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Thom-Sebastiani properties of Kohn-Rossi cohomology of compact connected strongly pseudoconvex CR manifolds

In memory of Professor LU QiKeng (1927–2015)

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Abstract Let X_1 and X_2 be two compact connected strongly pseudoconvex embeddable Cauchy-Riemann (CR) manifolds of dimensions $2m-1$ and $2n-1$ in $\mathbb{C}^{m+1}$ and $\mathbb{C}^{n+1}$, respectively. We introduce the Thom-Sebastiani sum $X = X_1 \oplus X_2$ which is a new compact connected strongly pseudoconvex embeddable CR manifold of dimension $2m+2n+1$ in $\mathbb{C}^{m+n+2}$. Thus the set of all codimension 3 strongly pseudoconvex compact connected CR manifolds in $\mathbb{C}^{n+1}$ for all $n \geqslant 2$ forms a semigroup. X is said to be an irreducible element in this semigroup if X cannot be written in the form $X_1 \oplus X_2$. It is a natural question to determine when X is an irreducible CR manifold. We use Kohn-Rossi cohomology groups to give a necessary condition of the above question. Explicitly, we show that if $X = X_1 \oplus X_2$, then the Kohn-Rossi cohomology of the X is the product of those Kohn-Rossi cohomology coming from X_1 and X_2 provided that X_2 admits a transversal holomorphic S^1-action.

Keywords CR manifold, Kohn-Rossi cohomology, isolated singularity

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1 Introduction

One of the natural fundamental questions of complex geometry is to study the boundaries of complex varieties. For example, the famous classical complex Plateau problem asks which odd-dimensional real sub-manifolds of $\mathbb{C}^N$ are boundaries of complex sub-manifolds in $\mathbb{C}^N$. In their beautiful seminal paper, Harvey and Lawson [5, 6] proved the following theorem.

Theorem 1.1 (Harvey-Lawson). *Let X be a compact connected strongly pseudoconvex embeddable CR manifold. Then there exists a unique complex variety V in $\mathbb{C}^N$ for some N such that the boundary $\partial V = X$ and V has only normal isolated singularities.*

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The above theorem is one of the deepest theorems in complex geometry. It relates the theory of strongly pseudoconvex CR manifolds on the one hand and the theory of isolated normal singularities on the other hand (see [10, 15]).

CR manifolds in $\mathbb{C}^N$ which bound varieties with isolated singularities behave quite differently to those CR manifolds in $\mathbb{C}^N$ which bound Stein manifolds. Let D and B^n be a domain and ball in $\mathbb{C}^n$, respectively. The CR manifold $X = \partial D$ is said to have spherical property if for each point $p \in X$, there is a biholomorphic map $f : U \to V$ such that $f(U \cap X) \subset V \cap \partial B^n$ where U and V are open neighborhoods of p and $f(p)$, respectively. It is well known in [2] that D is a simply connected bounded domain in $\mathbb{C}^n$ with spherical real analytic boundary $X = \partial D$, then every local biholomorphic map at boundary as above extends to a biholomorphic map from D to B^n. As a consequence, a local biholomorphic map between $X_1 = \partial D_1$ and $X_2 = \partial D_2$ where D_1, D_2 are simply connected domains in $\mathbb{C}^n$ with spherical real analytic boundaries can extend to a global biholomorphic map from D_1 onto D_2. In [7], Ji et al. showed that the above phenomenon is no longer true if the CR manifolds bound varieties with isolated singularities.

Therefore, it is of great interest to study the interior regularity of V, the Harvey-Lawson solution of complex Plateau problem. For this purpose one has to study CR-invariants. It seems to us that the first fundamental invariant of this kind was introduced by Kohn and Rossi [8], the so-called Kohn-Rossi $\overline{\partial}_b$-cohomology groups $H^{p,q}_{KR}(X)$ (for definition see Section 2). They proved the finite dimensionality of their cohomology groups under certain natural conditions. Of course it would be of interest to compute the dimension of these $\overline{\partial}_b$-cohomology groups. In general, a strongly pseudoconvex manifold M is a modification of a Stein space V with isolated singularities. In [8], Kohn-Rossi made the following conjecture. In general, either there is no Kohn-Rossi cohomology of X the boundary of M (or V) in degree (p,q), $q \neq 0$, $n-1$, or it must result from the interior singularities of V. The following theorem of Yau answers the Kohn-Rossi conjecture affirmatively.

