{-1}+\ldots+t^{-15}$
W_{13}	x^4+xy^4 $(16;4,3)$	(x^3+y^4,xy^3) $(12,13;4,3)$	$[1]{:}1,x,y,x^2,xy,y^2,x^2y,xy^2,y^3,x^2y^2;$ $[2]{:}1,x,y,x^2,xy,y^2,xy^2$ $-12,-8,-9,-4,-5,-6,-1,-2,-3,2;$ $-13,-9,-10,-5,-6,-7,-3$	$17\|12$	$1\|0$	$t^2+t^{-1}+\ldots+t^{-13}$

Proof. It is enough to analyze all quasihomogeneous isolated hypersurface singularities with Milnor number not greater than 13. One may use, for example, a list of normal forms presented in [10]. The genotypes K_{12} and K_{13} from this list we denote here by E_{12} and E_{13}, respectively. Then we transform the Jacobian ideals for each class of hypersurfaces to the *normal* form given in the third column of the list. The case of $\mathrm{grad}(A_k)$-singularity is trivial. The next four genotypes $\mathrm{grad}(D_k)$, $\mathrm{grad}(E_6)$, $\mathrm{grad}(E_7)$, $\mathrm{grad}(E_8)$ correspond to the simple singularities $F_{k-1}^{k-2,2}$, G_5, H_6, G_7 from the list [16, V, Tableau 4], respectively.

It should be underlined that for $\mathrm{grad}(E_6)$ and $\mathrm{grad}(E_8)$ singularities there are *various* gradings so that the Poincaré series of the vector space T^1 are also different. More precisely, the homogeneity type of $\mathrm{grad}(E_6)$ is equal to $(2a,3b;a,b)$, where $a,b\in\mathbb{Z}$ (see Example 4). That is why we choose a minimal grading. In a similar way, one can examine the classes $\mathrm{grad}(E_8)$, $\mathrm{grad}(X_9^0)$, $\mathrm{grad}(J_{10}^0)$, and others.

The gradient singularities associated with Q_{10}, Z_{11}, Q_{11}, S_{11}, E_{12}, Z_{12}, W_{12}, S_{12}, Q_{12}, U_{12}, E_{13}, Z_{13} and W_{13} are analyzed by easy computations as in Example 4. It is clear also that the grading for these singularities (with exception of Q_{10}, Q_{12}, E_{12}, W_{12} and U_{12}) is defined *uniquely.* □

REMARK 9. The following comments are very useful for further computations. In fact, any normal form is represented by a quasihomogeneous polynomial of type π whose support consists of monomials of equal weights. However, in order to describe *all* possible zero-dimensional gradient singularities corresponding to the type π, one must analyze all subsets of the *complete* set of monomials with given weighted degree. For example, let us consider the normal form of a Q_{14}-singularity from the list [10, 11.4], that is, the polynomial $F_0 = x^3 + yz^2 + y^6 + ax^2y^2$, where $4a^3 + 27 \neq 0$; its type is equal to $(12; 4, 2, 5)$. It is clear that the complete set of monomials of degree 12 contains also xy^4. Set $F_1 = x^3 + yz^2 + y^6$ and $F_2 = x^3 + yz^2 + y^6 + xy^4$. A direct computation shows that $\mu(\mathrm{grad}(F_1)) = \mu(\mathrm{grad}(F_2)) = 13$, $\tau(\mathrm{grad}(F_1)) = 21$, but $\tau(\mathrm{grad}(F_2)) = 19$. As a result, we conclude that there are *two different* zero-dimensional gradient singularities with the same type $(12; 4, 2, 5)$.

EXERCISE 2. One may ask: how to describe all deformations of a gradient singularity which are contact equivalent (or analytically isomorphic) to gradient ones? In other terms, how to compute the *gradient locus* which is contained in the base space of a (minimal) versal deformation of the singularity.

COROLLARY 7. *Gradient zero-dimensional singularities with inner modality 0 have the same contact modality.*

Proof. Using the list from [16, V, Tableau 4], one can verify that all singularities with $m_0 = 0$ have *contact* modality zero, $m = 0$. In the case $m_0 = 1$, we see that for $\mathrm{grad}(S_{11})$ and $\mathrm{grad}(W_{13})$ singularities one gets $m \geqslant 2$ (cf. the tables from [13] in the case of two variables and [14, Proposition 2.3] in the case of three variables). Next, the unimodular families P_8, X_9 and J_{10} produce also the three gradient singularities denoted by $\mathrm{grad}(P_8^0)$, $\mathrm{grad}(X_9^0)$ and $\mathrm{grad}(J_{10}^0)$ with modularity *zero*, $\wp = 0$. In these cases there exist an infinite number of various gradings, similarly to the case of a $\mathrm{grad}(E_6)$-singularity, which has been analyzed in Example 4. □

EXERCISE 3. Using tables from [13], [14], it is not difficult to complete the above list with *nongradient* singularities up to $\tau \leqslant 11$.

