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A sharp lower bound for the geometric genus and Zariski multiplicity question

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Abstract It is well known that the geometric genus and multiplicity are two important invariants for isolated singularities. In this paper we give a sharp lower estimate of the geometric genus in terms of the multiplicity for isolated hypersurface singularities. In 1971, Zariski asked whether the multiplicity of an isolated hypersurface singularity depends only on its embedded topological type. This problem remains unsettled. In this paper we answer positively Zariski's multiplicity question for isolated hypersurface singularity if Milnor number or geometric genus is small.

Keywords Geometric genus · Multiplicity · Isolated singularity

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1 Introduction

Let $(V, 0) \subset (\mathbb{C}^N, 0)$ be the analytic germ of an n-dimensional complex isolated complete intersection singularity (ICIS). One of the most important goals of singularity theory is the clarification of the subtle connections between some basic numerical invariants, the *Milnor number* μ, *multiplicity* ν and *geometric genus* p_g.

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For the surface case, it is already a difficult problem. In this case, if F_V is the Milnor fiber of $(V, 0)$ and (μ_+, μ_0, μ_-) are the Sylvester invariants of the symmetric intersection form of the middle integral homology $H_2(F_V, \mathbb{Z})$, then $2p_g = \mu_0 + \mu_+$ (Durfee [5]), while, obviously, $\mu = \mu_+ + \mu_0 + \mu_-$. Hence, numerical relations between μ and p_g can be rewritten in terms of the Sylvester invariants. In topology, the signature $\sigma := \mu_+ - \mu_-$ is well studied. In fact, for compact complex surfaces, the Euler number, Todd genus and the signature are the most important index-theoretical numerical invariants; their local analogs are the above integers μ, p_g and σ. The relation of μ, p_g was investigated intensively giving rise to several open problems as well. In particular, this was formulated in Durfee's Conjecture [5] and Yau's Conjecture [31] (for weighted homogeneous singularities) as follows:

Durfee's Conjecture: If $(V, 0)$ is an isolated complete intersection surface singularity, then $6p_g \leq \mu$.

Yau's Conjecture: Let $f : (\mathbb{C}^n, 0) \to (C, 0)$ be a weighted homogeneous polynomial with an isolated singularity at the origin. Let μ, p_g and ν be the Milnor number, geometric genus and multiplicity of the singularity $V = \{z\colon f(z) = 0\}$, then

$$\mu - p(\nu) \geq n!p_g,$$

where $p(\nu) = (\nu - 1)^n - \nu(\nu - 1) \ldots (\nu - n + 1)$, and equality holds if and only if f is a homogeneous polynomial.

The Yau's Conjecture are sharp estimate and can be viewed as an improvement of Durfee's Conjecture in the case of weighted homogeneous singularity, and it has some important applications in geometry.

The Durfee's Conjecture and Yau's Conjectures were studied systematically by the authors and their collaborators [4,13–15,29–31,33].

The relation of μ, ν was investigated in [31]. We proved $\mu \geq (\nu - 1)^n$ for isolated hypersurface singularity $(V, 0)$. It is natural to ask: what is the relation between ν and p_g? In [26], Yau give a lower bound for p_g in terms of multiplicity and some other number which depends on the equation of a hypersurface singularity as follows.

Theorem 1.1 (Yau [26]) . *Let*

$$f(z_1, \ldots, z_{n-1}, z_n) = z_n^m + a_1(z_1, \ldots z_{n-1})z_n^{m-1} + \cdots + a_m(z_1, \ldots, z_{n-1})$$

be holomorphic near $(0, \ldots, 0)$*. Let* d_i *be the order of the zero of* $a_i(z_1, \ldots, z_{n-1})$ *at* $(0, \ldots, 0)$*,* $d_i \geq i$*. Let* $d = min_{1\leq i\leq m}(\frac{d_i}{i})$*. Suppose that*

$$V = \{(z_1, \ldots, z_n) : f(z_1, \ldots, z_n) = 0\}$$

defined in a suitably small polydisc, has $p = (0, \ldots, 0)$ *as its only singularity. Then* $p_g > (m - 1)d - (n - 1)$*.*

In [32], we investigated the relation between p_g and irregularity for isolated complete intersection singularities with $\mathbb{C}^*$-action. In the following result, a lower bound for p_a in terms of the multiplicity ν for an isolated two dimensional hypersurface singularity was given.