Theorem 1.2 (See [16]). *Let M be a strongly pseudoconvex manifold of dimension n ($n \geqslant 3$) which is a modification of a Stein space V at the isolated singularities $s_1, \ldots, s_m$. Let $X = \partial M$. Then $\dim H^{p,q}_{KR}(X) = \sum_{i=1}^m b^{p,q+1}_{s_i}$ for $1 \leqslant q \leqslant n-2$, where $b^{p,q+1}_{s_i} = \dim H^{q+1}_{\{s_i\}}(V, \Omega^p_V)$ is a local invariant of the singularity s_i. Suppose that $s_1, \ldots, s_m$ are hypersurface singularities. Then for $1 \leqslant q \leqslant n-2$,*

$$\dim H^{p,q}_{KR}(X) = \begin{cases} 0, & \text{if } p+q \leqslant n-2 \ \text{ or } \ p+q \geqslant n+1, \\ \tau_1 + \cdots + \tau_m, & \text{if } p+q = n-1 \ \text{ or } \ p+q = n, \end{cases}$$

where τ_i is the number of moduli of V at s_i and can be computed explicitly.

Remark 1.3. Let f be a holomorphic function in $\mathbb{C}^{n+1}$. Suppose $V = \{f = 0\}$ has an isolated singularity at the origin. Then the Tjurina number τ of V at 0 is equal to

$$\tau = \dim \mathbb{C}\{x_0, x_1, \ldots, x_n\} \Big/ \left(f, \frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n}\right).$$

The Milnor number μ of V at 0 is equal to

$$\mu = \dim \mathbb{C}\{x_0, x_1, \ldots, x_n\} \Big/ \left(\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n}\right).$$

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a holomorphic function with an isolated critical point at the origin. Recall that a polynomial $f(x_0, x_1, \ldots, x_n)$ is weighted homogeneous of type $(w_0, w_1, \ldots, w_n)$, where $w_0, w_1, \ldots, w_n$ are fixed positive rational numbers, if it can be expressed as a linear combination of monomials $x_0^{i_0} x_1^{i_1} \cdots x_n^{i_n}$ for which $\frac{i_0}{w_0} + \frac{i_1}{w_1} + \cdots + \frac{i_n}{w_n} = 1$. On the other hand, for arbitrary holomorphic function $f(x_0, x_1, \ldots, x_n)$ with 0 as an isolated singularity of $V = \{x : f(x) = 0\}$. We say that V has quasi-homogeneous singularity if $f \in \left(\frac{\partial f}{\partial x_0}, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$, the ideal generated by $\frac{\partial f}{\partial x_0}, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$ in the local ring $\mathscr{O}_{V,0}$. Saito [12] proved that V has quasi-homogeneous singularity at 0, if and only if after a biholomorphic change of variables, f is a weighted homogeneous polynomial.

Definition 1.4. Let $(V_1, 0)$ and $(V_2, 0)$ be two germs of varieties in $(\mathbb{C}^N, 0)$. We say that $(V_1, 0)$ and $(V_2, 0)$ have the same analytic type (i.e., $(V_1, 0) \cong (V_2, 0)$) if there exists a germ of biholomorphism from $(\mathbb{C}^N, V_1, 0)$ to $(\mathbb{C}^N, V_2, 0)$.

By Theorem 1.1, any compact connected strongly pseudoconvex embeddable CR manifold X bounds a complex variety V in $\mathbb{C}^N$ with only isolated normal singularities at Y.

