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NON-EXISTENCE OF NEGATIVE WEIGHT DERIVATIONS ON POSITIVELY GRADED ARTINIAN ALGEBRAS

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Dedicated to Professor Shigefumi Mori on the occasion of his 65th birthday

Abstract. Let $R = \mathbb{C}[x_1, x_2, \ldots, x_n]/(f_1, \ldots, f_m)$ be a positively graded Artinian algebra. A long-standing conjecture in algebraic geometry, differential geometry, and rational homotopy theory is the non-existence of negative weight derivations on R. Alexsandrov conjectured that there are no negative weight derivations when R is a complete intersection algebra, and Yau conjectured there are no negative weight derivations on R when R is the moduli algebra of a weighted homogeneous hypersurface singularity. This problem is also important in rational homotopy theory and differential geometry. In this paper we prove the non-existence of negative weight derivations on R when the degrees of $f_1, \ldots, f_m$ are bounded below by a constant C depending only on the weights of $x_1, \ldots, x_n$. Moreover this bound C is improved in several special cases.

1. Introduction

Let $P = \mathbb{C}[x_1, \ldots, x_n]$ be the polynomial ring of n weighted variables $x_1, \ldots, x_n$ with positive integer weights $w_1, w_2, \ldots, w_n$. For a monomial $x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ in P, its weighted degree is defined to be $w_1 i_1 + \cdots + w_n i_n$. A polynomial $f \in P$ is called weighted homogeneous with respect to weights $w_1, \ldots, w_n$ if there exists a positive integer d such that $\sum a_i w_i = d$ for each monomial $\prod x_i^{a_i}$ appearing in f with a non-zero coefficient. The number d is called the (weighted homogeneous) degree of f and denoted by $\deg f$. For an ideal I generated by weighted homogeneous polynomials in P we have a graded quotient algebra $R = P/I = \bigoplus_{i=0}^{\infty} R_i$. Furthermore R is called a graded complete intersection algebra if I is generated by a regular sequence $f_1, \ldots, f_m, m \leq n$. When the Krull dimension of R is zero, R is a positively graded Artinian algebra.

Let $R = P/I$ be a positively graded algebra as above. Then the derivations of R are induced by derivations of P sending I to I. Let $Der(R)$ be the R-module of derivations of R. As R is graded, we have a natural grading on $Der(R) = \bigoplus_{k=-\infty}^{+\infty} Der(R)_k$ where $Der(R)_k = \{D \in Der(R) \colon D(R_i) \subset R_{i+k}$ for any $i\}$. In particular, the Euler derivation $\Delta = \sum w_i x_i \frac{\partial}{\partial x_i}$ has weight 0.

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Given a positively graded algebra R, a very natural and important question in algebraic geometry is whether there are no derivations on R of negative weight, i.e., $Der(R)_k = 0$ for any $k < 0$. Throughout the paper, we shall focus on positively graded Artinian algebras R. Those are interesting objects in algebraic geometry (cf. the overview [Ia]). When R is a positively graded Artinian complete intersection algebra, we have the following long-standing conjecture of Alexsandrov.

Aleksandrov Conjecture ([A2]). Let R be a positively graded Artinian complete intersection; i.e., all weights $w_i, i = 1, \ldots, n$ are positive. Then there are no negative weight derivations on R.

Remark 1.1. It was pointed out by Aleksandrov and Martin [AM] that the condition of "positively graded" in the above conjecture cannot be omitted. Assume that one of the weights w_i of x_i is negative. For example, let

$$R = \mathbb{C}[x, y, z]/(x^5 + y^5 + x^3y^3z, y^8, z^3).$$

Then R is $\mathbb{Z}$-graded with $wt(x) = 1, wt(y) = 1$, and $wt(z) = -1$. It is easy to see that $D = z^2x\frac{\partial}{\partial x} + z^2y\frac{\partial}{\partial y}$ defines a derivation on R and $wt(D) = -2$.

The Aleksandrov Conjecture was proved only for some special cases.

Theorem 1.1 ([AM, Theorem 4.3]). *Suppose $R = \mathbb{C}\{x, y\}/(f, g)$ is a $\mathbb{Z}$-graded Artinian complete intersection. Then there are no derivations of negative weight on R.*

Theorem 1.2 ([AM, Proposition 4.1]). *Let R be an Artinian homogeneous algebra, i.e., all weights $w_i = 1$. Then there are no derivations of negative weight on R.*

Theorem 1.3 ([PP1]). *Let $R = \mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ be a positively graded Artinian complete intersection algebra, and assume that $\deg(f_i) \le \deg(f_n), 1 \le i \le n-1$, and*

$$\mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_{n-1})$$

is reduced. Then there are no derivations of negative weight on R.

Theorem 1.4 ([Ch2]). *Assume that $R = \mathbb{C}[x_1, x_2, x_3]/(f_1, f_2, f_3)$ is a positively graded Artinian complete intersection algebra. Then there are no derivations of negative weight on R.*

In [A3], Alexsandrov claimed that his conjecture was also true in cases of three and four variables. However, very little progress has been made on the Alexsandrov Conjecture over the last twenty years. This problem has been regarded as notoriously hard.

There are many reasons why the problem of non-existence of negative weight derivations is important [PP2]. It is related to many questions arising in the deformation theory of singularities (cf. [A1, (6.3)], [A2, A3, AM]). In [PP1], the authors pointed out that the study of non-existence of negative weight derivations turns out to be very useful from the point of view of the moduli space theory of deformations. Furthermore, according to Pinkham [Pi1, Pi2], the existence of coarse moduli spaces for certain problems depends on a graded R having no derivations of negative degree. The point is that we can compactify $X = \mathrm{Spec}R$ (a variety with good $\mathbb{C}^*$-action) by considering $\bar{X} = \mathrm{Proj}R[T]$, where $\mathrm{wt}T = 1$. Then $\bar{X} - X = E$ is a Weil divisor, isomorphic to $\mathrm{Proj}R$. Then the derivations of weight ≤ 0 of R yield exactly the derivations of the natural compactification $\bar{X}$ which are logarithmic at

E. In section 2, we shall also give its applications in rational homology theory and differential geometry.

On the other hand, Yau has made the following conjecture in his work on the Lie algebras of derivations on the moduli algebras of isolated hypersurface singularities and especially his micro-local characterization (using only the Lie algebras of derivations on the moduli algebras) of weighted homogeneous isolated hypersurface singularities ([XY],[MY],[Ya1],[Ya2]):

Yau Conjecture. Let $(V,0) = \{(x_1, x_2, \ldots, x_n) \in \mathbb{C}^n : f(x_1, x_2, \ldots, x_n) = 0\}$ be an isolated singularity defined by the weighted homogeneous polynomial $f(x_1, x_2, \ldots, x_n)$. Then there are no non-zero negative weight derivations on the moduli algebra (= Milnor algebra here) $A(V) = \mathbb{C}[x_1, x_2, \ldots, x_n]/(f_{x_1}, f_{x_2}, \ldots, f_{x_n})$ where $f_{x_i} = \partial f/\partial x_i$; i.e., the Yau algebra $L(V) := Der(A(V))$ of V is a non-negatively graded algebra.

Though the Yau Conjecture is a special case of the Aleksandrov Conjecture, it is also of independent interest [YZ1]. The Yau Conjecture has only been proven in the low-dimensional case $n \leq 4$ ([CXY],[Ch1]) by explicit calculations. Recently, Yau and Zuo proved the following result.

Theorem 1.5 ([YZ2])**.** *Let $(V,0) = \{(x_1, x_2, \ldots, x_n) \in \mathbb{C}^n : f(x_1, x_2, \ldots, x_n) = 0\}$ be an isolated singularity defined by the weighted homogeneous polynomial $f(x_1, x_2, \ldots, x_n)$ of canonical weight type $(d : w_1, w_2, \ldots, w_n)$ (i.e., $d \geq 2w_1 \geq 2w_2 \geq \cdots \geq 2w_n > 0$). Let $A(V) = \mathbb{C}[x_1, x_2, \ldots, x_n]/(f_{x_1}, f_{x_2}, \ldots, f_{x_n})$ be the moduli algebra. If $w_n \geq w_1/2$, then $Der(A(V))_{<0} = 0$.*

It is interesting that in some special cases the complete intersection restriction in the Aleksandrov Conjecture is not necessary to show the non-existence of negative weight derivations (for example the Artinian homogeneous algebra; see Theorem 1.2). It is natural to try to relax the complete intersection restriction. Here we shall consider a more general question and formulate the following conjecture.

Conjecture 1.1. Let $R = \mathbb{C}[x_1, x_2, \ldots, x_n]/(f_1, f_2, \ldots, f_m), m \geq n$, be a positively graded Artinian algebra. If the weighted degree of each f_i is bounded below by a suitable constant C depending only on the weights of $x_1, \ldots, x_n$, then all derivations of negative degree of R vanish.

Remark 1.2. The assumption about C is not restrictive but necessary. In view of the following example, we see that the condition "the weighted degree of each f_i is bounded below by a suitable constant C" in Conjecture 1.1 cannot be omitted. Take the algebra $R = \mathbb{C}[x_1, x_2]/(x_1^2, x_1x_2, x_2^2)$, with $wt(x_1) = 1$ and $wt(x_2) = 2$. Then $D = x_1 \frac{\partial}{\partial x_2}$ induces a non-zero derivation on R and $wt(D) = -1$.

Remark 1.3. The Aleksandrov Conjecture is a special case of the above Conjecture 1.1 when the weighted degree of each f_i is bounded below by a suitable constant C.

Both the Alexandrov Conjecture and the Yau Conjecture are very hard to prove in general. In these cases, the variety defined by $I = (f_1, \ldots, f_m)$ consists of the origin alone, the geometry is poor, and the difficulty increases. Until now, we have not seen any other techniques for dealing with the general dimension case. The explicit calculation and proof by case analysis methods used in [AM],[PP1],[CXY],[Ch1, Ch2],[YZ2] are hard to generalize to higher dimensions.

However, in this paper, we have been able to develop novel techniques (see section 3) to solve these problems. We prove that the Aleksandrov Conjecture is true so long as the weighted degree of each f_i is bounded below by a suitable constant C depending only on the weights of $x_1, \dots, x_n$. In fact, we verify the more general Conjecture 1.1.

Main Theorem A (Conjecture 1.1)**.** *Let $P = \mathbb{C}[x_1, x_2, \dots, x_n]$ be the weighted polynomial ring of n weighted variables $x_1, x_2, \dots, x_n$ with positive integer weights $w_1 \geq w_2 \geq \cdots \geq w_n$. Suppose that $f_1, f_2, \dots, f_m$ are m ($m \geq n \geq 2$) weighted homogeneous polynomials with degrees greater than $(m-1)(w_1 w_2)^{n-1}$ and $R = P/(f_1, f_2, \dots, f_m)$ is an Artinian algebra. Then there are no non-zero negative weight derivations on R.*

Remark 1.4. In the example of Remark 1.2 above, the weights of the generators x_1 and x_2 are 1 and 2 respectively, $n = 2$ and $m = 3$. In Main Theorem A, the condition is that the weighted degree of each f_i is greater than $(m-1)(w_1 w_2)^{n-1}$, which is 4. But $(x_1^2, x_1 x_2, x_2^2)$ has weighted degrees $(2, 3, 4)$. So the condition in Main Theorem A is not satisfied. More generally, suppose the weights of the generators x_1 and x_2 are w_1 and w_2 respectively in this example. Then the condition in Main Theorem A is that the weighted degree of each f_i is greater than $(m-1)(w_1 w_2)^{n-1}$, which is $2w_1 w_2$. But $(x_1^2, x_1 x_2, x_2^2)$ have weighted degrees $(2w_1, w_1 + w_2, 2w_2)$. So the weighted degree of $x_1 x_2$ is $w_1 + w_2$, which is no more than $2w_1 w_2$. Thus no matter which weights one chooses for this example, it cannot satisfy the condition in Main Theorem A.

It is an interesting question whether or not the constant $(m-1)(w_1 w_2)^{n-1}$ can be made smaller. The following Main Theorem B tells us that this bound can be improved under the extra condition that any two of the weights $w_1, w_2, \dots, w_n$ are coprime.

Main Theorem B. *Let $P = \mathbb{C}[x_1, x_2, \dots, x_n]$ be the weighted polynomial ring of n weighted variables $x_1, x_2, \dots, x_n$ with positive integer weights $w_1 \geq w_2 \geq \cdots \geq w_n$ and let $f_1, f_2, \dots, f_m$ be m ($m \geq n \geq 2$) weighted homogeneous polynomials with degrees greater than $(m-1)w_1 w_2$. Suppose that any two of the weights $w_1, w_2, \dots, w_n$ are coprime and $R = P/(f_1, f_2, \dots, f_m)$ is an Artinian algebra. Then there are no non-zero negative weight derivations on R.*

Fewnomial singularities are an important class of weighted homogeneous singularities. The concept of fewnomial singularities was first introduced by Khovanski [Kho]. We say that a polynomial f in n variables is fewnomial if the number of monomials appearing in f does not exceed n. It is easy to show that, except for certain trivial cases, a fewnomial in n variables can define an isolated singularity only if it has exactly n monomials, in which case we speak of it as a fewnomial isolated singularity. In other words, fewnomial singularities are those which can be defined by n-nomials in n indeterminates. Simple singularities (i.e., ADE singularity) are obviously fewnomial in this sense. Fewnomial singularities play an important role in mirror symmetry theory [ET].

In some sense, the constant $(m-1)(w_1 w_2)^{n-1}$ and $(m-1)w_1 w_2$ in Main Theorems A and B can be further improved for fewnomial singularities. In particular, we confirm the Yau Conjecture for fewnomial singularities with multipliciy at least 5.

Let $f \in \mathbb{C}\{x_1, x_2, \dots, x_n\}$. Recall that the multiplicity of f (mult(f)) is defined to be the order of the lowest non-vanishing term in the power series Taylor expansion of f at 0.

Main Theorem C (Yau Conjecture)**.** *Let $f \in \mathbb{C}\{x_1, x_2, \dots, x_n\}$ be a weighted homogeneous fewnomial isolated singularity with positive weights $w_1, w_2, \dots, w_n$ and the multiplicity of f at least 5.*

$$A(f) = \mathbb{C}\{x_1, x_2, \dots, x_n\}/(f_{x_1}, f_{x_2} \dots, f_{x_n})$$

is the moduli algebra of f. Then there are no non-zero negative weight derivations on $A(f)$.

Remark 1.5. In some cases, the condition in Main Theorem C is less strict than the condition in Main Theorem A (resp. B). This can be seen from the following example. Let

$$A(f) = \mathbb{C}\{x_1, x_2, \dots, x_5\}/(f_{x_1}, f_{x_2} \dots, f_{x_5})$$

where $f = x_1^5 + \cdots + x_5^5$ with multiplicity 5 and weights $w_i = 1$, $i = 1, \dots, 5$, $n = 5$, $m = 5$ (with the same notation as above). In Main Theorem A (resp. B), the condition is that the weighted degree of each $f_i = f_{x_i} = 5x_i^4$ is greater than $(m-1)(w_1w_2)^{n-1}$ (resp. $(m-1)(w_1w_2)$), which is 4 (resp. 4). But $f_i = 5x_i^4$ has weighted degree 4. So this example satisfies the condition of Main Theorem C, but does not satisfy the condition in Main Theorem A (resp. B). However, we are not saying that the condition in Main Theorem C is always better than the condition in Main Theorem A (resp. B). For example, let $f = x_1^3x_2 + x_2^{30}x_3 + x_3^{31}$ with multiplicity 4 and weights $w_1 = 10, w_2 = 1, w_3 = 1$, $m = n = 3$. Then $(m-1)(w_1w_2) = 20$ and $\deg f_{x_1} = 21$, $\deg f_{x_2} = \deg f_{x_3} = 30$. Thus this example does not satisfy the condition in Main Theorem C, but it satisfies the condition in Main Theorem B. It is easy to check that $A(f)$ where $f = x_1^3x_2 + x_2^{31}$ does not satisfy the condition in Main Theorem C, but it satisfies the condition in Main Theorem A.

The paper is organized as follows. We start by discussing some applications of our results to rational homotopy theory and differential geometry in section 2. In section 3 we define and give the necessary properties for the main technical tool—a new weight type associated with a negative weight derivation on the weighted polynomial ring. Some lemmas and theorems which are used in the proofs of our Main Theorems A and B are introduced and are proved in section 4. We shall give the proofs of Main Theorems A and B in section 5 and 6. In sections 7, 8, and 9, we recall some preliminary knowledge which is needed in the proof of Main Theorem C. We give the proof of Main Theorem C in sections 10.

2. Applications

In this section, we give some applications of our results to rational homotopy theory and differential geometry.

A classic result of Borel [Bo] states that the Serre spectral sequence for the rational cohomology of the universal bundle $G/H \to B_H \to B_G$ collapses if G/H is a homogeneous space of equal rank pairs (G, H) of compact connected Lie groups. Halperin made a very general conjecture on the collapsing of the Serre spectral sequence on a general fibration, which is one of the most important open problems in rational homotopy theory ([FHT, p. 516], [Me]).

Recall that a finite simply-connected cell complex C is called elliptic if all but finitely many cohomology and homotopy groups of C are finite. If C is elliptic, then C has non-negative Euler characteristic.

Halperin Conjecture ([FHT, p. 516])**.** Suppose F is a rational elliptic space with non-zero Euler-Poincaré characteristic and $F \to E \to B$ is a Serre fibration of simply-connected spaces. Then the (rational) Serre spectral sequence for this fibration collapses at E_2.

In fact, for an elliptic space with positive Euler characteristic, all cohomology must be concentrated in even degrees; i.e., cohomology is zero in odd degree. Furthermore, the cohomology algebra is a complete intersection, i.e., the same number of generators as relations.

It was shown that the above conjecture is equivalent to the following conjecture about the non-existence of negative weight derivations ([Me, Theorem A on p. 329]).

Halperin Conjecture (Equivalent form)**.** Let $P = \mathbb{C}[x_1, x_2, \dots, x_n]$ be the polynomial ring of n weighted variables $x_1, x_2, \dots, x_n$ with positive even integer weights $w_1, w_2, \dots, w_n$ and let $f_1, f_2, \dots, f_n$ be weighted homogeneous polynomials in P. Suppose that R is an Artinian complete intersection algebra of the form

$$\mathbb{C}[x_1, x_2, \dots, x_n]/(f_1, f_2, \dots, f_n).$$

Then there are no non-zero negative weight derivations on R.