Theorem 1.2 [28] *Let* $(V, 0)$ *be an isolated two dimensional hypersurface singularities in* $\mathbb{C}^3$*. Then we have*

$$p_a \geq \frac{\nu(\nu-1)(\nu-3)}{8} + 1, \quad \text{for } \nu \text{ is odd and } \nu \geq 3,$$
$$p_a \geq \frac{\nu(\nu-2)^2}{8} + 1, \quad \text{for } \nu \text{ is even and } \nu \geq 3,$$

where p_a is the arithmetic genus (see Definition 2.6) and ν is the multiplicity of $(V, 0)$.

Thus a lower bound for p_g in terms of the multiplicity ν is also obtained as follows.

Corollary 1.1 *Let* $(V, 0)$ *be an isolated two dimensional hypersurface singularities in* $\mathbb{C}^3$. *Then we have*

$$p_g \geq \frac{\nu(\nu-1)(\nu-3)}{8} + 1, \quad \text{for } \nu \text{ is odd and } \nu \geq 3,$$
$$p_g \geq \frac{\nu(\nu-2)^2}{8} + 1, \quad \text{for } \nu \text{ is even and } \nu \geq 3,$$

where p_g is the geometric genus and ν is the multiplicity of $(V, 0)$.

Proof It follows from Theorem 1.2 and Proposition 2.1 immediately. □

It is easy to see that the new lower bound for p_g in Corollary 1.1 is better than the Yau's bound in Theorem 1.1. For example: let $f = z_1^5 + z_2^5 + z_3^5$, then by Theorem 1.1, we have $p_g > (5-1) - (3-1) = 2$ (notice that $m = 5, d = 1, n = 3$). However, if we use Corollary 1.1, we get $p_g \geq 6$. In fact, $p_g = 10$ for this singularity.

From the example above, we can see that the lower bound of p_g in Corollary 1.1 is far from sharp. One of the main goals of this paper is to construct a sharp lower estimate for p_g.

Main Theorem A *Let* $(V, 0)$ *be an isolated hypersurface singularities in* $\mathbb{C}^n$ *defined by a holomorphic function* f. *Then we have*

$$p_g \geq \frac{1}{n!} \prod_{i=0}^{n-1} (\nu - i), \tag{1.1}$$

where p_g is the geometric genus and ν is the multiplicity of $(V, 0)$.

Remark 1.1 If f in Main Theorem A is homogeneous or semi-homogeneous (see Definition 2.13), then it is easy to conclude that the "=" holds in (1.1) from the proof of the Main Theorem A. However, the inverse direction is not correct. For example, let $f = x^3 + y^3 + z^4$. Since $p_g = 1$ and $\nu = 3$, so $p_g = 1 = \frac{3(3-1)(3-2)}{6}$. Thus "=" holds in (1.1). However, f is not homogeneous and semi-homogeneous.

In other words, one can not expect the "=" holds in (1.1) implies f is homogeneous or semi-homogeneous.

If $n = 3$, we have the following immediate corollary.

Corollary 1.2 *Let* $(V, 0)$ *be an isolated two dimensional hypersurface singularities in* $\mathbb{C}^3$ *defined by a holomorphic function* f. *Then we have*

$$p_g \geq \frac{\nu(\nu-1)(\nu-2)}{6},$$

where p_g is the geometric genus and ν is the multiplicity of $(V, 0)$.

Remark 1.2 It is easy to check that the lower bound for p_g in Corollary 1.2 is greater than the bound in Corollary 1.1. Example: let $f = z_1^4 + z_2^4 + z_3^4$, then by Corollary 1.1, we have $p_g \geq 3$. However, if we use Corollary 1.2, we get $p_g \geq 4$. In fact, $p_g = 4$ for this singularity. From this example we can see that the lower bound in Corollary 1.2 and Main Theorem A are sharp.

Let $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be germs (at the origin) of holomorphic functions, $V_f := f^{-1}(0)$, $V_g := g^{-1}(0)$ the corresponding germs of hypersurfaces in $\mathbb{C}^n$, and ν_f, ν_g the multiplicities at 0 of V_f and V_g respectively.

Definition 1.1 One says that f and g have the same topological type (or are called topologically equivalent) if there is a germ of homeomorphism $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $\phi(V_f) = V_g$.

Zariski's multiplicity Question: (cf. [34]) If f and g are topologically equivalent, then is it true that $\nu_f = \nu_g$?