Definition 1.5. Let X_1 and X_2 be two compact connected strongly pseudoconvex embeddable CR manifolds of dimension $2n-1$ which bound normal varieties V_1 and V_2 with only isolated singularities at Y_1 and Y_2, respectively. We say that X_1 and X_2 are algebraically equivalent if $(V_1, Y_1) \cong (V_2, Y_2)$ as germs of varieties.

It is well known that the "number of moduli" of a "moduli space" of strongly pseudoconvex CR manifolds has to be infinite (see [1]). However, under the notion of algebraically equivalence of CR manifolds in the sense of Definition 1.5, then the number of moduli becomes a finite problem. Obviously, two analytically equivalent CR manifolds are automatically algebraically equivalent. In order to understand the classification problem of CR manifolds, a first step is to understand the classification problem of CR manifolds up to algebraic equivalence.

Let $V := \{(x_0, x_1, \ldots, x_n) : f(x_0, x_1, \ldots, x_n) = 0\}$ be a hypersurface with Y as a finite set of isolated singularities in $\mathbb{C}^{n+1}$. Let X_V be a CR manifold of dimension $2n-1$ sitting in V such that X_V bounds the variety V which contains all singularities Y. It is clear that all such CR manifolds X_V are algebraically equivalent to each other.

Let V_1 and V_2 be two hypersurfaces defined by $f(x_0, \ldots, x_m) = 0$ and $g(y_0, \ldots, y_n) = 0$ with singularities $Y_1 = \{p_1, \ldots, p_{n_1}\}$ and $Y_2 = \{q_1, \ldots, q_{n_2}\}$ in $\mathbb{C}^{m+1}$ and $\mathbb{C}^{n+1}$, respectively. Then it is easy to see $f(x_0, \ldots, x_m) + g(y_0, \ldots, y_n) = 0$ defines hypersurface in $\mathbb{C}^{m+n+2}$ with $n_1 n_2$ isolated singularities. We shall denote this hypersurface by $V_1 \oplus V_2$, the Thom-Sebastiani (see [13]), addition of two hypersurfaces.

Definition 1.6. Let V_1 and V_2 be two hypersurfaces with associate CR manifolds X_{V_1} and X_{V_2}. We define $X_{V_1} \oplus X_{V_2}$ as $X_{V_1 \oplus V_2}$, where $V_1 \oplus V_2$ is the Thom-Sebastiani: Sum as above.

Remark 1.7. For each hypersurface V, we can assume without loss of generality that X_V is a strongly pseudoconvex CR manifold.

Remark 1.8. The set of all codimension 3 strongly pseudoconvex compact connected CR manifolds in $\mathbb{C}^{n+1}$ for all $n \geqslant 2$ forms a semigroup under the addition defined in Definition 1.6. A CR manifold X is said to be irreducible if it cannot be written as the sum of two CR manifolds.

It is natural to ask the following question.

Question. *Given a real codimension* 3 *compact connected CR manifold X, how can one tell whether X is an irreducible element in the semi-group of real codimension* 3 *CR manifolds.*

Main Theorems A and B below, answer this question partially.

Main Theorem A. *Let X_1 and X_2 be two compact connected strongly pseudoconvex embeddable CR manifolds of dimensions $2m-1$ and $2n-1$ in $\mathbb{C}^{m+1}$ and $\mathbb{C}^{n+1}$ respectively. Assume that X_2 admits a transversal holomorphic S^1-action* (*see Definition* 2.3). *Let $X = X_1 \oplus X_2$. The following statements hold.*

(a) *If $1 \leqslant t \leqslant m+n-1$ and $s+t = m+n+1$ or $m+n$, then $\dim H^{s,t}_{KR}(X) \neq 0$ and $\dim H^{s,t}_{KR}(X) = \dim H^{a,b}_{KR}(X_1) \dim H^{c,d}_{KR}(X_2)$, where a, b, c, d satisfy the following conditions*:

(1) $1 \leqslant b \leqslant m-2$ *and* $a+b = m$ *or* $m-1$.

(2) $1 \leqslant d \leqslant n-2$ *and* $c+d = n$ *or* $n-1$.

