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On the new k-th Yau algebras of isolated hypersurface singularities

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Abstract

Let V be a hypersurface with an isolated singularity at the origin defined by the holomorphic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. The Yau algebra $L(V)$ is defined to be the Lie algebra of derivations of the moduli algebra $A(V) := \mathcal{O}_n/(f, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$, i.e., $L(V) = \mathrm{Der}(A(V), A(V))$ and plays an important role in singularity theory. It is known that $L(V)$ is a finite dimensional Lie algebra and its dimension $\lambda(V)$ is called Yau number. In this article, we generalize the Yau algebra and introduce a new series of k-th Yau algebras $L^k(V)$ which are defined to be the Lie algebras of derivations of the moduli algebras $A^k(V) = \mathcal{O}_n/(f, m^k J(f)), k \geq 0$, i.e., $L^k(V) = \mathrm{Der}(A^k(V), A^k(V))$ and where m is the maximal ideal of $\mathcal{O}_n$. In particular, it is Yau algebra when $k = 0$. The dimension of $L^k(V)$ is denoted by $\lambda^k(V)$. These numbers i.e., k-th Yau numbers $\lambda^k(V)$, are new numerical analytic invariants of an isolated singularity. In this paper we studied these new series of Lie algebras $L^k(V)$ and also compute the Lie algebras $L^1(V)$ for fewnomial isolated singularities. We also formulate a sharp upper estimate conjecture for the $\lambda^k(V)$ of weighted homogeneous isolated hypersurface singularities and we prove this conjecture in case of $k = 1$ for large class of singularities.

Keywords Isolated hypersurface singularity · Lie algebra · Moduli algebra

Mathematics Subject Classification 14B05 · 32S05

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1 Introduction

The algebra of germs of holomorphic functions at the origin of $\mathbb{C}^n$ is denoted as $\mathcal{O}_n$. Clearly, $\mathcal{O}_n$ can be naturally identified with the algebra of convergent power series in n indeterminates with complex coefficients. As a ring $\mathcal{O}_n$ has a unique maximal ideal m, the set of germs of holomorphic functions which vanish at the origin. Let $\mathbb{C}[x_1, \dots, x_n]$ be the polynomial ring. For any $f \in \mathbb{C}[x_1, \dots, x_n]$, we denote by $V = V(f)$ the germ at the origin of $\mathbb{C}^n$ of hypersurface $\{f = 0\} \subset \mathbb{C}^n$. In other words, if the origin is an isolated zero of the gradient of f, then V is a germ of isolated hypersurface singularity. According to Hilbert's Nullstellensatz for an isolated singularity $V = V(f) = \{f = 0\}$ the factor-algebra $A(V) = \mathcal{O}_n/(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$ is finite dimensional. This factor-algebra is called the moduli algebra of V and its dimension $\tau(V)$ is called Tyurina number. The Mather–Yau theorem stated that: Let V_1 and V_2 be two isolated hypersurface singularities and, $A(V_1)$ and $A(V_2)$ be the moduli algebra, then $(V_1, 0) \cong (V_2, 0) \iff A(V_1) \cong A(V_2)$.

The order of the lowest nonvanishing term in the power series expansion of f at 0 is called the multiplicity (denoted by $mult(f)$) of the singularity $(V, 0)$. It is well-known that a polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ is said to be weighted homogeneous if there exist positive rational numbers $w_1, \dots, w_n$ (weights of $x_1, \dots, x_n$) and d such that, $\sum a_i w_i = d$ for each monomial $\prod x_i^{a_i}$ appearing in f with nonzero coefficient. The number d is called weighted homogeneous degree (w-degree) of f with respect to weights w_j. The weight type of f is denoted as $(w_1, \cdots, w_n; d)$. Without loss of generality, we can assume that w-deg$f = 1$. According to [20,29] the weight types of 1 or 2-dimensional weighted homogeneous hypersurface singularities are toplogical invariants. The Milnor number of the isolated hypersurface singularity is defined by $\mu = \dim \mathbb{C}[x_1, \dots, x_n] \big/ (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$. The Milnor number in case of weighted homogeneous hypersurface singularity is calculated by: $\mu = (\frac{1}{w_1} - 1)(\frac{1}{w_2} - 1) \cdots (\frac{1}{w_n} - 1)$ [17]. In 1971, Saito was the first person who gave the necessary and sufficient numerical condition for V to be defined by a weighted homogeneous polynomial. His beautiful theorem says that f is a weighted homogeneous polynomial after a biholomorphic change of coordinates $\iff \mu = \tau$ [19].

Another important class of isolated hypersurface singularity is fewnomial singularities which is defined by Elashvili and Khimshiashvili [10]. A weighted homogeneous polynomial $f(x_1, \dots, x_n)$ is called fewnomial if number of variables coincides with number of monomials [10,15,16,32]. According to Ebeling and Takahashi [11], the fewnomial singularity, which is defined by a fewnomial polynomial, is also called an invertible singularity.

It is well-known that for any isolated hypersurface singularity $(V, 0) \subset (\mathbb{C}^n, 0)$ where $V = V(f) = \{f = 0\}$, based on the Mather–Yau theorem [18], one considers the Lie algebra of derivations of moduli algebra $A(V) := \mathcal{O}_n/(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$, i.e., $L(V) = \mathrm{Der}(A(V), A(V))$. It is known that $L(V)$ is a finite dimensional solvable Lie algebra [24, 25]. $L(V)$ is called the Yau algebra of V in [30] and [16] in order to distinguish from Lie algebras of other types appearing in singularity theory [1,3,5]. The Yau algerba play an improtant role in singularities. Yau and his collaborators have been systematically studying the Lie algebras of isolated hypersurface singularities begin from eighties (see, e.g., [4,5,7–9,13,14,22–28,31,32]). The Mather–Yau theorem was slightly generalized in ([12], Theorem 2.26) (without assuming isolated singularity):

Theorem 1.1 *Let $f, g \in m \subset \mathcal{O}_n$. The following are equivalent:*

(1) $(V(f), 0) \cong (V(g), 0)$*;*
(2) For all $k \geq 0$, $\mathcal{O}_n/(f, m^k J(f)) \cong \mathcal{O}_n/(g, m^k J(g))$ *as* $\mathbb{C}$*-algebra;*

(3) There is some $k \geq 0$ such that $\mathcal{O}_n/(f, m^k J(f)) \cong \mathcal{O}_n/(g, m^k J(g))$ as $\mathbb{C}$-algebra, where $J(f) = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$.

In particular, if $k = 0$ and $k = 1$ above, then the claim of the equivalence of (1) and (3) is exactly the Mather–Yau theorem [18].

Based on Theorem 1.1, it is natural for us to introduce the new series of k-th Yau algebras $L^k(V)$ (or $L^k((V, 0))$) which are defined to be the Lie algebra of derivations of the moduli algebra $A^k(V) = \mathcal{O}_n/(f, m^k J(f))$, $k \geq 0$, i.e., $L^k(V) = \mathrm{Der}(A^k(V), A^k(V))$ and where m is the maximal ideal. Its dimension is denoted as $\lambda^k(V)$ (or $\lambda^k((V, 0))$). This number $\lambda^k(V)$ is a new numerical analytic invariant. We call it k-th Yau number. We have reasons to believe that these new Lie algebras and numerical invariants will also play an important role in the study of singularities.

It is interesting to bound the Yau number with a number which depends on weight type. In [32], Yau and Zuo firstly proposed the sharp upper estimate conjecture that bound the Yau number. They also proved that this conjecture holds in case of binomial isolated hypersurface singularities. Furthermore, in [14], this conjecture was verified for trinomial singularities.

A natural interesting question is: whether one can give a sharp bound for the k-th Yau numbers of isolated hypersurface singularities. We proposed the following sharp upper estimate conjecture which is a generalization of the conjecture in [32].

Conjecture 1.1 *Assume that $\lambda^k(\{x_1^{a_1} + \cdots + x_n^{a_n} = 0\}) = h_k(a_1, \ldots, a_n)$, $(k \geq 0)$. Let $(V, 0) = \{(x_1, x_2, \ldots, x_n) \in \mathbb{C}^n : f(x_1, x_2, \ldots, x_n) = 0\}$, $(n \geq 2)$ be an isolated singularity defined by the weighted homogeneous polynomial $f(x_1, x_2, \ldots, x_n)$ of weight type $(w_1, w_2, \ldots, w_n; 1)$. Then $\lambda^k(V) \leq h_k(1/w_1, \ldots, 1/w_n)$.*

The conjecture was proved for binomial and trinomial singularities when $k = 0$ [14,32].

The main purpose of this paper is to prove the conjecture for binomial and trinomial singularities when $k = 1$. We obtain the following main results.

Main Theorem A *Let $(V, 0) = \{(x_1, x_2, \ldots, x_n) \in \mathbb{C}^n : x_1^{a_1} + \cdots + x_n^{a_n} = 0\}$, $(n \geq 2)$. Then*

$$\lambda^1(V) = h_1(a_1, \ldots, a_n) = \sum_{j=1}^{n} \frac{a_j - 2}{a_j - 1} \prod_{i=1}^{n} (a_i - 1) + n(n+1).$$

Main Theorem B *Let $(V, 0)$ be a binomial singularity defined by the weighted homogeneous polynomial $f(x_1, x_2)$ (see corollary 2.1) with weight type $(w_1, w_2; 1)$. Then*

$$\lambda^1(V) \leq h_1\left(\frac{1}{w_1}, \frac{1}{w_2}\right) = \sum_{j=1}^{2} \frac{\frac{1}{w_j} - 2}{\frac{1}{w_j} - 1} \prod_{i=1}^{2} \left(\frac{1}{w_i} - 1\right) + 6.$$

Main Theorem C *Let $(V, 0)$ be a fewnomial singularity defined by the weighted homogeneous polynomial $f(x_1, x_2, x_3)$ (see proposition 2.2) with weight type $(w_1, w_2, w_3; 1)$. Then*

$$\lambda^1(V) \leq h_1\left(\frac{1}{w_1}, \frac{1}{w_2}, \frac{1}{w_3}\right) = \sum_{j=1}^{3} \frac{\frac{1}{w_j} - 2}{\frac{1}{w_j} - 1} \prod_{i=1}^{3} \left(\frac{1}{w_i} - 1\right) + 12.$$

2 Generalities on derivation Lie algebras of isolated singularities

In this section we shall briefly defined the basic definitions and important results which are helpful to solve the problem. The following basic concepts and results will be used to compute the derivation Lie algebras of isolated hypersurface singularities.

Let A, B be associative algebras over $\mathbb{C}$. The subalgebra of endomorphisms of A generated by the identity element and left and right multiplications by elements of A is called multiplication algebra $M(A)$ of A. The centroid $C(A)$ is defined as the set of endomorphisms of A which commute with all elements of $M(A)$. Obviously, $C(A)$ is a unital subalgebra of End (A). The following statement is a particular case of a general result from Proposition 1.2 of [6]. Let $S = A \otimes B$ be a tensor product of finite dimensional associative algebras with units. Then

$$\mathrm{Der} S \cong (\mathrm{Der} A) \otimes C(B) + C(A) \otimes (\mathrm{Der} B).$$

We will only use this result for commutative associative algebras with unit, in which case the centroid coincides with the algebra itself and one has following result for commutative associative algebras A, B:

Theorem 2.1 [6] *For commutative associative algebras A, B,*

$$Der S \cong (Der A) \otimes B + A \otimes (Der B). \tag{2.1}$$

We shall use this formula in the sequel.

Definition 2.1 Let J be an ideal in an analytic algebra S. Then $\mathrm{Der}_J S \subseteq \mathrm{Der}_{\mathbb{C}} S$ is Lie subalgebra of all $\sigma \in \mathrm{Der}_{\mathbb{C}} S$ for which $\sigma(J) \subset J$.

We shall use the following well-known result to compute the derivations.

Theorem 2.2 [32] *Let J be an ideal in $R = \mathbb{C}\{x_1, \ldots, x_n\}$. Then there is a natural isomorphism of Lie algebras*

$$(Der_J R)/(J \cdot Der_{\mathbb{C}} R) \cong Der_{\mathbb{C}}(R/J).$$

Recall that a derivation of commutative associative algebra A is defined as a linear endomorphism D of A satisfying the Leibniz rule: $D(ab) = D(a)b + aD(b)$. Thus for such an algebra A one can consider the Lie algebra of its derivations $Der(A, A)$ with the bracket defined by the commutator of linear endomorphisms.

Definition 2.2 Let $f(x_1, \ldots, x_n)$ be a complex polynomial and $V = \{f = 0\}$ be a germ of an isolated hypersurface singularity at the origin in $\mathbb{C}^n$. Let $A^k(V) = \mathcal{O}_n/(f, m^k J(f))$, $1 \leq k \leq n$ be a moduli algebra. Then $Der(A^k(V), A^k(V))$ defined the derivation Lie algebras $L^k(V)$. The $\lambda^k(V)$ is the dimension of derivation Lie algebra $L^k(V)$.

It is noted that when $k = 0$, then derivation Lie algebra is called Yau algebra.

Definition 2.3 A polynomial $f \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ is called quasi-homogeneous (or weighted homogeneous) if there exist positive rational numbers $w_1, \ldots, w_n$ (called weights of indeterminates x_j) and d such that, for each monomial $\prod x_j^{k_j}$ appearing in f with non-zero coefficient, one has $\sum w_j k_j = d$. The number d is called the quasi-homogeneous degree (w-degree) of f with respect to weights w_j and is denoted deg f. The collection $(w; d) = (w_1, \ldots, w_n; d)$ is called the quasi-homogeneity type (qh-type) of f.

Definition 2.4 An isolated hypersurface singularity in $\mathbb{C}^n$ is fewnomial if it can be defined by a n-nomial in n variables and it is a weighted homogeneous fewnomial isolated singularity if it can be defined by a weighted homogeneous fewnomial. 3-nomial isolated hypersurface singularity is also called trinomial singularity.

Proposition 2.1 *Let f be a weighted homogeneous fewnomial isolated singularity with mult $(f) \geq 3$. Then f analytically equivalent to a linear combination of the following three series:*

Type A. $x_1^{a_1} + x_2^{a_2} + \cdots + x_{n-1}^{a_{n-1}} + x_n^{a_n}$, $n \geq 1$,
Type B. $x_1^{a_1} x_2 + x_2^{a_2} x_3 + \cdots + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n}$, $n \geq 2$,
Type C. $x_1^{a_1} x_2 + x_2^{a_2} x_3 + \cdots + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n} x_1$, $n \geq 2$.

Proposition 2.1 has an immediate corollary.

Corollary 2.1 *Each binomial isolated singularity is analytically equivalent to one from the three series:*

(A) $x_1^{a_1} + x_2^{a_2}$,
(B) $x_1^{a_1} x_2 + x_2^{a_2}$,
(C) $x_1^{a_1} x_2 + x_2^{a_2} x_1$.

Wolfgang and Atsushi [11] give the following classification of weighted homogeneous fewnomial singularities in case of three variables.

Proposition 2.2 [11] *Let $f(x_1, x_2, x_3)$ be a weighted homogeneous fewnomial isolated singularity with mult $(f) \geq 3$. Then f is analytically equivalent to following five types:*

Type 1. $x_1^{a_1} + x_2^{a_2} + x_3^{a_3}$,
Type 2. $x_1^{a_1} x_2 + x_2^{a_2} x_3 + x_3^{a_3}$,
Type 3. $x_1^{a_1} x_2 + x_2^{a_2} x_3 + x_3^{a_3} x_1$,
Type 4. $x_1^{a_1} + x_2^{a_2} + x_3^{a_3} x_2$,
Type 5. $x_1^{a_1} x_2 + x_2^{a_2} x_1 + x_3^{a_3}$.

3 Proof of main theorems

In order to prove the main theorems, we need to prove following propositions.