The question is, in general, still open despite more than four decades effort to prove it. Nevertheless, the answer is known to be true in the following special cases:

(i) if $n = 2$ (Zariski [35]);
(ii) if $\nu_f = 1$, that is, if 0 is not a critical point of f (A'Campo [1], Lê [11]);
(iii) if $n = 3$ and $\nu_f = 2$ (Navarro Aznar [18]);
(iv) if $n = 3$ and f and g are quasihomogeneous with an isolated critical point at the origin (Xu-Yau [25,27]).
(v) if $n = 3$ and f and g have an isolated critical point at the origin and the arithmetic genius p_a of V_f at 0 is ≤ 2 (Yau [28]).

There are several other partial positive answers to Zariski's question, the reader interested in this question can refer to Eyral's beautiful survey article [6].

The following result which gives positive answer to Zariski's question partially, is a corollary of Main Theorem A.

Corollary 1.3 *Let* $(V, 0)$ *and* $(W, 0)$ *be two isolated two dimensional hypersurface singularities in* $\mathbb{C}^3$ *having the same topological type. If* $p_g(V, 0) \leq 3$, *then* $\nu(V, 0) = \nu(W, 0)$ *where* $p_g(V, 0)$ *is the geometric genus of* $(V, 0)$, $\nu(V, 0)$ *and* $\nu(W, 0)$ *are the multiplicities of* $(V, 0)$ *and* $(W, 0)$ *respectively.*

Another purpose of this paper is to prove the following results.

Main Theorem B *Let* $f, g : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ *be germs of holomorphic functions with an isolated critical point at the origin,* V_f, V_g *the corresponding germs of zero locus in* $\mathbb{C}^3$, *and* ν_f, ν_g *the multiplicities at* 0 *of* V_f, V_g *respectively. Suppose that* f *and* g *are topologically equivalent. If the Milnor number of* V_f *at* 0 *is less than or equal to 26, then* $\nu_f = \nu_g$.

Main Theorem C *Let* $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, $n > 3$ *be germs of holomorphic functions with an isolated critical point at the origin,* V_f, V_g *the corresponding germs of hypersurfaces in* $\mathbb{C}^n$, *and* ν_f, ν_g *the multiplicities at* 0 *of* V_f, V_g *respectively. We suppose that* f *and* g *are topologically equivalent. If the Milnor number of* V_f *at* 0 *is less than or equal to* $2^n - 1$, *then* $\nu_f = \nu_g$.

In Sect. 2, we recall the necessary materials which are needed to prove the Main Theorems. In Sect. 3, we shall give the proofs of the Main Theorems.

2 Preliminary

2.1 Geometric geneus and arithmetic genus

Definition 2.1 Let $(V, 0)$ be a normal surface singularity. If there exist an open neighborhood U of 0 in V and a non-vanishing holomorphic 2-form on the deleted neighborhood $U\backslash\{0\}$, then we say that $(V, 0)$ is a Gorenstein singularity.

We shall consider normal surface Gorenstein singularities. Let V be a two-dimensional normal analytic space with only singular point 0 in it. Let $\pi : M \to V$ be a resolution of V, and let $A = \pi^{-1}(0)$ which is called the exceptional set associated to a resolution π. We always decompose A into irreducible components $A_1 \cup A_2 \cup \cdots A_n$. The weighted dual graph of $(V, 0)$ consists of the information of the genera of all A_i and the intersection matrix $(A_i \cdot A_j)$.

Definition 2.2 A positive cycle D on the exceptional set $A = A_1 \cup A_2 \cup \cdots A_n \Leftrightarrow D = d_1A_1 + d_2A_2 + \cdots + d_nA_n$ with d_i nonnegative integer and $(d_1, d_2, \ldots, d_n) \neq (0, 0, \ldots, 0)$.

Definition 2.3 Let $D_1 = d_1A_1 + d_2A_2 + \cdots + d_nA_n$ and $D_2 = d_1'A_1 + d_2'A_2 + \cdots + d_n'A_n$ be positive cycles on A. Then $D_1 + D_2 := (d_1 + d_1')A_1 + \cdots + (d_n + d_n')A_n$, and $D_1 \geq D_2 \Leftrightarrow d_i \geq d_i'$ for any $i = 1, \ldots, n$.