In this form the Halperin Conjecture becomes accessible to diverse algebraic methods, and its deep relationship to algebraic geometry becomes evident. This conjecture has been proven when the fiber is homogeneous ([Bo],[Me]) or when $n \leq 3$ ([Th1],[Th2],[Ch2]).

The Halperin Conjecture is a special case of Conjecture 1.1 when the weighted degree of each f_i is bounded below by a suitable constant C. Therefore the following corollary follows from Main Theorem A immediately, thus verifying the Halperin Conjecture.

Corollary 2.1. *Suppose the cohomology algebra of X is $R = P/(f_1, f_2, \dots, f_n)$ where $P = \mathbb{C}[x_1, x_2, \dots, x_n]$ is the weighted polynomial ring of n weighted variables $x_1, x_2, \dots, x_n$ with positive even integer weights $w_1 \geq w_2 \geq \cdots \geq w_n$. Suppose that $f_1, f_2, \dots, f_n$ are n ($n \geq 2$) weighted homogeneous polynomials with degrees greater than $(n-1)(w_1 w_2)^{n-1}$ and R is an Artinian complete intersection algebra. Then the Serre spectral sequence of any fibration with fiber X collapses.*

In [Me], Meier proved quite generally that collapsing of the Serre spectral sequence is closely related to vanishing of derivations of the cohomology algebra. It was shown that if the negative weight derivations of the cohomology algebra of X vanish, then the Serre spectral sequence of any (orientable) fibration with fiber X collapses. Conversely, Meier also proved that if X is a formal 1-connected space with $H^*(X, Q)$ of finite type and the Serre spectral sequence collapses whenever X is (rationally) the fiber and the base is a sphere, then $Der_{<0} H^*(X, Q) = 0$.

In view of the above result of Meier, we obtain the following more general result, which is an immediate corollary of Main Theorem A.

Corollary 2.2. *Suppose the cohomology algebra of X is $R = P/(f_1, f_2, \dots, f_m)$ where $P = \mathbb{C}[x_1, x_2, \dots, x_n]$ is the weighted polynomial ring of n weighted variables $x_1, x_2, \dots, x_n$ with positive even integer weights $w_1 \geq w_2 \geq \cdots \geq w_n$. Suppose that*

$f_1, f_2, \ldots, f_m$ are n ($n \geq 2$) weighted homogeneous polynomials with degrees greater than $(m-1)(w_1 w_2)^{n-1}$ and R is an Artinian algebra. Then the Serre spectral sequence of any fibration with fiber X collapses.

Remark 2.1. In view of the following example due to Gregory Lupton [Lu], we see that the condition "the weighted degree of each f_i is bounded below by $(m-1)(w_1 w_2)^{n-1}$" in Corollary 2.2 cannot be omitted. Take the algebra

$$R = \mathbb{C}[x_1, x_2]/(x_1^2, x_1 x_2, x_2^2),$$

with $wt(x_1) = 2$ and $wt(x_2) = 4$. This is the cohomology algebra of $S^2 \vee S^4$ (2-sphere wedge with 4-sphere, which is not elliptic since its rational homotopy is not finite-dimensional). Then $D = x_1 \frac{\partial}{\partial x_2}$ induces a non-zero derivation on R and $wt(D) = -2$.

The Halperin Conjecture also plays an interesting role in differential geometry [BK2] and is related to the obstructions for vector bundles to admit complete non-negatively curved metrics. In 1972 Cheeger and Gromoll [CG] proved their famous theorem in differential geometry, which asserts that a complete open manifold M of non-negative sectional curvature is diffeomorphic to the total space of the normal bundle of a compact totally geodesic submanifold $S \hookrightarrow M$, called the soul. The natural converse open question becomes: Which vector bundles admit complete non-negatively curved metrics? In 2001 Belegradek and Kapovitch [BK1] (see also [Wil]) proved that up to a finite cover, a soul S splits as $S = C \times T$, where T is a torus, C is a simply-connected non-negative sectional curvature manifold, and the normal bundle splits as $\xi_C \times T$, where ξ_C is a vector bundle over C with total space $E(\xi_C)$ having non-negative curvature. Thus, in 2002, Belegradek and Kapovitch [BK2] attacked the converse question by concentrating on vector bundles over $C \times T$. They say that a vector bundle virtually comes from C if the pullback bundle under a map $\mathrm{id} \times p \colon C \times T \to C \times T$ (where $p \colon T \to T$ is a finite cover) is $\xi_C \times T$.

If ξ virtually comes from C, then no known method can rule out the existence of a complete metric with non-negative sectional curvature on the total space $E(\xi)$ of ξ, and potentially all such bundles might be non-negatively curved. In [BK2], it was shown that, under various assumptions on C, if ξ is a vector bundle over $C \times T$ such that $E(\xi)$ admits a complete metric with non-negative sectional curvature, then ξ virtually comes from C. This happens for any C if ξ has rank two. More generally, "most" vector bundles over $C \times T$ do not virtually come from C, at least when $\dim(T)$ is large enough (see [BK1, 4.4, 4.6] and Lemma B.1 of [BK2]).

Let ξ be a vector bundle over $C \times T$. We say that ξ satisfies $(*)$ if

$(*)$ $E(\xi)$ has a finite cover diffeomorphic to the product of T and the total space of a vector bundle over a closed simply-connected manifold.

To understand how assumption $(*)$ restricts ξ, Belegradek and Kapovitch found conditions on C ensuring that if ξ satisfies $(*)$, then ξ virtually comes from C.

Definition 2.1. A triple (C, T, k), where $k > 0$ is an integer, is called splitting rigid if any rank k vector bundle ξ over $C \times T$ that satisfies $(*)$ virtually comes from C.

Belegradek and Kapovitch [BK2] showed that if $\sec(E(\xi)) \geq 0$, then ξ satisfies $(*)$. Consequently they got the following proposition.

Proposition 2.1. *If (C, T, k) is splitting rigid and ξ is a rank k vector bundle over $C \times T$ such that $E(\xi)$ has a complete metric with* $\sec(E(\xi)) \geq 0$, *then ξ virtually comes from C.*

Thus if (C, T, k) is splitting rigid, then the total spaces of "most" rank k vector bundles over $C \times T$ do not admit complete metrics with non-negative sectional curvature. Therefore, finding splitting rigid triples is crucial to obtaining a necessary condition for non-negative curvature.

To find splitting rigid triples, Belegradek and Kapovitch made use of rational homotopy theory. Recall that one can associate a differential graded algebra M_X (called the minimal model of X) to each simply-connected space X such that the rational homotopy type of X is perfectly algebraically reflected in M_X. In particular, $H^*(M_X) = H^*(X; \mathbb{Q})$. It was shown in [BK2] that splitting rigidity of (C, T, k) often results from the vanishing of certain negative degree derivations of $H^*(C; \mathbb{Q})$.

3. New weight type

Our main idea for the proofs of Main Theorems A and B is as follows. Suppose $R = P/I$ is an Artinian algebra and there exists a negative weight derivation D on R with respect to weight type $(w_1, w_2, \ldots, w_n)$ where $w_1 \geq w_2 \geq \cdots \geq w_n \geq 1$. We can think of D as a negative weight derivation on the weighted polynomial ring $P = \mathbb{C}[x_1, x_2, \ldots, x_n]$ which preserves the ideal I. It is well known that D is of the form

$$D = p_1 \partial/\partial x_1 + p_2 \partial/\partial x_2 + \cdots + p_n \partial/\partial x_n, \tag{3.1}$$

where $p_1, p_2, \ldots, p_n$ are weighted homogeneous polynomials with the degrees $w_1 + wt\,D, w_2 + wt\,D, \ldots, w_n + wt\,D$. Let weighted homogeneous polynomials $f_1, f_2, \ldots, f_m$ generate the ideal I and without loss of generality assume that $\deg f_1 \geq \deg f_2 \geq \cdots \geq \deg f_m$. Since $D(f_1, f_2, \ldots, f_m) \subset (f_1, f_2, \ldots, f_m)$ and $\deg f_1 \geq \deg f_2 \geq \cdots \geq \deg f_m$, we have

$$\begin{aligned} Df_1 &= \ell_1^2 f_2 + \ell_1^3 f_3 + \cdots + \ell_1^m f_m, \\ Df_2 &= \ell_2^3 f_3 + \ell_2^4 f_4 + \cdots + \ell_2^m f_m, \\ &\cdots\cdots\cdots\cdots \\ Df_{m-1} &= \ell_{m-1}^m f_m, \\ Df_m &= 0, \end{aligned} \tag{3.2}$$

where ℓ_j^i's are weighted homogeneous polynomials. Our main technical tool is to associate with any negative weight derivation D on P as in (3.1) some families of new weight types $(\ell_1, \ell_2, \ldots, \ell_n)$ controlled by the parameters ϵ_i (see Definition 3.1). Then in Theorem 4.1, we prove that if we can choose suitable parameters ϵ_i to make the new weight type $(\ell_1, \ell_2, \ldots, \ell_n)$ satisfy the following three conditions:

(1) there is only one index $i_0 \in \{1, 2, \ldots, n\}$ such that

$$\ell_{i_0}/w_{i_0} = \max\{\ell_i/w_i \colon i = 1, 2, \ldots, n\};$$

(2) $\epsilon_{i_0} = \epsilon_{\min}$, where $\epsilon_{\min} = \min\{\epsilon_i$ for i such that p_i is a non-zero polynomial$\}$;

(3) p_{i_0} is a non-zero polynomial,

where p_i is the coefficient of $\partial/\partial x_i$ in D for $i = 1, 2, \ldots n$, then we claim that this contradicts the condition in Main Theorem A that the degree of each f_j is greater than $(m-1)(w_1 w_2)^{n-1}$. Thus D doesn't exist, and there are no negative weight derivations on R. So the key point is to choose suitable parameters for a given negative weight derivation D preserving the ideal $(f_1, f_2, \ldots, f_m)$, so that the above three conditions are satisfied. First we let

$$\epsilon_i = \begin{cases} \epsilon & p_i \text{ is a non-zero polynomial,} \\ 0 & p_i \text{ is the zero polynomial} \end{cases}$$

where ϵ is a positive real number. Then we have $\epsilon_{\min} = \epsilon$ and $\ell_i = 0$ for i such that p_i is the zero polynomial. Let $I_{\max} = \{e\colon \ell_e/w_e$ is the maximum among all ℓ_i/w_i for $i = 1, 2, \ldots, n\}$. It's easy to see that $\epsilon_i = \epsilon_{\min}$ and p_i is a non-zero polynomial for any $i \in I_{\max}$. Thus if $I_{\max}$ has only one element, then conditions (1), (2), and (3) are satisfied. But in the general case $I_{\max}$ might have more than one element. So in this case we need to adjust the parameters ϵ_i in order to separate $\{\ell_i/w_i\colon i \in I_{\max}\}$ such that these numbers have only one maximum. Then we adjust the parameters as follows: pick an index $i_1 \notin I_{\max}$ and replace the parameter ϵ_{i_1} with $\epsilon_{i_1} + \epsilon/(w_1 w_2)$ with the new weight type and $I_{\max}$ changing accordingly. Then pick an index $i_2 \notin I_{\max}$ and replace the parameter ϵ_{i_2} with $\epsilon_{i_2} + \epsilon/(w_1 w_2)^2$. Repeat this process, and Theorem 6.1 tells us that within finite steps we will accomplish our goal and the theorem will be proved. We speculate that this new technique of decomposing equations according to the new weight type might be useful for attacking other problems about singularities.

Let $P = \mathbb{C}[x_1, x_2, \ldots, x_n]$, $w_1 \geq w_2 \geq \cdots \geq w_n > 0$ as above and let D be a non-zero negative weight derivation on P. It is well known that D is of the form

$$D = p_1 \partial/\partial x_1 + p_2 \partial/\partial x_2 + \cdots + p_n \partial/\partial x_n, \tag{3.3}$$

where p_i is a weighted homogeneous polynomial of degree $w_i + wt\,D$ with respect to the weight type $(w_1, w_2, \ldots, w_n)$ or the zero polynomial for $i = 1, 2, \ldots, n$. Since $wt\,D < 0$, we know that p_i is a polynomial of only variables $x_{i+1}, x_{i+2}, \ldots, x_n$ for $1 \leq i \leq n$. Thus p_n is a constant polynomial (in fact it is zero; see Remark 4.1). We define a new weight type associated with D as follows.

Definition 3.1. Let D be a non-zero negative weight derivation on the weighted polynomial ring P as in (3.3). The following weight type $(\ell_1, \ell_2, \ldots, \ell_n)$ controlled by the given n parameters $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ is called the new weight type associated with D, where the ϵ_i's are non-negative real parameters. Set

$$\ell_n = \epsilon_n.$$

If $\ell_n, \ell_{n-1}, \ldots, \ell_{q+1}$ are defined, then ℓ_q is defined as follows:

(i) If the coefficient $p_q(x_{q+1}, \ldots, x_n)$ of $\partial/\partial x_q$ in D is the zero-polynomial, then

$$\ell_q = \epsilon_q. \tag{3.4}$$

(ii) If the coefficient $p_q(x_{q+1}, \ldots, x_n)$ of $\partial/\partial x_q$ in D is a non-zero polynomial, then

$$\ell_q = \epsilon_q + \max\{\ell_{q+1} i_{q+1} + \ell_{q+2} i_{q+2} + \cdots + \ell_n i_n \mid \text{monomial } x_{q+1}^{i_{q+1}} x_{q+2}^{i_{q+2}} \ldots x_n^{i_n} \text{ appears in the expansion of } p_q\}, \tag{3.5}$$

where p_i is the coefficient of $\partial/\partial x_i$ in D for $i = 1, 2, \ldots, n$.

It is clear that when

$$\epsilon_i = \begin{cases} -\,wt\,D & p_i \text{ is a non-zero polynomial,} \\ w_i & p_i \text{ is the zero polynomial,} \end{cases}$$

the new weight type $(\ell_1, \ell_2, \dots, \ell_n)$ is just the original weight type $(w_1, w_2, \dots, w_n)$.

Definition 3.2. The Q-degree of a monomial $x^\alpha = x_1^{i_1} x_2^{i_2} \dots x_n^{i_n}$ is defined to be $\ell_1 i_1 + \ell_2 i_2 + \cdots + \ell_n i_n$, and the Q-degree of a polynomial f is defined as

$$\text{Q-deg } f := \max\{\text{Q-degrees of monomials in the expansion of } f\}.$$

Thus $\ell_i = \epsilon_i + \text{Q-deg } p_i$ for $i = 1, 2, \dots, n$ such that p_i is a non-zero polynomial, where p_i is the coefficient of $\partial/\partial x_i$ in D.

Definition 3.3. For any polynomial f in P, we denote by $f_{\max}$ the sum of terms in the expansion of f with maximum Q-degree with respect to $(\ell_1, \ell_2, \dots, \ell_n)$; i.e., if we write

$$f = \sum_{\alpha \in I} c_\alpha x^\alpha$$

where I is a finite set, then

$$f_{\max} := \sum_{\alpha \in I \text{ and Q-deg } x^\alpha = \text{Q-deg } f} c_\alpha x^\alpha.$$

Definition 3.4. With the same notation as above, we define

$$d_{\max}(D) := \max\{\text{the Q-degree of } (p_j)_{\max} \partial/\partial x_j \mid p_j \text{ is a non-zero polynomial}\}$$

and

$$D_{\max} := \sum_{\substack{\text{for } j \text{ such that} \\ (p_j)_{\max}\partial/\partial x_j \text{ has Q-degree } d_{\max}(D)}} (p_j)_{\max} \partial/\partial x_j, \tag{3.6}$$

where the Q-degree of $(p_j)_{\max} \partial/\partial x_j$ is defined to be Q-deg $(p_j)_{\max} - \ell_j$.

Proposition 3.1. *With the same notation as above, we have*

$$D_{\max} = \sum_{\substack{\text{for } j \text{ such that} \\ p_j \text{ is a non-zero polynomial and } \epsilon_j = \epsilon_{\min}}} (p_j)_{\max} \partial/\partial x_j \tag{3.7}$$

where

$$\epsilon_{\min} = \min\{\epsilon_i \text{ for } i \text{ such that } p_i \text{ is a non-zero polynomial}\}.$$

It follows that the Q-degree of $D_{\max} = -\epsilon_{\min}$.

Proof. It is clear from the definition of the new weight type and $D_{\max}$. □

Proposition 3.2. *Let $D, D_{\max}, \epsilon_{\min}$ be as above and let g be an arbitrary polynomial in P. We have either*

(i) $D_{\max} g_{\max} = 0$, *in this case* $Q - \deg (Dg)_{\max} < Q - \deg g_{\max} - \epsilon_{\min}$ *or*
(ii) $D_{\max} g_{\max} = (Dg)_{\max}$.

Proof. Write

$$g = g_{\max} + \text{lower Q-deg terms} = g_{\max} + g_r + g_{r-1} + \cdots$$

and

$$D = D_{\max} + \text{lower Q-deg terms} = D_{\max} + D_s + D_{s-1} + \cdots,$$

where

$$\cdots < \text{Q-deg } g_{r-1} < \text{Q-deg } g_r < \text{Q-deg } g_{\max}$$

and

$$\cdots < \text{Q-deg } D_{s-1} < \text{Q-deg } D_s < \text{Q-deg } D_{\max}.$$

Then we have

$$Dg = D_{\max}g_{\max} + D_{\max}g_r + D_s g_{\max} + D_s g_r + \cdots .$$

If $D_{\max}g_{\max} \neq 0$, then $Dg = D_{\max}g_{\max} + \text{lower Q-deg terms}$. Thus we have

$$D_{\max}g_{\max} = (Dg)_{\max}.$$

If $D_{\max}g_{\max} = 0$, then $Dg = D_{\max}g_r + D_s g_{\max} + D_s g_r + \cdots$. Thus

$$\text{Q-deg } (Dg)_{\max} \leq \max\{\text{Q-deg } D_{\max} + \text{Q-deg } g_r, \text{Q-deg } D_s + \text{Q-deg } g_{\max}\}.$$

Since Q-degree $D_{\max} = -\epsilon_{\min}$ by Proposition 3.1, we have

$$\text{Q-deg } D_{\max} + \text{Q-deg } g_r < \text{Q-deg } D_{\max} + \text{Q-deg } g_{\max} = \text{Q-deg } g_{\max} - \epsilon_{\min}$$

and

$$\text{Q-deg } D_s + \text{Q-deg } g_{\max} < \text{Q-deg } D_{\max} + \text{Q-deg } g_{\max} = \text{Q-deg } g_{\max} - \epsilon_{\min}.$$

Therefore Q-deg $(Dg)_{\max} <$ Q-deg $g_{\max} - \epsilon_{\min}$. □

Corollary 3.1. *Let D and g be as above. If $Dg = 0$, then $D_{\max}g_{\max} = 0$.*

Proof. This is an immediate conclusion of Proposition 3.2. □

4. Some lemmas for the proof of Main Theorems A and B

In sections 4, 5, and 6, $P = \mathbb{C}[x_1, x_2, \ldots, x_n]$ is the weighted polynomial ring of n weighted variables $x_1, x_2, \ldots, x_n$ with positive integer weights $w_1 \geq w_2 \geq \cdots \geq w_n$. Let

$$D = p_1\partial/\partial x_1 + p_2\partial/\partial x_2 + \cdots + p_n\partial/\partial x_n$$

be a fixed non-zero negative weight derivation on P, and let $(\ell_1, \ell_2, \ldots, \ell_n)$ be the new weight type associated with D controlled by non-negative parameters ϵ_i.