Proposition 3.1 *Let $(V, 0)$ be a weighted homogeneous fewnomial isolated singularity which is defined by $f = x_1^{a_1} + x_2^{a_2} + \cdots + x_n^{a_n}$ ($a_i \geq 3, i = 1, 2, \ldots, n$) with weight type $(\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n}; 1)$. Then*

$$\lambda^1(V) = \sum_{j=1}^{n} \frac{a_j - 2}{a_j - 1} \prod_{i=1}^{n} (a_i - 1) + n(n+1).$$

Proof It follows that the generalized moduli algebra

$$A^1(V) = \mathbb{C}\{x_1, x_2, \ldots, x_n\}/(f, m.J(f)),$$

has dimension $(a_1 - 1)(a_2 - 1)(a_3 - 1) \ldots (a_n - 1) + n$ and has a monomial basis of the form (cf. [2], Theorem 13.1)

$$\{x_1^{i_1}x_2^{i_2}\dots x_n^{i_n}, 0\le i_1\le a_1-2, 0\le i_2\le a_2-2,\dots,0\le i_n\le a_n-2;$$
$$x_1^{a_1-1}, x_2^{a_2-1}, x_3^{a_3-1},\dots,x_n^{a_n-1}\},$$

with following relations:

$$\begin{aligned}
a_1x_1^{a_1} &= a_1x_1^{a_1-1}x_2 = a_1x_1^{a_1-1}x_3=\dots=a_1x_1^{a_1-1}x_n=0,\\
a_2x_1x_2^{a_2-1} &= a_2x_2^{a_2} = a_2x_2^{a_2-1}x_3 = a_2x_2^{a_2-1}x_4=\dots=a_2x_2^{a_2-1}x_n=0,\\
a_3x_1x_3^{a_3-1} &= a_3x_2x_3^{a_3-1} = a_3x_3^{a_3} = a_3x_3^{a_3-1}x_4=\dots=a_3x_3^{a_3-1}x_n=0,\\
&\vdots\\
a_nx_1x_n^{a_n-1} &= a_nx_2x_n^{a_n-1} = a_nx_3x_n^{a_n-1}=\dots=a_nx_n^{a_n}=0.
\end{aligned}$$

In order to compute a derivation D of $A^1(V)$ it suffices to indicate its values on the generators $x_1, x_2,\dots,x_n$ which can be written in terms of the monomial basis. Without loss of generality, we write

$$\begin{aligned}
Dx_j &= \sum_{i_1=0}^{a_1-2}\sum_{i_2=0}^{a_2-2}\dots\sum_{i_n=0}^{a_n-2} c^j_{i_1,i_2,\dots,i_n}x_1^{i_1}x_2^{i_2}\dots x_n^{i_n} + c^j_{a_1-1,0,0,\dots,0}x_1^{a_1-1}\\
&+ c^j_{0,a_2-1,0,0,\dots,0}x_2^{a_2-1}+\dots+c^j_{0,0,0,\dots,a_n-1}x_n^{a_n-1},\ \ j=1,2,\dots,n.
\end{aligned}$$

We obtain the following description of the Lie algebras in question [24,32]. The following derivations form a basis of Der $A^1(V)$:

$$\begin{aligned}
&x_1^{i_1}x_2^{i_2}\dots x_n^{i_n}\partial_1,\ \ 1\le i_1\le a_1-2, 0\le i_2\le a_2-2, 0\le i_3\le a_3-2,\dots,0\le i_n\le a_n-2;\\
&\qquad x_1^{a_1-1}\partial_1, x_2^{a_2-1}\partial_1,\dots,x_n^{a_n-1}\partial_1;\\
&\qquad (x_2^{a_2-2}x_3^{a_3-2}x_4^{a_4-2}\dots x_n^{a_n-2})\partial_1;\\
&x_1^{i_1}x_2^{i_2}\dots x_n^{i_n}\partial_2,\ \ 0\le i_1\le a_1-2, 1\le i_2\le a_2-2, 0\le i_3\le a_3-2,\\
&\qquad\ 0\le i_4\le a_4-2,\dots,0\le i_n\le a_n-2;\\
&\qquad x_1^{a_1-1}\partial_2, x_2^{a_2-1}\partial_2,\dots,x_n^{a_n-1}\partial_2;\\
&\qquad (x_1^{a_1-2}x_3^{a_3-2}x_4^{a_4-2}\dots x_n^{a_n-2})\partial_2;\\
&x_1^{i_1}x_2^{i_2}\dots x_n^{i_n}\partial_3,\ \ 0\le i_1\le a_1-2, 0\le i_2\le a_2-2, 1\le i_3\le a_3-2,\\
&\qquad\ 0\le i_4\le a_4-2,\dots,0\le i_n\le a_n-2;\\
&\qquad x_1^{a_1-1}\partial_3, x_2^{a_2-1}\partial_3,\dots,x_n^{a_n-1}\partial_3;\\
&\qquad (x_1^{a_1-2}x_2^{a_2-2}x_4^{a_4-2}\dots x_n^{a_n-2})\partial_3;\\
&\vdots\\
&x_1^{i_1}x_2^{i_2}\dots x_n^{i_n}\partial_n,\ \ 0\le i_1\le a_1-2, 0\le i_2\le a_2-2, 0\le i_3\le a_3-2,\dots,1\le i_n\le a_n-2;\\
&\qquad x_1^{a_1-1}\partial_n, x_2^{a_2-1}\partial_n,\dots,x_n^{a_n-1}\partial_n;\\
&\qquad (x_1^{a_1-2}x_2^{a_2-2}x_3^{a_3-2}\dots x_{n-1}^{a_{n-1}-2})\partial_n.
\end{aligned}$$

Therefore we have the following formula

$$\lambda^1(V) = \sum_{j=1}^{n} \frac{a_j - 2}{a_j - 1} \prod_{i=1}^{n} (a_i - 1) + n(n+1).$$

□

Proposition 3.2 *Let* $(V, 0)$ *be a weighted homogeneous fewnomial isolated singularity of type A which is defined by* $f = x_1^{a_1} + x_2^{a_2}$ $(a_1 \geq 2,\ a_2 \geq 2)$ *with weight type* $(\frac{1}{a_1}, \frac{1}{a_2}; 1)$. *Then*

$$\lambda^1(V) = \begin{cases} 2a_1a_2 - 3(a_1 + a_2) + 10; & a_1 \geq 3, a_2 \geq 3 \\ a_1 + 2; & a_1 \geq 2, a_2 = 2. \end{cases}$$

Proof It follows that the generalized moduli algebra

$$A^1(V) = \mathbb{C}\{x_1, x_2\}/(f, m.J(f)),$$

has dimension $a_1a_2 - (a_1 + a_2) + 3$ and has a monomial basis of the form (cf. [2], Theorem 13.1)

$$(1) \text{ if } a_1 \geq 3,\ \{x_1^{i_1}x_2^{i_2}, 0 \leq i_1 \leq a_1 - 2; 0 \leq i_2 \leq a_2 - 2; x_1^{a_1-1}; x_2^{a_2-1}\}, \tag{3.1}$$

$$(2) \text{ if } a_2 = 2,\ \{x_1^{i_1}, 0 \leq i_1 \leq a_1 - 1; x_2\}, \tag{3.2}$$

with the following relations:

$$x_1^{a_1} + x_2^{a_2} = 0, \tag{3.3}$$

$$a_1x_1^{a_1} = 0, \tag{3.4}$$

$$a_1x_1^{a_1-1}x_2 = 0, \tag{3.5}$$

$$a_2x_2^{a_2} = 0, \tag{3.6}$$

$$a_2x_2^{a_2-1}x_1 = 0. \tag{3.7}$$

In order to compute a derivation D of $A^1(V)$ it suffices to indicate its values on the generators x_1, x_2 which can be written in terms of the basis (3.1) or (3.2). Without loss of generality, we write

$$Dx_j = \sum_{i_1=0}^{a_1-2} \sum_{i_2=0}^{a_2-2} c_{i_1,i_2}^j x_1^{i_1}x_2^{i_2} + c_{a_1-1,0}^j x_1^{a_1-1} + c_{0,a_2-1}^j x_2^{a_2-1},\ \ j = 1, 2.$$

Using the relations (3.3)–(3.7) one easily finds the necessary and sufficient conditions defining a derivation of $A^1(V)$ as follows:

$$c_{0,0}^1 = c_{0,1}^1 = \cdots = c_{0,a_2-3}^1 = 0; \tag{3.8}$$

$$c_{0,0}^2 = c_{1,0}^2 = \cdots = c_{a_1-3,0}^2 = 0. \tag{3.9}$$

Using (3.8)–(3.9) we obtain the following description of the Lie algebras in question. The following derivations form a basis of $\mathrm{Der}A^1(V)$:

$$x_1^{i_1}x_2^{i_2}\partial_1, 1 \leq i_1 \leq a_1 - 2, 0 \leq i_2 \leq a_2 - 2; x_1^{a_1-1}\partial_1; x_2^{a_2-2}\partial_1; x_2^{a_2-1}\partial_1;$$
$$x_1^{i_1}x_2^{i_2}\partial_2, 0 \leq i_1 \leq a_1 - 2, 1 \leq i_2 \leq a_2 - 2; x_1^{a_1-1}\partial_2; x_1^{a_1-2}\partial_2; x_2^{a_2-1}\partial_2.$$

Therefore we have the following formula

$$\lambda^1(V) = 2a_1a_2 - 3(a_1 + a_2) + 10.$$

In case of

$$a_1 \geq 2, a_2 = 2,$$

we have following derivations which form a basis of $L^1(V)$:

$$x_1^{i_1}\partial_1, 1 \leq i_1 \leq a_1 - 1; x_2\partial_1, x_2\partial_2, x_1^{a_1-1}\partial_2.$$

Therefore we get following formula

$$\lambda^1(V) = a_1 + 2.$$

□

Proposition 3.3 *Let $(V, 0)$ be a binomial isolated singularity of type B which is defined by $f = x_1^{a_1}x_2 + x_2^{a_2}$ $(a_1 \geq 1,\ a_2 \geq 2)$ with weight type $(\frac{a_2-1}{a_1a_2}, \frac{1}{a_2}; 1)$. Then*

$$\lambda^1(V) = \begin{cases} 2a_1a_2 - 2a_1 - 3a_2 + 11; & a_1 \geq 2, a_2 \geq 3 \\ 2a_1 + 2; & a_1 \geq 2, a_2 = 2 \\ 4; & a_1 = 1, a_2 \geq 2. \end{cases}$$

Furthermore, if $mult(f) \geq 3$, then our proposed conjecture is true, i.e.,

$$2a_1a_2 - 2a_1 - 3a_2 + 11 \leq \frac{2a_1a_2^2}{a_2 - 1} - 3\left(\frac{a_1a_2}{a_2 - 1} + a_2\right) + 10.$$

Proof It is easy to see that the generalized moduli algebra

$$A^1(V) = \mathbb{C}\{x_1, x_2\}/(f, mJ(f)),$$

has dimension $a_2(a_1 - 1) + 3$ and has a monomial basis of the form (cf. [2], Theorem 13.1)

$$\text{(1) if } a_1 \geq 2,\ \{x_1^{i_1}x_2^{i_2}, 0 \leq i_1 \leq a_1 - 2; 0 \leq i_2 \leq a_2 - 1; x_1^{i_1}, a_1 - 1 \leq i_1 \leq a_1; x_1^{a_1-1}x_2\}, \tag{3.10}$$

$$\text{(2) if } a_2 = 2,\ \{x_1^{i_1}, 0 \leq i_1 \leq 2a_1 - 1; x_2\}, \tag{3.11}$$

$$\text{(3) if } a_1 = 1,\ \{1; x_1; x_2\}, \tag{3.12}$$

with the following relations:

$$x_1^{a_1}x_2 + x_2^{a_2} = 0, \tag{3.13}$$
$$a_1x_1^{a_1}x_2 = 0, \tag{3.14}$$
$$a_1x_1^{a_1-1}x_2^2 = 0, \tag{3.15}$$
$$x_1^{a_1+1} + a_2x_1x_2^{a_2-1} = 0, \tag{3.16}$$
$$x_1^{a_1}x_2 + a_2x_2^{a_2} = 0. \tag{3.17}$$

Using the relations (3.13)–(3.17) we get

$$x_1^i = 0,\ i \geq 2a_1 - 1, \tag{3.18}$$
$$x_2^i = 0,\ i \geq a_2. \tag{3.19}$$

In order to compute a derivation D of $A^1(V)$ it suffices to indicate its values on the generators x_1, x_2 which can be written in terms of the basis (3.10), (3.11) or (3.12). Thus we can write

$$Dx_j = \sum_{i_1=0}^{a_1-2}\sum_{i_2=0}^{a_2-1} c^j_{i_1,i_2} x_1^{i_1} x_2^{i_2} + \sum_{i_1=a_1-1}^{a_1} c^j_{i_1,0} x_1^{i_1} + c^j_{a_1-1,1} x_1^{a_1-1} x_2,\ \ j = 1, 2.$$

Using the relations (3.13)–(3.19) one finds the conditions defining a derivation of $A^1(V)$ as follows:

$$c^1_{0,0} = c^1_{0,1} = \cdots = c^1_{0,a_2-3} = 0; \tag{3.20}$$

$$c^2_{0,0} = c^2_{1,0} = \cdots = c^2_{a_1-2,0} = 0; \tag{3.21}$$

$$c^1_{1,0} = c^2_{0,1}, c^1_{2,0} = c^2_{1,1}, \ldots, c^1_{a_1-2,0} = c^2_{a_1-4,1}. \tag{3.22}$$

Using (3.20)–(3.22) we obtain the following description of the Lie algebra in question. The following derivations form a basis of $\mathrm{Der}A^1(V)$:

$$\begin{aligned}
&x_1^{i_1} x_2^{i_2}\partial_1, 1 \le i_1 \le a_1-2, 2 \le i_2 \le a_2-1; x_1^{i_1}\partial_1, a_1-1 \le i_1 \le a_1;\\
&x_1^{i_1} x_2\partial_1, 1 \le i_1 \le a_1-1; x_2^{i_2}\partial_1, a_2-2 \le i_2 \le a_2-1;\\
&x_1^{i_1} x_2\partial_2, a_1-2 \le i_1 \le a_1-1; x_1^{i_1}\partial_2, a_1-1 \le i_1 \le a_1;\\
&x_1^{i_1}\partial_1 + x_1^{i_1-1} x_2\partial_2, 1 \le i_1 \le a_1-2;\\
&x_1^{i_1} x_2^{i_2}\partial_2, 0 \le i_1 \le a_1-2, 2 \le i_2 \le a_2-1.
\end{aligned}$$

Therefore we have following formula

$$\lambda^1(V) = 2a_1a_2 - 2a_1 - 3a_2 + 11.$$

In case of

$$a_1 \ge 2, a_2 = 2,$$

we obtain the conditions defining a derivation of $A^1(V)$ as follows:

$$c^1_{0,0} = 0, a_1c^1_{1,0} + 2c^2_{a_1,0} = c^2_{0,1}; \tag{3.23}$$

$$c^2_{0,0} = c^2_{1,0} = \cdots = c^2_{a_1-1,0} = 0; \tag{3.24}$$

$$a_1c^1_{i_1,0} + c^2_{a_1-1+i_1,0} = 0, 2 \le i_1 \le a_1-1. \tag{3.25}$$

Using (3.23)–(3.25) we obtain the following description of the Lie algebra in question. The following derivations form a basis of $\mathrm{Der}A^1(V)$:

$$\begin{aligned}
&x_1^{2a_1-1}\partial_2, x_1^{i_1}\partial_1, a_1 \le i_1 \le 2a_1-1;\\
&2x_1^{i_1}\partial_1 - a_1x_1^{i_1-1+a_1}\partial_2, 2 \le i_1 \le a_1-1;\\
&x_1\partial_1 + a_1x_2\partial_2, -2x_1\partial_1 + a_1x_1^{a_1}\partial_2, x_2\partial_1.
\end{aligned}$$

Therefore we have the following formula

$$\lambda^1(V) = 2a_1 + 2. \tag{3.26}$$

In case of $a_1 = 1, a_2 \ge 2$, we have following derivations form a basis of $\mathrm{Der}A^1(V)$:

$$x_1\partial_1; x_2\partial_1; x_1\partial_2; x_2\partial_2.$$

Therefore we have the following formula

$$\lambda^1(V) = 4. \tag{3.27}$$

$$2a_1a_2 - 2a_1 - 3a_2 + 11 \leq \frac{2a_1a_2^2}{a_2-1} - 3\left(\frac{a_1a_2}{a_2-1} + a_2\right) + 10. \tag{3.28}$$