Definition 2.4 Let $D_1 = d_1A_1 + d_2A_2 + \cdots + d_nA_n$ be a positive cycle on A. The sheaf of germs of holomorphic functions on M is denoted by $\mathcal{O}$. The sheaf of germs of holomorphic functions on M which vanish on A_i of order at least d_i, for each i, is denoted by $\mathcal{O}(-D)$. The $\mathcal{O}(-D)$ is a subsheaf of $\mathcal{O}$. Let $\mathcal{O}_D := \mathcal{O}/\mathcal{O}(-D)$. The $\mathcal{O}_D$ is a coherent analytic sheaf on M.

Definition 2.5 Let D be a positive cycle on A. The virtual genus of D, denoted by $p(D)$, is defined by $p(D) = 1 - h^0(M, \mathcal{O}_D) + h^1(M, \mathcal{O}_D)$. By Riemann-Roch Theorem, we can see that $p(D) = 1 + \frac{1}{2}(D^2 + DK)$ where K is the canonical divisor on M. From this, one can see that $p(D_1 + D_2) = p(D_1) + p(D_2) + D_1 \cdot D_2 - 1$.

Definition 2.6 The arithmetic genus of the A, denoted by $p_a(A)$, is defined by

$$\sup\{p(D) : D \text{ is a positive cycle on } A\}.$$

This is independent of the choice of the resolution, so this is an invariant of a normal surface singularity $(V, 0)$. Thus we denote this simply by p_a.

Definition 2.7 Let $\pi : M \to V$ be a resolution of an isolated n-dimensional singularity $(V, 0)$. The geometric genus p_g is defined by the dimension of the direct image sheaf: $p_g := \dim_{\mathbb{C}}(R^{n-1}\pi_*\mathcal{O}_M)_0$.

Without loss of generality, we may assume that V is a Stein space, and the resolution M is a strictly pseudoconvex manifold. Then $p_g = h^{n-1}(M, \mathcal{O}_M)$.

In [9,26], Laufer and Yau proved that $p_g = \dim_{\mathbb{C}} H^0(V^*, \mathcal{O}(K_{V^*}))/L^2(V^*)$ where $V^* := V\backslash\{0\}$ and $L^2(V^*)$ is a vector subspace of $H^0(V^*, \mathcal{O}(K_{V*}))$ consisting of square-integrable 2-forms on deleted neighborhood of the singularity. This says that p_g is independent of the choice of the resolution. So p_g is an invariant of $(V, 0)$.

Proposition 2.1 (Artin [3]) *Let* $(V, 0)$ *be a 2-dimensional normal singularity, then*

(1) $0 \leq p_a \leq p_g$,
(2) $p_a = 0 \Leftrightarrow p_g = 0$.

2.2 Newton polyhedron and non-degenerate

Let $f(z_1, \ldots, z_n)$ be a germ of an analytic function at the origin such that $f(0) = 0$. Suppose f has an isolated critical point at the origin. f can be developed in a convergent Taylor series

$f(z_1, \dots, z_n) = \sum a_\lambda z^\lambda$ where $z^\lambda = z_1^{\lambda_1} \cdots z_n^{\lambda_n}$. Recall that the Newton boundary $\Gamma(f)$ is the union of compact faces of $\Gamma_+(f)$ where $\Gamma_+(f)$ is the convex hull of the union of subsets $\{\lambda + \mathbb{R}_+^n\}$ for λ such that $a_\lambda \neq 0$. Let $\Gamma_-(f)$, the Newton polyhedron of f, be the cone over $\Gamma(f)$ with cone point at 0. For any closed face Δ in $\Gamma(f)$, we associate the polynomial $f_\Delta(z) = \sum_{\lambda \in \Delta} a_\lambda z^\lambda$. The principal part of f is a polynomial consisting of monomials, the indices of which lie in the $\Gamma(f)$. In other words, the principal part of f is the sum of $f_\Delta(z)$ for all Δ in $\Gamma(f)$.

Definition 2.8 f is called non-degenerate if f_Δ has no critical point in $(\mathbb{C}^*)^n$ for any $\Delta \in \Gamma(f)$ where $\mathbb{C}^* = \mathbb{C} \backslash \{0\}$.

Definition 2.9 f is called convenient if $\Gamma_+(f)$ meets each of the coordinate axes.

Definition 2.10 Let X be a complex analytic space. A subset $C \subset X$ is called constructible if one can find an $m \in \mathbb{N}$ and analytic subsets $A_1, \dots, A_m$ and $B_1, \dots, B_m$ of X such that

$$C = \bigcup_{i=1}^{m} (A_i \backslash B_i).$$

Theorem 2.1 [2, Lemma 6.1] *The set of degenerate principal parts is a proper constructible subset in the space of all principal parts corresponding to a given Newton polyhedron, the complement of which is everywhere dense.*

We say that a point p of the integral lattice $\mathbb{Z}^n$ in $\mathbb{R}^n$ is positive if all coordinates of p are positive. The following beautiful theorem is due to Merle-Teissier.