(3) $a+b+n = c+d+m = s+t-1$.

(b) *If $1 \leqslant t \leqslant m+n-1$ and $s+t \neq m+n+1$ and $s+t \neq m+n$, then $\dim H^{s,t}_{KR}(X) = 0$.*

If the condition that X_2 admits a transversal holomorphic S^1-action is deleted in Main Theorem A, then the equality should be replaced by inequality as follows.

Main Theorem B. *Let X_1 and X_2 be two compact connected strongly pseudoconvex embeddable CR manifolds of dimensions $2m-1$ and $2n-1$ in $\mathbb{C}^{m+1}$ and $\mathbb{C}^{n+1}$, respectively. Let $X = X_1 \oplus X_2$. The following statements hold.*

(a) *If $1 \leqslant t \leqslant m+n-1$ and $s+t = m+n+1$ or $m+n$, then $\dim H^{s,t}_{KR}(X) \neq 0$ and $\dim H^{s,t}_{KR}(X) \geqslant \dim H^{a,b}_{KR}(X_1) \dim H^{c,d}_{KR}(X_2)$, where a, b, c, d satisfy the following conditions*:

(1) $1 \leqslant b \leqslant m-2$ *and* $a+b=m$ *or* $m-1$.
(2) $1 \leqslant d \leqslant n-2$ *and* $c+d=n$ *or* $n-1$.
(3) $a+b+n=c+d+m=s+t-1$.
(b) *If* $1 \leqslant t \leqslant m+n-1$ *and* $s+t \neq m+n+1$ *and* $s+t \neq m+n$, *then* $\dim H_{KR}^{s,t}(X)=0$.

In Section 2, we present some basic notation and facts about CR manifolds. We also recall the definitions of Kohn-Rossi cohomology groups. In Section 3, we give the proof of our main theorems. Section 4 presents some concluding remarks.

2 Preliminaries

In 1965, Kohn and Rossi [8] defined their cohomology on CR manifold. Following Tanaka [14], we reformulate the definition in a way independent of the interior manifold.

Definition 2.1. Let X be a connected orientable manifold of real dimension $2n-1$. A CR structure on X is an $(n-1)$-dimensional subbundle S of $\mathbb{C}T(X)$ (complexified tangent bundle) such that
(1) $S \cap \bar{S} = \{0\}$,
(2) if L, L' are local sections of S, then so is $[L, L']$.

Such a manifold with a CR structure is called a CR manifold.

Definition 2.2. Let X be a CR manifold with structures S as in Definition 2.1. Since $S \cap \bar{S} = \{0\}$, there is a unique subbundle $\mathcal{H}$ of $T(X)$ such that

$$\mathbb{C}\mathcal{H} = S \oplus \bar{S},$$

i.e., $\mathcal{H}$ is the real part of $S \oplus \bar{S}$. Furthermore, there is a unique homomorphism $J : \mathcal{H} \to \mathcal{H}$ such that

$$J^2 = -1, \quad 1 = \text{identity}.$$

The pair $(\mathcal{H}, J)$ is called the real expression of S.

Definition 2.3. With the notation in the above definition, a smooth S^1-action on X is said to be holomorphic if it preserves the subbundle $\mathcal{H} \subset T(X)$ and commutes with J. It is said to be transversal if, in addition, the vector field $\mathcal{V}$ which generates the action is transversal to $\mathcal{H}$ at all points of X .