It is follows from proposition 3.2 we have

$$h_1(a_1, a_2) = \begin{cases} 2a_1a_2 - 3(a_1+a_2) + 10; & a_1 \geq 3, a_2 \geq 3 \\ a_1 + 2; & a_1 \geq 2, a_2 = 2. \end{cases}$$

After putting the weight type $(\frac{a_2-1}{a_1a_2}, \frac{1}{a_2}; 1)$ of binomial isolated singularity of type B we have

$$h_1\left(\frac{1}{w_1}, \frac{1}{w_2}\right) = \begin{cases} \frac{2a_1a_2^2}{a_2-1} - 3\left(\frac{a_1a_2}{a_2-1} + a_2\right) + 10; & a_1 \geq 2, a_2 \geq 3 \\ 2a_1 + 2; & a_1 \geq 1, a_2 = 2. \end{cases}$$

Finally we need to show that After solving 3.28 we have $(a_2-2)(a_1-1)-1 \geq 0$. It is noted that $(a_2-2) \geq 1$ and $(a_1-1) \geq 1$. It is also noted that when $a_1 \geq 1, a_2 = 2$ then

$$h_1\left(\frac{1}{w_1}, \frac{1}{w_2}\right) = \lambda^1(V)$$

□

Proposition 3.4 *Let $(V, 0)$ be a binomial isolated singularity of type C which is defined by $f = x_1^{a_1}x_2 + x_2^{a_2}x_1$ $(a_1 \geq 1, a_2 \geq 1)$ with weight type $(\frac{a_2-1}{a_1a_2-1}, \frac{a_1-1}{a_1a_2-1}; 1)$. Then*

$$\lambda^1(V) = \begin{cases} 2a_1a_2 - 2a_1 - 2a_2 + 12; & a_1 \geq 3, a_2 \geq 3 \\ 2a_1 + 6; & a_1 \geq 2, a_2 = 2 \\ 4; & a_1 \geq 1, a_2 = 1 \\ 4; & a_1 = 1, a_2 \geq 2. \end{cases}$$

Furthermore, we need to show that when $a_1 \geq 3, a_2 \geq 3$ then

$$2a_1a_2 - 2(a_1+a_2) + 12 \leq \frac{2(a_1a_2-1)^2}{(a_1-1)(a_2-1)} - 3(a_1a_2-1)\left(\frac{a_1+a_2-2}{(a_1-1)(a_2-1)}\right) + 10.$$

Proof It is easy to see that when $a_1 \geq a_2 \geq 3$ then generalized moduli algebra

$$A^1(V) = \mathbb{C}\{x_1, x_2\}/(f, mJ(f)),$$

has a dimension $a_1a_2 + 2$ and has monomial basis of the form (cf. [2], Theorem 13.1)

$$\left\{x_1^{i_1}x_2^{i_2}, 1 \leq i_1 \leq a_1 - 2; 1 \leq i_2 \leq a_2 - 1; x_2^{i_2}, 1 \leq i_2 \leq 2a_2 - 2; \right.$$
$$\left. x_1^{a_1-1}x_2; x_1^{i_1}, 0 \leq i_1 \leq a_1 \right\}. \tag{3.29}$$

But if $a_1 \geq a_2 = 2$, then we have

$$\left\{x_1^{i_1}, 0 \leq i_1 \leq 2a_1 - 2; x_2, x_2^2, x_1x_2\right\}, \tag{3.30}$$

with the following relations:

$$x_1^{a_1}x_2 + x_2^{a_2}x_1 = 0, \tag{3.31}$$

$$a_1x_1^{a_1}x_2 + x_2^{a_2}x_1 = 0, \tag{3.32}$$
$$a_1x_1^{a_1-1}x_2^2 + x_2^{a_2+1} = 0, \tag{3.33}$$
$$x_1^{a_1+1} + a_2x_2^{a_2-1}x_1^2 = 0, \tag{3.34}$$
$$x_1^{a_1}x_2 + a_2x_2^{a_2}x_1 = 0. \tag{3.35}$$

Using the relations (3.31)–(3.35) we get

$$x_1^i = 0,\ i \geq 2a_1 - 2, \tag{3.36}$$
$$x_2^i = 0,\ i \geq 2a_2 - 1. \tag{3.37}$$

In order to compute a derivation D of $A^1(V)$ it suffices to indicate its values on the generators x_1, x_2 which can be written in the basis (3.29) or (3.30). Thus we can write

$$Dx_j = \sum_{i_1=1}^{a_1-2}\sum_{i_2=1}^{a_2-1} c_{i_1,i_2}^j x_1^{i_1}x_2^{i_2} + \sum_{i_2=1}^{2a_2-2} c_{0,i_2}^j x_2^{i_2} + c_{a_1-1,1}^j x_1^{a_1-1}x_2 + \sum_{i_1=0}^{a_1} c_{i_1,0}^j x_1^{i_1}\ , j = 1, 2.$$

Using the relations (3.31)–(3.37) one easily finds the necessary and sufficient conditions defining a derivation of $A^1(V)$ as follows:

$$c_{0,0}^1 = c_{0,1}^1 = \cdots = c_{0,a_2-3}^1 = 0; \tag{3.38}$$
$$c_{0,0}^2 = c_{1,0}^2 = \cdots = c_{a_1-3,0}^2 = 0;\ a_1c_{0,a_2-2}^1 = a_2c_{a_1-2,0}^2; \tag{3.39}$$
$$(a_1-1)c_{1,0}^1 = (a_2-1)c_{0,1}^2, \ldots, (a_1-1)c_{a_1-2,0}^1 = (a_2-1)c_{a_1-3,1}^2; \tag{3.40}$$
$$(a_1-1)c_{1,1}^1 = (a_2-1)c_{0,2}^2, \ldots, (a_1-1)c_{1,a_2-3}^1 = (a_2-1)c_{0,a_2-2}^2. \tag{3.41}$$

Using (3.38)–(3.41) we obtain the following description of the Lie algebra in question. The following derivations form a basis of $\mathrm{Der}A^1(V)$:

$$\begin{aligned}
&x_1^{i_1}x_2^{i_2}\partial_1,\ 2 \leq i_1 \leq a_1-2, 1 \leq i_2 \leq a_2-1;\\
&x_1x_2^{i_2}\partial_1,\ a_2-2 \leq i_2 \leq a_2-1;\ x_1^{i_1}\partial_1,\ a_1-1 \leq i_1 \leq a_1;\\
&x_2^{i_2}\partial_1,\ a_2-1 \leq i_2 \leq 2a_2-2;\\
&x_1^{i_1}x_2^{i_2}\partial_2,\ 1 \leq i_1 \leq a_1-2, 2 \leq i_2 \leq a_2-1;\\
&x_2^{i_2}\partial_2,\ a_2-1 \leq i_2 \leq 2a_2-2;\\
&x_1^{a_1-1}x_2\partial_1;\ x_1^{i_1}\partial_2,\ a_1-1 \leq i_1 \leq a_1;\ x_1^{i_1}x_2\partial_2,\ a_1-2 \leq i_1 \leq a_1;\\
&a_2x_2^{a_2-2}\partial_1 + a_1x_1^{a_1-2}\partial_2;\\
&(a_2-1)x_1x_2^{i_2}\partial_1 + (a_1-1)x_2^{i_2+1}\partial_2,\ 1 \leq i_2 \leq a_2-3;\\
&x_1^{i_1}\partial_1 + x_1^{i_1-1}x_2\partial_2,\ 1 \leq i_1 \leq a_1-2.
\end{aligned}$$

Therefore we have the following formula

$$\lambda^1(V) = 2a_1a_2 - 2(a_1+a_2) + 12. \tag{3.42}$$

In case of

$$a_1 \geq a_2 = 2,$$

Similarly one easily finds the necessary and sufficient conditions defining a derivation of $A^1(V)$ as follows:

$$c^1_{0,0} = 0; \tag{3.43}$$
$$c^2_{0,0} = c^2_{1,0} = \cdots = c^2_{a_1-2,0} = 0; \tag{3.44}$$
$$c^2_{0,1} = (a_2-1)c^1_{1,0} + 2c^2_{a_1-1,0};\ c^2_{1,1} = (a_2-1)c^1_{2,0} + 2c^2_{a_1,0}; \tag{3.45}$$
$$\begin{aligned}(a_2-1)c^1_{3,0} &= -2c^2_{(a_1-2)+3,0},\ (a_2-1)c^1_{4,0} = -2c^2_{(a_1-2)+4,0}, \ldots, (a_2-1)c^1_{a_1-2,0}\\ &= -2c^2_{2a_1-4,0}.\end{aligned} \tag{3.46}$$

Similarly one obtains the basis of derivation represented by following derivations form a basis of $\mathrm{Der}A^1(V)$:

$$\begin{aligned}&x_2\partial_1;\ x_1x_2\partial_1;\ x_2^2\partial_1;\ x_2^2\partial_1;\ x_1\partial_1 - (a_1-1)x_2\partial_2;\\ &x_1^{i_1}\partial_2;\ \ 2a_1-3 \le i_1 \le 2a_1-2;\ x_1^{i_1}\partial_1;\ \ a_1-1 \le i_1 \le 2a_1-2;\\ &x_1^2\partial_1 - (a_1-1)x_1x_2\partial_2;\ 2x_1^{i_1}\partial_1 - (a_1-1)x_1^{i_1+(a_1-2)}\partial_2,\ \ 1 \le i_1 \le a_1-2.\end{aligned}$$

Therefore we get the following formula

$$\lambda^1(V) = 2a_1 + 6.$$

In case of

$$a_1 \ge 1, a_2 = 1,$$

one obtains the basis of derivation represented by following derivations form a basis of $\mathrm{Der}A^1(V)$:

$$(x_1^{a_1} + x_1)\partial_1;\ x_2\partial_1;\ x_2\partial_2;\ (x_1^{a_1} + x_1)\partial_2.$$

Therefore we get the following formula

$$\lambda^1(V) = 4.$$

In case of

$$a_1 = 1, a_2 \ge 2,$$

one obtains the basis of derivation represented by following derivations form a basis of $\mathrm{Der}A^1(V)$:

$$(x_2^{a_2} + x_2)\partial_1;\ x_1\partial_1;\ x_1\partial_2;\ (x_2^{a_2} + x_2)\partial_2.$$

Therefore we get the following formula

$$\lambda^1(V) = 4.$$

It is follows from proposition 3.2 and weight types of proposition 3.4 we have $h_1(\frac{1}{w_1}, \frac{1}{w_2}) = \frac{2(a_1a_2-1)^2}{(a_1-1)(a_2-1)} - 3(a_1a_2-1)(\frac{a_1+a_2-2}{(a_1-1)(a_2-1)}) + 10$; $a_1 \ge 3, a_2 \ge 3$. Finally we need to show that

$$2a_1a_2 - 2(a_1+a_2) + 12 \le \frac{2(a_1a_2-1)^2}{(a_1-1)(a_2-1)} - 3(a_1a_2-1)\left(\frac{a_1+a_2-2}{(a_1-1)(a_2-1)}\right) + 10. \tag{3.47}$$

After solving 3.47 it is easy to see that

$$a_1(a_2-1)(a_1-3) + a_2(a_1-1)(a_2-3) + 2(a_1-3) + 2a_1 + 4a_2 \ge 0.$$

□

Proposition 3.5 *Let $(V, 0)$ be a fewnomial surface isolated singularity of type 1 which is defined by $f = x_1^{a_1} + x_2^{a_2} + x_3^{a_3}$ ($a_1 \geq 2, a_2 \geq 2, a_3 \geq 2$) with weight type $(\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}; 1)$. Then*

$$\lambda^1(V) = \begin{cases} a_1 + 7; & a_1 \geq 2, a_2 = 2, a_3 = 2 \\ a_2 + 7; & a_1 = 2, a_2 \geq 3, a_3 = 2 \\ a_3 + 7; & a_1 = 2, a_2 = 2, a_3 \geq 3 \\ 3a_1a_2a_3 + 5(a_1 + a_2 + a_3) & \\ \quad -4(a_1a_2 + a_1a_3 + a_2a_3) + 6; & Otherwise. \end{cases}$$

Proof It is easy to see that the moduli algebra $A^1(V) = \mathbb{C}\{x_1, x_2, x_3\}/(f, m.J(f))$ has dimension $(a_1a_2a_3 - a_1a_2 - a_1a_3 - a_2a_3 + a_1 + a_2 + a_3 + 2)$ and has a monomial basis of the form (cf. [2], Theorem 13.1)

$\{x_1^{i_1}x_2^{i_2}x_3^{i_3}, 1 \leq i_1 \leq a_1 - 2; 1 \leq i_2 \leq a_2 - 2; 0 \leq i_3 \leq a_3 - 2; x_1^{i_1}, 1 \leq i_1 \leq a_1 - 1;$
$x_1^{i_1}x_3^{i_3}, 1 \leq i_1 \leq a_1 - 2; 1 \leq i_3 \leq a_3 - 2; x_2^{i_2}x_3^{i_3}, 1 \leq i_2 \leq a_2 - 2; 0 \leq i_3 \leq a_3 - 2; x_2^{a_2-1};$
$x_3^{i_3}, 0 \leq i_3 \leq a_3 - 1\}.$

In order to compute a derivation D of $A^1(V)$ it suffices to indicate its values on the generators x_1, x_2, x_3 which can be written in terms of the basis. Thus we can write

$$Dx_j = \sum_{i_1=1}^{a_1-2}\sum_{i_2=1}^{a_2-2}\sum_{i_3=0}^{a_3-2} c^j_{i_1,i_2,i_3}x_1^{i_1}x_2^{i_2}x_3^{i_3} + \sum_{i_1=1}^{a_1-1} c^j_{i_1,0,0}x_1^{i_1} + \sum_{i_3=0}^{a_3-1} c^j_{0,0,i_3}x_3^{i_3} + \sum_{i_1=1}^{a_1-2}\sum_{i_3=1}^{a_3-2} c^j_{i_1,0,i_3}x_1^{i_1}x_3^{i_3}$$
$$+ \sum_{i_2=1}^{a_2-2}\sum_{i_3=0}^{a_3-2} c^j_{0,i_2,i_3}x_2^{i_2}x_3^{i_3} + c^j_{0,a_2-1,0}x_2^{a_2-1}, \ j = 1, 2, 3.$$

Using the above derivations we obtain the following description of Lie algebras in question. The derivations represented by the following vector fields form a basis in $\mathrm{Der}A^1(V)$:

$$\begin{aligned} &x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_1, \ 1 \leq i_1 \leq a_1 - 2, 0 \leq i_2 \leq a_2 - 2, 0 \leq i_3 \leq a_3 - 2; \\ &\quad x_2^{a_2-2}x_3^{a_3-2}\partial_1; x_1^{a_1-1}\partial_1; x_2^{a_2-1}\partial_1; x_3^{a_3-1}\partial_1; \\ &x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_2, \ 0 \leq i_1 \leq a_1 - 2, 1 \leq i_2 \leq a_2 - 2, 0 \leq i_3 \leq a_3 - 2; \\ &\quad x_1^{a_1-2}x_3^{a_3-2}\partial_2; x_1^{a_1-1}\partial_2; x_2^{a_2-1}\partial_2; x_3^{a_3-1}\partial_2; \\ &x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_3, \ 0 \leq i_1 \leq a_1 - 2, 0 \leq i_2 \leq a_2 - 2, 1 \leq i_3 \leq a_3 - 2; \\ &\quad x_1^{a_1-2}x_2^{a_2-2}\partial_3; x_1^{a_1-1}\partial_3; x_2^{a_2-1}\partial_3; x_3^{a_3-1}\partial_3. \end{aligned}$$

Therefore we have

$$\lambda^1(V) = 3a_1a_2a_3 + 5(a_1 + a_2 + a_3) - 4(a_1a_2 + a_1a_3 + a_2a_3) + 6.$$