Lemma 4.1. *Let I be the ideal generated by weighted homogeneous polynomials $f_1, f_2, \ldots, f_m$ with respect to weight type $(w_1, w_2, \ldots, w_n)$ as above and let P/I be a non-zero Artinian algebra. Let m be the maximal ideal generated by $x_1, x_2, \ldots, x_n$. Then we have $m^r \subseteq I$ for some integer $r > 0$, and P/I is a local Artinian algebra.*

Proof. Let d_i be the degree of f_i with respect to $(w_1, w_2, \ldots, w_n)$ for $i = 1, 2, \ldots, m$. Then for any point $(x_1, x_2, \ldots, x_n) \in \mathbb{C}^n$, we have

$$f_i(\alpha^{w_1}x_1, \alpha^{w_2}x_2, \ldots, \alpha^{w_n}x_n) = \alpha^{d_i}f_i(x_1, x_2, \ldots, x_n)$$

for any $i = 1, 2, \ldots, m$ and any $\alpha \in \mathbb{C}$. We claim that $Z(I) = \{0\}$, where $Z(I)$ is the zero locus of I in $\mathbb{C}^n$, because if not, then there is a point $(x_1, x_2, \ldots, x_n) \in Z(I)$ and $(x_1, x_2, \ldots, x_n) \neq 0$. Thus $\{(\alpha^{w_1}x_1, \alpha^{w_2}x_2, \ldots, \alpha^{w_n}x_n), \alpha \in \mathbb{C}\} \subseteq Z(I)$ has dimension one, which contradicts that P/I is an Artinian algebra. Thus $Z(I) = \{0\}$, which follows that $m^r \subseteq I$ for some integer $r > 0$. Hence, for any maximal ideal m' in P such that $I \subseteq m'$, we have $m^r \subseteq m'$, which implies $m = m'$. So P/I has only one maximal ideal; thus P/I is a local Artinian algebra. □

Lemma 4.2. *Let $f_1, f_2, \ldots, f_m \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ be weighted homogeneous polynomials. Suppose that $\mathbb{C}[x_1, x_2, \ldots, x_n]/(f_1, f_2, \ldots, f_m)$ is a non-zero Artinian algebra. Then for any given index $i \in \{1, 2, \ldots, n\}$ there exists an index $j \in \{1, 2, \ldots, m\}$ such that $f_j(x_1, x_2, \ldots, x_n)$ contains a term $x_i^{a_i}$ (with a_i a positive integer) in its expansion.*

Proof. Suppose the conclusion is not true; thus the ideal $(f_1, f_2, \ldots, f_m)$ would have to be contained within the ideal $(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. On the other hand, by Lemma 4.1, there exists some integer $r > 0$ such that

$$(x_1, x_2, \ldots, x_n)^r \subseteq (f_1, f_2, \ldots, f_m).$$

Thus

$$(x_1, x_2, \ldots, x_n)^r \subseteq (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n),$$

which is absurd. The lemma is proved. □

The following observation, based on the assumption that $\{\ell_i/w_i : i = 1, \ldots, n\}$ has the unique maximum, plays an important role in our proofs for the main theorems.

Lemma 4.3. *Suppose that there is only one index $i_0 \in \{1, 2, \ldots, n\}$ such that $\beta = \ell_{i_0}/w_{i_0} = \max\{\ell_i/w_i : i = 1, 2, \ldots, n\}$. Let $f \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ be a weighted homogeneous polynomial with respect to both the original weight type $(w_1, w_2, \ldots, w_n)$ and the new weight type $(\ell_1, \ell_2, \ldots, \ell_n)$. Suppose that the degree of f and the Q-degree of f satisfy the following:*

(i)

$$\deg f > M/(\beta - \gamma), \tag{4.1}$$

(ii)

$$\text{Q-deg} f \geq \beta \deg f - M, \tag{4.2}$$

where M is a fixed constant and

$$\gamma = \max\{\ell_i/w_i : i = 1, 2, \ldots, i_0 - 1, i_0 + 1, \ldots, n\}.$$

Then f can be divided by x_{i_0}.

Proof. Suppose that some monomial x^a in the expansion of $f(x_1, x_2, \ldots, x_n)$ cannot be divided by x_{i_0}. Write $x^a = x_1^{a_1} \cdots x_{i_0-1}^{a_{i_0-1}} x_{i_0+1}^{a_{i_0+1}} \cdots x_n^{a_n}$.

By the definition of γ, we conclude that

$$\begin{aligned} \text{Q-deg} f &= \text{Q-deg} x^a \\ &= a_1\ell_1 + \cdots + a_{i_0-1}\ell_{i_0-1} + a_{i_0+1}\ell_{i_0+1} + \cdots + a_n\ell_n \\ &\leq \gamma(a_1 w_1 + \cdots + a_{i_0-1}w_{i_0-1} + a_{i_0+1}w_{i_0+1} + \cdots + a_n w_n) \\ &= \gamma \deg x^a = \gamma \deg f. \end{aligned} \tag{4.3}$$

Combining (4.3) with (4.2), we get

$$\beta \deg f - M \leq \text{Q-deg} f \leq \gamma \deg f. \tag{4.4}$$

This implies that

$$\deg f \leq M/(\beta - \gamma), \tag{4.5}$$

which contradicts (4.1). Thus the lemma is proved. □

Lemma 4.4. *If the coefficient p_{i_0} of $\partial/\partial x_{i_0}$ in D is a non-zero polynomial and f be a polynomial which can be divided by x_{i_0}, then $Df \neq 0$.*

Proof. Consider the expansion of $f(x_1, x_2, \ldots, x_n)$ in powers of x_{i_0}:

$$f(x_1, x_2, \ldots, x_n) = b_q x_{i_0}^q + b_{q-1} x_{i_0}^{q-1} + \cdots + b_h x_{i_0}^h \text{ with } b_h \neq 0, \tag{4.6}$$

where $h \le q$ and $b_q, b_{q-1}, \ldots, b_h$ are polynomials of $x_1, x_2, \ldots, x_{i_0-1}, x_{i_0+1}, \ldots, x_n$. From the condition of Lemma 4.4 we know that $h \ge 1$. We have that

$$Df = (D_1 + p_{i_0} \partial/\partial x_{i_0}) f, \tag{4.7}$$

where $D_1 = D - p_{i_0}\partial/\partial x_{i_0}$. Therefore $D_1 f = x_{i_0}^h D_1(b_q x_{i_0}^{q-h} + \cdots + b_h)$ and

$$\begin{aligned} Df = {} & x_{i_0}^h D_1 \left(b_q x_{i_0}^{q-h} + \cdots + b_h\right) \\ & + p_{i_0}\left(q b_q x_{i_0}^{q-1} + \cdots + h b_h x_{i_0}^{h-1}\right) \\ = {} & x_{i_0}^h D_1 \left(b_q x_{i_0}^{q-h} + \cdots + b_h\right) \\ & + x_{i_0}^h p_{i_0}\left(q b_q x_{i_0}^{q-h-1} + \cdots + (h+1) b_{h+1}\right) + h x_{i_0}^{h-1} p_{i_0} b_h. \end{aligned} \tag{4.8}$$

It is clear that $p_{i_0} b_h$ is a non-zero polynomial of $x_1, x_2, \ldots, x_{i_0-1}, x_{i_0+1}, \ldots, x_n$. Hence the last term in (4.8) can only be divided by $x_{i_0}^{h-1}$. Thus Df is a non-zero polynomial. □

Lemma 4.5. *If $\ell_{i_0}/w_{i_0} = \max\{\ell_i/w_i : i = 1, 2 \ldots, n\}$ (not necessarily the unique maximum) and the coefficient p_{i_0} of $\partial/\partial x_{i_0}$ in D is a non-zero polynomial, then $\ell_{i_0}/w_{i_0} \le \epsilon_{i_0}/(-\,wt\,D)$; that is to say, $\ell_i/w_i \le \epsilon_{i_0}/(-\,wt\,D)$ for $i = 1, 2, \ldots, n$.*

Proof. Assume that $\ell_{i_0}/w_{i_0} > \epsilon_{i_0}/(-\,wt\,D)$. Then by the definition of the new weight type and the fact that $wt\,D = \deg p_{i_0} - w_{i_0}$, we have

$$\frac{\text{Q-deg } (p_{i_0})_{\max} + \epsilon_{i_0}}{\deg(p_{i_0})_{\max} - wt\,D} = \frac{\ell_{i_0}}{w_{i_0}}.$$

Combining with the assumption that $\epsilon_{i_0}/(-\,wt\,D) < \ell_{i_0}/w_{i_0}$, we conclude that

$$\frac{\text{Q-deg } (p_{i_0})_{\max}}{\deg(p_{i_0})_{\max}} > \frac{\ell_{i_0}}{w_{i_0}}. \tag{4.9}$$

However $(p_{i_0})_{\max}$ is a polynomial of x_t for $t > i_0$, and we have $\ell_t/w_t \le \ell_{i_0}/w_{i_0}$ for $t > i_0$. Thus

$$\frac{\text{Q-deg } (p_{i_0})_{\max}}{\deg(p_{i_0})_{\max}} \le \frac{\ell_{i_0}}{w_{i_0}},$$

which contradicts (4.9). Thus the conclusion is proved. □

The following theorem is critical to the proofs of Main Theorems A and B.

Theorem 4.1. *Let $f_1, f_2, \ldots, f_m$ be m weighted homogeneous polynomials with respect to the weight type $(w_1, w_2, \ldots, w_n)$ such that*

$$R = \mathbb{C}[x_1, x_2, \ldots, x_n]/(f_1, f_2, \ldots, f_m)$$

is a non-zero Artinian algebra. Suppose that the negative weight derivation D on P preserves the ideal $(f_1, f_2, \ldots, f_m)$. If we can choose suitable parameters ϵ_i' to make the new weight type $(\ell_1, \ell_2, \ldots, \ell_n)$ satisfy the following three conditions:

(1) *there is only one index $i_0 \in \{1, 2, \ldots, n\}$ such that*

$$\beta = \ell_{i_0}/w_{i_0} = \max\{\ell_i/w_i : i = 1, 2, \ldots, n\},$$

(2) $\epsilon_{i_0} = \epsilon_{\min}$, *where* $\epsilon_{\min} = \min\{\epsilon_i$ *for* i *such that* p_i *is a non-zero polynomial*$\}$,

(3) p_{i_0} *is a non-zero polynomial,*
where p_i *is the coefficient of* $\partial/\partial x_i$ *in* D *for* $i = 1, 2, \dots n$, *then there exists* $j \in \{1, 2, \dots, m\}$ *such that*

$$\deg f_j \leq \frac{(m-1)\epsilon_{\min}}{\beta - \gamma},$$

where $\gamma = \max\{\ell_i/w_i \colon i = 1, 2, \dots, i_0 - 1, i_0 + 1, \dots, n\}$.

Proof. Without loss of generality we can assume that $\deg f_1 \geq \deg f_2 \geq \cdots \geq \deg f_m$. By comparing degrees we find that

$$\begin{aligned} Df_1 &= \ell_1^2 f_2 + \cdots + \ell_1^m f_m, \\ Df_2 &= \ell_2^3 f_3 + \cdots + \ell_2^m f_m, \\ &\cdots\cdots\cdot \\ Df_{m-1} &= \ell_{m-1}^m f_m, \\ Df_m &= 0, \end{aligned} \tag{4.10}$$

where the ℓ_j^i's with $i > j$ are weighted homogeneous polynomials with respect to the original weight type $(w_1, w_2, \dots, w_n)$.

By Lemma 4.2 we can find an f_{j_0} which contains a term of the form $x_{i_0}^a$ (with a a positive integer) in its expansion, which follows that Q-deg $(f_{j_0})_{\max} \geq a\ell_{i_0} = \beta \deg f_{j_0}$.

We construct a sequence $j_0 < j_1 < \cdots$ as follows. If $j_0, j_1, \dots, j_i$ are defined, then by Proposition 3.2 we have either $D_{\max}(f_{j_i})_{\max} = 0$ or $D_{\max}(f_{j_i})_{\max} = (Df_{j_i})_{\max}$. If the former, let the sequence end. If the latter, by the j_i-th equation in (4.10), there is an index $j_{i+1} \in \{j_i + 1, \dots, m\}$ such that

$$\text{Q-deg}\left(\ell_{j_i}^{j_{i+1}} f_{j_{i+1}}\right)_{\max} = \text{Q-deg}\left(D_{\max}(f_{j_i})_{\max}\right). \tag{4.11}$$

Now we prove that the sequence has the following proposition by induction:

$$\text{Q-deg}(f_{j_i})_{\max} \geq -i(\beta\, wt\, D + \epsilon_{\min}) + \beta \deg f_{j_i}. \tag{4.12}$$

We have already proved it for $i = 0$. Suppose (4.12) holds for i; we will prove it holds for $i + 1$. By (4.11) and Proposition 3.1, we can get

$$\text{Q-deg}\left(\ell_{j_i}^{j_{i+1}}\right)_{\max} + \text{Q-deg}(f_{j_{i+1}})_{\max} = -\epsilon_{\min} + \text{Q-deg}(f_{j_i})_{\max}.$$

Using the fact that $\deg f_{j_i} + wt\, D = \deg \ell_{j_i}^{j_{i+1}} + \deg f_{j_{i+1}}$ and that $\beta \deg \ell_{j_i}^{j_{i+1}} \geq \text{Q-deg}(\ell_{j_i}^{j_{i+1}})_{\max}$, we get

$$\begin{aligned} \text{Q-deg}(f_{j_{i+1}})_{\max} =& -\epsilon_{\min} + \text{Q-deg}(f_{j_i})_{\max} - \text{Q-deg}(\ell_{j_i}^{j_{i+1}})_{\max} \\ \geq& -\epsilon_{\min} - i(\beta\, wt\, D + \epsilon_{\min}) + \beta \deg f_{j_i} - \text{Q-deg}(\ell_{j_i}^{j_{i+1}})_{\max} \\ =& -\epsilon_{\min} - i(\beta\, wt\, D + \epsilon_{\min}) + \beta(\deg \ell_{j_i}^{j_{i+1}} + \deg f_{j_{i+1}} - wt\, D) \\ & - \text{Q-deg}(\ell_{j_i}^{j_{i+1}})_{\max} \\ \geq& -(i+1)(\beta\, wt\, D + \epsilon_{\min}) + \beta \deg f_{j_{i+1}}. \end{aligned} \tag{4.13}$$

Because of the last equation of (4.10), $Df_m = 0$, it follows from Corollary 3.1 that

$$D_{\max}(f_m)_{\max} = 0.$$

From this point and the fact that $j_i < j_{i+1}$, we find that the sequence will end within $(m-1)$ steps. That is to say, there is an index $t \in \{1, 2, \ldots, m-1\}$ such that

$$D_{\max}(f_{j_t})_{\max} = 0, \tag{4.14}$$

$$\text{Q-deg}(f_{j_t})_{\max} \geq -t(\epsilon_{\min} + \beta\, wt\, D) + \beta \deg f_{j_t}. \tag{4.15}$$

By Lemma 4.5, we have $(\epsilon_{\min} + \beta\, wt\, D) \geq 0$. Notice that $t \leq m-1$, so we have

$$\text{Q-deg}(f_{j_t})_{\max} \geq -(m-1)(\epsilon_{\min} + \beta\, wt\, D) + \beta \deg f_{j_t}. \tag{4.16}$$

Assume that

$$\deg(f_{j_t})_{\max} > \frac{(m-1)\epsilon_{\min}}{\beta - \gamma}. \tag{4.17}$$

Using the fact that $wt\, D < 0$, we have

$$\deg(f_{j_t})_{\max} > \frac{(m-1)(\epsilon_{\min} + \beta\, wt\, D)}{\beta - \gamma}. \tag{4.18}$$

By Lemma 4.3 (here $M = (m-1)(\epsilon_{\min} + \beta\, wt\, D)$ and notice that $\deg(f_{j_t}) = \deg(f_{j_t})_{\max}$) we know that $(f_{j_t})_{\max}$ is divisible by x_{i_0}. Since $\epsilon_{i_0} = \epsilon_{\min}$, Proposition 3.1 tells us that the coefficient of $\partial/\partial x_{i_0}$ in $D_{\max}$ is $(p_{i_0})_{\max}$. Also p_{i_0} is a non-zero polynomial, so $(p_{i_0})_{\max}$ is a non-zero polynomial. Thus we know that $D_{\max}(f_{j_t})_{\max} \neq 0$ from Lemma 4.4, which contradicts (4.14). Therefore the assumption (4.17) is false. Thus

$$\deg f_{j_t} = \deg(f_{j_t})_{\max} \leq \frac{(m-1)\epsilon_{\min}}{\beta - \gamma},$$

and the conclusion is proved. $\square$

Lemma 4.6. *Let $(\ell_1, \ell_2, \ldots, \ell_n)$ be the new weight type associated with D controlled by non-negative parameters ϵ_i'. If there exists a positive real number ε such that all parameters ϵ_i' can be divided by ε, that is to say, $\epsilon_i = b_i\varepsilon$ where b_i is a non-negative integer for $i = 1, 2, \ldots, n$, then we have*

(i) *$\ell_i = q_i\varepsilon$, where q_i is a non-negative integer for $i = 1, 2, \ldots, n$;*
(ii) *for any $i, j \in \{1, 2, \ldots, n\}$, if $\ell_i/w_i > \ell_j/w_j$, then*

$$\ell_i/w_i - \ell_j/w_j \geq \varepsilon/(w_1 w_2).$$

Proof. (i) We prove it by induction on i. If $i = n$, then, by Definition 3.1, $\ell_n = \epsilon_n = b_n\varepsilon$, and the lemma holds. Suppose it holds for $i = k+1, \ldots, n$; we prove it for $i = k$. If p_k is the zero polynomial, then $\ell_k = \epsilon_k$, and the lemma obviously holds. If p_k is a non-zero polynomial, for any term $x_{k+1}^{a_{k+1}} \ldots x_n^{a_n}$ in the expansion of p_k, we have

$$\text{Q-deg}\, x_{k+1}^{a_{k+1}} \ldots x_n^{a_n} = (a_{k+1}q_{k+1} + \cdots + a_n q_n)\varepsilon.$$

By Definition 3.1, we have

$$\begin{aligned}\ell_k &= \epsilon_k + \max\{\text{Q-degrees of monomials } x_{k+1}^{a_{k+1}} \ldots x_n^{a_n} \text{ in the expansion of } p_k\} \\ &= \Big(b_k + \max\{a_{k+1}q_{k+1} + \cdots + a_n q_n : \\ &\qquad \text{the monomial } x_{k+1}^{a_{k+1}}, \ldots, x_n^{a_n} \text{ appears in the expansion of } p_k\}\Big)\varepsilon.\end{aligned}$$

Thus the lemma for case $i = k$ is proved.