EXAMPLE 5. It should be also remarked that sometimes the grading of $\mathrm{grad}(F)$, *induced* by the natural grading of F, is *not* minimal. Indeed, the class $\mathrm{grad}(X_9^0)$ is defined by the polynomial $F = x^4 + y^4$, we see that $\pi(F) = (4; 1, 1)$. On the other hand, $\mathrm{grad}(F) = (x^3, y^3)$, that is, $\pi(\mathrm{grad}(F)) = (3, 3; 1, 1)$. And there exist two basis vector-monomials of non-negative (more precisely, zero) weights in this grading, that is, $\#(\pi) = 2$, $\wp_0 = 2$ and $m_0 = 2$. However, a minimal grading is, in fact, $\pi_{\min}(\mathrm{grad}(F)) = (9, 12; 3, 4)$, so that, in this grading one has $\wp_0 = 0$ and $m_0 = 1$.

For completeness, let us compute the primitive ideals and the first cotangent homology modules for singularities from the list of Theorem 3. For $\mathrm{grad}(A_k)$ and $\mathrm{grad}(D_k)$ the corresponding primitive ideals are described in Proposition 4 and Proposition 5, respectively. The cases $\mathrm{grad}(E_6)$, $\mathrm{grad}(E_8)$, $\mathrm{grad}(X_9^0)$, $\mathrm{grad}(J_{10}^0)$, $\mathrm{grad}(E_{12})$, $\mathrm{grad}(W_{12})$ and $\mathrm{grad}(U_{12})$ are examined in Proposition 2. Next, at first remark that in the case where F is a *quasihomogeneous* polynomial the primitive ideal of $\mathrm{Jac}(F)$

contains at least one generator, namely, the function F. We shall denote it by φ_0. Let us write down other generators, φ_i, $i = 1, 2, \dots$, for the remaining types of the gradient singularities from the list of Theorem 3.

Further computations show that for a grad(E_7)-singularity we obtain 2 generators of the primitive ideal and $\varphi_1 = x^2y^2 + \frac{2}{15}y^5$; in the case of grad($Z_{11}$) there is a system of 3 generators containing $\varphi_1 = y^6$ and $\varphi_2 = x^5 + \frac{25}{2}x^2y^4$. In the cases of grad($Z_{12}$) and grad($E_{13}$) we get systems of 2 generators involving $\varphi_1 = x^4 + 8x^2y^3 + \frac{4}{3}y^6$ and $\varphi_1 = x^2y^4 + \frac{4}{27}y^9$, respectively. For grad(Z_{13}) and grad(W_{13}) we find 3 generators, $\varphi_1 = y^7$, $\varphi_2 = x^5 + 15x^2y^5$, and $\varphi_1 = x^5$, $\varphi_2 = x^3y^3 + \frac{3}{28}y^7$, respectively.

Next, for grad(S_{11}) there is a more complicated system of 4 generators: $\varphi_1 = y^5$, $\varphi_2 = xyz^2$, $\varphi_3 = x^3y^2 + 2xy^3z - xz^3$; for grad($S_{12}$) there is a system of 3 generators: $\varphi_1 = x^3y + y^5 + 6xy^2z + \frac{2}{3}z^3$, $\varphi_2 = x^3z - y^4z$. Finally, for grad(Q_{12}) there are 3 generators of the corresponding primitive ideal also, namely, $\widetilde{\varphi_0} = y^5 + yz^2$, $\varphi_1 = x^3$, $\varphi_2 = y^4z + \frac{1}{15}z^3$, so that $\varphi_0 = \widetilde{\varphi_0} + \varphi_1$.

Remark 10. It is a very useful exercise to compute the modules of derivations of all singularities from the list of Theorem 3, taking into account the dualities of Theorem 2. On the other hand, in the case of gradient zero-dimensional singularities associated with *simple* hypersurface singularities the corresponding Lie algebras have been already investigated in detail three decades ago (see [12], [21]). Further results in this direction and interesting applications one can find also in [15].

It should be mentioned the following simple properties of zero-dimensional *semi*-quasihomogeneous complete intersections.

Proposition 7. *Assume that X_0 is a complete intersection with an isolated singularity determined by a flat quasihomogeneous map $F_0 = (f_1, \dots, f_k)\colon \mathbb{C}^m \to \mathbb{C}^k$, $1 \leqslant k \leqslant m$. Let X be the semi-quasihomogeneous germ defined by the function $F = F_0 + \sum c_i e_i$, where $c_i \in \mathbb{C}$ and $e_i \in \sum_{\nu>0} \mathrm{T}^1(X_0)_\nu$. Then*

$$\mu(X_0) = \mu(X), \quad \tau(X_0) > \tau(X).$$

Proof. The proof goes similarly to the well-known line of reasoning for complete intersections of positive dimension. □

Remark 11. Using standard considerations, one may complete the list from Theorem 3 with the corresponding classes of gradient *semi*-quasihomogeneous singularities. It is also clear, that an adjacency of two hypersurface singularities induces at least one adjacency of the corresponding gradient singularities.

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