In case of $a_1 \geq 2, a_2 = 2, a_3 = 2$, we obtain the basis of derivation represented by following derivations form a basis of $\mathrm{Der}A^1(V)$:

$$x_1^{i_1}\partial_1, \ 1 \leq i_1 \leq a_1 - 1; x_3\partial_1; x_2\partial_1; x_3\partial_2; x_2\partial_2; x_1^{a_1-1}\partial_2; x_2\partial_3; x_3\partial_3; x_1^{a_1-1}\partial_3.$$

Therefore we have

$$\lambda^1(V) = a_1 + 7.$$

In case of $a_1 = 2, a_2 \geq 3, a_3 = 2$, we obtain the basis of derivation represented by following derivations form a basis of Der$A^1(V)$:

$$x_2^{i_2}\partial_2,\ 1 \leq i_2 \leq a_2 - 1; x_3\partial_1; x_2^{a_2-1}\partial_1; x_3\partial_2; x_1\partial_1; x_1\partial_2; x_3\partial_3; x_1\partial_3; x_2^{a_2-1}\partial_3.$$

Therefore we have

$$\lambda^1(V) = a_2 + 7.$$

In case of $a_1 = 2, a_2 = 2, a_3 \geq 3$, we obtain the basis of derivation represented by following derivations form a basis of Der$A^1(V)$:

$$x_3^{i_3}\partial_3,\ 1 \leq i_3 \leq a_3 - 1; x_3^{a_3-1}\partial_1; x_2\partial_1; x_1\partial_1; x_3^{a_3-1}\partial_2; x_2\partial_2; x_1\partial_2; x_2\partial_3; x_1\partial_3.$$

Therefore we have

$$\lambda^1(V) = a_3 + 7.$$

□

Proposition 3.6 *Let $(V, 0)$ be a fewnomial surface isolated singularity of type 2 which is defined by $f = x_1^{a_1}x_2 + x_2^{a_2}x_3 + x_3^{a_3}$ $(a_1 \geq 1, a_2 \geq 1, a_3 \geq 1)$ with weight type $(\frac{1-a_3+a_2a_3}{a_1a_2a_3}, \frac{a_3-1}{a_2a_3}, \frac{1}{a_3}; 1)$. Then*

$$\lambda^1(V) = \begin{cases} 5a_3 + 7; & a_1 = 2, a_2 = 2, a_3 \geq 2 \\ 4a_1a_3 - 2a_1 - 3a_3 + 11; & a_1 \geq 3, a_2 = 2, a_3 \geq 3 \\ a_1a_3 + 7; & a_1 \geq 1, a_2 = 1, a_3 \geq 2 \\ a_2(a_1 - 2); & a_1 \geq 2, a_2 \geq 1, a_3 = 1 \\ 0; & a_1 = 1, a_2 \geq 1, a_3 = 1 \\ 3a_1a_2a_3 - 2a_1a_2 - 2a_1a_3 & \\ \quad -4a_2a_3 + 2a_1 + 2a_2 + 6a_3 + 5; & Otherwise. \end{cases}$$

Furthermore, we need to show that when $a_1 \geq 3, a_2 \geq 3, a_3 \geq 3$, then

$$3a_1a_2a_3 - 2a_1a_2 - 2a_1a_3 - 4a_2a_3 + 2a_1 + 2a_2 + 6a_3 + 5 \leq \frac{3a_1a_2^2a_3^3}{(1 - a_3 + a_2a_3)(a_3 - 1)} - 4\left(\frac{a_1a_2^2a_3^3}{(1 - a_3 + a_2a_3)(a_3 - 1)} + \frac{a_1a_2a_3^2}{1 - a_3 + a_2a_3} + \frac{a_2a_3^2}{a_3 - 1}\right) + 5\left(\frac{a_1a_2a_3}{1 - a_3 + a_2a_3} + \frac{a_2a_3}{a_3 - 1} + a_3\right) + 6.$$

Proof It is easy to see that the moduli algebra $A^1(V) = \mathbb{C}\{x_1, x_2, x_3\}/(f, m.J(f))$ has dimension $(a_1a_2a_3 - a_2a_3 + a_3 + 2)$ and has a monomial basis of the form (cf. [2], Theorem 13.1)

$\{x_1^{i_1}x_2^{i_2}x_3^{i_3}, 1 \leq i_1 \leq a_1 - 2; 1 \leq i_2 \leq a_2 - 1; 0 \leq i_3 \leq a_3 - 1; x_1^{a_1-1}x_3^{i_3}, 0 \leq i_3 \leq a_3 - 2;$
$x_2^{i_2}x_3^{i_3}, 0 \leq i_2 \leq a_2 - 1; 0 \leq i_3 \leq a_3 - 1; x_1^{i_1}x_3^{i_3}, 1 \leq i_1 \leq a_1 - 2; 0 \leq i_3 \leq a_3 - 1;$
$x_2^{a_2}; x_1^{a_1}; x_1^{a_1-1}x_2\}.$

In order to compute a derivation D of $A^1(V)$ it suffices to indicate its values on the generators x_1, x_2, x_3 which can be written in terms of the basis. Thus we can write

$$Dx_j=\sum_{i_1=1}^{a_1-2}\sum_{i_2=1}^{a_2-1}\sum_{i_3=0}^{a_3-1}c^j_{i_1,i_2,i_3}x_1^{i_1}x_2^{i_2}x_3^{i_3}+\sum_{i_3=0}^{a_3-2}c^j_{a_1-1,0,i_3}x_1^{a_1-1}x_3^{i_3}+c^j_{a_1-1,1,0}x_1^{a_1-1}x_2+c^j_{a_1,0,0}x_1^{a_1}$$
$$+\sum_{i_2=0}^{a_2-1}\sum_{i_3=0}^{a_3-1}c^j_{0,i_2,i_3}x_2^{i_2}x_3^{i_3}+\sum_{i_1=1}^{a_1-2}\sum_{i_3=0}^{a_3-1}c^j_{i_1,0,i_3}x_1^{i_1}x_3^{i_3}+c^j_{0,a_2,0}x_2^{a_2},\ j=1,2,3.$$

Using the above derivations we obtain the following description of Lie algebras in question. The derivations represented by the following vector fields form a basis in $\mathrm{Der}A^1(V)$:

$$x_3^{a_3-1}\partial_1;\ x_2x_3^{i_3}\partial_1,\ 1\le i_3\le a_3-1;\ x_2^2\partial_1;\ (1+a_3)x_1\partial_1+2(a_3-1)x_2\partial_2+4x_3\partial_3;$$
$$x_1x_3\partial_1+2x_3^2\partial_3;\ x_1x_3^{i_3}\partial_1,\ 2\le i_3\le a_3-2;\ x_1x_2\partial_1;\ x_1^2\partial_1;\ x_3^{a_3-1}\partial_2;\ x_2^2\partial_3;$$
$$x_2x_3\partial_2-x_3^2\partial_3;\ x_2x_3^{i_3}\partial_2,\ 2\le i_3\le a_3-1;\ x_2^2\partial_2;\ x_1x_3^{a_3-2}\partial_2;\ x_1x_2\partial_2;\ x_1^{a_1-2}\partial_3;$$
$$x_1^2\partial_2+2x_3^{a_3-2}\partial_3;\ x_3^{a_3-1}\partial_3;\ x_2x_3^{i_3}\partial_3,\ 1\le i_3\le a_3-1;\ x_1x_3^{i_3}\partial_3,\ 1\le i_3\le a_3-2;\ x_1x_2\partial_3.$$

Therefore we have

$$\lambda^1(V)=5a_3+7.$$

In case of $a_1\ge 3, a_2=2, a_3\ge 3$, we obtain the basis of derivation represented by following derivations form a basis of $\mathrm{Der}A^1(V)$:

$$(1+a_3)x_1^{i_1}\partial_1+a_1(a_3-1)x_1^{i_1-1}x_2\partial_2+2a_1x_1^{i_1-1}x_3\partial_3,\ 1\le i_1\le a_1-1;$$
$$x_2x_3^{i_3}\partial_1,\ 1\le i_3\le a_3-1;\ x_1^{a_1}\partial_1;$$
$$x_3^{a_3-1}\partial_1;\ x_2^2\partial_1;\ x_1^{i_1}x_3^{i_3}\partial_1+a_1x_1^{i_1-1}x_3^{i_3+1}\partial_3,\ 1\le i_1\le a_1-1;\ 1\le i_3\le a_3-2;$$
$$x_1^{i_1}x_3^{a_3-1}\partial_1,\ 1\le i_1\le a_1-2;\ x_1^{i_1}x_2x_3^{i_3}\partial_1,\ 1\le i_1\le a_1-2;\ 0\le i_3\le a_3-1;\ x_1^{a_1-1}x_2\partial_1;$$
$$x_3^{a_3-1}\partial_2+\frac{1}{a_3(a_3-1)}x_1^{a_1}\partial_3;\ x_2x_3^{i_3}\partial_2-x_3^{i_3+1}\partial_3,\ 1\le i_3\le a_3-2;\ x_2x_3^{a_3-1}\partial_2;$$
$$x_1^{i_1}x_3^{a_3-1}\partial_2,\ 1\le i_1\le a_1-2;\ x_1^{i_1}x_2x_3^{i_3}\partial_2-x_1^{i_1}x_3^{i_3+1}\partial_3,\ 1\le i_1\le a_1-2;\ 1\le i_3\le a_3-1;$$
$$x_2^2\partial_2-\frac{1}{a_1}x_1^{a_1}\partial_3;\ x_1^{a_1-1}x_3^{a_3-2}\partial_2;\ x_1^{a_1-1}x_2\partial_2;\ x_1^{a_1}\partial_2+2x_3^2\partial_3;\ \left(\frac{1}{a_3}x_2^2+x_3^{a_3-1}\right)\partial_3;$$
$$x_2x_3^{i_3}\partial_3,\ 2\le i_3\le a_3-1;\ x_1^{i_1}x_2x_3^{i_3}\partial_3,\ 1\le i_1\le a_1-2;\ 2\le i_3\le a_3-1;\ x_1^{a_1-2}x_3^{a_3-1}\partial_3;$$
$$x_1^{a_1-2}x_2x_3\partial_3;\ x_1^{a_1-1}x_3^{i_3}\partial_3,\ 1\le i_3\le a_3-2;\ x_1^{a_1-1}x_2\partial_3;\ (\frac{1}{2}x_1^{a_1}+x_2x_3)\partial_3.$$

Therefore we have

$$\lambda^1(V)=4a_1a_3-2a_1-3a_3+11.$$

In case of $a_1\ge 1, a_2=1, a_3\ge 2$, we obtain the basis of derivation represented by following derivations form a basis of $\mathrm{Der}A^1(V)$:

$$x_3^{i_3}\partial_1-a_1x_1^{a_1-1}x_3^{i_3}\partial_3,\ 1\le i_3\le a_3-2;\ x_1\partial_1+(a_1-1)(a_3-1)x_2\partial_2-a_1x_1^{a_1}\partial_3;$$
$$x_1^{i_1}x_3^{i_3}\partial_1+a_1x_1^{i_1-1}x_3^{i_3+1}\partial_3,\ 1\le i_1\le a_1-1;\ 1\le i_3\le a_3-2;\ x_1^{i_1}x_3^{a_3-1}\partial_3,\ 1\le i_1\le a_1-1;$$
$$x_1^{a_1}\partial_1+a_1x_1^{a_1-1}x_3\partial_3;\ (x_1^{a_1}+x_3)\partial_1;\ \left(\frac{1}{a_3}x_2+x_3^{a_3-1}\right)\partial_2;\ x_1^{a_1-1}x_3^{a_3-1}\partial_2;\ (x_1^{a_1}+x_3)\partial_3;$$
$$x_1^{a_1-1}x_3^{a_3-1}\partial_3;\ x_1^{i_1}\partial_1-a_1a_3(a_3-1)x_1^{i_1-1}x_3^{a_3-1}\partial_2+a_1x_1^{i_1-1}x_3\partial_3\ 2\le i_1\le a_1-1.$$
$$x_3^{a_3-1}\partial_1;\ x_2\partial_2;\ \left(\frac{1}{a_3}x_2+x_3^{a_3-1}\right)\partial_3.$$

Therefore we have

$$\lambda^1(V) = a_1a_3 + 7.$$

In case of $a_1 \geq 2, a_2 \geq 1, a_3 = 1$, we obtain the basis of derivation represented by following derivations form a basis of $\mathrm{Der}A^1(V)$:

$$x_1^{i_1}x_2^{i_2}\partial_1,\ 1 \leq i_1 \leq a_1 - 2; 0 \leq i_2 \leq a_2 - 1.$$

Therefore we have

$$\lambda^1(V) = a_2(a_1 - 2).$$

In last case we obtain the basis of derivation represented by following derivations form a basis of $\mathrm{Der}A^1(V)$:

$$\begin{aligned}
&\left(\frac{1}{a_3}x_2^{a_2} + x_3^{a_3-1}\right)\partial_1;\ x_2^{a_2-2}x_3^{i_3}\partial_1 + \frac{a_1-1}{a_2}x_1^{a_1-1}x_3^{i_3-1}\partial_2,\ 1 \leq i_3 \leq a_3 - 2;\ x_2^{a_2-2}x_3^{a_3-1}\partial_1;\\
&(1 - a_3 + a_2a_3)x_1^{i_1}\partial_1 + a_1(a_3-1)x_1^{i_1-1}x_2\partial_2 + a_1a_2x_1^{i_1-1}x_3\partial_3,\ 1 \leq i_1 \leq a_1 - 1;\\
&x_1^{i_1}x_3^{i_3}\partial_1 + a_1x_1^{i_1-1}x_3^{i_3+1}\partial_3,\ 1 \leq i_1 \leq a_1 - 2; 1 \leq i_3 \leq a_3 - 2;\\
&x_1^{a_1-1}x_3^{i_3}\partial_1 - \frac{a_1}{a_3}x_1^{a_1-2}x_3^{i_3-1}\partial_3,\ 1 \leq i_3 \leq a_3 - 3;\\
&x_1^{i_1}x_3^{a_3-1}\partial_1,\ 1 \leq i_1 \leq a_1 - 2;\ x_1^{a_1-1}x_3^{a_3-2}\partial_1;\ x_1^{a_1-1}x_2\partial_1;\ x_1^{a_1}\partial_1;\ x_2^{a_2-1}x_3^{i_3}\partial_1,\ 1 \leq i_3 \leq a_3 - 1;\\
&x_3^{a_3-1}\partial_2 + \frac{1}{a_3(a_3-1)}x_1^{a_1}\partial_3;\ x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_1,\ 1 \leq i_1 \leq a_1 - 2; 1 \leq i_2 \leq a_2 - 1; 0 \leq i_3 \leq a_3 - 1;\\
&x_2x_3^{i_3}\partial_2 - a_1x_3^{i_3+1}\partial_3,\ 1 \leq i_3 \leq a_3 - 2;\ x_2x_3^{a_3-1}\partial_2;\ x_2^{i_2+1}\partial_2 + \frac{a_2}{a_3-1}x_2^{i_2}x_3\partial_3,\ 1 \leq i_2 \leq a_2 - 2;\\
&x_2^{i_2}x_3^{i_3}\partial_2,\ 2 \leq i_2 \leq a_2 - 1; 1 \leq i_3 \leq a_3 - 1;\ x_2^{a_2}\partial_2 - \frac{1}{a_3-1}x_1^{a_1}\partial_3;\\
&x_1^{i_1}x_3^{a_3-1}\partial_2 - \frac{a_2}{a_3(a_3-1)}x_1^{i_1}x_2^{a_2-1}x_3\partial_3,\ 1 \leq i_1 \leq a_1 - 2;\ x_1^{a_1-1}x_3^{a_3-2}\partial_2;\ x_1^{a_1-1}x_2\partial_2;\ x_1^{a_1}\partial_2;\\
&x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_2,\ 1 \leq i_1 \leq a_1 - 2; 2 \leq i_2 \leq a_2 - 1; 1 \leq i_3 \leq a_3 - 1;\ \left(\frac{1}{a_2}x_2^{a_2} + x_3^{a_3-1}\right)\partial_3;\\
&x_1^{i_1}x_2^{i_2}\partial_1 + \frac{a_2}{a_3-1}x_1^{i_1}x_2^{i_2-1}x_3\partial_3,\ 1 \leq i_1 \leq a_1 - 2; 2 \leq i_2 \leq a_2 - 1;\ \frac{1}{a_2}x_1^{a_1}\partial_2 + x_2^{a_2-1}x_3\partial_3;\\
&x_2^{i_2}x_3^{i_3}\partial_3,\ 1 \leq i_2 \leq a_2 - 1; 2 \leq i_3 \leq a_3 - 1;\ x_1^{a_1-1}x_3^{i_3}\partial_3,\ 1 \leq i_3 \leq a_3 - 2;\ x_1^{a_1-2}x_2^{a_2-1}x_3\partial_3;\\
&x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_3,\ 1 \leq i_1 \leq a_1 - 2; 1 \leq i_2 \leq a_2 - 1; 2 \leq i_3 \leq a_3 - 1;\ x_1^{a_1-1}x_2\partial_3;\\
&x_1^{i_1}x_2x_3^{i_3}\partial_2 - a_1x_1^{i_1}x_3^{i_3+1}\partial_3,\ 1 \leq i_1 \leq a_1 - 2; 1 \leq i_3 \leq a_3 - 1;\ x_1^{a_1-2}x_3^{a_3-1}\partial_3.
\end{aligned}$$