(ii) By (i), we have

$$\ell_i/w_i - \ell_j/w_j = (\ell_i w_j - \ell_j w_i)/(w_i w_j) = (q_i w_j - q_j w_i)\varepsilon/(w_i w_j).$$

Notice that $\ell_i/w_i > \ell_j/w_j$; thus $q_i w_j - q_j w_i > 0$. Since $q_i w_j - q_j w_i$ is an integer, $q_i w_j - q_j w_i \geq 1$. Notice that $w_1 \geq w_2 \geq \cdots \geq w_n \geq 1$; hence

$$\ell_i/w_i - \ell_j/w_j \geq \varepsilon/(w_i w_j) \geq \varepsilon/(w_1 w_2).$$

□

Lemma 4.7. *Let R be a commutative local Artinian algebra. Let $D \in Der(R, R)$ be any derivation of R. Then D preserves the m-adic filtration of R; i.e., $D(m) \subseteq m$ where m is the maximal ideal of R.*

Proof. If the assertion is false, then there exists $a \in m$ such that $D(a) \notin m$. Since R is an Artinian algebra, we can find a smallest integer k such that $a^k \neq 0$ but $a^{k+1} = 0$. Then $0 = D(a^{k+1}) = (k+1)a^k D(a)$, which implies that $a^k = 0$ because $D(a)$ is a unit. This leads to a contradiction. □

Lemma 4.8. *Let P and I be as in Lemma* 4.1. *Suppose that*

$$D = p_1 \partial/\partial x_1 + p_2 \partial/\partial x_2 + \cdots + p_n \partial/\partial x_n$$

is a derivation on P/I. Then $p_i(0) = 0$ for $1 \leq i \leq n$.

Proof. It follows from Lemma 4.1 that P/I is a local Artinian algebra and has a unique maximal ideal m. Assume that $p_i(0) \neq 0$ for some $1 \leq i \leq n$; thus p_i is not in m. However, it follows from Lemma 4.7 that $D(x_i) = p_i \in m$. This leads to a contradiction. Thus the lemma is proved. □

Remark 4.1. Suppose that D is a non-zero negative weight derivation on P/I as before. From Lemma 4.8, we know that each coefficient p_i of $\partial/\partial x_i$ in D does not contain any constant term. Notice that p_n is a constant polynomial, so p_n is the zero polynomial.

5. Proof of Main Theorem B

We first prove Main Theorem B. Main Theorem A is proved in the next section.

Theorem 5.1 (Main Theorem B)**.** *Let $P = \mathbb{C}[x_1, x_2, \ldots, x_n]$ be the weighted polynomial ring of n weighted variables $x_1, x_2, \ldots, x_n$ with positive integer weights $w_1 \geq w_2 \geq \cdots \geq w_n$ and let $f_1, f_2, \ldots, f_m$ be m $(m \geq n \geq 2)$ weighted homogeneous polynomials with degrees greater than $(m-1)w_1 w_2$. Suppose that any two of the original weights $w_1, w_2, \ldots, w_n$ are coprime and $R = P/(f_1, f_2, \ldots, f_m)$ is an Artinian algebra. Then there are no non-zero negative weight derivations on R.*

Proof. If the conclusion is not true, suppose D is a non-zero negative weight derivation on R or equivalently a non-zero negative weight derivation on P which preserves the ideal $(f_1, f_2, \ldots, f_m)$ as in (3.3). We take the new weight type $(\ell_1, \ldots, \ell_n)$ of D controlled by the parameters ϵ_i'. Here

$$\epsilon_i = \begin{cases} \epsilon & p_i \text{ is a non-zero polynomial,} \\ 0 & p_i \text{ is the zero polynomial,} \end{cases}$$

where ϵ is a positive real number. Let $I_{\max} = \{e\colon \ell_e/w_e$ is the maximum among all ℓ_i/w_i for $i = 1, 2, \ldots, n\}$. We claim that $I_{\max}$ has only one element for the following reason. It's clear that $\ell_i > 0$ for any i such that p_i is a non-zero polynomial and

$\ell_i = 0$ for any i such that p_i is the zero polynomial. Thus p_i is a non-zero polynomial for any $i \in I_{\max}$, which follows that $\epsilon_i = \epsilon$ for any $i \in I_{\max}$. Since ϵ_i can be divided by ϵ for all $i = 1, 2, \ldots, n$, by Lemma 4.6 we have $\ell_i = q_i\epsilon$, where q_i is a non-negative integer for $i = 1, 2, \ldots, n$. Now we prove that $q_i < w_i$ for all i by induction on i. If $i = n$, then we know p_n is the zero polynomial (see Remark 4.1); thus $q_n = 0 < w_n$. Suppose $q_i < w_i$ for $i = k+1, k+2, \ldots, n$; we prove that $q_k < w_k$. If p_k is the zero polynomial, then $q_k = \epsilon_k = 0 < w_k$. If p_k is a non-zero polynomial, by Lemma 4.8 we have $p_k(0) = 0$; thus p_k is a non-constant polynomial. We have

$$\ell_k = \epsilon_k + \text{Q-deg } p_k = \epsilon + \text{Q-deg } p_k,$$

and notice that

$$w_k = -\operatorname{wt} D + \deg p_k$$

and p_k is a non-constant polynomial of variables $x_{k+1}, \ldots, x_n$ and $\ell_i = q_i\epsilon < w_i\epsilon$ for $i > k$. Thus we have Q-deg $p_k < \epsilon \deg p_k$; hence

$$\ell_k = \epsilon + \text{Q-deg } p_k < (1 + \deg p_k)\epsilon \leq (-\operatorname{wt} D + \deg p_k)\epsilon = w_k\epsilon,$$

which follows that $q_k < w_k$. Thus $q_i < w_i$ for $i = 1, 2, \ldots, n$. Suppose that $I_{\max}$ has more than one element. Then for any $i, j \in I_{\max}$ such that $i \neq j$, since $0 < q_i < w_i$, $0 < q_j < w_j$, and w_i, w_j are coprime, $q_i/w_i \neq q_j/w_j$. It follows that $\ell_i/w_i \neq \ell_j/w_j$, which contradicts $i, j \in I_{\max}$. Thus the claim that $I_{\max}$ has only one element is proved. Write $I = \{i_0\}$. Let $\beta = \ell_{i_0}/w_{i_0}$ and let

$$\gamma = \max\{\ell_i/w_i \colon i = 1, 2, \ldots, i_0 - 1, i_0 + 1, \ldots, n\}.$$

Since ϵ_i can be divided by ϵ for all i, by Lemma 4.6 we have $\beta - \gamma \geq \epsilon/(w_1w_2)$. Let $\epsilon_{\min} = \min\{\epsilon_i$ for i such that p_i is a non-zero polynomial$\}$, and it's clear that $\epsilon_{\min} = \epsilon$. Then by Theorem 4.1 we know that there exist $j \in \{1, 2, \ldots, m\}$ such that $\deg f_j \leq (m-1)(w_1w_2)$, which contradicts the condition that $\deg f_j > (m-1)(w_1w_2)$ for all j. So the conclusion is proved. □

6. Proof of Main Theorem A

In this section we give the proof for Main Theorem A. In order to use Theorem 4.1, we need to choose suitable parameters ϵ_i' to make the new weight type $(\ell_1, \ldots, \ell_n)$ satisfy the following conditions in Theorem 4.1:

(1) there is only one index $i_0 \in \{1, 2, \ldots, n\}$ such that

$$\ell_{i_0}/w_{i_0} = \max\{\ell_i/w_i \colon i = 1, 2, \ldots, n\};$$

(2) $\epsilon_{i_0} = \epsilon_{\min}$, where $\epsilon_{\min} = \min\{\epsilon_i$ for i such that p_i is a non-zero polynomial$\}$;

(3) p_{i_0} is a non-zero polynomial.

First we let

$$\epsilon_i = \begin{cases} \epsilon & p_i \text{ is a non-zero polynomial,} \\ 0 & p_i \text{ is the zero polynomial,} \end{cases}$$

where ϵ is a positive real number. Let $(\ell_1, \ldots, \ell_n)$ be the new weight type associated with a non-zero negative weight derivation D and controlled by parameter ϵ_i. Then we have $\epsilon_{\min} = \epsilon$ and $\ell_i = 0$ for i such that p_i is the zero polynomial. Let $I_{\max} = \{e \colon \ell_e/w_e$ is the maximum among all ℓ_i/w_i for $i = 1, 2, \ldots, n\}$. It's easy to see that $\epsilon_i = \epsilon_{\min}$ and p_i is a non-zero polynomial for any $i \in I_{\max}$. Thus if $I_{\max}$ has only one element, then the conditions (1), (2), and (3) in Theorem 4.1 are satisfied. But in the general case $I_{\max}$ might have more than one element. So the key thing we need to do is to adjust the parameters ϵ_i' in order to separate $\{\ell_i/w_i\colon i \in I_{\max}\}$ such that these numbers have only one maximum.

Lemma 6.1. *Let D be a non-zero negative weight derivation such that $p_i(0) = 0$ for $1 \le i \le n$, where p_i is the coefficient of $\partial/\partial x_i$ in D. Suppose there exists a positive real number ε such that all parameters ϵ_i' can be divided by ε. Fix an index $j_0 \in \{1, 2, \dots, n\}$, and define another group of parameters (ϵ_i') as follows:*

$$\epsilon_i' = \begin{cases} \epsilon_i + \varepsilon/(w_1 w_2) & i = j_0, \\ \epsilon_i & i \neq j_0. \end{cases}$$

Let $(\ell_1, \dots, \ell_n)$ and $(\ell_1', \dots, \ell_n')$ be new weight types associated with D and controlled by parameters ϵ_i' and (ϵ_i') respectively. Then we have the following:

(i) *For any $i, j = 1, 2, \dots, n$ such that both p_i and p_j are non-zero polynomials, we have*
$$\ell_i/w_i < \ell_j/w_j \Rightarrow \ell_i'/w_i < \ell_j'/w_j.$$

(ii) *For any $i, j = 1, 2, \dots, n$ such that both p_i and p_j are non-zero polynomials, for any term t_i in the expansion of p_i and any term t_j in the expansion of p_j, we have*
$$\begin{aligned}(Q\text{-deg } t_i + \epsilon_i)/w_i < (Q\text{-deg } t_j + \epsilon_j)/w_j \\ \Rightarrow (Q'\text{-deg } t_i + \epsilon_i')/w_i < (Q'\text{-deg } t_j + \epsilon_j')/w_j.\end{aligned}$$

(iii) *For any $i = 1, 2, \dots, n$ such that p_i is a non-zero polynomial, for any terms t_1 and t_2 in the expansion of p_i, we have*
$$Q\text{-deg } t_1 < Q\text{-deg } t_2 \Rightarrow Q'\text{-deg } t_1 < Q'\text{-deg } t_2.$$

Here Q-deg and Q'-deg denote the degrees with respect to the new weight types $(\ell_1, \dots, \ell_n)$ and $(\ell_1', \dots, \ell_n')$ respectively.

Proof. We claim that $0 \le \ell_i' - \ell_i \le w_i\varepsilon/(w_1w_2)$ for all i and $0 \le \ell_i' - \ell_i < w_i\varepsilon/(w_1w_2)$ for i such that p_i is a non-zero polynomial. We prove the claim by induction on i.

If $i = n$, then $\ell_n = \epsilon_n$, $\ell_n' = \epsilon_n'$, and p_n is the zero polynomial. By the definition of ϵ_i', we know that $0 \le \epsilon_n' - \epsilon_n \le \varepsilon/(w_1w_2) \le w_n\varepsilon/(w_1w_2)$. Thus the claim holds for $i = n$.

Suppose the claim holds for $i = k+1, \dots, n$. We prove it holds for $i = k$. There are the following two cases:

(1) p_k is the zero polynomial. With the same argument as when $i = n$, we know the claim holds.

(2) p_k is a non-zero polynomial, since $p_k(0) = 0$, so there is no constant term in the expansion of p_k. There are two subcases as follows.

(a) $k = j_0$. Pick any $s > k$. If p_s is a non-zero polynomial, then by the inductive assumption we have $0 \le \ell_s' - \ell_s < w_s\varepsilon/(w_1w_2)$. If p_s is the zero polynomial, then $\ell_s = \epsilon_s$ and $\ell_s' = \epsilon_s'$. Notice that $s \ne k = j_0$. We have $\epsilon_s = \epsilon_s'$, which follows that $\ell_s' - \ell_s = 0 < w_s\varepsilon/(w_1w_2)$. Thus $0 \le \ell_s' - \ell_s < w_s\varepsilon/(w_1w_2)$ for all $s > k$. For any term $t = x_{k+1}^{a_{k+1}} \dots x_n^{a_n}$ in the expansion of p_k ($a_{k+1}, \dots, a_n$ are not all zero), using the fact $a_{k+1}w_{k+1} + \cdots + a_nw_n = w_k + wt\,D$, we have

$$\begin{aligned} 0 \le \text{Q}'\text{-deg } t - \text{Q-deg } t &= a_{k+1}(\ell_{k+1}' - \ell_{k+1}) + \cdots + a_n(\ell_n' - \ell_n) \\ &< (a_{k+1}w_{k+1} + \cdots + a_nw_n)\varepsilon/(w_1w_2) \\ &= (w_k + wt\,D)\varepsilon/(w_1w_2). \end{aligned}$$

Therefore

$$\mathrm{Q}'\text{-deg } t < \mathrm{Q}\text{-deg } t + (w_k + wt\,D)\varepsilon/(w_1w_2) \le \mathrm{Q}\text{-deg } p_k + (w_k + wt\,D)\varepsilon/(w_1w_2),$$

for any term t in the expansion of p_k; thus

$$\mathrm{Q}'\text{-deg } p_k < \mathrm{Q}\text{-deg } p_k + (w_k + wt\,D)\varepsilon/(w_1w_2). \tag{6.1}$$

Since

$$\mathrm{Q}\text{-deg } t \le \mathrm{Q}'\text{-deg } t \le \mathrm{Q}'\text{-deg } p_k,$$

for any term t in the expansion of p_k, we have

$$\mathrm{Q}\text{-deg } p_k \le \mathrm{Q}'\text{-deg } p_k. \tag{6.2}$$

Combining (6.1) with (6.3), we have

$$0 \le \mathrm{Q}'\text{-deg } p_k - \mathrm{Q}\text{-deg } p_k < (w_k + wt\,D)\varepsilon/(w_1w_2).$$

By definition, we have $\epsilon'_k - \epsilon_k = \varepsilon/(w_1w_2)$. Thus

$$0 \le \ell'_k - \ell_k = \epsilon'_k + \mathrm{Q}'\text{-deg } p_k - (\epsilon_k + \mathrm{Q}\text{-deg } p_k) < (w_k + wt\,D + 1)\varepsilon/(w_1w_2).$$

Since $wt\,D$ is a negative integer, we have $wt\,D + 1 \le 0$, and the claim is proved.

(b) $k \ne j_0$, so $\epsilon'_k = \epsilon_k$. For any term $t = x_{k+1}^{a_{k+1}} \dots x_n^{a_n}$ in the expansion of p_k ($a_{k+1}, \dots, a_n$ are not all zero), using the fact that $a_{k+1}w_{k+1} + \dots + a_nw_n = w_k + wt\,D$ and the inductive assumption that $0 \le \ell'_s - \ell_s \le w_s\varepsilon/(w_1w_2)$ for $s > k$, we have

$$\begin{aligned} 0 \le \mathrm{Q}'\text{-deg } t - \mathrm{Q}\text{-deg } t &= a_{k+1}(\ell'_{k+1} - \ell_{k+1}) + \dots + a_n(\ell'_n - \ell_n) \\ &\le (a_{k+1}w_{k+1} + \dots + a_nw_n)\varepsilon/(w_1w_2) \\ &= (w_k + wt\,D)\varepsilon/(w_1w_2). \end{aligned}$$

Therefore

$$\mathrm{Q}'\text{-deg } t \le \mathrm{Q}\text{-deg } t + (w_k + wt\,D)\varepsilon/(w_1w_2) \le \mathrm{Q}\text{-deg } p_k + (w_k + wt\,D)\varepsilon/(w_1w_2)$$

for any term t in the expansion of p_k; thus

$$\mathrm{Q}'\text{-deg } p_k \le \mathrm{Q}\text{-deg } p_k + (w_k + wt\,D)\varepsilon/(w_1w_2). \tag{6.3}$$

Since

$$\mathrm{Q}\text{-deg } t \le \mathrm{Q}'\text{-deg } t \le \mathrm{Q}'\text{-deg } p_k,$$

for any term t in the expansion of p_k, we have

$$\mathrm{Q}\text{-deg } p_k \le \mathrm{Q}'\text{-deg } p_k. \tag{6.4}$$

Combining (6.3) with (6.4), we have

$$0 \le \mathrm{Q}'\text{-deg } p_k - \mathrm{Q}\text{-deg } p_k \le (w_k + wt\,D)\varepsilon/(w_1w_2).$$

Also $\epsilon'_k - \epsilon_k = 0$, so by the definition of the new weight type, we have $0 \le \ell'_k - \ell_k \le (w_k + wt\,D)\varepsilon/(w_1w_2)$. Since $wt\,D$ is a negative integer, we have $w_k + wt\,D < w_k$. Thus $0 \le \ell'_k - \ell_k < w_k\varepsilon/(w_1w_2)$, and the claim is proved.