Theorem 2.2 (Merle-Teissier, [16]) *Let* $(V, 0)$ *be an isolated hypersurface singularity defined by a non-degenerate holomorphic function* $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. *Then the geometric genus* $p_g = \#\{p \in \mathbb{Z}^n \cap \Gamma_-(f) : p \text{ is positive}\}$.

Definition 2.11 Let $(V, 0)$ be an isolated hypersurface singularity defined by a holomorphic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. The Milnor number μ of the singularity $(V, 0)$ is defined respectively by

$$\mu = \dim_{\mathbb{C}} \mathbb{C}\{z_1, z_2, \dots, z_n\} \big/ (f_{z_1}, \dots, f_{z_n}).$$

Definition 2.12 The multiplicity of the singularity $(V_f, 0)$ is defined to be the order of the lowest non-vanishing term in the power series expansion of f at 0.

The following result about nonsingular germs is well-known.

Theorem 2.3 (A'Campo [1] and Lê [11]) *Let* $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ *be germs of holomorphic functions and let* ν_f *and* ν_g *be the multiplicities at the origin of* f *and* g *respectively. We suppose that* f *and* g *are topologically equivalent. If* $\nu_f = 1$, *so is* ν_g.

The following result is a corollary of A'Campo's work.

Theorem 2.4 (Navarro Aznar [18]) *Let* $f, g : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ *be germs of holomorphic functions and let* ν_f *and* ν_g *be the multiplicities at the origin of* f *and* g *respectively. Suppose that* f *and* g *are topologically equivalent. If* $\nu_f = 2$, *so is* ν_g.

2.3 Weighted homogeneous singularities

Recall that a polynomial $f(z_1, \ldots, z_n)$ is weighted homogeneous of type $(w_1, \ldots, w_n; 1)$, where $w_1, \ldots, w_n$ are fixed positive rational numbers, if it can be expressed as a linear combination of monomials $z_1^{i_1} \cdots z_n^{i_n}$ for which $i_1 w_1 + \cdots + i_n w_n = 1$. As a consequence of the theorem of Merle-Teissier (see Theorem 2.2), for an isolated singularity defined by a weighted homogeneous polynomial, computing the geometric genus is equivalent to counting the number of positive integral points in the tetrahedron defined by $x_1 w_1 + \cdots + x_n w_n \leq 1$, $x_1 \geq 0, \ldots, x_n \geq 0$. We have the following result.

Theorem 2.5 (Milnor-Orlik [17]) *Let $f(z_1, \ldots, z_n)$ be a weighted homogeneous polynomial of type $(w_1, \ldots, w_n)$ with isolated singularity at the origin. Then the Milnor number $\mu = (\frac{1}{w_1} - 1) \cdots (\frac{1}{w_n} - 1)$.*

Let $f \in \mathbb{C}\{z_1, \ldots, z_n\}$ defines an isolated singularities at the origin. Let the weight w_i of z_i be a positive integer for all i. We have the weighted Taylor expansion $f = f_\rho + f_{\rho+1} + \cdots$ with respect to $w = (w_1, \ldots, w_n)$ and $f_\rho \neq 0$, where f_k is a weighted homogeneous of type $(w_1, \ldots, w_n; k)$, for $k \geq \rho$. Here f_k is linear combination of monomials $z_1^{i_1} \cdots z_n^{i_n}$ for which $i_1 w_1 + \cdots + i_n w_n = k$.

Definition 2.13 With the notation as above. Suppose that f defines an isolated singularity at the origin. f is called a semi-quasihomogeneous function if the initial term f_ρ defines an isolated singularity at the origin. Furthermore, if f is semi-quasihomogeneous with respect to $w_1 = \cdots w_n = 1$, then f is called semi-homogeneous.