Therefore we have

$$\lambda^1(V) = 3a_1a_2a_3 - 2a_1a_2 - 2a_1a_3 - 4a_2a_3 + 2a_1 + 2a_2 + 6a_3 + 5.$$

It is follows from proposition 3.5 and weight types of proposition 3.6 we have

$$h_1\left(\frac{1}{w_1}, \frac{1}{w_2}, \frac{1}{w_3}\right) = \begin{cases} 2a_1 + 7; & a_1 \geq 1, a_2 = 1, a_3 = 2\\ \frac{8a_1a_2^2}{2a_2-1} - 3\left(\frac{2a_1a_2}{2a_2-1} + 2a_2\right) + 16; & a_1 \geq 3, a_2 \geq 2, a_3 = 2\\ 3\frac{a_1a_2a_3}{1-a_3+a_2a_3}\frac{a_2a_3}{a_3-1}a_3 & \\ -4\left(\frac{a_1a_2a_3}{1-a_3+a_2a_3}\frac{a_2a_3}{a_3-1}\right. & \\ \left.+\frac{a_1a_2a_3}{1-a_3+a_2a_3}a_3 + \frac{a_2a_3}{a_3-1}a_3\right) & \\ +5\left(\frac{a_1a_2a_3}{1-a_3+a_2a_3} + \frac{a_2a_3}{a_3-1} + a_3\right) + 6; & a_1 \geq 3, a_2 \geq 3, a_3 \geq 3. \end{cases}$$

It is easy to see that our conjecture is true in following cases:

(i) $a_1 \geq 3, a_2 = 2, a_3 = 2,$
(ii) $a_1 \geq 1, a_2 = 1, a_3 = 2,$
(iii) $a_1 \geq 3, a_2 \geq 3, a_3 \geq 3,$
(iv) $a_1 \geq 3, a_2 \geq 3, a_3 = 2.$

It is easy to prove proposed conjecture in first two cases and last two cases following as:
Case (iii) when $a_1 \geq 3, a_2 \geq 3, a_3 \geq 3$, then $3a_1a_2a_3 - 2a_1a_2 - 2a_1a_3 - 4a_2a_3 + 2a_1 + 2a_2 + 6a_3 + 5 \leq 3\frac{a_1a_2^2a_3^3}{(1-a_3+a_2a_3)(a_3-1)} - 4(\frac{a_1a_2^2a_3^3}{(1-a_3+a_2a_3)(a_3-1)} + \frac{a_1a_2a_3^2}{1-a_3+a_2a_3} + \frac{a_2a_3^2}{a_3-1}) + 5(\frac{a_1a_2a_3}{1-a_3+a_2a_3} + \frac{a_2a_3}{a_3-1} + a_3) + 6.$
After simplification we get $(a_3-1)^3(a_2-3) + (a_1-1)a_2a_3((a_3-1)(a_2-3) + (a_3-2)) + a_2(3a_3-2)(a_1-1) + a_2(a_1-1) \geq 0$. Case(iv) when $a_1 \geq 3, a_2 \geq 3, a_3 = 2$ then

$$4a_1a_2 - 2a_1 - 6a_2 + 17 \leq \frac{8a_1a_2^2}{2a_2-1} - 3(\frac{2a_1a_2}{2a_2-1} + 2a_2) + 16,$$

after simplification we get $a_2(a_1-2) + a_1(a_2-2) + 1 \geq 0$. □

Proposition 3.7 *Let* $(V, 0)$ *be a fewnomial surface isolated singularity of type 3 which is defined by* $f = x_1^{a_1}x_2 + x_2^{a_2}x_3 + x_3^{a_3}x_1$ $(a_1 \geq 1, a_2 \geq 1, a_3 \geq 1)$ *with weight type*

$$\left(\frac{1-a_3+a_2a_3}{1+a_1a_2a_3}, \frac{1-a_1+a_1a_3}{1+a_1a_2a_3}, \frac{1-a_2+a_1a_2}{1+a_1a_2a_3}; 1\right).$$

Then

$$\lambda^1(V) = \begin{cases} a_2a_3 + 8; & a_1 = 1, a_2 \geq 1, a_3 \geq 1 \\ a_1a_3 + 8; & a_1 \geq 2, a_2 = 1, a_3 \geq 2 \\ a_1a_2 + 8; & a_1 \geq 2, a_2 \geq 2, a_3 = 1 \\ 24; & a_1 = 2, a_2 = 2, a_3 = 2 \\ 3a_1a_2a_3 + 2(a_1 + a_2 + a_3) & \\ \quad -2(a_1a_2 + a_1a_3 + a_2a_3) + 11; & Otherwise. \end{cases}$$

Furthermore, we need to show that when $a_1 \geq 3, a_2 \geq 3, a_3 \geq 3$, *then* $3a_1a_2a_3 + 2(a_1 + a_2 + a_3) - 2(a_1a_2 + a_1a_3 + a_2a_3) + 11 \leq \frac{3(1+a_1a_2a_3)^3}{(1-a_3+a_2a_3)(1-a_1+a_1a_3)(1-a_2+a_1a_2)} + 5(\frac{1+a_1a_2a_3}{1-a_3+a_2a_3} + \frac{1+a_1a_2a_3}{1-a_1+a_1a_3} + \frac{1+a_1a_2a_3}{1-a_2+a_1a_2}) - 4(\frac{(1+a_1a_2a_3)^2}{(1-a_3+a_2a_3)(1-a_1+a_1a_3)} + \frac{(1+a_1a_2a_3)^2}{(1-a_1+a_1a_3)(1-a_2+a_1a_2)} + \frac{(1+a_1a_2a_3)^2}{(1-a_3+a_2a_3)(1-a_2+a_1a_2)}) + 6.$

Proof It is easy to see that the moduli algebra $A^1(V) = \mathbb{C}\{x_1, x_2, x_3\}/(f, m.J(f))$ has dimension $(a_1a_2a_3 + 3)$ and has a monomial basis of the form (cf. [2], Theorem 13.1)

$\{x_1^{i_1}x_2^{i_2}x_3^{i_3}, 1 \leq i_1 \leq a_1 - 2; 1 \leq i_2 \leq a_2 - 1; 0 \leq i_3 \leq a_3 - 2; x_1^{i_1}, 1 \leq i_1 \leq a_1;$
$x_1^{i_1}x_3^{i_3}, 1 \leq i_1 \leq a_1 - 1; 1 \leq i_3 \leq a_3 - 2; x_2^{i_2}x_3^{i_3}, 0 \leq i_2 \leq a_2 - 2; 0 \leq i_3 \leq 2a_3 - 2, x_1x_3^{a_3-1};$
$x_1^{i_1}x_2^{i_2}, 0 \leq i_1 \leq a_1 - 3; a_2 \leq i_2 \leq 2a_2 - 1; x_1^{a_1-2}x_2^{a_2}; x_1^{a_1-1}x_2; x_2^{a_2-1}x_3^{i_3}, 0 \leq i_3 \leq a_3 - 1\}.$

In order to compute a derivation D of $A^1(V)$ it suffices to indicate its values on the generators x_1, x_2, x_3 which can be written in terms of the basis. Thus we can write

$$
\begin{aligned}
Dx_j =& \sum_{i_1=1}^{a_1-2}\sum_{i_2=1}^{a_2-1}\sum_{i_3=0}^{a_3-2} c^j_{i_1,i_2,i_3} x_1^{i_1}x_2^{i_2}x_3^{i_3} + \sum_{i_1=1}^{a_1} c^j_{i_1,0,0}x_1^{i_1} + \sum_{i_3=0}^{a_3-1} c^j_{0,a_2-1,i_3}x_2^{a_2-1}x_3^{i_3}\\
&+ \sum_{i_2=0}^{a_2-2}\sum_{i_3=0}^{2a_3-1} c^j_{0,i_2,i_3}x_2^{i_2}x_3^{i_3} + \sum_{i_1=0}^{a_1-3}\sum_{i_3=a_2}^{2a_2-1} c^j_{i_1,i_2,0}x_1^{i_1}x_2^{i_2} + c^j_{1,0,a_3-1}x_1x_3^{a_3-1}\\
&+ c^j_{a_1-2,a_2,0}x_1^{a_1-2}x_2^{a_2} + c^j_{a_1-1,1,0}x_1^{a_1-1}x_2 + \sum_{i_1=1}^{a_1-1}\sum_{i_3=1}^{a_3-2} c^j_{i_1,0,i_3}x_1^{i_1}x_3^{i_3},\ \ j=1,2,3.
\end{aligned}
$$

Using the above derivations we obtain the following description of Lie algebras in question. The derivations represented by the following vector fields form a basis in $\mathrm{Der}A^1(V)$:

$(1-a_3+a_2a_3)x_1^{i_1}\partial_1 + (1-a_1+a_1a_3)x_1^{i_1-1}x_2\partial_2 + (1-a_2+a_1a_2)x_1^{i_1-1}x_3\partial_3,\ 1 \le i_1 \le a_1-1;$

$(1-a_3+a_2a_3)x_1x_3^{i_3}\partial_1 + (1-a_1+a_1a_3)x_2x_3^{i_3-1}\partial_2 + (1-a_2+a_1a_2)x_3^{i_3+1}\partial_3,\ 1 \le i_3 \le a_3-2;$

$(1-a_3+a_2a_3)x_1x_2^{i_2}\partial_1 + (1-a_1+a_1a_3)x_2^{i_2+1}\partial_2 + (1-a_2+a_1a_2)x_2^{i_2-1}x_3\partial_3,\ 1 \le i_2 \le a_2-2;$

$x_2^{a_2-2}x_3^{a_3-1}\partial_1 + \frac{a_1}{a_2}x_1^{a_1-1}x_3^{a_3-2}\partial_2 + \frac{a_1}{a_3}x_1^{a_1-2}x_2^{a_2-1}\partial_3;\ x_2^{i_2}\partial_1, 2a_2-2 \le i_2 \le 2a_2-1;$

$x_2^{i_2}x_3^{i_3}\partial_1,\ 0 \le i_2 \le a_2-2;\ a_3+1 \le i_3 \le 2a_3-1;\ x_1^{i_1}x_2^{a_2}\partial_2 - \frac{1}{a_3-1}x_1^{i_1+1},\ 0 \le i_1 \le a_1-3;$

$x_1^{i_1}x_2^{i_2}\partial_1,\ 1 \le i_1 \le a_1-3;\ a_2+1 \le i_2 \le 2a_2-1;\ x_1x_2^{a_2-1}\partial_1 + \frac{1}{a_2(a_3-1)}x_1^{a_1}\partial_3;$

$x_2^{a_2-2}x_3^{a_3}\partial_1;\ x_2^{a_2-1}x_3^{a_3-1}\partial_1;\ x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_1,\ 2 \le i_1 \le a_1-2;\ 1 \le i_2 \le a_2-1;\ 1 \le i_3 \le a_3-2;$

$x_1x_2^{a_2-1}x_3^{i_3}\partial_1,\ 1 \le i_3 \le a_3-2;\ x_1^{i_1}x_2^{a_2}\partial_1,\ 1 \le i_1 \le a_1-2;$

$x_2^{i_2-1}x_3^{a_3}\partial_1 + \frac{a_1}{a_3-1}x_1^{a_1-2}x_2^{i_2}x_3\partial_3,\ 1 \le i_2 \le a_2-2;$

$x_2^{i_2+(a_2-1)}\partial_1 + a_1(a_1-1)x_1^{a_1-1}x_2^{i_2}\partial_3;\ 1 \le i_2 \le a_2-2;$

$x_1x_2^{i_2}x_3^{i_3}\partial_1 + \frac{a_1-1}{a_3}x_2^{i_2}x_3^{i_3+1}\partial_3,\ 1 \le i_2 \le a_2-2;\ 1 \le i_3 \le a_3-2;$

$x_1^{i_1}x_3^{i_3}\partial_1 + a_1x_1^{i_1-1}x_3^{i_3+1}\partial_3,\ 2 \le i_1 \le a_1-1;\ 1 \le i_3 \le a_3-2;$

$x_1^{i_1}x_2^{i_2}\partial_1 - \frac{1}{a_3-1}x_1^{i_1-1}x_2^{i_2}x_3\partial_3,\ 2 \le i_1 \le a_1-2;\ 1 \le i_2 \le a_2-1;$

$x_3^{i_3+(a_3-1)}\partial_1 + a_1(a_2-1)x_1^{a_1-1}x_3^{i_3}\partial_3,\ 1 \le i_3 \le a_3-2;\ x_1^{a_1}\partial_1 + a_1x_1^{a_1-1}x_3\partial_3;$

$x_3^{i_3}\partial_2,\ 2a_3-2 \le i_3 \le 2a_3-1;\ x_2^{i_2}x_3^{i_3}\partial_2,\ 1 \le i_2 \le a_2-2;\ \ a_3 \le i_3 \le 2a_3-1;$

$x_2^{i_2}x_3^{i_3}\partial_2 + \frac{1}{a_3}x_2^{i_2-1}x_3^{i_3+1}\partial_3,\ 2 \le i_2 \le a_2-1;\ \ 1 \le i_3 \le a_3-2;$

$x_2^{i_2}x_3^{a_3-1}\partial_2 + \frac{1}{a_3}x_2^{i_2-1}x_3^{a_3},\ 1 \le i_2 \le a_2-2;\ x_1^{a_1-1}x_2\partial_1 - \frac{1}{a_3-1}x_1^{a_1-2}x_2x_3\partial_3;$

$x_1^{i_1}x_2^{i_2}\partial_2,\ 0 \le i_1 \le a_1-3;\ \ a_2+1 \le i_2 \le 2a_2-1;\ x_2^{a_2-2}x_3^{a_3-1}\partial_2;$

$x_1^{i_1}x_2x_3^{i_3}\partial_2 - (a_2-1)x_1^{i_1}x_3^{i_3+1}\partial_3,\ 1 \le i_1 \le a_1-2;\ \ 1 \le i_3 \le a_3-2;$

$x_1x_3^{a_3-1}\partial_2 + \frac{1}{a_3(a_3-1)}x_1^{a_1}\partial_3;\ x_2^{a_2-1}x_3^{i_3}\partial_1 - \frac{a_1}{a_2}x_1^{a_1-1}x_3^{i_3}\partial_3,\ 1 \le i_3 \le a_3-2;$

$x_1^{i_1}x_2^{i_2}\partial_2 + \frac{a_2}{a_3-1}x_1^{i_1}x_2^{i_2-1}x_3\partial_3,\ 1 \le i_1 \le a_1-2;\ \ 2 \le i_2 \le a_2-1;$

$x_1^{a_1-2}x_2^{a_2}\partial_2;\ (a_1x_1^{a_1-1} + x_3^{a_3})\partial_3;\ x_1x_3^{a_3-1}\partial_1 - \frac{a_1(a_1-1)}{a_3}x_1^{a_1-1}x_2\partial_3;$

$x_1^{a_1-1}x_2\partial_2 - (a_2-1)x_1^{a_1-1}x_3\partial_3;\ x_1^{a_1}\partial_2 - \frac{a_2}{a_3}x_2^{a_2-2}x_3^2\partial_3;$