From the argument above, we also know that for any i such that p_i is a non-zero polynomial and for any term t in the expansion of p_i, we have

$$0 \le (\mathrm{Q}'\text{-deg } t + \epsilon'_i) - (\mathrm{Q}\text{-deg } t + \epsilon_i) < w_i\varepsilon/(w_1w_2). \tag{6.5}$$

(i) For any i, j such that both p_i and p_j are non-zero polynomials, if $\ell_i/w_i < \ell_j/w_j$, by Lemma 4.6 we have $\ell_j/w_j - \ell_i/w_i \ge \varepsilon/(w_1w_2)$. By the claim above, we have $0 \le \ell'_i/w_i - \ell_i/w_i < w_i\varepsilon/(w_1w_2w_i) = \varepsilon/(w_1w_2)$ and $0 \le \ell'_j/w_j - \ell_j/w_j$. Therefore we have $\ell'_i/w_i < \ell_i/w_i + \varepsilon/(w_1w_2) \le \ell_j/w_j \le \ell'_j/w_j$. Thus (i) is proved.

(ii) For any i, j such that both p_i and p_j are non-zero polynomials and for any term t_i in the expansion of p_i and any term t_j in the expansion of p_j, by Lemma 4.6, we know all ℓ_k, $k = 1, \ldots, n$ are divisible by ε; thus Q-deg $t_i + \epsilon_i$ and Q-deg $t_j + \epsilon_j$ are divisible by ε. Write Q-deg $t_i + \epsilon_i$ and Q-deg $t_j + \epsilon_j$ as the forms $q_i\varepsilon$ and $q_j\varepsilon$ respectively where q_i and q_j are integers. If (Q-deg $t_i+\epsilon_i)/w_i <$ (Q-deg $t_j+\epsilon_j)/w_j$, then $q_i w_j < q_j w_i$. Notice that $q_i w_j$ and $q_j w_i$ are integers, so $q_j w_i - q_i w_j \geq 1$. Thus

$$(\text{Q-deg } t_j + \epsilon_j)/w_j - (\text{Q-deg } t_i + \epsilon_i)/w_i = (q_j w_i - q_i w_j)\varepsilon/(w_i w_j) \geq \varepsilon/(w_1 w_2).$$

By equation (6.5), we have

$$(\text{Q}'\text{-deg } t_i + \epsilon_i')/w_i - (\text{Q-deg } t_i + \epsilon_i)/w_i < w_i\varepsilon/(w_1 w_2 w_i) = \varepsilon/(w_1 w_2).$$

Combining these two inequalities and noticing that Q-deg $t_j + \epsilon_j \leq$ Q$'$-deg $t_j + \epsilon_j'$ (see equation (6.5)), we have

$$(\text{Q}'\text{-deg } t_i + \epsilon_i')/w_i < (\text{Q-deg } t_j + \epsilon_j)/w_j \leq (\text{Q}'\text{-deg } t_j + \epsilon_j')/w_j.$$

(iii) Using (ii) for the case that $i = j$, we can get that for any i such that p_i is a non-zero polynomial and for any terms t_1 and t_2 in the expansion of p_i, we have

$$\begin{aligned}(\text{Q-deg } t_1 + \epsilon_i)/w_i &< (\text{Q-deg } t_2 + \epsilon_i)/w_i \\ &\Rightarrow (\text{Q}'\text{-deg } t_1 + \epsilon_i')/w_i < (\text{Q}'\text{-deg } t_2 + \epsilon_i')/w_i.\end{aligned}$$

Thus we have

$$\text{Q-deg } t_1 < \text{Q-deg } t_2 \Rightarrow \text{Q}'\text{-deg } t_1 < \text{Q}'\text{-deg } t_2.$$

□

Theorem 6.1. *Let $f_1, f_2, \ldots, f_m$ be m weighted homogeneous polynomials with respect to the weight type $(w_1, w_2, \ldots, w_n)$ such that*

$$R = \mathbb{C}[x_1, x_2, \ldots, x_n]/(f_1, f_2, \ldots, f_m)$$

is a non-zero Artinian algebra. D is a non-zero negative weight derivation as in (3.3) on P which preserves the ideal $(f_1, \ldots, f_m)$. $(\ell_1, \ldots, \ell_n)$ is the new weight type associated with D and controlled by parameters ϵ_i'. Fix a subset I of $\{1, 2, \ldots, n\}$ $(n \geq 2)$ such that I has more than one element. Suppose the parameter $(\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$ satisfies the following condition:

$$\epsilon_i = \begin{cases} \epsilon & i \in I \text{ and } p_i \text{ is a non-zero polynomial,} \\ 0 & i \in I \text{ and } p_i \text{ is the zero polynomial,} \\ \epsilon + \epsilon/(w_1 w_2)^{b_i} & i \notin I \text{ and } p_i \text{ is a non-zero polynomial,} \\ \epsilon/(w_1 w_2)^{b_i} & i \notin I \text{ and } p_i \text{ is the zero polynomial,} \end{cases} \tag{6.6}$$

where ϵ is a positive real number, k is the number of elements in I $(k \geq 2)$, and $b : i \mapsto b_i$ is a one-to-one map from $\{1, 2, \ldots, n\} \setminus I$ to $\{1, 2, \ldots, n-k\}$. Let $I_{\max} = \{e \colon \ell_e/w_e$ is the maximum among all ℓ_i/w_i for $i = 1, 2, \ldots, n\}$. If $I_{\max} \subseteq I$ and p_i is a non-zero polynomial for any $i \in I_{\max}$, then there exists $j \in \{1, 2, \ldots, m\}$ such that

$$\deg f_j \leq (m-1)(w_1 w_2)^{n-1}.$$

Proof. Consider the following two cases: Case 1: $I_{\max} = I$, and Case 2: $I_{\max}$ is a proper subset of I. We first prove that Case 1 can be reduced to Case 2. Then we only need to consider Case 2.

Assume that $I_{\max} = I$. Write $I = I_{\max} = \{i_1, \dots, i_k\}$, where $i_1 < i_2 < \cdots < i_k$, $k \geq 2$. Since p_i is a non-zero polynomial for any $i \in I_{\max} = I$, by (6.6) we can see that $\epsilon_i = \epsilon$ for any $i \in I_{\max} = I$.

We shall first prove the following proposition.

Proposition 6.1. *For any term t_0 in the expansion of $(p_{i_{k-1}})_{\max}$ and for any term t_1 in the expansion of $(p_{i_k})_{\max}$, we have $t_0 = cx_{i_k}^a t_1$, where a is a non-negative integer and c is a non-zero constant coefficient.*

Proof of Proposition 6.1. Let $h : \{1, 2, \dots, n-k\} \to \{1, \dots, n\} \setminus I$ be the inverse function of the map $b : i \mapsto b_i$; that is to say, $b_{h(i)} = i$ for $i = 1, 2, \dots, n-k$. Define a group of parameters $\epsilon_i^{(0)}, \epsilon_i^{(1)}, \dots, \epsilon_i^{(n-k)}$, $i = 1, \cdots, n$ by induction as follows:

$$\epsilon_i^{(0)} = \begin{cases} \epsilon & p_i \text{ is a non-zero polynomial,} \\ 0 & p_i \text{ is the zero polynomial.} \end{cases}$$

Assume that the $(j-1)$-th group of parameter $(\epsilon_1^{(j-1)}, \dots, \epsilon_n^{(j-1)})$ has been defined. Then we define

$$\epsilon_i^{(j)} = \begin{cases} \epsilon_i^{(j-1)} + \epsilon/(w_1 w_2)^j & i = h(j), \\ \epsilon_i^{(j-1)} & i \neq h(j). \end{cases}$$

By this definition, it is clear that $\epsilon_i^{(j)} = \epsilon$ for any $i \in I_{\max} = I$ and any $j = 0, 1, \dots, n-k$. In particular $\epsilon_i^{(n-k)} = \epsilon = \epsilon_i$, for $i \in I_{\max} = I$. On the other hand, for $i \notin I_{\max}$, there exists a unique $j \in \{1, \dots, n-k\}$ such that $h(j) = i$; hence $b_i = j$. Thus $\epsilon_i^{(n-k)} = \epsilon_i^j = \epsilon_i^{(j-1)} + \epsilon/(w_1 w_2)^{b_i} = \epsilon_i^{(0)} + \epsilon/(w_1 w_2)^{b_i} = \epsilon_i$. Thus $(\epsilon_1^{(n-k)}, \dots, \epsilon_n^{(n-k)}) = (\epsilon_1, \dots, \epsilon_n)$. Let $(\ell_1^{(j)}, \dots, \ell_n^{(j)})$ be the new weight type controlled by parameter $(\epsilon_1^{(j)}, \dots, \epsilon_n^{(j)})$ for $j = 0, 1, \dots, n-k$, and let Q(j)-deg mean the associated degree. For convenience, we write $i_{k-1} = s$ and $i_k = t$, then $s < t$. Since $s, t \in I_{\max} = I$, p_s and p_t are not the zero polynomials. Pick any term t_0 in the expansion of $(p_s)_{\max}$ and pick any term t_1 in the expansion of $(p_t)_{\max}$. Notice that $s, t \in I_{\max}$. We have $\ell_s/w_s = \ell_t/w_t$; thus

$$(\text{Q-deg } t_0 + \epsilon_s)/w_s = (\text{Q-deg } t_1 + \epsilon_t)/w_t.$$

Since $(\epsilon_1, \dots, \epsilon_n) = (\epsilon_1^{(n-k)}, \dots, \epsilon_n^{(n-k)})$, we have

$$\ell_s^{(n-k)}/w_s = \ell_t^{(n-k)}/w_t, \tag{6.7}$$

$$(\text{Q(n-k)-deg } t_0 + \epsilon_s^{(n-k)})/w_s = (\text{Q(n-k)-deg } t_1 + \epsilon_t^{(n-k)})/w_t. \tag{6.8}$$

We claim that

$$\ell_s^{(j)}/w_s = \ell_t^{(j)}/w_t \tag{6.9}$$

for $j = 0, 1, \dots, n-k$, because if there exists e such that $\ell_s^{(e)}/w_s \neq \ell_t^{(e)}/w_t$, notice that both p_s and p_t are not the zero polynomials by Lemma 6.1(i) (here we set $\varepsilon = \epsilon/(w_1 w_2)^e$) we have $\ell_s^{(e+1)}/w_s \neq \ell_t^{(e+1)}/w_t$. Similarly, $\ell_s^{(e+1)}/w_s \neq \ell_t^{(e+1)}/w_t$ can imply $\ell_s^{(e+2)}/w_s \neq \ell_t^{(e+2)}/w_t$. Continuing this process, finally it will imply that $\ell_s^{(n-k)}/w_s \neq \ell_t^{(n-k)}/w_t$, which contradicts (6.7). Hence (6.9) is proved. Likewise, using Lemma 6.1(ii) and equation (6.8), we have

$$(\text{Q(j)-deg } t_0 + \epsilon_s^{(j)})/w_s = (\text{Q(j)-deg } t_1 + \epsilon_t^{(j)})/w_t. \tag{6.10}$$

From above, since $\epsilon_s^{(j)} = \epsilon_t^{(j)} = \epsilon$ for $j = 0, \dots, n-k$, so (6.10) implies that

$$(\text{Q(j)-deg } t_0 + \epsilon)/w_s = (\text{Q(j)-deg } t_1 + \epsilon)/w_t \tag{6.11}$$

for $j = 0, 1, \dots, n-k$. We claim that t_0 is independent of x_i for $i = s+1, \dots, t-1$ for the following reason. Assume the opposite, that there exist $e \in \{s+1, \dots, t-1\}$ such that t_0 depends on x_e. Let $j = b_e$; then $h(j) = e$. Thus by definition we have $\epsilon_i^{(j-1)} = \epsilon_i^{(j)}$ for $i \neq e$ and $\epsilon_e^{(j-1)} < \epsilon_e^{(j)}$, which follows that $\ell_i^{(j-1)} = \ell_i^{(j)}$ for $i > e$ and $\ell_e^{(j-1)} < \ell_e^{(j)}$ and $\ell_i^{(j-1)} \leq \ell_i^{(j)}$ for $i < e$. Notice that t_1 is a monomial of only variables $x_{t+1}, \dots, x_n$ and $t+1 > e$, so we have

$$\text{Q(j-1)-deg } t_1 = \text{Q(j)-deg } t_1. \tag{6.12}$$

Notice that t_0 depends on x_e, so we have

$$\text{Q(j-1)-deg } t_0 < \text{Q(j)-deg } t_0. \tag{6.13}$$

Equation (6.12) and inequality (6.13) contradict (6.11); thus the claim that t_0 is independent of x_i for $i = s+1, \dots, t-1$ is proved. So t_0 can be written as the form $cx_t^a t_2$, where t_2 is a monomial of variables $x_{t+1}, \dots, x_n$ and a is a non-negative integer and c is a constant coefficient. Next, we will prove $t_2 = t_1$ up to a scale by two steps. Write t_1 and t_2 as $c_1 x_{t+1}^{a_{t+1}} \dots x_n^{a_n}$ and $c_2 x_{t+1}^{b_{t+1}} \dots x_n^{b_n}$ respectively.

Step 1. We first prove that

$$a_i/b_i = (\deg t_1 - wt\, D)/(\deg t_2 - wt D) \tag{6.14}$$

for $i = t+1, \dots, n$.

Since the term t_1 appears in the expansion of $(p_t)_{\max}$, for any term g in the expansion of p_t we have Q-deg $t_1 \geq$ Q-deg g, i.e., Q(n-k)-deg $t_1 \geq$ Q(n-k)-deg g. Using Lemma 6.1(iii), we can get Q(j)-deg $t_1 \geq$ Q(j)-deg g for any $j = 0, 1, \dots, n-k$ and for any term g in the expansion of p_t. It follows that

$$\ell_t^{(j)} = \text{Q(j)-deg } t_1 + \epsilon, \tag{6.15}$$

for $j = 0, 1, \dots, n-k$. By (6.11), (6.15), and the facts that $w_s = \deg t_0 - wt\, D$, $w_t = \deg t_1 - wt\, D$, and $t_0 = cx_t^a t_2$, we have

$$\begin{aligned}\frac{\text{Q(j)-deg } t_0 + \epsilon}{\deg t_0 - wt\, D} &= \frac{a\ell_t^{(j)} + \text{Q(j)-deg } t_2 + \epsilon}{aw_t + \deg t_2 - wt\, D} \\ &= \frac{\text{Q(j)-deg } t_1 + \epsilon}{\deg t_1 - wt\, D} = \frac{\ell_t^{(j)}}{w_t}\end{aligned}$$

for $j = 0, \dots, n-k$. It follows that

$$\frac{\text{Q(j)-deg } t_2 + \epsilon}{\deg t_2 - wt\, D} = \frac{\text{Q(j)-deg } t_1 + \epsilon}{\deg t_1 - wt\, D} \tag{6.16}$$

for $j = 0, \dots, n-k$. We prove the claim that $a_i/b_i = (\deg t_1 - wt\, D)/(\deg t_2 - wtD)$ for $i = t+1, \dots, n$ by induction. If $i = t+1$, let $j = b_{t+1}$; then $h(j) = t+1$. Thus $\ell_{t+1}^{(j)} - \ell_{t+1}^{(j-1)} > 0$ and $\ell_{t+2}^{(j)} - \ell_{t+2}^{(j-1)} = \dots = \ell_n^{(j)} - \ell_n^{(j-1)} = 0$. Hence we have

$$\text{Q(j)-deg } t_1 = \text{Q(j-1)-deg } t_1 + a_{t+1}(\ell_{t+1}^{(j)} - \ell_{t+1}^{(j-1)}),$$

$$\text{Q(j)-deg } t_2 = \text{Q(j-1)-deg } t_2 + b_{t+1}(\ell_{t+1}^{(j)} - \ell_{t+1}^{(j-1)}).$$

By (6.16) we can get $a_{t+1}/b_{t+1} = (\deg t_1 - wt\, D)/(\deg t_2 - wtD)$; thus the claim holds for $t+1$.

Suppose (6.14) holds for $t+1, t+2, \ldots, i-1$. Let's prove it for i. Let $j = b_i$. Then $h(j) = i$; thus we have $\ell_i^{(j)} - \ell_i^{(j-1)} > 0$ and $\ell_{i+1}^{(j)} - \ell_{i+1}^{(j-1)} = \cdots = \ell_n^{(j)} - \ell_n^{(j-1)} = 0$. Thus we have

$$\text{Q(j)-deg } t_1 = \text{Q(j-1)-deg } t_1 + a_{t+1}(\ell_{t+1}^{(j)} - \ell_{t+1}^{(j-1)}) + \cdots + a_i(\ell_i^{(j)} - \ell_i^{(j-1)})$$

and

$$\text{Q(j)-deg } t_2 = \text{Q(j-1)-deg } t_2 + b_{t+1}(\ell_{t+1}^{(j)} - \ell_{t+1}^{(j-1)}) + \cdots + b_i(\ell_i^{(j)} - \ell_i^{(j-1)}).$$

By inductive assumption and (6.16) we can get

$$a_i/b_i = (\deg t_1 - wt\, D)/(\deg t_2 - wtD).$$

Step 2. We shall prove that $\deg t_1 - wt\, D = \deg t_2 - wtD$. Assume that $\deg t_1 - wt\, D > \deg t_2 - wtD$. Then by (6.14), we have $a_i > b_i$ for $i = t+1, \ldots, n$. Let $t_3 = x_{t+1}^{a_{t+1}-b_{t+1}} \ldots x_n^{a_n - b_n}$; then $t_1 = t_2 t_3$ up to a scale. Thus we have

$$\frac{\ell_t}{w_t} = \frac{\text{Q-deg } t_1 + \epsilon}{\deg t_1 - wtD} = \frac{\text{Q-deg } t_2 + \text{Q-deg } t_3 + \epsilon}{\deg t_2 + \deg t_3 - wtD}. \tag{6.17}$$

By the fact that $(\epsilon_1^{(n-k)}, \ldots, \epsilon_n^{(n-k)}) = (\epsilon_1, \ldots, \epsilon_n)$ and the equation (6.16) for $j = n-k$, we have

$$\frac{\text{Q-deg } t_1 + \epsilon}{\deg t_1 - wtD} = \frac{\text{Q-deg } t_2 + \epsilon}{\deg t_2 - wtD}. \tag{6.18}$$

By (6.17) and (6.18) we can get $\ell_t/w_t = \text{Q-deg } t_3/\deg t_3$. Since $t \in I_{\max}$ and $t+1, \ldots, n \notin I_{\max}$, we have $\ell_t/w_t > \ell_{t+1}/w_{t+1}, \ldots, \ell_t/w_t > \ell_n/w_n$. Also t_3 is a monomial of $x_{t+1}, \ldots, x_n$, so $\text{Q-deg } t_3/\deg t_3 < \ell_t/w_t$, which contradicts $\ell_t/w_t = \text{Q-deg } t_3/\deg t_3$. So the assumption $\deg t_1 - wt\, D > \deg t_2 - wtD$ is invalid. Similarly we can prove the assumption $\deg t_1 - wt\, D < \deg t_2 - wtD$ is invalid. Thus $\deg t_1 - wt\, D = \deg t_2 - wtD$. It follows that $a_i = b_i$ for $i = t+1, \ldots, n$; thus $t_1 = t_2$ up to a scale. So Proposition 6.1 is proved.