Theorem 2.4 (See [9]). *Let X be a strongly pseudoconvex CR manifold of dimension $2n-1>1$, and suppose that X admits a transversal holomorphic S^1-action. Then there exists a holomorphic equivariant embedding $X \hookrightarrow V$ as a hypersurface in an n-dimensional algebraic variety $V \subset \mathbb{C}^N$ with a linear $\mathbb{C}^*$-action.*

Definition 2.5. Let $L_1, \ldots, L_{n-1}$ be a local frame of the CR structure S on X so that $\bar{L}_1, \ldots, \bar{L}_{n-1}$ is a local frame of $\bar{S}$. Since $S \oplus \bar{S}$ has complex codimension one in $\mathbb{C}T(X)$, we may choose a local section N of $\mathbb{C}T(X)$ such that $L_1, \ldots, L_{n-1}, \bar{L}_1, \ldots, \bar{L}_{n-1}$, N span $\mathbb{C}T(X)$. We may assume that N is purely imaginary. Then the matrix (c_{ij}) defined by

$$[L_i, \bar{L}_j] = \sum_k a_{i,j}^k L_k + \sum_k b_{i,j}^k \bar{L}_k + c_{i,j} N$$

is Hermitian, and is called the Levi form of X.

Proposition 2.6. *The number of non-zero eigenvalues and the absolute value of the signature of (c_{ij}) at each point are independent of the choice of $L_1, \ldots, L_{n-1}, N$.*

Definition 2.7. *X is said to be strongly pseudoconvex if the Levi form is positive definite at each point of X.*

Let $\{\mathscr{A}^k(X), d\}$ be the De Rham complex of X with complex coefficients, and let $H^k(X)$ be the De Rham cohomology groups. There is a natural filtration of the De Rham complex as follows. For any integer p and k, put $A^k(X) = \wedge^k(\mathbb{C}T(X)^*)$ and denote by $F^p(A^k(X))$ the subbundle of $A^k(X)$ consisting of all $\phi \in A^k(X)$ which satisfy the equality

$$\phi(Y_1, \dots, Y_{p-1}, \bar{Z}_1, \dots, \bar{Z}_{k-p+1}) = 0$$

for all $Y_1, \dots, Y_{p-1} \in \mathbb{C}T(X)_0$ and $Z_1, \dots, Z_{k-p+1} \in S_0$, 0 being the origin of ϕ. Then

$$A^k(X) = F^0(A^k(X)) \supset F^1(A^k(X)) \supset \cdots \supset F^k(A^k(X)) \supset F^{k+1}(A^k(X)) = 0.$$

Setting $F^p(\mathscr{A}^k(X)) = \Gamma(F^p(A^k(X)))$, we have

$$\mathscr{A}^k(X) = F^0(\mathscr{A}^k(X)) \supset F^1(\mathscr{A}^k(X)) \supset \cdots \supset F^k(\mathscr{A}^k(X)) \supset F^{k+1}(\mathscr{A}^k(X)) = 0.$$

Since clearly $dF^p(\mathscr{A}^k(X)) \subseteq F^p(\mathscr{A}^{k+1}(X))$, the collection $\{F^p(\mathscr{A}^k(X))\}$ gives a filtration of the De Rham complex.

We denote by $H^{p,q}_{KR}(X)$ the groups $E^{p,q}_1(X)$ of the spectral sequence $\{E^{p,q}_r(X)\}$ associated with the filtration $\{F^p(\mathscr{A}^k(X))\}$. We call $H^{p,q}_{KR}(X)$ the Kohn-Rossi cohomology group of type (p, q). More explicitly, let

$$A^{p,q}(X) = F^p(A^{p+q}(X)), \quad \mathscr{A}^{p,q}(X) = \Gamma(A^{p,q}(X)),$$
$$C^{p,q}(X) = A^{p,q}(X)/A^{p+1,q-1}(X), \quad \mathscr{C}^{p,q}(X) = \Gamma(C^{p,q}(X)).$$

Since $d : \mathscr{A}^{p,q}(X) \to \mathscr{A}^{p,q+1}(X)$ maps $\mathscr{A}^{p+1,q-1}(X)$ into $\mathscr{A}^{p+1,q}(X)$, it induces an operator $d'' : \mathscr{C}^{p,q}(X) \to \mathscr{C}^{p,q+1}(X)$. $H^{p,q}_{KR}(X)$ are then the cohomology groups of the complex $\{\mathscr{C}^{p,q}(X), d''\}$.

In our proof of the main theorems, we need the following result in commutative algebra.