$x_2^{i_2}x_3^{i_3}\partial_3,\ 0\le i_2\le a_2-2;\ a_3+1\le i_3\le 2a_3-1;$
$x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_2,\ 1\le i_1\le a_1-2;\ 2\le i_2\le a_2-1; 1\le i_3\le a_3-2; x_2^{a_2-2}x_3^{a_3}\partial_3;$
$\left(\frac{1}{a_2}x_1^{a_1}+x_2^{a_2-1}x_3\right)\partial_3; x_2^{a_2-1}x_3^{i_3}\partial_3,\ 2\le i_3\le a_3-1; x_2^{i_2}\partial_3,\ a_2+1\le i_2\le 2a_2-1;$
$x_1^{i_1}x_2^{i_2}\partial_3,\ 1\le i_1\le a_1-3;\ a_2+1\le i_2\le 2a_2-1; x_1^{a_1-2}x_2^{a_2}\partial_3; x_1^{a_1-2}x_2^{a_2-1}x_3\partial_3;$
$x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_3,\ 1\le i_1\le a_1-2;\ 1\le i_2\le a_2-1; 2\le i_3\le a_3-2; (a_3x_1x_3^{a_3-1}+x_2^{a_2})\partial_3.$

Therefore we have

$$\lambda^1(V)=3a_1a_2a_3+2(a_1+a_2+a_3)-2(a_1a_2+a_1a_3+a_2a_3)+11.$$

In case of $a_1=1, a_2\ge 1, a_3\ge 1$, we obtain the basis of derivation represented by following derivations form a basis of $\mathrm{Der}A^1(V)$:

$(x_2+x_3^{a_3})\partial_1; (a_2(1-a_3+a_2a_3))x_3^{2a_3+i_3}\partial_1-a_3x_2x_3^{i_3-1}\partial_2-x_3^{i_3}\partial_3,\ 1\le i_3\le a_3-1;$
$-x_2x_3^{2a_3}\partial_1; (1-a_3+a_2a_3)x_1\partial_1+a_3x_2\partial_2+x_3\partial_3; (x_2+x_3^{a_3})\partial_2; -x_2x_3^{2a_3}\partial_2; (x_2+x_3^{a_3})\partial_3;$
$a_3x_3^{2a_3+(i_3-1)}\partial_2+x_2^{i_2}x_3^{i_3}\partial_3,\ 1\le i_2\le a_2-2; 0\le i_3\le a_3-1; a_3x_1\partial_2+a_2x_2^{2a_3-2}\partial_3;$
$x_1\partial_3; -x_2x_3^{i_3}\partial_3,\ a_3+1\le i_3\le 2a_3; a_3x_3^{2a_3+(a_3-1)}\partial_2+x_2x_3^{a_3}\partial_3.$

Therefore we have

$$\lambda^1(V)=a_2a_3+8.$$

Similarly it is easy to compute the basis of $\mathrm{Der}A^1(V)$ in case of $a_1\ge 2, a_2=1, a_3\ge 2$ and $a_1\ge 2, a_2\ge 2, a_3=1$. In case of $a_1=2, a_2=2, a_3=2$, we obtain the basis of derivation represented by following derivations form a basis of $\mathrm{Der}A^1(V)$:

$x_3\partial_1+x_1\partial_2+x_2\partial_3; x_3^2\partial_1; x_3^3\partial_1; x_2\partial_1+x_3\partial_2+x_1\partial_3; x_2x_3\partial_1; x_2^2\partial_1; x_1\partial_1+x_2\partial_2+x_3\partial_3;$
$x_1^2\partial_1; x_3^2\partial_2; x_3^3\partial_2; x_2x_3\partial_2; x_2^2\partial_2; x_1x_3\partial_2; x_1x_2\partial_2; x_1^2\partial_2; x_3^2\partial_3; x_3^3\partial_3; x_2x_3\partial_3; x_2^2\partial_3; x_1x_3\partial_3;$
$x_1x_3\partial_1; x_1x_2\partial_1; x_1x_2\partial_3; x_1^2\partial_3.$

Therefore we have

$$\lambda^1(V)=24.$$

It is easy to see that when $a_1\ge 3, a_2\ge 3, a_3\ge 3$, then $h_1(\frac{1}{w_1},\frac{1}{w_2},\frac{1}{w_3})=\frac{3(1+a_1a_2a_3)^3}{(1-a_3+a_2a_3)(1-a_1+a_1a_3)(1-a_2+a_1a_2)}+5(\frac{1+a_1a_2a_3}{1-a_3+a_2a_3}+\frac{1+a_1a_2a_3}{1-a_1+a_1a_3}+\frac{1+a_1a_2a_3}{1-a_2+a_1a_2})-4(\frac{(1+a_1a_2a_3)^2}{(1-a_3+a_2a_3)(1-a_1+a_1a_3)}+\frac{(1+a_1a_2a_3)^2}{(1-a_1+a_1a_3)(1-a_2+a_1a_2)}+\frac{(1+a_1a_2a_3)^2}{(1-a_3+a_2a_3)(1-a_2+a_1a_2)})+6$. We need to show that $3a_1a_2a_3+2(a_1+a_2+a_3)-2(a_1a_2+a_1a_3+a_2a_3)+11\le\frac{3(1+a_1a_2a_3)^3}{(1-a_3+a_2a_3)(1-a_1+a_1a_3)(1-a_2+a_1a_2)}+5(\frac{1+a_1a_2a_3}{1-a_3+a_2a_3}+\frac{1+a_1a_2a_3}{1-a_1+a_1a_3}+\frac{1+a_1a_2a_3}{1-a_2+a_1a_2})-4(\frac{(1+a_1a_2a_3)^2}{(1-a_3+a_2a_3)(1-a_1+a_1a_3)}+\frac{(1+a_1a_2a_3)^2}{(1-a_1+a_1a_3)(1-a_2+a_1a_2)}+\frac{(1+a_1a_2a_3)^2}{(1-a_3+a_2a_3)(1-a_2+a_1a_2)})+6$. After simplification we get $6(a_1a_2+a_2a_3+a_1a_3)+a_1(a_2-3)+a_2(a_3-3)+a_3(a_1-3)+2a_1^2[a_2(a_3-3)+a_3(a_2-3)]+2a_2^2[a_1(a_3-3)+a_3(a_1-3)]+2a_3^2[a_1(a_2-3)+a_2(a_1-3)]+2(a_1^2+a_2^2+a_3^2)+2(a_1^3a_2+a_2^3a_3+a_3^3a_1)+a_1^2a_2^2a_3^2+10(a_1a_2^2a_3+a_1a_2a_3^2)+4a_1^2a_2a_3+2a_1a_2a_3[3a_1-8]+a_1^3a_2a_3^2(a_3-3)(a_2-3)+a_1^2a_3^2(a_3-3)(a_1a_2-2)+2a_1^2a_2a_3^2(a_3+a_2-6)+2a_1a_2a_3^3(a_1-3)+a_1^2a_2^3a_3(a_3-3)(a_1-3)+a_1^2a_2^2(a_1-3)(a_2a_3-2)+2a_1^3a_2a_3(a_2-3)+2a_1^2a_2^2a_3(a_1-3+(a_3-3))+a_1a_2^2a_3^3(a_2-3)(a_1-3)+a_2^2a_3^2(a_2-3)(a_1a_3-2)+2a_1a_2^2a_3^2(a_1+a_2-6)+2a_2^3a_3a_1(a_3-3)+1\ge 0$. □

Proposition 3.8 *Let* $(V, 0)$ *be a fewnomial surface isolated singularity of type* 4 *which is defined by* $f = x_1^{a_1} + x_2^{a_2} + x_3^{a_3} x_2$ $(a_1 \geq 2, a_2 \geq 1, a_3 \geq 1)$ *with weight type* $(\frac{1}{a_1}, \frac{1}{a_2}, \frac{a_2-1}{a_2 a_3}; 1)$. *Then*

$$\lambda^1(V) = \begin{cases} a_1 + 7; & a_1 \geq 2, a_2 = 2, a_3 = 1 \\ a_3(a_1 - 2); & a_1 \geq 2, a_2 = 1, a_3 \geq 1 \\ 5a_1a_2 - a_1 - 7a_2 + 15; & a_1 \geq 3, a_2 \geq 3, a_3 = 3 \\ 3a_2 + 11; & a_1 = 2, a_2 \geq 3, a_3 = 3 \\ 9a_1 - 2; & a_1 \geq 3, a_2 = 2, a_3 = 3 \\ 5a_1 + 4; & a_1 \geq 3, a_2 = 2, a_3 = 2 \\ a_2 + 13; & a_1 = 2, a_2 \geq 3, a_3 = 2 \\ 2a_3 + 7; & a_1 = 2, a_2 = 2, a_3 \geq 3 \\ 2a_2a_3 - 3a_2 - 2a_3 + 17; & a_1 = 2, a_2 \geq 3, a_3 \geq 3 \\ 4a_1a_3 - 3a_1 - 6a_3 + 16; & a_1 \geq 3, a_2 = 2, a_3 \geq 3 \\ 11; & a_1 = 2, a_2 = 2, a_2 = 2 \\ 3a_1a_2a_3 - 4a_1a_2 - 3a_1a_3 - 4a_2a_3 & \\ \quad + 8a_1 + 5a_2 + 5a_3 - 1; & Otherwise. \end{cases}$$

Furthermore, we need to show that when $a_1 \geq 3, a_2 \geq 3, a_3 \geq 3$, *then* $3a_1a_2a_3 - 4a_1a_2 - 3a_1a_3 - 4a_2a_3 + 8a_1 + 5a_2 + 5a_3 - 1 \leq \frac{3a_1a_2^2a_3}{a_2-1} + 5(a_1 + a_2 + \frac{a_2a_3}{a_2-1}) - 4(a_1a_2 + \frac{a_1a_2a_3}{a_2-1} + \frac{a_2^2a_3}{a_2-1}) + 6.$

Proof It is easy to see that the moduli algebra $A^1(V) = \mathbb{C}\{x_1, x_2, x_3\}/(f, m.J(f))$ has dimension $(a_1a_2a_3 - a_1a_2 - a_2a_3 + a_1 + a_2 + 2)$ and has a monomial basis of the form (cf. [2], Theorem 13.1)

$\{x_1^{i_1} x_2^{i_2} x_3^{i_3}, 0 \leq i_1 \leq a_1 - 2; 0 \leq i_2 \leq a_2 - 2; 0 \leq i_3 \leq a_3 - 2; x_3^{i_3}, a_3 - 1 \leq i_3 \leq 2a_3 - 2;$
$x_1^{a_1-1}; x_2^{a_2-1}; x_2x_3^{a_3-1}; x_1^{i_1}x_3^{i_3}, 1 \leq i_1 \leq a_1 - 2; a_3 - 1 \leq i_3 \leq 2a_3 - 2\}.$

In order to compute a derivation D of $A^1(V)$ it suffices to indicate its values on the generators x_1, x_2, x_3 which can be written in terms of the basis. Thus we can write

$$\begin{aligned} Dx_j = & \sum_{i_1=0}^{a_1-2} \sum_{i_2=0}^{a_2-2} \sum_{i_3=0}^{a_3-2} c^j_{i_1,i_2,i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3} + \sum_{i_3=a_3-1}^{2a_3-2} c^j_{0,0,i_3} x_3^{i_3} + \sum_{i_1=1}^{a_1-2} \sum_{i_3=a_3-1}^{2a_3-2} c^j_{i_1,0,i_3} x_1^{i_1} x_3^{i_3} + c^j_{a_1-1,0,0} x_1^{a_1-1} \\ & + c^j_{0,a_2-1,0} x_2^{a_2-1} + c^j_{0,1,a_3-1} x_2 x_3^{a_3-1}, \ j = 1, 2, 3. \end{aligned}$$

In case of $a_1 \geq 2, a_2 = 2, a_3 = 1$, we obtain the basis of derivation represented by following derivations form a basis of $\mathrm{Der}A^1(V)$:

$$\begin{aligned} & x_1^{a_1-1}\partial_1; x_1^{i_1}\partial_1, \ 1 \leq i_1 \leq a_1 - 1; x_2\partial_1; x_3\partial_2; x_2\partial_2; x_1^{a_1-1}\partial_2; \\ & x_3\partial_3; x_2\partial_3; x_1^{a_1-1}\partial_3. \end{aligned}$$

Therefore we have

$$\lambda^1(V) = a_1 + 7.$$

In case of $a_1 \geq 2, a_2 = 1, a_3 \geq 1$, we obtain the basis of derivation represented by following derivations form a basis of $\mathrm{Der}A^1(V)$:

$$x_1^{i_1} x_3^{i_3} \partial_1, \ 1 \leq i_1 \leq a_1 - 2; 0 \leq i_3 \leq a_3 - 1.$$

Therefore we have

$$\lambda^1(V) = a_3(a_1 - 2).$$

In case of $a_1 \geq 3, a_2 \geq 3, a_3 = 3$, we obtain the basis of derivation represented by following derivations form a basis of $\mathrm{Der}A^1(V)$:

$$\begin{aligned}
&x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_1,\ 1 \leq i_1 \leq a_1 - 2, 1 \leq i_2 \leq a_2 - 2, 0 \leq i_3 \leq 1;\\
&x_1^{i_1}x_3^{i_3}\partial_1,\ 1 \leq i_1 \leq a_1 - 2, 1 \leq i_3 \leq 4; x_2x_3^2\partial_1; x_3^4\partial_1;\\
&x_1^{i_1}\partial_1,\ 1 \leq i_1 \leq a_1 - 1; (a_2x_2^{a_2-1} + x_3^3)\partial_1; x_3^2\partial_2 + \frac{a_2}{2}x_2^{a_2-2}\partial_3; x_3^{i_3}\partial_2,\ 3 \leq i_3 \leq 4;\\
&x_2x_3^{i_3-1}\partial_2 + \frac{a_1}{a_3}x_3^{i_3}\partial_3,\ 1 \leq i_3 \leq 2; x_2x_3^2\partial_2; x_2^{i_2}x_3^{i_3}\partial_2,\ 2 \leq i_2 \leq a_2 - 2,\ 0 \leq i_3 \leq 1;\\
&x_2^{a_2-1}\partial_2; x_1^{i_1}x_3^{i_3}\partial_2,\ 1 \leq i_1 \leq a_1 - 2,\ 3 \leq i_3 \leq 4; x_1x_2\partial_2 + \frac{a_1}{3}x_1x_3\partial_3;\\
&x_1x_2x_3\partial_2; x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_2,\ 1 \leq i_1 \leq a_1 - 2, 2 \leq i_2 \leq a_2 - 2, 0 \leq i_3 \leq 1; x_1^{a_1-1}\partial_2;\\
&x_3^{i_3}\partial_3,\ 3 \leq i_3 \leq 4; x_2^{i_2}x_3\partial_3,\ 1 \leq i_2 \leq a_2 - 2,\ x_2x_3^2\partial_3; x_2^{a_2-1}\partial_3;\\
&x_1^{i_1}x_3^{i_3}\partial_3,\ 1 \leq i_1 \leq a_1 - 2,\ 2 \leq i_3 \leq 4; x_1^{a_1-1}\partial_3; x_1^{a_1-2}x_2^{a_2-2}\partial_3;\\
&x_1^{i_1}x_2^{i_2}x_3\partial_3,\ 1 \leq i_1 \leq a_1 - 2, 1 \leq i_2 \leq a_2 - 2.
\end{aligned}$$

Therefore we have

$$\lambda^1(V) = 5a_1a_2 - a_1 - 7a_2 + 15.$$

In case of $a_1 = 2, a_2 \geq 3, a_3 = 3$, we obtain the basis of derivation represented by following derivations form a basis of $\mathrm{Der}A^1(V)$:

$$\begin{aligned}
&\left(a_2x_2^{a_2-1} + x_3^3\right)\partial_1; x_3^4\partial_1; x_1\partial_1; x_2x_3^2\partial_1; x_3^{i_3}\partial_2,\ 2 \leq i_3 \leq 4; x_2\partial_2 + \frac{a_2 - 1}{3}x_3\partial_3;\\
&x_2^{i_2}x_3\partial_2,\ 1 \leq i_2 \leq a_2 - 2; x_2^{i_2}\partial_2,\ 2 \leq i_2 \leq a_2 - 1; x_2x_3^2\partial_2; x_1\partial_2; x_3^{i_3}\partial_3,\ 2 \leq i_3 \leq 4;\\
&x_2^{i_2}x_3\partial_3,\ 1 \leq i_2 \leq a_2 - 2; x_2^{i_2}\partial_3,\ a_2 - 2 \leq i_2 \leq a_2 - 1; x_2x_3^2\partial_3; x_1\partial_3.
\end{aligned}$$