2.4 Integral points in simplex

Let Δ_n be an n-dimensional real right-angled simplex defined by the inequality

$$\frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_n}{a_n} \leq 1,$$

where $x_1 \geq 0, \ldots, x_n \geq 0$ and $a_1 \geq a_2 \geq \cdots \geq a_n > 0$. Define $P_n(a_1, \ldots, a_n)$ to be the number of positive integral points in Δ_n, as shown below:

$$P_n(a_1, \ldots, a_n) = \# \left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{Z}_+^n \mid \frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_n}{a_n} \leq 1 \right\}.$$

Define $Q_n(a_1, \ldots, a_n)$ to be the number of nonnegative integral points in Δ_n, as shown below:

$$Q_n(a_1, \ldots, a_n) = \# \left\{ (x_1, x_2, \ldots, x_n) \in (\mathbb{Z}_+ \cup \{0\})^n \mid \frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_n}{a_n} \leq 1 \right\}.$$

These two different numbers are tied together through the equation (see [14])

$$P_n(a_1, a_2, \ldots, a_n) = Q_n(a_1(1-a), a_2(1-a), \ldots, a_n(1-a)), \tag{2.1}$$

where $a = \frac{1}{a_1} + \cdots + \frac{1}{a_n}$.

Theorem 2.6 (Lehmer [12]) *If $a_1 = a_2 = \cdots = a_n = a \geq 0$, then*

$$Q_n(a, \ldots, a) = \binom{[a] + n}{n}$$

where $[a]$ is Gauss symbol, i.e., the integral part of a.

Remark 2.1 If $a = 0$ in Theorem 2.6, then

$$Q_n(0, \ldots, 0) = 1 = \#\left\{(x_1, x_2, \ldots, x_n) \in (\mathbb{Z}_+ \cup \{0\})^n \mid x_1 + x_2 + \cdots + x_n \leq 0\right\}.$$

2.5 Plurigenus

Definition 2.14 Let V be an analytic space with the singular locus S. A proper birational morphism $\pi : \tilde{V} \to V$ is called a good resolution of the singularities on X, if

(1) $\tilde{V}$ is non-singular,
(2) the restriction $\tilde{V} \backslash \pi^{-1}(S) \to V \backslash S$ of π is an isomorphism, and
(3) the reduced fiber $\pi^{-1}(S)_{red}$ is a divisor of simple normal crossings.

Definition 2.15 For a normal isolated singularity $(V, 0)$ of dimension $n \geq 2$, we define a plurigenera $\{\delta_m\}_{m \in \mathbb{Z}_+}$ by

$$\delta_m(V, 0) = \dim_{\mathbb{C}} \Gamma(V \backslash \{0\}, \mathcal{O}(mK))/L^{2/m}(V \backslash \{0\}),$$

where $L^{2/m}(V \backslash \{0\})$ denotes the set of all $L^{2/m}$-integrable m-ple holomorphic n-form on $V \backslash \{0\}$.

Remark 2.2 It is shown in [9,26] that $p_g(V, 0) = \delta_1(V, 0)$.

Proposition 2.2 (Watanabe [24]) *The plurigenus $\delta_m(V, 0)$ is represented as*

$$\begin{aligned}\delta_m(V, 0) &= \dim_{\mathbb{C}} \Gamma(\tilde{V} \backslash E, \mathcal{O}(mK_{\tilde{V}}))/\,\Gamma(\tilde{V}, \mathcal{O}(mK_{\tilde{V}} + (m-1)E)) \\ &= \dim_{\mathbb{C}} \mathcal{O}(mK_V)/\pi_* \mathcal{O}(mK_{\tilde{V}} + (m-1)E),\end{aligned}$$

where $\pi : \tilde{V} \to V$ is a good resolution of the singularity and $E = \pi^{-1}(0)_{red}$.

Proposition 2.3 (Okuma [20]) *Let $(V, 0)$ be a normal Gorenstein surface singularity. Then $\delta_m(V, 0)$ is determined by $p_g(V, 0)$ and the weighted dual graph.*

2.6 Deformation

Let $f : Y \to S$ be a morphism of complex analytic spaces and $\mathcal{F}$ a coherent $\mathcal{O}_Y$-module. The sheaf $\mathcal{F}$ is said to be flat over S, if the stalk $\mathcal{F}_y$ is a flat $\mathcal{O}_{S, f(y)}$-module for every $y \in Y$. The morphism f is said to be flat, if $\mathcal{O}_Y$ is flat over S. Let $s \in S$ be a point and m_s the maximal ideal of the point s. The fiber Y_s is the complex analytic space $(f^{-1}(s), \mathcal{O}_{Y_s})$, where $\mathcal{O}_{Y_s} = \mathcal{O}_Y/m_s\mathcal{O}_Y$. We write as $Y_s = f^{-1}(s)$. The coherent $\mathcal{O}_{Y_s}$-module $\mathcal{F}/m_s\mathcal{F}$ is denoted by $\mathcal{F}_s$. Let $f : Y \to S$ be a flat surjective morphism and $o \in S$ a distinguished point. If V is a complex analytic space such that $Y_o \cong V$, then f is called a deformation of V, and Y and S are called a total space and a base space, respectively. If the base space S is a nonsingular curve, then the deformation f is called a 1-parameter deformation.