If $\{p_1, \dots, p_m\}$ is a finite subset of $\mathbb{C}^{n+1}$, and $M_i = I(\{p_i\})$ is the maximal ideal of $\mathbb{C}^{n+1}$ corresponding to p_i, we will write

$$\mathbb{C}[x_0, \dots, x_n]_{M_i} = \{f/g : g(p_i) \neq 0\} = \mathcal{O}_i$$

for simplicity of notation. $\mathcal{O}_i$ is called the local ring of $\mathbb{C}[x_0, \dots, x_n]$ at p_i.

Theorem 2.8 (See [3, Theorem 2.2]). *Let I be a zero-dimensional ideal in $\mathbb{C}[x_0, \dots, x_n]$ and let $V(I) = \{p_1, \dots, p_m\}$. Then, there is an isomorphism between $\mathbb{C}[x_0, \dots, x_n]/I$ and the direct product of the rings $A_i = \mathcal{O}_i/I\mathcal{O}_i$, for $i = 1, \dots, m$.*

3 Proofs of the main theorems

Proof of Main Theorem A. Since X_1 and X_2 are two compact connected strongly pseudoconvex embeddable CR manifolds of dimensions $2m-1$ and $2n-1$ in $\mathbb{C}^{m+1}$ and $\mathbb{C}^{n+1}$, it follows from Theorem 1.1 that there exist two corresponding hypersurfaces V_1 and V_2, which are bounded by X_1 and X_2 in $\mathbb{C}^{m+1}$ and $\mathbb{C}^{n+1}$, respectively, having isolated singularities $p_1, \dots, p_{n_1}$ and $q_1, \dots, q_{n_2}$. Let $f(x_0, \dots, x_m)$ and $g(y_0, \dots, y_n)$ be the defining equations for V_1 and V_2, respectively. Since X_2 admits a transversal holomorphic S^1-action, it follows from Theorem 2.4 that g is a weighted homogeneous polynomial. We know that $f(x_0, \dots, x_m) + g(y_0, \dots, y_n)$ defines hypersurface in $\mathbb{C}^{m+n+2}$ with n_1n_2 isolated singularities $\{a_1, \dots, a_{n_1n_2}\}$. By Theorem 1.2, for $1 \leqslant t \leqslant m+n-1$, we have

$$\dim H^{s,t}_{KR}(X) = \begin{cases} 0, & \text{if } s+t \leqslant m+n-1 \text{ or } \ s+t \geqslant m+n+2, \\ \tau_1 + \cdots + \tau_{n_1n_2}, & \text{if } s+t = m+n \text{ or } \ p+q = m+n+1, \end{cases}$$

where τ_i is the Tjurina number of $V := V(f+g)$ at a_i for $i = 1, \dots, n_1n_2$, i.e., $\tau_i = \dim \mathcal{O}_i/(f+g, \frac{\partial(f+g)}{\partial x_0}, \dots, \frac{\partial(f+g)}{\partial y_n})\mathcal{O}_i$, where $\mathcal{O}_i$ is the local ring of $\mathbb{C}[x_0, \dots, x_m, y_0, \dots, y_n]$ at a_i. It follows from Theorem 2.8 that

$$\tau_1 + \cdots + \tau_{n_1n_2} = \dim \mathbb{C}[x_0, \dots, x_m, y_0, \dots, y_n] \Big/ \left(f+g, \frac{\partial(f+g)}{\partial x_0}, \dots, \frac{\partial(f+g)}{\partial y_n}\right).$$

Thus we have

$$\dim H_{KR}^{s,t}(X) = \begin{cases} 0, \quad \text{if } s+t \leqslant m+n-1 \text{ or } \ s+t \geqslant m+n+2, \\ \dim \mathbb{C}[x_0,\dots,x_m,y_0,\dots,y_n] \Big/ \left(f+g, \dfrac{\partial(f+g)}{\partial x_0},\dots,\dfrac{\partial(f+g)}{\partial y_n}\right), \\ \text{if } s+t=m+n \text{ or } \ p+q=m+n+1. \end{cases}$$