Therefore we have

$$\lambda^1(V) = 3a_2 + 11.$$

In case of $a_1 \geq 3, a_2 = 2, a_3 = 3$, we obtain the basis of derivation represented by following derivations form a basis of $\mathrm{Der}A^1(V)$:

$$\begin{aligned}
&\left(2x_2 + x_3^3\right)\partial_1; x_1^{a_1-1}\partial_1; x_3^{i_3}\partial_1,\ 4 \leq i_3 \leq 5; x_1^{i_1}x_3^{i_3}\partial_1,\ 1 \leq i_1 \leq a_1 - 2; 0 \leq i_3 \leq 4;\\
&x_1^{a_1-2}x_3^3\partial_2 - \frac{2}{3}x_1^{a_1-2}x_3\partial_3; x_3^{i_3+2}\partial_2 - \frac{2}{3}x_3^{i_3}\partial_3,\ 1 \leq i_3 \leq 2; x_3^5\partial_2; x_2\partial_2 + \frac{1}{3}x_3\partial_3;\\
&x_1^{a_1-2}x_3^2\partial_2 - \frac{2}{3}x_1^{a_1-2}\partial_3; x_1^{i_1}x_3^{i_3+2}\partial_2 - \frac{2}{3}x_1^{i_1}x_3^{i_3}\partial_3,\ 1 \leq i_1 \leq a_1 - 3; 1 \leq i_3 \leq 2;\\
&x_1^{a_1-2}x_3^4\partial_2; x_1^{a_1-1}\partial_2; x_3^{i_3}\partial_3,\ 3 \leq i_3 \leq 5; x_1^{i_1}x_3^{i_3}\partial_3,\ 1 \leq i_1 \leq a_1 - 3; 3 \leq i_3 \leq 4;\\
&x_1^{a_1-2}x_3^{i_3}\partial_3,\ 2 \leq i_3 \leq 4; x_1^{a_1-1}\partial_3; x_2\partial_3.
\end{aligned}$$

Therefore we have

$$\lambda^1(V) = 9a_1 - 2.$$

In case of $a_1 \geq 3, a_2 = 2, a_3 = 2$, we obtain the basis of derivation represented by following derivations form a basis of Der$A^1(V)$:

$$x_3^2\partial_1;\ x_3^3\partial_1;\ x_2\partial_1;\ x_1^{i_1}x_3^{i_3}\partial_1,\ 1 \leq i_1 \leq a_1 - 2; 0 \leq i_3 \leq 2;\ x_1^{a_1-1}\partial_1;\ x_3^2\partial_2 - x_3\partial_3;$$
$$x_3^3\partial_2;\ x_2\partial_2 + \frac{1}{2}x_3\partial_3;\ x_1^{a_1-2}x_3^2\partial_3;\ x_1^{a_1-1}\partial_2;\ x_3^2\partial_3;\ x_3^3\partial_3;\ x_2\partial_3;$$
$$x_1^{a_1-1}\partial_3;\ x_1^{i_1}x_3^2\partial_3,\ 1 \leq i_1 \leq a_1 - 2;\ x_1^{a_1-2}x_3\partial_3;\ x_1^{i_1}x_3^2\partial_2 - x_1^{i_1}x_3\partial_3,\ 1 \leq i_1 \leq a_1 - 3;$$
$$x_1^{a_1-2}x_3\partial_2 - x_1^{a_1-2}\partial_3.$$

□

Therefore we have

$$\lambda^1(V) = 5a_1 + 4.$$

In case of $a_1 = 2, a_2 \geq 3, a_3 = 2$, we obtain the basis of derivation represented by following derivations form a basis of Der$A^1(V)$:

$$x_3^2\partial_1;\ x_2x_3\partial_1;\ x_2^{a_2-1}\partial_1;\ x_1\partial_1;\ x_3\partial_2;\ x_3^2\partial_2;\ x_2^{i_2}\partial_2,\ 1 \leq i_2 \leq a_2 - 1;\ x_2x_3\partial_2;\ x_1\partial_2;$$
$$x_3\partial_3;\ x_3^2\partial_3;\ x_2x_3\partial_3;\ x_1\partial_3;\ x_2^{i_2}\partial_3,\ a_2 - 2 \leq i_2 \leq a_2 - 1.$$

Therefore we have

$$\lambda^1(V) = a_2 + 13.$$

In case of $a_1 = 2, a_2 = 2, a_3 \geq 3$, we obtain the basis of derivation represented by following derivations form a basis of Der$A^1(V)$:

$$\left(2x_2 + x_3^{a_3}\right)\partial_1;\ x_3^{2a_3-1}\partial_1;\ x_1\partial_1;\ x_3^{2a_3-1}\partial_2;\ x_2\partial_2 + \frac{1}{a_3}x_3\partial_3;\ x_1\partial_2;\ x_3^{i_3}\partial_3,\ a_3 \leq i_3 \leq 2a_3 - 1;$$
$$x_2\partial_3;\ x_1\partial_3;\ x_3^{i_3+a_3-1}\partial_2 - \frac{2}{a_3}x_3^{i_3}\partial_3,\ 1 \leq i_3 \leq a_3 - 1.$$

Therefore we have

$$\lambda^1(V) = 2a_3 + 7.$$

In case of $a_1 = 2, a_2 \geq 3, a_3 \geq 3$, we obtain the basis of derivation represented by following derivations form a basis of Der$A^1(V)$:

$$\left(a_2x_2^{a_2-1} + x_3^{a_3}\right)\partial_1;\ x_3^{2a_3-2}\partial_1;\ x_2x_3^{a_3-1}\partial_1;\ x_1\partial_1;\ x_2x_3^{i_3-1}\partial_2 + \frac{a_2-1}{a_3}x_3^{i_3}\partial_3,\ 1 \leq i_3 \leq a_3 - 2;$$
$$x_3^{i_3}\partial_2,\ a_3 - 1 \leq i_3 \leq 2a_3 - 2;\ x_1\partial_2;\ x_2^{i_2}x_3^{i_3}\partial_2,\ 2 \leq i_2 \leq a_2 - 2; 1 \leq i_3 \leq a_3 - 2;$$
$$x_2x_3^{i_3}\partial_2,\ a_3 - 2 \leq i_3 \leq a_3 - 1;\ x_2^{i_2}x_3^{i_3}\partial_3,\ 1 \leq i_2 \leq a_2 - 2; 1 \leq i_3 \leq a_3 - 2;$$
$$x_2^{i_2}\partial_3,\ a_2 - 2 \leq i_2 \leq a_2 - 1;\ x_1\partial_3;\ x_2x_3^{a_3-1}\partial_3;$$
$$x_2^{i_2}\partial_2,\ 2 \leq i_2 \leq a_2 - 1;\ x_3^{i_3}\partial_3,\ a_3 - 1 \leq i_3 \leq 2a_3 - 2.$$

Therefore we have

$$\lambda^1(V) = 2a_2a_3 - 3a_2 - 2a_3 + 17.$$

In case of $a_1 \geq 3, a_2 = 2, a_3 \geq 3$, we obtain the basis of derivation represented by following derivations form a basis of Der$A^1(V)$:

$\left(2x_2+x_3^{a_3}\right)\partial_1;\, x_3^{i_3}\partial_1,\ 2a_3-2\le i_3\le 2a_3-1;\, x_1^{i_1}x_3^{i_3}\partial_1,\ 1\le i_1\le a_1-2;\, 0\le i_3\le 2a_3-2;$
$x_3^{a_3-1+i_3}\partial_2-\dfrac{2}{a_3}x_3^{i_3}\partial_3,\ 1\le i_3\le a_3-1;\, x_2\partial_2+\dfrac{1}{a_3}x_3\partial_3;\, x_1^{a_1-2}x_3^{2a_3-2}\partial_2;\, x_1^{a_1-1}\partial_2;$
$x_1^{a_1-1}\partial_1;\, x_3^{2a_3-1}\partial_2;\, x_1^{i_1}x_3^{i_3}\partial_1-\dfrac{2}{a_3}x_1^{i_1}x_3^{i_3-a_3+1}\partial_3,\ 1\le i_1\le a_1-3;\, a_3\le i_3\le 2a_3-2;$
$x_1^{a_1-2}x_3^{i_3}\partial_1-\dfrac{2}{a_3}x_1^{a_1-2}x_3^{i_3-a_3+1}\partial_3,\ a_3-1\le i_3\le 2a_3-3;\, x_3^{i_3}\partial_3,\ a_3\le i_3\le 2a_3-1;\, x_2\partial_3;$
$x_1^{a_1-1}\partial_3;\, x_1^{i_1}x_3^{i_3}\partial_3,\ 1\le i_1\le a_1-2;\, a_3\le i_3\le 2a_3-2;\, x_1^{a_1-2}x_3^{a_3-1}\partial_3.$

Therefore we have

$$\lambda^1(V)=4a_1a_3-3a_1-6a_3+16.$$

In case of $a_1=2, a_2=2, a_3=2$, we obtain the basis of derivation represented by following derivations form a basis of $\mathrm{Der}A^1(V)$:

$$\begin{aligned}&\left(2x_2+x_3^2\right)\partial_1;\, x_3^3\partial_1;\, x_1\partial_1;\, x_3^2\partial_2-x_3\partial_3;\, x_3^3\partial_2;\, x_1\partial_2;\, x_3^2\partial_3;\, x_3^3\partial_3;\\ &x_2\partial_2+\frac{1}{2}x_3\partial_3;\, x_2\partial_3;\, x_1\partial_3.\end{aligned}$$

Therefore we have

$$\lambda^1(V)=11.$$

In last case we obtain the basis of derivation represented by following derivations form a basis of $\mathrm{Der}A^1(V)$:

$x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_1,\ 1\le i_1\le a_1-2, 1\le i_2\le a_2-2, 0\le i_3\le a_3-2;$
$x_1^{i_1}x_3^{i_3}\partial_1,\ 1\le i_1\le a_1-2, 1\le i_3\le 2a_3-2;\, x_2x_3^{a_3-1}\partial_1;\, x_3^{2a_3-2}\partial_1;$
$x_1^{i_1}\partial_1,\ 1\le i_1\le a_1-1;\, \left(a_2x_2^{a_2-1}+x_3^{a_3}\right)\partial_1;\, x_3^{a_3-1}\partial_2+\dfrac{a_2}{a_3-1}x_2^{a_2-2}\partial_3;\, x_3^{i_3}\partial_2,\ a_3\le i_3\le 2a_3-2;$
$x_2x_3^{i_3-1}\partial_2+\dfrac{a_1}{a_3}x_3^{i_3}\partial_3,\ 1\le i_3\le a_3-1;\, x_2x_3^{a_3-1}\partial_2;\, x_2^{i_2}x_3^{i_3}\partial_2,\ 2\le i_2\le a_2-2,\ 0\le i_3\le a_3-2;$
$x_2^{a_2-1}\partial_2;\, x_1^{i_1}x_3^{i_3}\partial_2,\ 1\le i_1\le a_1-2,\ a_3-1\le i_3\le 2a_3-2;$
$x_1x_2x_3^{i_3-1}\partial_2+\dfrac{a_1}{a_3}x_1x_3^{i_3}\partial_3,\ 1\le i_3\le a_3-2;$
$x_1x_2x_3^{a_3-2}\partial_2;\, x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_2,\ 1\le i_1\le a_1-2, 2\le i_2\le a_2-2, 0\le i_3\le a_3-2;\, x_1^{a_1-1}\partial_2;$
$x_3^{i_3}\partial_3,\ a_3\le i_3\le 2a_3-2;\, x_2^{i_2}x_3^{i_3}\partial_3,\ 1\le i_2\le a_2-2,\ 1\le i_3\le a_3-2;\, x_2x_3^{a_3-1}\partial_3;\, x_2^{a_2-1}\partial_3;$
$x_1^{i_1}x_3^{i_3}\partial_3,\ 1\le i_1\le a_1-2,\ a_3-1\le i_3\le 2a_3-2;\, x_1^{a_1-1}\partial_3;\, x_1^{a_1-2}x_2^{a_2-2}\partial_3;$
$x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_3,\ 1\le i_1\le a_1-2, 1\le i_2\le a_2-2, 1\le i_3\le a_3-2.$

Therefore we have

$$\lambda^1(V)=3a_1a_2a_3-4a_1a_2-3a_1a_3-4a_2a_3+8a_1+5a_2+5a_3-1.$$

It is easy to see that we have

$$h_1(\frac{1}{w_1}, \frac{1}{w_2}, \frac{1}{w_3}) = \begin{cases} a_1 + 7; & a_1 \geq 2, a_2 = 2, a_3 = 1 \\ 2a_3 + 7; & a_1 = 2, a_2 = 2, a_3 \geq 2 \\ 4a_1a_3 - 3(a_1 + a_3) + 16; & a_1 \geq 3, a_2 = 2, a_3 \geq 2 \\ \frac{2a_2^2a_3}{a_2-1} - 3\left(a_2 + \frac{a_2a_3}{a_2-1}\right) + 16; & a_1 = 2, a_2 \geq 3, a_3 \geq 2 \\ \frac{3a_1a_2^2a_3}{a_2-1} + 5\left(a_1 + a_2 + \frac{a_2a_3}{a_2-1}\right) & \\ -4\left(a_1a_2 + \frac{a_1a_2a_3}{a_2-1} + \frac{a_2^2a_3}{a_2-1}\right) + 6; & a_1 \geq 3, a_2 \geq 3, a_3 \geq 2. \end{cases}$$

We need to shows that when $a_1 \geq 3, a_2 \geq 3, a_3 \geq 3$, then $3a_1a_2a_3 - 4a_1a_2 - 3a_1a_3 - 4a_2a_3 + 8a_1 + 5a_2 + 5a_3 - 1 \leq \frac{3a_1a_2^2a_3}{a_2-1} + 5(a_1 + a_2 + \frac{a_2a_3}{a_2-1}) - 4(a_1a_2 + \frac{a_1a_2a_3}{a_2-1} + \frac{a_2^2a_3}{a_2-1}) + 6$. After simplification we get

$$3a_1 + (a_2 - 1) + (a_1 - 2)(a_2 - 1)(a_3 - 3) + a_1(a_3 - 3) + (a_1 - 2)(a_2 - 3)a_3 \geq 0.$$

It is easy to see that in following cases:

(i) $a_1 \geq 2, a_2 = 2, a_3 = 1$,
(ii) $a_1 = 2, a_2 = 2, a_3 \geq 3$,
(iii) $a_1 = 2, a_2 = 2, a_3 = 2$,
(iv) $a_1 \geq 3, a_2 = 2, a_3 = 2$,
(v) $a_1 \geq 3, a_2 = 2, a_3 \geq 3$,
(vi) $a_1 = 2, a_2 \geq 3, a_3 \geq 3$,
(vii) $a_1 \geq 3, a_2 \geq 3, a_3 = 2$,

our proposed conjecture also holds. The first five cases is easy to prove. The last two cases is following as: Case(vi) when $a_1 = 2, a_2 \geq 3, a_3 \geq 3$, then

$$2a_2a_3 - 3a_2 - 2a_3 + 17 \leq \frac{2a_2^2a_3}{a_2 - 1} - 3\left(a_2 + \frac{a_2a_3}{a_2 - 1}\right) + 16,$$

after simplification we get

$$(a_2 - 2)(a_3 - 2) + (a_2 - 3) \geq 0.$$

Case (vii) when $a_1 \geq 3, a_2 \geq 3, a_3 = 2$, then

$$2a_1a_2 - 2a_1 - 3a_2 + 9 \leq \frac{6a_1a_2^2}{a_2 - 1} + 5a_1 + 5a_2 + \frac{10a_2}{a_2 - 1} - 4a_1a_2 - \frac{8a_1a_2}{a_2 - 1} - \frac{8a_2^2}{a_2 - 1} + 6,$$

after simplification we get

$$a_1(3a_2 - 1) + (a_2 - 3)(2a_1 - 1) \geq 0.$$

Proposition 3.9 *Let $(V, 0)$ be a fewnomial surface isolated singularity of type 5 which is defined by $f = x_1^{a_1}x_2 + x_2^{a_2}x_1 + x_3^{a_3}$ $(a_1 \geq 1, a_2 \geq 1, a_3 \geq 2)$ with weight type $(\frac{a_2-1}{a_1a_2-1}, \frac{a_1-1}{a_1a_2-1}, \frac{1}{a_3}; 1)$. Then*

$$\lambda^1(V) = \begin{cases} 4a_2a_3 - 6a_2 + 12; & a_1 = 2, a_2 \geq 2, a_3 \geq 2 \\ a_2(a_3 - 2) + 9; & a_1 = 1, a_2 \geq 1, a_3 \geq 2 \\ 3a_1a_2a_3 - 4a_1a_2 - 2a_2a_3 - 2a_1a_3 & \\ \quad +2a_1 + 2a_2 + 6a_3 + 6; & Otherwise. \end{cases}$$