Now we come back to the proof of Theorem 6.1.

Fix a term t_0 in the expansion of $(p_{i_{k-1}})_{\max}$. For any two terms t_1, t_2 in the expansion of $(p_{i_k})_{\max}$, by Proposition 6.1, we have $t_0 = c_1 x_{i_k}^{a_1} t_1$ and $t_0 = c_2 x_{i_k}^{a_2} t_2$, where c_1, c_2 are non-zero constant coefficients and a_1, a_2 are non-negative integers. So $c_1 x_{i_k}^{a_1} t_1 = c_2 x_{i_k}^{a_2} t_2$. Notice that t_1, t_2 are monomials of variables $x_{i_k+1}, \ldots, x_n$, so $t_1 = t_2$ up to a scale. So there is only one term in the expansion of $(p_{i_k})_{\max}$.

Fix a term t_2 in the expansion of $(p_{i_k})_{\max}$. For any two terms t_0, t_1 in the expansion of $(p_{i_{k-1}})_{\max}$, by Proposition 6.1, we have $t_0 = c_0 x_{i_k}^{a_0} t_2$ and $t_1 = c_1 x_{i_k}^{a_1} t_2$, where c_0, c_1 are non-zero constant coefficients and a_0, a_1 are non-negative integers. Because $p_{i_{k-1}}$ is a weighted homogeneous polynomial with respect to the original weight type $(w_1, \ldots, w_n)$, we have $\deg t_0 = \deg t_1$; thus $a_0 = a_1$. Since $t_0 = t_1$ up to a scale, it follows that there is only one term in the expansion of $(p_{i_{k-1}})_{\max}$. Hence

$$(p_{i_{k-1}})_{\max} = c x_{i_k}^a (p_{i_k})_{\max},$$

where c is a non-zero constant coefficient and a is a non-negative integer. Noticing that $\deg(p_{i_{k-1}})_{\max} = \deg p_{i_{k-1}} = w_{i_{k-1}} + wt\, D$ and $\deg(p_{i_k})_{\max} = \deg p_{i_k} = w_{i_k} + wt\, D$, we have $w_{i_{k-1}} + wt\, D = a w_{i_k} + w_{i_k} + wt\, D$, which follows that

$$w_{i_{k-1}} = (a+1) w_{i_k}. \tag{6.19}$$

And since $i_{k-1}, i_k \in I_{\max}$, so $\ell_{i_{k-1}}/w_{i_{k-1}} = \ell_{i_k}/w_{i_k}$, we have that

$$\ell_{i_{k-1}} = (a+1)\ell_{i_k}. \tag{6.20}$$

In the following, we will take a coordinate change which preserves the original weight type $(w_1, w_2, \ldots, w_n)$. The coordinate change is of the following form:

$$\begin{aligned} x_1 &= x_1', \\ &\cdots\cdots \\ x_{i_{k-1}} &= x_{i_{k-1}}' + c(x_{i_k}')^{a+1}/(a+1), \\ &\cdots\cdots \\ x_n &= x_n'. \end{aligned} \tag{6.21}$$

Calculating the transformation of derivations in this coordinate change (6.21) we have

$$\begin{aligned} \frac{\partial}{\partial x_1'} &= \frac{\partial}{\partial x_1}, \\ &\cdots\cdots \\ \frac{\partial}{\partial x_{i_{k-1}}'} &= \frac{\partial}{\partial x_{i_{k-1}}}, \\ \frac{\partial}{\partial x_{i_k}'} &= \frac{\partial}{\partial x_{i_k}} + c(x_{i_k}')^a \frac{\partial}{\partial x_{i_{k-1}}}, \\ &\cdots\cdots \\ \frac{\partial}{\partial x_n'} &= \frac{\partial}{\partial x_n}. \end{aligned} \tag{6.22}$$

Write the expression of the negative weight derivation D in the new coordinate system as

$$D' = p_1' \frac{\partial}{\partial x_1'} + p_2' \frac{\partial}{\partial x_2'} + \cdots + p_n' \frac{\partial}{\partial x_n'}.$$

It is clear that $p_t' = p_t$ for $t \neq i_{k-1}$. Only

$$p_{i_{k-1}}' = p_{i_{k-1}} - c(x_{i_k}')^a p_{i_k} = p_{i_{k-1}} - c x_{i_k}^a p_{i_k}$$

is changed. Let $(\ell_1', \ldots, \ell_n')$ be the new weight type associated with D' in the new coordinate system and controlled by the origin parameter $(\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$ and let Q$'$-deg mean the associating degree. For any $t > i_{k-1}$, we have $p_t' = p_t$ and p_t is independent of $x_{i_{k-1}}$. Thus the expression of p_t in the origin coordinate system is the same as the expression of p_t' in the new coordinate system (since the coordinate change only happens on $x_{i_{k-1}}$), which follows that $\ell_t' = \ell_t$ for all $t > i_{k-1}$. We claim that $\ell_{i_{k-1}}' < \ell_{i_{k-1}}$ for the following reason. Since $(p_{i_{k-1}})_{\max} = c x_{i_k}^a (p_{i_k})_{\max}$, we have $(p_{i_{k-1}} - c x_{i_k}^a p_{i_k})$ is the zero polynomial or Q-deg $(p_{i_{k-1}} - c x_{i_k}^a p_{i_k}) <$ Q-deg $p_{i_{k-1}}$. If the former, then $p_{i_{k-1}}'$ is the zero polynomial and it's clear that $\ell_{i_{k-1}}' < \ell_{i_{k-1}}$. If the latter, notice that $p_{i_{k-1}}' = p_{i_{k-1}} - c x_{i_k}^a p_{i_k}$ is a polynomial of x_t for $t > i_{k-1}$ and $\ell_t' = \ell_t$ for $t > i_{k-1}$. We have Q$'$-deg $p_{i_{k-1}}' =$ Q-deg $(p_{i_{k-1}} - c x_{i_k}^a p_{i_k})$. Thus Q$'$-deg $p_{i_{k-1}}' <$ Q-deg $p_{i_{k-1}}$, which implies that $\ell_{i_{k-1}}' < \ell_{i_{k-1}}$. Now we claim that $\ell_t' \leq \ell_t$ for all $t = 1, 2, \ldots, n$ and we will prove it by induction on t. From the above argument we already know it holds for $t \geq i_{k-1}$. Assume the claim holds for

$t+1, t+2, \ldots, n$, and we will prove it holds for t (here $t < i_{k-1}$). For any term $g = x_{t+1}^{a_{t+1}} \ldots x_n^{a_n}$ in the expansion of p_t,

$$g = (x'_{t+1})^{a_{t+1}} \ldots \left(x'_{i_{k-1}} + \frac{c}{a+1}(x'_{i_k})^{a+1}\right)^{a_{i_{k-1}}} \ldots (x'_n)^{a_n}$$

in terms of the new coordinate system. By the fact that Q$'$-deg$(x'_{i_k})^{a+1} = (a+1)\ell'_{i_k} = (a+1)\ell_{i_k} = \ell_{i_{k-1}}$ (by equation 6.20) and the inductive assumption, we can get Q $'$-deg $g \leq$ Q-deg g for any term g in the expression of p_t. Also because $p'_t = p_t$ (because $t < i_{k-1}$), we can get Q$'$-deg $p'_t \leq$ Q-deg p_t, which follows that $\ell'_t \leq \ell_t$, and the claim is proved.

Let $I'_{\max} = \{e\colon \ell'_e/w_e$ is the maximum among all ℓ'_i/w_i for $i = 1, 2, \ldots, n\}$. From the above argument we know that for any $i \notin I_{\max}$, $\ell'_i/w_i \leq \ell_i/w_i < \ell_{i_k}/w_{i_k} = \ell'_{i_k}/w_{i_k}$, which follows that $i \notin I'_{\max}$. Thus $I'_{\max} \subseteq I_{\max}$. Notice that $\ell'_{i_{k-1}}/w_{i_{k-1}} < \ell_{i_{k-1}}/w_{i_{k-1}} = \ell_{i_k}/w_{i_k} = \ell'_{i_k}/w_{i_k}$, so we have $I'_{\max} \subseteq I_{\max} \setminus \{i_{k-1}\}$, which follows that $I'_{\max}$ is a proper subset of $I_{\max} = I$. And for any $i \in I'_{\max}$, we have $i \in I_{\max}$ and $i \neq i_{k-1}$, so that p_i is a non-zero polynomial and $p'_i = p_i$. Thus the condition that p'_i is a non-zero polynomial for any $i \in I'_{\max}$ is satisfied. Thus the case that $I_{\max} = I$ can be reduced to the case that $I_{\max}$ is a proper subset of I.

Then we prove Theorem 6.1 by induction on the number k of elements of I. If $k = 2$, for the above reason we may assume that $I_{\max}$ is a proper subset of I; thus $I_{\max}$ has only one element. Assume that $I_{\max} = \{i_0\}$. Let $\beta = \ell_{i_0}/w_{i_0}$, $\gamma = \max\{\ell_i/w_i\colon i = 1, 2, \ldots, i_0 - 1, i_0 + 1, \ldots, n\}$, and $\epsilon_{\min} = \min\{\epsilon_i$ for i such that p_i is a non-zero polynomial$\}$. Since $i_0 \in I_{\max} \subset I$, we know that p_{i_0} is a non-zero polynomial and $\epsilon_{i_0} = \epsilon$. Also $\epsilon_i \geq \epsilon = \epsilon_{i_0}$ for any i such that p_i is a non-zero polynomial; thus $\epsilon_{\min} = \epsilon = \epsilon_{i_0}$. Since all ϵ_i are divisible by $\epsilon/(w_1w_2)^{n-k} = \epsilon/(w_1w_2)^{n-2}$, by Lemma 4.6 we have $\beta - \gamma \geq \epsilon/(w_1w_2)^{n-1}$. Since $i_0 \in I$, we know that $\epsilon_{i_0} = \epsilon_{\min} = \epsilon$ and p_{i_0} is a non-zero polynomial. By Theorem 4.1, there exists $j \in \{1, 2, \ldots, m\}$ such that

$$\deg f_j \leq \frac{(m-1)\epsilon_{\min}}{\beta - \gamma} \leq (m-1)(w_1w_2)^{n-1}.$$

Suppose that the conclusion holds for $2, \ldots, k-1$; we prove it for k. If $I_{\max}$ has only one element, then using a similar argument as above we can get the conclusion. Thus we may assume that $I_{\max}$ has more than one element, and we may assume that $I_{\max}$ is a proper subset of I. Then we pick an index $j_0 \in I \setminus I_{\max}$. Define another parameter (ϵ'_i) as follows:

$$\epsilon'_i = \begin{cases} \epsilon_i + \epsilon/(w_1w_2)^{n-k+1} & i = j_0, \\ \epsilon_i & i \neq j_0. \end{cases}$$

Consider the new weight type $(\ell'_1, \ldots, \ell'_n)$ controlled by parameters $(\epsilon'_1, \ldots, \epsilon'_n)$, and let $I'_{\max} = \{e\colon \ell'_e/w_e$ is the maximum among all ℓ'_i/w_i for $i = 1, 2, \ldots, n\}$. We claim that $I'_{\max} \subseteq I_{\max}$ for the following reason. For any $i \notin I_{\max}$, we need to consider the following two cases.

(1) p_i is a non-zero polynomial. Fix an index $j \in I_{\max}$; then $\ell_i/w_i < \ell_j/w_j$. By Lemma 6.1(i) (here we set $\varepsilon = \epsilon/(w_1w_2)^{n-k}$) we have $\ell'_i/w_i < \ell'_j/w_j$, which follows that $i \notin I'_{\max}$.

(2) p_i is the zero polynomial. Then $\epsilon'_i \leq \epsilon/(w_1w_2)$; thus $\ell'_i \leq \epsilon/(w_1w_2)$. For any $t \in I_{\max} \subset I$, p_t is a non-zero polynomial, so $\epsilon_t = \epsilon$. Also $t \neq j_0$; thus $\epsilon'_t = \epsilon_t = \epsilon$, which follows that $\ell'_t = \epsilon'_t +$ Q$'$-deg $p_t \geq \epsilon$ (here the equality may hold because

Q$'$-deg p_t may be equal to zero). Assume that $i \in I'_{\max}$. Then we have

$$\epsilon/w_t \leq \ell'_t/w_t \leq \ell'_i/w_i \leq \epsilon/(w_1 w_2 w_i), \quad \text{for any } t \in I_{\max}.$$

Thus $w_1 w_2 w_i \leq w_t$ for any $t \in I_{\max}$, concluding that $w_2 = w_i = 1$ and $w_1 = w_t$ for any $t \in I_{\max}$. Since $I_{\max}$ has more than one element by assumption, there exists $t_0 \in I_{\max}$ such that $t_0 \geq 2$, so it follows that $w_{t_0} \leq w_2$. Thus $w_1 = w_{t_0} \leq w_2 = 1$, so that $w_1 = 1$; that is to say, $w_1 = w_2 = \cdots = w_n$. Notice that $\deg p_i < w_i$ and $p_i(0) = 0$ (by Lemma 4.8) for i such that p_i is a non-zero polynomial. Thus p_i has to be the zero polynomial for $i = 1, 2, \ldots, n$, i.e., $D = 0$, and this leads to a contradiction. Hence the assumption $i \in I'_{\max}$ is wrong.

Thus $i \notin I'_{\max}$ for all $i \notin I_{\max}$, which follows that $I'_{\max} \subseteq I_{\max} \subseteq I \backslash \{j_0\}$. For any $i \in I'_{\max}$, we have $i \in I_{\max}$; thus p_i is a non-zero polynomial. Let $I' = I \backslash \{j_0\}$. Then the number of elements of I' is $k-1$ and $I'_{\max} \subseteq I'$. By the inductive assumption, the conclusion is proved. $\square$

Main Theorem A. Let $P = \mathbb{C}[x_1, x_2, \ldots, x_n]$ be the weighted polynomial ring of n weighted variables $x_1, x_2, \ldots, x_n$ with positive integer weights $w_1 \geq w_2 \geq \cdots \geq w_n$. Suppose that $f_1, f_2, \ldots, f_m$ are m ($m \geq n \geq 2$) weighted homogeneous polynomials with degrees greater than $(m-1)(w_1 w_2)^{n-1}$ and $R = P/(f_1, f_2, \ldots, f_m)$ is an Artinian algebra. Then there are no non-zero negative weight derivations on $R = P/(f_1, f_2, \ldots, f_m)$.

Proof of Main Theorem A. If the conclusion is not true, suppose D is a non-zero negative weight derivation on R or equivalently a non-zero negative weight derivation on P which preserves the ideal $(f_1, f_2, \ldots, f_m)$ as in (3.3). We take the new weight type $(\ell_1, \ldots, \ell_n)$ of D controlled by the parameters ϵ_i. Here

$$\epsilon_i = \begin{cases} \epsilon & p_i \text{ is a non-zero polynomial,} \\ 0 & p_i \text{ is the zero polynomial,} \end{cases}$$

where ϵ is a positive real number. It's clear that $\ell_i > 0$ for any i such that p_i is a non-zero polynomial and $\ell_i = 0$ for any i such that p_i is the zero polynomial. Thus p_i is a non-zero polynomial for any $i \in I_{\max}$. Let $I = \{1, 2, \ldots, n\}$, and it's clear that $I_{\max} \subseteq I$. Then by Theorem 6.1 we know that there exists $j \in \{1, 2, \ldots, m\}$ such that $\deg f_j \leq (m-1)(w_1 w_2)^{n-1}$, which contradicts the condition that $\deg f_j > (m-1)(w_1 w_2)^{n-1}$ for all j. So the conclusion is proved. $\square$

7. Duality for zero-dimensional singularities

In this section we recall the duality theorem for zero-dimensional, singularities, which is crucial in our proof of Main Theorem C.

Let A be a local analytic $\mathbb{C}$-algebra, i.e., a quotient algebra of the convergent power series ring $H = \mathbb{C}\{x_1, \ldots, x_n\}$ in n variables over $\mathbb{C}$. We shall denote the maximal ideal of A by m_A and the module of regular holomorphic differential 1-forms of H by $\Omega^1_{H/\mathbb{C}} \cong H\{dx_1, \ldots, dx_n\}$.

Let I be an ideal of H and let $A = H/I$ be a quotient algebra of H. One can define the module $\Omega^1_{A/\mathbb{C}}$ of Kähler differentials of A over $\mathbb{C}$ by the standard exact sequence of A-modules

$$I/I^2 \xrightarrow{d} \Omega^1_{H/\mathbb{C}} \otimes_H A \to \Omega^1_{A/\mathbb{C}} \to 0, \tag{7.1}$$

where d is given by $d(\bar{f}) = d_{H/\mathbb{C}}(\bar{f}) \otimes 1$ for $\bar{f} \in I/I^2$ and $d_{H/\mathbb{C}} : H \to \Omega^1_{H/\mathbb{C}}$ is the universal differential.

Let us consider the functor $\mathrm{Der}_{\mathbb{C}}(A, -)$ of $\mathbb{C}$-derivations (from A-modules to A-modules). There is a fundamental functorial isomorphism $\mathrm{Der}_{\mathbb{C}}(A, -) \cong \mathrm{Hom}_A(\Omega^1_{A/\mathbb{C}}, -)$. Let $\mathrm{Der}(A) = \mathrm{Der}_{\mathbb{C}}(A, A)$. $\mathrm{Der}(A)$ is the A-module of $\mathbb{C}$-derivations of A.