Similarly, for $1 \leqslant b \leqslant m-2$ we have

$$\dim H_{KR}^{a,b}(X_1) = \begin{cases} 0, \quad \text{if } a+b \leqslant m-2 \text{ or } \ a+b \geqslant m+1, \\ \dim \mathbb{C}[x_0,\dots,x_m] \Big/ \left(f, \dfrac{\partial f}{\partial x_0},\dots,\dfrac{\partial f}{\partial x_m}\right), \\ \text{if } a+b=m-1 \text{ or } \ a+b=m, \end{cases}$$

and for $1 \leqslant d \leqslant n-2$ we have

$$\dim H_{KR}^{c,d}(X_2) = \begin{cases} 0, \quad \text{if } c+d \leqslant n-2 \text{ or } \ c+d \geqslant n+1, \\ \dim \mathbb{C}[y_0,\dots,y_n] \Big/ \left(g, \dfrac{\partial g}{\partial y_0},\dots,\dfrac{\partial g}{\partial y_n}\right), \\ \text{if } c+d=n-1 \text{ or } \ c+d=n. \end{cases}$$

It follows from Theorem 1.2 that if $1 \leqslant t \leqslant m+n-1$ and $s+t=m+n$ or $m+n+1$, then $\dim H_{KR}^{s,t}(X) \neq 0$ and we have

$$\begin{aligned} \dim H_{KR}^{s,t}(X) &= \dim \mathbb{C}[x_0,\dots,x_m,y_0,\dots,y_n] \Big/ \left(f+g, \frac{\partial(f+g)}{\partial x_0},\dots,\frac{\partial(f+g)}{\partial y_n}\right) && \text{(3.1a)} \\ &= \dim \mathbb{C}[x_0,\dots,x_m,y_0,\dots,y_n] \Big/ \left(f+g, \frac{\partial f}{\partial x_0},\dots,\frac{\partial f}{\partial x_m},\frac{\partial g}{\partial y_0},\dots,\frac{\partial g}{\partial y_n}\right) && \text{(3.1b)} \\ &= \dim \mathbb{C}[x_0,\dots,x_m,y_0,\dots,y_n] \Big/ \left(f, \frac{\partial f}{\partial x_0},\dots,\frac{\partial f}{\partial x_m},\frac{\partial g}{\partial y_0},\dots,\frac{\partial g}{\partial y_n}\right) && \text{(3.1c)} \\ &= \dim \mathbb{C}[x_0,\dots,x_m] \Big/ \left(f, \frac{\partial f}{\partial x_0},\dots,\frac{\partial f}{\partial x_m}\right) \otimes \mathbb{C}[y_0,\dots,y_n] \Big/ \left(\frac{\partial g}{\partial y_0},\dots,\frac{\partial g}{\partial y_n}\right). && \text{(3.1d)} \end{aligned}$$

The third equality above comes from the fact that g is weighted homogeneous while the last equality follows from [4, Korollars 1 and 2, p. 181]. By Theorem 1.2 it is easy to see that when the following conditions are satisfied,

(1) $1 \leqslant b \leqslant m-2$ and $a+b=m$ or $m-1$;
(2) $1 \leqslant d \leqslant n-2$ and $c+d=n$ or $n-1$;
(3) $a+b+n=c+d+m=s+t-1$,

then $\dim H_{KR}^{c,d}(X_1) \neq 0$ and $\dim H_{KR}^{c,d}(X_2) \neq 0$, and

$$\begin{aligned} \dim H_{KR}^{s,t}(X) &= \dim \mathbb{C}[x_0,\dots,x_m] \Big/ \left(f, \frac{\partial f}{\partial x_0},\dots,\frac{\partial f}{\partial x_m}\right) \otimes \mathbb{C}[y_0,\dots,y_n] \Big/ \left(\frac{\partial g}{\partial y_0},\dots,\frac{\partial g}{\partial y_n}\right) \\ &= \dim \mathbb{C}[x_0,\dots,x_m] \Big/ \left(f, \frac{\partial f}{\partial x_0},\dots,\frac{\partial f}{\partial x_m}\right) \dim \mathbb{C}[y_0,\dots,y_n] \Big/ \left(\frac{\partial g}{\partial y_0},\dots,\frac{\partial g}{\partial y_n}\right) \\ &= \dim H_{KR}^{c,d}(X_1) \dim H_{KR}^{c,d}(X_2). \end{aligned}$$