Furthermore, we need to show that when $a_1 \geq 3, a_2 \geq 3, a_3 \geq 3$, then $3a_1a_2a_3 - 4a_1a_2 - 2a_2a_3 - 2a_1a_3 + 2a_1 + 2a_2 + 6a_3 + 6 \leq \frac{3(a_1a_2-1)^2}{a_3(a_2-1)(a_1-1)} + 5(\frac{a_1a_2-1}{a_2-1} + \frac{a_1a_2-1}{a_1-1} + a_3) - 4(\frac{(a_1a_2-1)^2}{(a_2-1)(a_1-1)} + \frac{a_3(a_1a_2-1)}{a_1-1} + \frac{a_3(a_1a_2-1)}{a_2-1}) + 6.$

Proof It is easy to see that the moduli algebra $A^1(V) = \mathbb{C}\{x_1, x_2, x_3\}/(f, m.J(f))$ has dimension $(a_1a_2a_3 - a_1a_2 + 3)$ and has a monomial basis of the form [2]

$$\{x_1^{i_1}x_2^{i_2}x_3^{i_3}, 1 \leq i_1 \leq a_1 - 2; 1 \leq i_2 \leq a_2 - 1; 0 \leq i_3 \leq a_3 - 2; x_3^{a_3-1}; x_1^{a_1-1}x_2; x_1^{a_1};$$
$$x_1^{i_1}x_3^{i_3}, 1 \leq i_1 \leq a_1 - 1; 0 \leq i_3 \leq a_3 - 2; x_2^{i_2}x_3^{i_3}, 0 \leq i_2 \leq 2a_2 - 2; 0 \leq i_3 \leq a_3 - 2\}.$$

In order to compute a derivation D of $A^1(V)$ it suffices to indicate its values on the generators x_1, x_2, x_3 which can be written in terms of the basis. Thus we can write

$$Dx_j = \sum_{i_1=1}^{a_1-2}\sum_{i_2=1}^{a_2-1}\sum_{i_3=0}^{a_3-2} c^j_{i_1,i_2,i_3}x_1^{i_1}x_2^{i_2}x_3^{i_3} + \sum_{i_1=1}^{a_1-1}\sum_{i_3=0}^{a_3-2} c^j_{i_1,0,i_3}x_1^{i_1}x_3^{i_3} + \sum_{i_2=0}^{2a_2-2}\sum_{i_3=0}^{a_3-2} c^j_{0,i_2,i_3}x_2^{i_2}x_3^{i_3}$$
$$+ c^j_{0,0,a_3-1}x_3^{a_3-1} + c^j_{a_1-1,1,0}x_1^{a_1-1}x_2 + c^j_{a_1,0,0}x_1^{a_1}, \;\; j = 1, 2, 3.$$

Using the above derivations we obtain the following description of Lie algebras in question. The derivations represented by the following vector fields form a basis in Der $A^1(V)$:

$$x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_1, \;\; 2 \leq i_1 \leq a_1 - 2, 1 \leq i_2 \leq a_2 - 1, 0 \leq i_3 \leq a_3 - 2;$$
$$x_3^{a_3-1}\partial_1; x_2^{a_2-2}x_3^{i_3}\partial_1 + \frac{a_1}{a_2}x_1^{a_1-2}x_3^{i_3}\partial_2, \;\; 0 \leq i_3 \leq a_3 - 2;$$
$$x_2^{a_2-1}x_3^{i_3}\partial_1 + \frac{a_1}{a_2}x_1^{a_1-2}x_2x_3^{i_3}\partial_2, \;\; 0 \leq i_3 \leq a_3 - 3; x_2^{a_2-1}x_3^{a_3-2}\partial_1;$$
$$x_2^{i_2}x_3^{i_3}\partial_1, \;\; a_2 \leq i_2 \leq 2a_2 - 2; 0 \leq i_3 \leq a_3 - 2; ;$$
$$x_1^{i_1}x_3^{i_3}\partial_1 + \frac{a_1 - 1}{a_2 - 1}x_1^{i_1-1}x_2x_3^{i_3}\partial_2, \;\; 1 \leq i_1 \leq a_1 - 2; 0 \leq i_3 \leq a_3 - 2;$$
$$x_1^{a_1-1}x_3^{i_3}\partial_1 + \frac{a_1 - 1}{a_2 - 1}x_1^{a_1-2}x_2x_3^{i_3}\partial_2, \;\; 0 \leq i_3 \leq a_3 - 3;$$
$$x_1x_2^{i_2}x_3^{i_3}\partial_1 + \frac{a_1 - 1}{a_2 - 1}x_2^{i_2+1}x_3^{i_3}\partial_2, \;\; 1 \leq i_2 \leq a_2 - 3; 0 \leq i_3 \leq a_3 - 2;$$
$$x_1x_2^{a_2-2}x_3^{i_3}\partial_1 + \frac{a_1}{a_2}x_1^{a_1-1}x_3^{i_3}\partial_2, \;\; 0 \leq i_3 \leq a_3 - 3; x_1x_2^{a_2-2}x_3^{a_3-2}\partial_1;$$
$$x_1x_2^{a_2-1}x_3^{i_3}\partial_1, \;\; 0 \leq i_3 \leq a_3 - 2; x_1^{a_1-1}x_3^{a_3-2}\partial_1; x_1^{a_1-1}x_2\partial_1; x_1^{a_1}\partial_1;$$
$$\left(-\frac{a_1(a_2 - 1)}{a_2(a_1 - 1)}x_1^{a_1-1}x_3^{i_3} + x_2^{a_2-1}x_3^{i_3}\right)\partial_2, \;\; 0 \leq i_3 \leq a_3 - 3; x_2^{a_2-1}x_3^{a_3-2}\partial_2;$$
$$x_2^{i_2}x_3^{i_3}\partial_2, \;\; a_2 \leq i_2 \leq 2a_2 - 2; 0 \leq i_3 \leq a_3 - 2; x_1^{a_1-1}x_3^{a_3-2}\partial_2; x_1^{a_1-1}x_2\partial_2;$$
$$x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_2, \;\; 1 \leq i_1 \leq a_1 - 2, 2 \leq i_2 \leq a_2 - 1, 0 \leq i_3 \leq a_3 - 2;$$
$$x_1^{a_1}\partial_2; x_1^{a_1-2}x_2x_3^{a_3-2}\partial_2; x_3^{i_3}\partial_3, \;\; 1 \leq i_3 \leq a_3 - 1; (a_1x_1^{a_1-1}x_2 + x_2^{a_2})\partial_3;$$
$$\left(\frac{1}{a_2}x_1^{a_1} + x_1x_2^{a_2-1}\right)\partial_3; x_1^{a_1-1}x_3^{i_3}\partial_3, \;\; 1 \leq i_3 \leq a_3 - 2; x_2^{2a_2-2}x_3^{i_3}\partial_3, \;\; 0 \leq i_3 \leq a_3 - 2;$$
$$x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_3, \;\; 1 \leq i_1 \leq a_1 - 2, 0 \leq i_2 \leq a_2 - 1, 1 \leq i_3 \leq a_3 - 2;$$
$$x_2^{i_2}x_3^{i_3}\partial_3, \;\; 1 \leq i_2 \leq 2a_2 - 3; 1 \leq i_3 \leq a_3 - 2.$$

Therefore we have

$$\lambda^1(V) = 3a_1a_2a_3 - 4a_1a_2 - 2a_2a_3 - 2a_1a_3 + 2a_1 + 2a_2 + 6a_3 + 6.$$

In case of $a_1 = 2, a_2 \geq 2, a_3 \geq 2$, we obtain the basis of derivation represented by following derivations form a basis of $\mathrm{Der}A^1(V)$:

$$\begin{aligned}
&x_3^{a_3-1}\partial_1;\ x_2^{a_2-1}x_3^{a_3-2}\partial_1 - \frac{2}{a_2-1}x_2x_3^{a_3-2}\partial_2;\\
&x_2^{i_2}x_3^{i_3}\partial_1 - \frac{2}{a_2-1}x_2^{i_2-2}x_3^{i_3}\partial_2,\ a_2 \leq i_2 \leq 2a_2-3, 0 \leq i_3 \leq a_3-2;\\
&x_2^{2a_2-2}x_3^{i_3}\partial_1,\ 0 \leq i_3 \leq a_3-2;\ x_1x_3^{i_3}\partial_1 + \frac{1}{a_2-1}x_2x_3^{i_3}\partial_2, 0 \leq i_3 \leq a_3-2;\\
&x_1x_2\partial_1 + \frac{1}{a_2-1}x_1x_2^2\partial_1;\ x_1^2\partial_1;\ x_3^{a_3-1}\partial_2;\ x_2^{a_2-1}x_3^{a_3-2}\partial_2;\ x_1x_3^{a_3-2}\partial_2;\\
&\left(-\frac{2(a_2-1)}{a_2}x_1x_3^{i_3} + x_2^{a_2-1}x_3^{i_3}\right)\partial_2,\ 0 \leq i_3 \leq a_3-3;\ x_1x_2\partial_2;\ x_1^2\partial_2;\\
&x_2^{i_2}x_3^{i_3}\partial_2,\ a_2 \leq i_2 \leq 2a_2-2; 0 \leq i_3 \leq a_3-2;\ x_3^{i_3}\partial_3,\ 1 \leq i_3 \leq a_3-1;\\
&x_2^{i_2}x_3^{i_3}\partial_3,\ 1 \leq i_2 \leq 2a_2-2;\ 1 \leq i_3 \leq a_3-2;\ x_2^{2a_2-2}\partial_3;\ x_1^2\partial_3;\\
&x_1x_3^{i_3}\partial_3,\ 1 \leq i_3 \leq a_3-2;\ (2x_1x_2 + x_2^{a_2})\partial_3.
\end{aligned}$$

Therefore we have

$$\lambda^1(V) = 4a_2a_3 - 6a_2 + 12.$$

In case of $a_1 = 1, a_2 \geq 1, a_3 \geq 2$, we obtain the basis of derivation represented by following derivations form a basis of Der $A^1(V)$:

$$\begin{aligned}
&x_3^{a_3-1}\partial_1;\ (x_2^{a_2}+x_2)\partial_1;\ x_1\partial_1;\ x_3^{a_3-1}\partial_2;\ (x_2^{a_2}+x_2)\partial_2;\ x_1\partial_2;\ x_3^{a_3-1}\partial_3;\ (x_2^{a_2}+x_2)\partial_3;\ x_1\partial_3;\\
&x_2^{i_2}x_3^{i_3}\partial_3,\ 0 \leq i_2 \leq a_2-1, 1 \leq i_3 \leq a_3-2.
\end{aligned}$$

Therefore we have

$$\lambda^1(V) = a_2(a_3-2) + 9.$$

We have

$$h_1\left(\frac{1}{w_1}, \frac{1}{w_2}, \frac{1}{w_3}\right) = \begin{cases} a_3 + 7;\ a_1 = 1, a_2 = 1, a_3 \geq 3 \\ 2\frac{(a_1a_2-1)^2}{(a_2-1)(a_1-1)} - 3(\frac{a_1a_2-1}{a_2-1} + \frac{a_1a_2-1}{a_1-1}) + 16;\ a_1 \geq 3, a_2 \geq 3, a_3 = 2 \\ \frac{3(a_1a_2-1)^2}{a_3(a_2-1)(a_1-1)} + 5(\frac{a_1a_2-1}{a_2-1} + \frac{a_1a_2-1}{a_1-1} + a_3) - 4(\frac{(a_1a_2-1)^2}{(a_2-1)(a_1-1)} + \frac{a_3(a_1a_2-1)}{a_1-1} \\ + \frac{a_3(a_1a_2-1)}{a_2-1}) + 6;\ a_1 \geq 3, a_2 \geq 3, a_3 \geq 3. \end{cases}$$

It is noted that in following cases:

(i) $a_1 \geq 3, a_2 \geq 3, a_3 \geq 3$,
(ii) $a_1 \geq 3, a_2 \geq 3, a_3 = 2$,
(iii) $a_1 = 1, a_2 = 1, a_3 \geq 3$,

our proposed Conjecture holds. Case(i) We need to show that $3a_1a_2a_3 - 4a_1a_2 - 2a_2a_3 - 2a_1a_3 + 2a_1 + 2a_2 + 6a_3 + 6 \leq \frac{3(a_1a_2-1)^2}{a_3(a_2-1)(a_1-1)} + 5(\frac{a_1a_2-1}{a_2-1} + \frac{a_1a_2-1}{a_1-1} + a_3) - 4(\frac{(a_1a_2-1)^2}{(a_2-1)(a_1-1)} + \frac{a_3(a_1a_2-1)}{a_1-1} + \frac{a_3(a_1a_2-1)}{a_2-1}) + 6$. After simplification we get

$$[a_1(a_2-2)(a_1-3) + a_2(a_1-2)(a_2-3) + (a_1-3) + (a_2-3)]\,(a_3-1) \geq 0.$$

Case (ii) We need to show that

$$2a_1a_2 - 2(a_1+a_2) + 18 \leq \frac{2(a_1a_2-1)^2}{(a_1-1)(a_2-1)} - 3(\frac{a_1a_2-1}{a_2-1} + \frac{a_1a_2-1}{a_1-1}) + 16,$$

after simplification we get $a_2(a_1-2)(a_2-3) + a_1(a_2-2)(a_1-3) + (a_1-3) + (a_2-3) \geq 0$. It is easy to see that when $a_1 = 1, a_2 = 1, a_3 \geq 3$, then our conjecture is also true. □

Proof of Main Theorem A It is an immediate corollary of proposition 3.1. □

Proof of Main Theorem B Let $f \in \mathbb{C}\{x_1, x_2\}$ be a weighted homogeneous fewnomial isolated singularity. Then f can be divided into the following three types:

Type A. $x_1^{a_1} + x_2^{a_2}$,
Type B. $x_1^{a_1}x_2 + x_2^{a_2}$,
Type C. $x_1^{a_1}x_2 + x_2^{a_2}x_1$.

The main theorem B is an immediate corollary of Propositions 3.2, 3.3 and 3.4 . □

Proof of Main Theorem C Let $f \in \mathbb{C}\{x_1, x_2, x_3\}$ be a weighted homogeneous fewnomial isolated surface singularity. Then f can be divided into the following five types:

Type 1. $x_1^{a_1} + x_2^{a_2} + x_3^{a_3}$,
Type 2. $x_1^{a_1}x_2 + x_2^{a_2}x_3 + x_3^{a_3}$,
Type 3. $x_1^{a_1}x_2 + x_2^{a_2}x_3 + x_3^{a_3}x_1$,
Type 4. $x_1^{a_1} + x_2^{a_2} + x_3^{a_3}x_2$,
Type 5. $x_1^{a_1}x_2 + x_2^{a_2}x_1 + x_3^{a_3}$.

The main theorem C is an immediate corollary of propositions 3.5, 3.6, 3.7, 3.8 and 3.9 . □

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