Remark 7.1. Suppose that the ideal I is generated by the sequence of functions $f_1, \ldots, f_k \in H$. Then

$$\Omega^1_A \cong \Omega^1_H / (\sum_{j=1}^{k} f_j \cdot \Omega^1_H + H \cdot df_j).$$

The local $\mathbb{C}$-algebra $A = H/I$ corresponds to the germ $X \subset (\mathbb{C}^n, 0)$ with the dual analytic $\mathbb{C}$ algebra $\mathcal{O}_X \cong A$. The modules Ω^1_A and $\mathrm{Der}(A)$ are usually referred to as the module of regular holomorphic differential 1-forms and the module of holomorphic vector fields on the germ X respectively. They are also denoted by Ω^1_X and $\mathrm{Der}(X)$.

We will also consider the tangent and cotangent functors of analytic algebras (see [Pa]) denoted by T_i and T^i respectively, $i \geq 0$. By definition, for any A-module M there exist the isomorphisms

$$T_0(A/\mathbb{C}, M) \cong \Omega^1_{A/\mathbb{C}} \otimes_A M, \; T^0(A/\mathbb{C}, M) \cong \mathrm{Der}_{\mathbb{C}}(A, M)$$

and exact sequences of A-modules

$$0 \to T_1(A/\mathbb{C}, M) \to I/I^2 \otimes_A M \xrightarrow{d \otimes 1_m} \Omega^1_{H/\mathbb{C}} \otimes_H M \to T_0(A/\mathbb{C}, M) \to 0,$$

$$\begin{aligned} 0 \to T^0(A/\mathbb{C}, M) &\to \mathrm{Hom}_A(\Omega^1_{H/\mathbb{C}} \otimes_H M, M) \to \mathrm{Hom}_A(I/I^2, M) \\ &\to T^1(A/\mathbb{C}, M) \to 0. \end{aligned}$$

The first sequence is obtained by tensoring (7.1) with M over A. Applying the functor $\mathrm{Hom}_A(-, M)$ to (7.1), we get the second sequence.

For brevity we shall denote the tangent and cotangent modules $T_i(A/\mathbb{C}, A)$ and $T^i(A/\mathbb{C}, A)$ by $T_i(A)$ and $T^i(A)$ respectively.

Hence

$$T_0(A) \cong \Omega^1_{A/\mathbb{C}}, \qquad T^0(A) \cong \mathrm{Der}(A), \qquad T_1(A) \cong \mathrm{Ker}(d).$$

Moreover, if $A = A_{red}$ is reduced and an A module M has no embedded associative primes, then there is an isomorphism (see [KL, (1.4.3)])

$$T^1(A/\mathbb{C}, M) \cong \mathrm{Ext}^1_A(\Omega^1_{A/\mathbb{C}}, M).$$

This is also true if instead of the above condition on A we assume that the analytic $\mathbb{C}$-algebra A corresponding to the germ $X \subset (\mathbb{C}^m, 0)$ has positive depth along its singular locus.

For convenience we also recall the following assertion.

Proposition 7.1 ([AM, (2.2)]). *Let $A = H/I$ be an Artinian complete intersection; that is, I is generated by a regular H sequence $f_1, \ldots, f_n \in H$. Then the tangent* *and cotangent module $T_i(A)$ and $T^j(A)$ have the same dimension for $i,j=0,1$, i.e.,*

$$\dim_{\mathbb{C}} \Omega^1_{A/\mathbb{C}} = \dim_{\mathbb{C}} T_1(A/\mathbb{C},A) = \dim_{\mathbb{C}} Der(A,A) = \dim_{\mathbb{C}} T^1(A/\mathbb{C},A).$$

Definition 7.1. Let $f \in H$ be an analytic function with an isolated critical point at the origin. The ideal I generated by partial derivatives $f_{x_i} = \partial f/\partial x_i, i=1,\dots,n$, is called the gradient ideal. In this case the H-sequence $f_{x_1},\dots,f_{x_n}$ is regular so that $A=H/I$ is an Artinian complete intersection.

Theorem 7.1 ([A2]). *Let I be the gradient ideal defined by an analytic function $f \in H$ with an isolated critical point at the origin, $A=H/I$. Then there exist two canonical non-degenerate pairings*

$$T^0(A)\times T^1(A) \longrightarrow \mathbb{C},$$
$$T_0(A)\times T_1(A) \longrightarrow \mathbb{C}.$$

Remark 7.2. Using elementary properties of tangent cohomology one may easily calculate an explicit representation of the pairings from the above theorem:

$$\begin{gathered}\mathrm{Der}(A)\times A^n/(\mathrm{Hess}(f)\cdot A^n) \longrightarrow A \longrightarrow \mathbb{C}\\ (\nu_1,\dots,\nu_n)\times(a_1,\dots,a_n) \longmapsto \sum_i \nu_i a_i \mapsto \mathbb{C}\end{gathered} \tag{7.2}$$

where

$$\mathrm{Hess}(f) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1\,\partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1\,\partial x_n}\\ \frac{\partial^2 f}{\partial x_2\,\partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2\,\partial x_n}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\partial^2 f}{\partial x_n\,\partial x_1} & \frac{\partial^2 f}{\partial x_n\,\partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2}\end{bmatrix}$$

and the last arrow is the projection on the one-dimensional component of the algebra A, which is called the socle that corresponds to the class of Det(Hess(f))$=df_{x_1}\wedge\cdots\wedge df_{x_n}/dx_1\wedge\cdots\wedge dx_n$.

8. Some related useful results

Theorem 8.1, which we want to use later, was proved by K. Saito in [Sa2]; see also [Wie].

Theorem 8.1 ([Sa2]). *Let $f \in \mathscr{O}_{\mathbb{C}^n,0}$ be a germ of a holomorphic function which defines a hypersurface with isolated singularity at 0. Then*

(1)
$$\mathrm{Det}(\partial^2 f/\partial x_i\partial x_j)_{i,j=1,\dots,n} \notin (f_{x_1},\dots,f_{x_n})\mathscr{O}_{\mathbb{C}^n,0}$$

and

$$m\mathrm{Det}(\partial^2 f/\partial x_i\partial x_j)_{i,j=1,\dots,n} \in (f_{x_1},\dots,f_{x_n})\mathscr{O}_{\mathbb{C}^n,0}$$

where m is the maximal ideal of $\mathscr{O}_{\mathbb{C}^n,0}$.

(2) *For each $g\in\mathscr{O}_{\mathbb{C}^n,0}$ with*

$$g \notin (f_{x_1},\dots,f_{x_n})\mathscr{O}_{\mathbb{C}^n,0}$$

there is an $h \in \mathscr{O}_{\mathbb{C}^n,0}$ *such that*

$$hg - \mathrm{Det}(\partial^2 f/\partial x_i \partial x_j)_{i,j=1,\dots,n} \in (f_{x_1}, \dots, f_{x_n})\mathscr{O}_{\mathbb{C}^n,0}.$$

(3) *If* f *is a weighted homogeneous polynomial with weights* $(w_1, \dots, w_n)$ *and degree* d*, then*

$$\mathrm{Det}(\partial^2 f/\partial x_i \partial x_j)_{i,j=1,\dots,n}$$

is also a weighted homogeneous polynomial with the same weights $(w_1, \dots, w_n)$ *as* f *and of degree* $nd - 2\sum_{i=1}^n w_i$.

The following concepts and results enable one to compute the Yau algebras of many concrete singularities. Let A, B be associative algebras over a field k. Recall that the multiplication algebra $M(A)$ of A is defined as the subalgebra of endomorphisms of A generated by the identity element and left and right multiplications by elements of A. The centroid $C(A)$ is the set of endomorphisms of A which commute with all elements of $M(A)$. Clearly, $C(A)$ is a unital subalgebra of $\mathrm{End}(A)$. The following statement is a particular case of a general result from [Bl, Proposition 1.2]. (cf. [Bl]). Let $S = A \otimes B$ be the tensor product of finite-dimensional unital associative algebras. Then

$$\mathrm{Der}S \cong (\mathrm{Der}A) \otimes C(B) + C(A) \otimes (\mathrm{Der}B).$$

We will only use this result for commutative unital associative algebras, in which case the centroid coincides with the algebra itself. Thus for commutative associative algebras A, B one has:

Theorem 8.2. *For commutative associative algebras* A, B*,*

$$DerS \cong (DerA) \otimes B + A \otimes (DerB). \tag{8.1}$$

This formula will be used in what follows.

For an ideal J in an analytic algebra S, denote by $\mathrm{Der}_J S \subseteq \mathrm{Der}_{\mathbb{C}} S$ the Lie subalgebra of all $\sigma \in \mathrm{Der}_{\mathbb{C}} S$ for which $\sigma(J) \subset J$. We shall use the following well-known result to compute the derivations.

Theorem 8.3. *Let* J *be an ideal in* $R = \mathbb{C}\{x_1, \dots, x_n\}$*. Then there is a natural isomorphism of Lie algebras*

$$(Der_J R)/(J \cdot Der_{\mathbb{C}} R) \cong Der_{\mathbb{C}}(R/J).$$

Proof. By definition, there is a map $\varphi : \mathrm{Der}_J R \to \mathrm{Der}_{\mathbb{C}}(R/J)$ whose kernel contains $J \cdot \mathrm{Der}_{\mathbb{C}} R$. Note that $\mathrm{Der}_{\mathbb{C}} R$ is a free R-module with basis $\partial/\partial x_1, \dots, \partial/\partial x_n$ and that the coefficient of $\partial/\partial x_i$ in $\sigma \in \mathrm{Der}_{\mathbb{C}} R$ is $\sigma(x_i)$. So if $\sigma \in \mathrm{Ker}\varphi$, then $\sigma(x_i) \in J$, and hence $\sigma \in J \cdot \mathrm{Der}_{\mathbb{C}} R$. This proves injectivity. By a result of Scheja and Wiebe [SW], any $\bar{\sigma} \in \mathrm{Der}_{\mathbb{C}}(R/J)$ lifts to a $\sigma \in \mathrm{Der}_{\mathbb{C}} R$, which is then necessarily in $\mathrm{Der}_J R$. This proves surjectivity, and the claim follows. □

9. Weighted homogenous fewnomial isolated singularities

We first recall the setting of the so-called fewnomials introduced in [Kho]. Let us first establish precise terminology which will be different from the setting of [Kho], where the term fewnomial was first introduced. Let P be a polynomial in n variables. We shall say that P is a fewnomial if the number of monomials appearing in P does not exceed n. Obviously, the number of monomials in P may depend on the system of coordinates. In order to obtain a rigorous concept we shall only allow linear changes of coordinates and say that P (or rather its germ at the origin) is a k-nomial if k is the smallest natural number such that P becomes a k-nomial after (possibly) a linear change of coordinates. For linguistic flexibility it is convenient to say in such a case that the nomiality of P, abbreviated as nom P, is equal to k. Nomiality may be considered as a sort of elementary complexity measure of polynomials which appears to be relevant in some problems of enumerative algebraic geometry [Kho]. An isolated hypersurface singularity $(V, 0)$ is called k-nomial if there exists an IHS Y analytically isomorphic to V which can be defined by a k-nomial and k is the smallest such number. It turns out that, except for some non-interesting cases, a singularity defined by a fewnomial P can be isolated only if nom $P = n$, i.e., if P is a n-nomial in n variables. We formulate this result separately for further reference.

Lemma 9.1 ([Sa1]). *We fix an index $i \in \{1, \dots, n\}$. For an isolated singularity f, at least one of the monomials of the form $x_i^a x_j, a \geq 1, j = 1, \dots, n$, appears in the series f with a non-zero coefficient.*

Lemma 9.2 ([Khi]). *A k-nomial P in n variables which does not contain monomials of degree less than three cannot have an isolated critical point at the origin if $k < n$.*

Proof. By Lemma 9.1, for each i, there exists a monomial $x_i^a x_j$ in P. For each i, fix a monomial of such form with the minimal $j = j(i)$. Since there are no monomials of degree two, two monomials of such type chosen for two different numbers $i_1 \neq i_2$ cannot coincide. This obviously implies that the number of monomials in P cannot be less than the number of coordinates n. This gives the conclusion. □

Remark 9.1. Using terminology of [AVZ], the requirement that there are no quadratic terms can be expressed by saying that P is of (maximal) corank n at the origin. The reason why we have to exclude quadratic terms is that otherwise the formulation given above would not be correct. Indeed, a stabilization of an A_1 singularity can be defined by a polynomial in $2k$ variables of the form $x_1x_2 + \cdots + x_{2k-1}x_{2k}$ which contains only k monomials. Notice also that Pham polynomials give evident examples of n-nomials with isolated singularity at the origin of $\mathbb{C}^n$.

We introduce some terminology.

Definition 9.1. We say that an IHS in $\mathbb{C}^n$ is a fewnomial if it can be defined by an n-nomial in n variables, and we say that it is a weighted homogenous fewnomial isolated singularity if it can be defined by a weighted homogenous fewnomial.

Notice that a direct sum of weighted homogenous fewnomial isolated singularity is also a weighted homogenous fewnomial isolated singularity. Moreover, according to (8.1) derivation algebras of direct sums can be easily computed. For this reason our strategy will be to prove the main theorem for certain series of weighted homogeneous fewnomial isolated singularities and then extend it to direct sums of singularities from those series. Theorem 9.1 may be deduced from [KS]. For the sake of the convenience to the reader, we include a short and elementary proof below.

Theorem 9.1. *Let f be a weighted homogeneous fewnomial isolated singularity with mult$(f) \geq 3$. Then f is analytically equivalent to a linear combination of the* *following three series:*

Type A. $x_1^{a_1} + x_2^{a_2} + \cdots + x_{n-1}^{a_{n-1}} + x_n^{a_n}$, $n \geq 1$.

Type B. $x_1^{a_1} x_2 + x_2^{a_2} x_3 + \cdots + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n} x_1$, $n \geq 2$.

Type C. $x_1^{a_1} x_2 + x_2^{a_2} x_3 + \cdots + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n}$, $n \geq 2$.

Proof. Let us first introduce a lemma which is a crucial part of the proof of the theorem.

Lemma 9.3. *Let $f(x_1, \ldots, x_n)$ be a weighted homogeneous fewnomial which defines an isolated singularity at origin. Then $x_{i_1}^{a_{i_1}} x_{j(i_1)}$, $x_{i_2}^{a_{i_2}} x_{j(i_2)}$, where $j(i_1) \neq i_1$, $j(i_2) \neq i_2$, $i_1 \neq i_2$, and $j(i_1) = j(i_2)$, cannot appear in the support of f simultaneously.*

Proof. By Lemma 9.1, for any $i \in \{1, \ldots, n\}$, either $x_i^{a_i}, a_i > 1$, or $x_i^{a_i} x_{j(i)}, j(i) \neq i$, appear in the support of f. Since $f(x_1, \ldots, x_n)$ is a fewnomial, and if there exists $i_1, i_2 \in \{1, \cdots, n\}$, such that $x_{i_1}^{a_{i_1}} x_{j(i_1)}$, $x_{i_2}^{a_{i_2}} x_{j(i_2)}$, where $j(i_1) \neq i_1$, $j(i_2) \neq i_2$, $i_1 \neq i_2$, and $j(i_1) = j(i_2)$, appear in the Supp(f), then $f = c_1 x_{i_1}^{a_{i_1}} x_{j(i_1)} + c_2 x_{i_2}^{a_{i_2}} x_{j(i_2)} + \sum_{k=3}^{n} c_k x_{i_k}^{a_{i_k}} x_{j(i_k)}$, where $c_i \neq 0$, $1 \leq i \leq n$, and $i_k \in \{1, \cdots, n\} \backslash \{i_1, i_2\}$, $3 \leq i \leq n$. We claim that f can't have isolated singularities. This is because when we calculate $\partial f / \partial x_i$, $1 \leq i \leq n$, and set x_{i_k}, $3 \leq k \leq n$, to 0, we end up with at most one non-trivial equation $\partial f / \partial x_{j(i_1)} = 0$ for the 2 variables $\{x_{i_1}, x_{i_2}\}$, because $j(i_1) \neq i_1$, $j(i_2) \neq i_2$, and $j(i_1) = j(i_2)$, so $x_{j(i_1)} = x_{j(i_2)} = 0$. Therefore f does not define an isolated singularity, which contradicts the hypothesis. □

Therefore Theorem 9.1 is an immediate corollary of Lemmas 9.1, 9.2, and 9.3 up to non-zero coefficients. We can then rescale so all the coefficients can be 1. □

10. Proof of Main Theorem C

Proposition 10.1. *Let $f = x_1^{a_1} + x_2^{a_2} + \cdots + x_n^{a_n}$ be a weighted homogeneous fewnomial isolated singularity of type A and let $A(f) = \mathbb{C}\{x_1, \ldots, x_n\}/(f_{x_1}, f_{x_2}, \ldots, f_{x_n})$ be the moduli algebra of f. Then there are no non-zero negative weight derivations on $A(f)$.*

Proof. Since

$$\begin{aligned} A(f) :=& \mathbb{C}\{x_1, \ldots, x_n\}/(f_{x_1}, f_{x_2}, \ldots, f_{x_n}) \\ =& \mathbb{C}\{x_1, \ldots, x_n\}/(a_1 x_1^{a_1 - 1}, a_2 x_2^{a_2 - 1}, \ldots, a_n x_n^{a_n - 1}) \\ \cong& \mathbb{C}\{x_1\}/(x_1^{a_1 - 1}) \otimes \mathbb{C}\{x_2\}/(x_2^{a_2 - 1}) \otimes \cdots \otimes \mathbb{C}\{x_n\}/(x_1^{a_n - 1}). \end{aligned}$$

By (8.1), it suffices to show that $\mathbb{C}\{x\}/(x^{a_k - 1})$ has no negative weight derivations. By Theorem 8.3, it is easy to compute $\mathrm{Der}(\mathbb{C}\{x\}/(x^{a_k - 1}))$ and see that it is spanned by $x^i \partial x$, $1 \leq i \leq a_k - 2$, $1 \leq k \leq n$. Each of these generators has non-negative weight. Thus there are no non-zero negative weight derivations on $A(f)$. □

Proposition 10.2. *Let $f = x_1^{a_1} x_2 + x_2^{a_2} x_3 + \cdots + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n} x_1$, $n \geq 2$, be a weighted homogeneous fewnomial isolated singularity of type B and mult$(f) \geq 4$.*

$$A(f) = \mathbb{C}\{x_1, \ldots, x_n\}/(f_{x_1}, f_{x_2}, \ldots, f_{x_n})$$

is the moduli algebra of f. Then there are no non-zero negative weight derivations on $A(f)$.