Part (b) of Main Theorem A follows from Theorem 1.2 directly. □

Proof of Main Theorem B. The proof of Main Theorem B is the same as the proof of Main Theorem A. We only need to replace (3.1a)–(3.1d) by the following (3.2a)–(3.2e):

$$\begin{aligned} \dim H_{KR}^{s,t}(X) &= \dim \mathbb{C}[x_0,\dots,x_m,y_0,\dots,y_n] \Big/ \left(f+g, \frac{\partial(f+g)}{\partial x_0},\dots,\frac{\partial(f+g)}{\partial y_n}\right) && \text{(3.2a)} \\ &= \dim \mathbb{C}[x_0,\dots,x_m,y_0,\dots,y_n] \Big/ \left(f+g, \frac{\partial f}{\partial x_0},\dots,\frac{\partial f}{\partial x_m},\frac{\partial g}{\partial y_0},\dots,\frac{\partial g}{\partial y_n}\right) && \text{(3.2b)} \end{aligned}$$

$$\geqslant \dim \mathbb{C}[x_0, \ldots, x_m, y_0, \ldots, y_n] \Big/ \left(f+g, g, \frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_m}, \frac{\partial g}{\partial y_0}, \ldots, \frac{\partial g}{\partial y_n} \right) \tag{3.2c}$$

$$= \dim \mathbb{C}[x_0, \ldots, x_m, y_0, \ldots, y_n] \Big/ \left(f, \frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_m}, g, \frac{\partial g}{\partial y_0}, \ldots, \frac{\partial g}{\partial y_n} \right) \tag{3.2d}$$

$$= \dim \mathbb{C}[x_0, \ldots, x_m] \Big/ \left(f, \frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_m} \right) \otimes \mathbb{C}[y_0, \ldots, y_n] \Big/ \left(g, \frac{\partial g}{\partial y_0}, \ldots, \frac{\partial g}{\partial y_n} \right). \tag{3.2e}$$

The proof is complete. □

4 Concluding remarks

The ultimate goal in CR geometry is the following: Given two strongly pseudoconvex CR manifolds, determine whether they are CR biholomorphically equivalent. This is certainly a very difficult problem. In order to solve the classification problem of CR manifolds, one can first investigate the irreducibility of a hypersurface type CR manifold, i.e., given a real codimension 3 compact connected CR manifold X, how can one tell whether X is an irreducible element in the semi-group of real codimension 3 CR manifolds. In Main Theorems A and B, we give a partial answer to this question in terms of the Kohn-Rossi cohomology groups which depend only on the information of the CR manifold. In our future work, we shall try to answer this question by means of singularity theory. In 1995, Luk and Yau [11] introduced many CR invariants to CR manifolds just using the theory of resolution of singularity. For example, when the CR manifold X has dimension 3, those invariants introduced in [11] include m_Z, p_f, p_a, p_g, q etc., where p_g is called geometric genus and can also be defined for CR manifolds of any dimension. Some of these invariants are proved to be very useful in studying the existence problem of non-trivial CR morphisms between strongly pseudoconvex CR manifolds. In [17], Yau proved that there is no non-constant CR morphism from X_1 to X_2 if $p_g(X_1) < p_g(X_2)$. Recently, Lin et al. [10] generalized this definition and got a series of CR invariants p_m which is called plurigenera of compact connected strongly pseudoconvex CR manifolds. Their p_1 coincides with previously defined p_g. We hope that one can also use the CR invariants which are defined from singularity theory or others [18] to give a complete answer to the irreducibility question in the near future.

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