Definition 2.16 Let $(V, 0)$ be a singularity. A flat surjective morphism of germ $f : (Y, 0) \to (S, o)$ is called a deformation of the singularity $(V, 0)$ if the fiber $(Y_o, 0)$ is isomorphic to $(V, 0)$. Then we say that $(V, 0)$ is deformed to the fiber $f^{-1}(s)$, or that $f^{-1}(s)$ is a deformation of $(V, 0)$, for $s \in S \backslash \{0\}$.

Theorem 2.7 (Ishii [8], upper semi-continuity of δ_m) *Let $(V, 0)$ be a normal isolated singularity and $f : (Y, 0) \to (S, o)$ be a 1-parameter deformation of $(V, 0)$. Then, for each $m \in \mathbb{N}$,*

$$\delta_m(Y,0) \geq \sum_{p \in Sing(Y_s)} \delta_m(Y_s,p) \quad \text{for} \;\; s \in S \text{ near } 0,$$

where $Sing(Y_s)$ denotes the set of singular points of Y_s.

3 Proof of the main theorems

Proof of the Main Theorem A

We fist prove the following Lemma.

Lemma 3.1 *Let $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be two isolated hypersurface singularities and suppose that f, g are non-degenerate with $\Gamma_+(f) \subseteq \Gamma_+(g)$. Then $p_g(f) \geq p_g(g)$.*

Proof Since f, g are non-degenerate, so by Theorem 2.2, we have $p_g(g) = \#\{p \in \mathbb{Z}_+^n \cap \Gamma_-(g)\}$ and $p_g(f) = \#\{p \in \mathbb{Z}_+^n \cap \Gamma_-(f)\}$. $\Gamma_+(f) \subseteq \Gamma_+(g)$ implies that $\Gamma_-(g) \subseteq \Gamma_-(f)$. Thus we have $\mathbb{Z}_+^n \cap \Gamma_-(g) \subseteq \mathbb{Z}_+^n \cap \Gamma_-(f)$, this implies $p_g(f) \geq p_g(g)$. □

When $2 \leq \nu \leq n-1$, the Main Theorem A is automatically true. So one only needs to consider the case that $\nu \geq n$. We first assume that f is a non-degenerate (see Definition 2.8) and convenient singularity (see Definition 2.9). In fact the condition "convenient" is not a restriction. This is because by adding z_i^N for a sufficient large N, the isomorphism class of f does not change. Hereafter we shall assume that f is convenient.

Let $g = z_1^\nu + \cdots + z_n^\nu$ where ν is the multiplicity of f. It is obvious that g is non-degenerate and convenient. It is also easy to see that $\Gamma_+(f) \subseteq \Gamma_+(g)$. By Lemma 3.1, we conclude that $p_g(f) \geq p_g(g)$. It follows from Theorem 2.2 and (2.1) that $p_g(g) = P_n(\nu, \ldots, \nu) = Q_n(\nu - n, \ldots, \nu - n)$. Since $\nu \geq n$, so by Theorem 2.6, we have

$$p_g(g) = Q_n(\nu - n, \ldots, \nu - n) = \frac{1}{n!} \prod_{i=0}^{n-1} (\nu - i).$$

This completes the proof the theorem.

For general f, by Theorem 2.8 we can construct a 1-parameter deformation of $(V(f), 0)$ fixing the Newton polyhedron such that the generic fiber is a non-degenerate hypersurface. This means that f can deform to a non-degenerate singularity g, such that $\Gamma(f) = \Gamma(g)$ which implies $\nu(g) = \nu(f) = \nu$. Then it follows from the upper semi-continuity of p_g (cf. Theorem 2.7 for $m = 1$) that $p_g(f) \geq p_g(g)$. Since g is non-degenerate singularity, so we have $p_g(g) \geq \frac{1}{n!} \prod_{i=0}^{n-1} (\nu - i)$. Thus $p_g(f) \geq \frac{1}{n!} \prod_{i=0}^{n-1} (\nu - i)$. This completes the proof. □

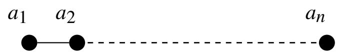
Proof of the Corollary 1.3

We need the following proposition which says that plurigenera $\{\delta_m\}_{m \in \mathbb{Z}_+}$ are topological invariants. For $m = 1$, this was proven in [28].

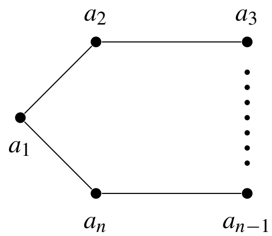
Proposition 3.1 *Let $(V, 0)$ and $(W, 0)$ be two isolated two dimensional hypersurface singularities in $\mathbb{C}^3$ having the same topological type. Then $\delta_m(V, 0) = \delta_m(W, 0)$.*

Proof Since $(V, 0)$ and $(W, 0)$ have the same topological type, the fundamental groups of the links of $(V, 0)$ and $(W, 0)$ are isomorphic (see for example [21]). Thus, by the result of Neumann [19], the minimal resolution graph Γ_V of $(V, 0)$ is the same as the minimal resolution graph Γ_W of $(W, 0)$ except the following two cases:

Case 1. Both Γ_V and Γ_W are exactly those of the form below with all $a_i \leq -2$



Case 2. Both Γ_V and Γ_W are exactly those of the form below with $a_i \leq -2$ and one $a_i \leq -3$.



In Case 1, we have $\delta_m(V, 0) = \delta_m(W, 0) = 0$, while in Case 2 we have $\delta_m(V, 0) = \delta_m(W, 0) = 1$ (see [7]).

In order to finish the proof of the proposition, we may assume that Γ_V is the same as Γ_W. It is well known that Milnor number μ of an isolated hypersurface singularity is an invariant of topological type (see for example [23]). Therefore, $\mu(V, 0) = \mu(W, 0)$. On the other hand, Laufer's formula [10], says that

$$1 + \mu = K^2 + \chi_T(A) + 12p_g,$$

where $\chi_T(A)$ is the topological Euler characteristic of A. Since K^2 and $\chi_T(A)$ can be computed from the resolution graph, it follows that $p_g(V, 0) = p_g(W, 0)$. By Proposition 2.3, $\delta_m(V, 0) = \delta_m(W, 0)$. □

Remark 3.1 The Proposition 3.1 is also correct for isolated Gorenstein surface singularities. The proof is the same as above. Notice that the Laufer's formula was generalized to isolated Gorenstein surface singularities by Steenbrink [22].

Since $(V, 0)$ and $(W, 0)$ have the same topological type and $p_g(V, 0) \leq 3$, so by Proposition 3.1, we have $p_g(V, 0) = p_g(W, 0) \leq 3$. In view of Corollary 1.2, we see that $\nu(V, 0) \leq 3$ and $\nu(W, 0) \leq 3$. Using the deep Theorem 2.3 and Theorem 2.4 that a surface singularity in $\mathbb{C}^3$ having multiplicity 2 cannot have the same topological type at 0 as another surface of multiplicity different from 2. It follows immediately that $\nu(V, 0) = \nu(W, 0)$. This completes the proof. □

We need the following proposition which is one of crucial steps in our proof of our Main Theorems B and C.

Proposition 3.2 [31] *Let* $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ *be a holomorphic germ defining an isolated hypersurface singularity* $V = \{z : f(z) = 0\}$ *at the origin. Let* μ *and* ν *be the Milnor number and multiplicity of* $(V, 0)$ *respectively. Then*

$$\mu \geq (\nu - 1)^n \tag{3.1}$$

and the equality in (3.1) *holds if and only if* f *is a semi-homogeneous function after a biholomorphic change of coordinates.*

Proof of the Main Theorem B

By Proposition 3.2, $\mu_f \geq (\nu_f - 1)^3$. Since $\mu_f \leq 26$, we have $\nu_f \leq 3$. Notice that Milnor number is an invariant of the topological type, we have also $\nu_g \leq 3$. By the same argument in the proof of Corollary 1.3, we have $\nu_f = \nu_g$. □

Proof of the Main Theorem C

By Proposition 3.2, $\mu_f \geq (\nu_f - 1)^n$. Since $\mu_f \leq 2^n - 1$, we have $\nu_f \leq 2$. Moreover, the Milnor number is an invariant of the topological type, we have also $\nu_g \leq 2$. By Theorem 2.3, it follows immediately $\nu_f = \nu_g$. □

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