In order to prove Proposition 10.2, we first introduce some lemmas.

In the moduli algebra $A(f) = \mathbb{C}\{x_1, x_2, \ldots, x_n\}/(f_{x_1}, f_{x_2}, \ldots, f_{x_n})$ we have the following relations:

$$
\begin{aligned}
x_1^{a_1} &= -a_2 x_2^{a_2-1} x_3,\\
x_2^{a_2} &= -a_3 x_3^{a_3-1} x_4,\\
&\vdots\\
x_i^{a_i} &= -a_{i+1} x_{i+1}^{a_{i+1}-1} x_{i+2},\\
&\vdots\\
x_{n-1}^{a_{n-1}} &= -a_n x_n^{a_n-1} x_1,\\
x_n^{a_n} &= -a_1 x_1^{a_1-1} x_2
\end{aligned}
\tag{10.1}
$$

and

$$
\begin{aligned}
x_1^{a_1} x_2 &= 0,\\
x_2^{a_2} x_3 &= 0,\\
&\vdots\\
x_{n-1}^{a_{n-1}} x_n &= 0,\\
x_n^{a_n} x_1 &= 0.
\end{aligned}
\tag{10.2}
$$

It is easy to see that $A(f) = \mathbb{C}\{x_1, x_2, \ldots, x_n\}/(f_{x_1}, f_{x_2}, \ldots, f_{x_n})$ is a vector space spanned by the monomial basis $\{x_1^{k_1} x_2^{k_2} \cdots x_{n-1}^{k_{n-1}} x_n^{k_n}\}$, where $(k_1, k_2, \ldots, k_n) \in \{0 \leq k_i \leq a_i - 1, 1 \leq i \leq n\}$.

Lemma 10.1. *Let $f = x_1^{a_1} x_2 + x_2^{a_2} x_3 + \cdots + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n} x_1$ be a weighted homogeneous fewnomial isolated singularity of type B and $mult(f) \geq 4$, with positive weights $w_1, \ldots, w_n$. Then $w_i < \frac{d}{3}$ for $1 \leq i \leq n$.*

Proof. Since $mult(f) \geq 4$, $a_i \geq 3$ for $1 \leq i \leq n$. It follows from $aw_i + w_{i+1} = d, a_i \geq 3$ that $w_i < \frac{d}{3}$ for $1 \leq i \leq n$. $\square$

The main idea of the proof of Proposition 10.2 is to use duality (see Theorem 7.1 and Remark 7.2). Since the pairing map in Theorem 7.1 is non-degenerate, it suffices to prove that for any negative weight derivation in $\mathrm{Der}(A)$, its pairing with all elements in $A(f)^n/\mathrm{Hess}(f)A(f)^n$ maps to 0.

First we simplify $A(f)^n/\mathrm{Hess}(f)A(f)^n$. We can see that the Hessian matrix $\mathrm{Hess}(f)$ of f is of the following form:

$$
\begin{pmatrix}
a_1(a_1-1)x_1^{a_1-2}x_2 & a_1x_1^{a_1-1} & 0 & \cdots & 0 & a_nx_n^{a_n-1}\\
a_1x_1^{a_1-1} & a_2(a_2-1)x_2^{a_2-2}x_3 & a_2x_2^{a_2-1} & \cdots & 0 & 0\\
0 & a_2x_2^{a_2-1} & a_3(a_3-1)x_3^{a_3-2}x_4 & \cdots & 0 & 0\\
0 & 0 & a_3x_3^{a_3-1} & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots\\
0 & 0 & 0 & \cdots & a_{n-1}(a_{n-1}-1)x_{n-1}^{a_{n-1}-2}x_n & a_{n-1}x_{n-1}^{a_{n-1}-1}\\
a_nx_n^{a_n-1} & 0 & 0 & \cdots & a_{n-1}x_{n-1}^{a_{n-1}-1} & a_n(a_n-1)x_n^{a_n-2}x_1
\end{pmatrix}
$$

By Theorem 8.1(3), the maximal weight of the monomial base of $A(f)$ is the class of $\mathrm{Det}(\mathrm{Hess}(f))$, which is the socle and has weight $nd - 2(w_1 + w_2 + \cdots + w_n)$.

Then we use relations (10.1) and (10.2) to simplify the $A(f)^n/(\mathrm{Hess}(f)\cdot A(f)^n)$. We get

$$A(f)^n/(\mathrm{Hess}(f)\cdot A(f)^n)\subseteq A(f)/I_1\oplus A(f)/I_2\oplus\cdots\oplus A(f)/I_n$$

where $I_i=\langle x_i^{a_i-1}x_{i+1}^2x_{i-1},\ x_i^{a_i-1}x_{i+1}^2x_{i+2},\ x_{i-2}x_{i-1}x_i^{a_i-1}x_{i+1}\rangle$, $1\le i\le n$, and $x_0=x_n, x_{-1}=x_{n-1}, x_{n+1}=x_1$, and $x_{n+2}=x_2$.

We describe in detail how we get $I_1=\langle x_1^{a_1-1}x_2^2x_n, x_1^{a_1-1}x_2^2x_3, x_{n-1}x_nx_1^{a_1-1}x_2\rangle$. For other I_i, $2\le i\le n$, the argument is similar. In this case, there are only 3 columns with non-zero first coordinate, i.e., the first, second, and last columns. We consider the first column, which is

$$(a_1(a_1-1)x_1^{a_1-2}x_2, a_1x_1^{a_1-1}, 0,\dots,0, a_nx_n^{a_n-1})^T.$$

By relation (10.2), in $A(f)$, $a_1x_1^{a_1-1}$ is killed by x_1x_2 and $a_nx_n^{a_n-1}$ is killed by x_nx_1, so we multiply $\mathrm{lcm}(x_1x_2, x_1x_n)$=$x_1x_2x_n$ with this column to get $(a_1(a_1-1)x_1^{a_1-1}x_2^2x_n, 0,\dots,0)^T$. Similarly, from the second column, we can get $(a_1x_1^{a_1-1}x_2^2x_3,0,\dots,0)^T$, and from the last column we can get $(a_nx_{n-1}x_n^{a_n+1},0,\dots,0)^T$, which is $(a_nx_1^{a_1-1}x_2x_{n-1}x_n, 0,\dots,0)^T$ by relation (10.1). Thus we can take $I_1=\langle x_1^{a_1-1}x_2^2x_n, x_1^{a_1-1}x_2^2x_3, x_{n-1}x_nx_1^{a_1-1}x_2\rangle$.

Proposition 10.3. *Let $f=x_1^{a_1}x_2+x_2^{a_2}x_3+\cdots+x_{n-1}^{a_{n-1}}x_n+x_n^{a_n}x_1$ $n\ge 2$ be a weighted homogeneous fewnomial isolated singularity of type B with positive weights $w_1,\dots,w_n$. Suppose $mult(f)\ge 4$. Then w_i + maximal weight of monomial bases of $A(f)/I_i\le nd-2(w_1+w_2+\cdots+w_n)$, $1\le i\le n$.*

Proof. We give the detailed description for $i=1$. For other indices $2\le i\le n$, the argument is similar.

From the grading on $A(f)/I_1=A(f)/(x_1^{a_1-1}x_2^2x_n, x_1^{a_1-1}x_2^2x_3, x_{n-1}x_nx_1^{a_1-1}x_2)$, since we already know the monomial bases of $A(f)$, it is easy to obtain the candidates for maximal weight of graded bases of $A(f)/I_1$ as follows:

$case(B)$ 1.1. $x_1^{a_1-2}x_2^{a_2-1}\cdots x_n^{a_n-1}$,

$case(B)$ 1.2. $x_1^{a_1-1}x_2x_3^{a_3-1}\cdots x_{n-1}^{a_{n-1}-1}$,

$case(B)$ 1.3. $x_1^{a_1-1}x_2x_3^{a_3-1}\cdots x_{n-2}^{a_{n-2}-1}x_n^{a_n-1}$,

$case(B)$ 1.4. $x_1^{a_1-1}x_2^{a_2-1}x_4^{a_4-1}\cdots x_{n-1}^{a_{n-1}-1}$,

$case(B)$ 1.5. $x_1^{a_1-1}x_3^{a_3-1}\cdots x_{n-1}^{a_{n-1}-1}x_n^{a_n-1}$.

Lemma 10.2. *For case*(B) 1.1–*case*(B) 1.5 *above, the weight of the monomial bases is less than or equal to $nd-2(w_1+\cdots+w_n)-wt(x_1)$.*

Proof. Case(B) 1.1. It is obvious that $wt(x_1^{a_1-2}x_2^{a_2-1}\cdots x_n^{a_n-1})+wt(x_1)=nd-2(w_1+\cdots+w_n)$. So the conclusion is true in this case.

Case(B) 1.2. We want to show that

$$wt(x_1^{a_1-1}x_2x_3^{a_3-1}\cdots x_{n-1}^{a_{n-1}-1})+wt(x_1)\le nd-2(w_1+\cdots+w_n),$$

i.e., $(a_1-1)w_1+w_2+(a_3-1)w_3+\cdots+(a_{n-1}-1)w_{n-1}+w_1\le nd-2(w_1+\cdots+w_n)$. It follows that $2w_1+2w_2+w_3+w_n\le 2d$ from $a_iw_i+w_{i+1}=d$, for $1\le i\le n$. We need to show that $2w_1+2w_2+w_3+w_n\le 2d$. This follows from Lemma 10.1.

Case(B) 1.3. We want to show that $wt(x_1^{a_1-1}x_2x_3^{a_3-1}\cdots x_{n-2}^{a_{n-2}-1}x_n^{a_n-1})+wt(x_1)$ $\le nd-2(w_1+\cdots+w_n)$. Simplifying it as in case 1.2 we get $w_1+2w_2+w_3+$

$w_{n-1}+w_n \leq 2d$. We need to show that $w_1+2w_2+w_3+w_{n-1}+w_n \leq 2d$. This follows from Lemma 10.1.

Case(B) 1.4. We want to show that $wt(x_1^{a_1-1}x_2^{a_2-1}x_4^{a_4-1}\cdots x_{n-1}^{a_{n-1}-1})+wt(x_1) \leq nd-2(w_1+\cdots+w_n)$. Simplifying it as in case 1.2 we get $2w_1+w_3+w_4+w_n \leq 2d$. We need to show that $2w_1+w_3+w_4+w_n \leq 2d$. This follows from Lemma 10.1.

Case(B) 1.5. We want to show that $wt(x_1^{a_1-1}x_3^{a_3-1}\cdots x_{n-1}^{a_{n-1}-1}x_n^{a_n-1})+wt(x_1) \leq nd-2(w_1+\cdots+w_n)$. Simplifying it as in case 1.2 we get $w_1+w_2+w_3 \leq d$. We need to show that $w_1+w_2+w_3 \leq d$. This follows from Lemma 10.1. □

For $2 \leq i \leq n$, the argument is similar to the above. The candidates for maximal weight of graded bases of

$$A(f)/I_i = A(f)/(x_{i-1}x_i^{a_i-1}x_{i+1}^2, x_i^{a_i-1}x_{i+1}^2x_{i+2}, x_{i-2}x_{i-1}x_i^{a_i-1}x_{i+1}),\ 2 \leq i \leq n$$

are as follows:

$$\begin{aligned}
&case(B)\, i.1.\ x_1^{a_1-1}\cdots x_{i-1}^{a_{i-1}-1}x_i^{a_i-2}x_{i+1}^{a_{i+1}-1}\cdots x_n^{a_n-1},\\
&case(B)\, i.2.\ x_i^{a_i-1}x_{i+1}x_{i+2}^{a_{i+2}-1}\cdots x_{i+n-2}^{a_{i+n-2}-1},\\
&case(B)\, i.3.\ x_i^{a_i-1}x_{i+1}x_{i+2}^{a_{i+2}-1}\cdots x_{i+n-3}^{a_{i+n-3}-1}x_{i+n-1}^{a_{i+n-1}-1},\\
&case(B)\, i.4.\ x_i^{a_i-1}x_{i+1}^{a_{i+1}-1}x_{i+3}^{a_{i+3}-1}\cdots x_{i+n-2}^{a_{i+n-2}-1},\\
&case(B)\, i.5.\ x_1^{a_1-1}\cdots x_{i-1}^{a_{i-1}-1}x_i^{a_i-1}x_{i+2}^{a_{i+2}-1}\cdots x_n^{a_n-1}.
\end{aligned}$$

Remark 10.1. Here $a_i = a_j$ and $x_i = x_j$ for $i \equiv j \pmod n$.

Lemma 10.3. *For case(B) i.1–case(B) i.5 above, the weight of the monomial bases is less than or equal to $nd-2(w_1+\cdots+w_n)-wt(x_1)$.*

Proof. The proof is the same as the above for Lemma 10.2. □

Proposition 10.3 follows from Lemma 10.3 immediately. □

Now we can give the proof of Proposition 10.2.

Proof. Let $D \in \mathrm{Der}(A(f))$ be a negative weight derivation. We can write $D = \sum_{i=1}^n g_i\partial x_i$, where $g_i \in A(f)$ and $wt(g_i) < w_i$. By Proposition 10.3, we have $wt(g_i)$+maximal weight of monomial bases of

$$A(f)/I_i < nd-2(w_1+w_2+\cdots+w_n),\ 1 \leq i \leq n.$$

By Theorem 8.1(3), the weight of socle is $nd-2(w_1+w_2+\cdots+w_n)$. Thus $wt(g_i)$+maximal weight of monomial bases of $A(f)/I_i < wt(socle)$, which means the projection map (7.2) is zero. By Theorem 7.1 we conclude that $D = 0$. Thus $\mathrm{Der}(A(f))$ has no negative weight derivation. □

Proposition 10.4. *Let $f = x_1^{a_1}x_2+x_2^{a_2}x_3+\cdots+x_{n-1}^{a_{n-1}}x_n+x_n^{a_n}$, $n \geq 2$, be a weighted homogeneous fewnomial isolated singularity of type C. Suppose $mult(f) \geq 5$ and n is even. $A(f) = \mathbb{C}\{x_1,\ldots,x_n\}/(f_{x_1},f_{x_2},\ldots,f_{x_n})$ is the moduli algebra of f. Then there are no non-zero negative weight derivations on $A(f)$.*

Remark 10.2. Since our main theorem is true for $2 \leq n \leq 4$ (see [Ch1]), we can assume $n \geq 5$ in the proof of Propositions 10.4 and 10.5. However, for $n \leq 4$ the proof is almost the same.

In the moduli algebra $A(f) = \mathbb{C}\{x_1, x_2, \ldots, x_n\}/(f_{x_1}, f_{x_2}, \ldots, f_{x_n})$ we have the following relations:

$$
\begin{aligned}
x_1^{a_1} &= -a_2 x_2^{a_2-1} x_3,\\
x_2^{a_2} &= -a_3 x_3^{a_3-1} x_4,\\
&\ \ \vdots\\
x_i^{a_i} &= -a_{i+1} x_{i+1}^{a_{i+1}-1} x_{i+2},\\
&\ \ \vdots\\
x_{n-2}^{a_{n-2}} &= -a_{n-1} x_{n-1}^{a_{n-1}-1} x_n,\\
x_{n-1}^{a_{n-1}} &= -a_n x_n^{a_n-1}
\end{aligned}
\tag{10.3}
$$

and

$$
\begin{aligned}
x_1^{a_1-1} x_2 &= 0,\\
x_2^{a_2} x_3 &= 0,\\
&\ \ \vdots\\
x_{n-1}^{a_{n-1}} x_n &= 0,\\
x_n^{a_n} &= 0.
\end{aligned}
\tag{10.4}
$$

It is also well known that $A(f) = \mathbb{C}\{x_1, x_2, \ldots, x_n\}/(f_{x_1}, f_{x_2}, \ldots, f_{x_n})$ is a vector space spanned by the monomial basis $\{x_1^{k_1} x_2^{k_2} \cdots x_{n-1}^{k_{n-1}} x_n^{k_n}\}$, where $(k_1, k_2, \ldots, k_n) \in \{0 \le k_1 \le a_1 - 2, 0 \le k_j \le a_j - 1 \text{ for } 2 \le j \le n\} \cup \{k_1 = a_1 - 1, k_2 = 0, 0 \le k_3 \le a_3 - 2, 0 \le k_j \le a_j - 1 \text{ for } 4 \le j \le n\} \cup \cdots \cup \{k_{2j-1} = a_{2j-1} - 1, k_{2j} = 0, \text{ for } 1 \le j \le i \le \frac{n}{2}, 0 \le k_{2i+1} \le a_{2i+1} - 2, 0 \le k_j \le a_j - 1, \text{ for } 1 \le 2i+1 < j \le n\} \cup \cdots \cup \{k_{2j-1} = a_{2j-1} - 1 \text{ and } k_{2j} = 0 \text{ for } 1 \le j \le \frac{n}{2}\}$.

Hess(f) is of the following form:

$$
\begin{pmatrix}
a_1(a_1-1)x_1^{a_1-2}x_2 & a_1x_1^{a_1-1} & 0 & \cdots & 0 & 0\\
a_1x_1^{a_1-1} & a_2(a_2-1)x_2^{a_2-2}x_3 & a_2x_2^{a_2-1} & \cdots & 0 & 0\\
0 & a_2x_2^{a_2-1} & a_3(a_3-1)x_3^{a_3-2}x_4 & \cdots & 0 & 0\\
0 & 0 & a_3x_3^{a_3-1} & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots\\
0 & 0 & 0 & \cdots & a_{n-2}x_{n-2}^{a_{n-2}-1} & 0\\
0 & 0 & 0 & \cdots & a_{n-1}(a_{n-1}-1)x_{n-1}^{a_{n-1}-2}x_n & a_{n-1}x_{n-1}^{a_{n-1}-1}\\
0 & 0 & 0 & \cdots & a_{n-1}x_{n-1}^{a_{n-1}-1} & a_n(a_n-1)x_n^{a_n-2}
\end{pmatrix}
$$

Lemma 10.4. *Let $f = x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}$, $n \ge 2$, be a weighted homogeneous fewnomial isolated singularity of type C with positive weights $w_1, \ldots, w_n$.*

Suppose $mult(f) \geq 3$ and n is even. Then

$$\begin{aligned}
&wt(x_1^{a_1-2}x_2^{a_2-1}x_3^{a_3-1}x_4^{a_4-1}x_5^{a_5-1}x_6^{a_6-1}\cdots x_{n-1}^{a_{n-1}-1}x_n^{a_n-1})\\
&\geq wt(x_1^{a_1-1}x_2^{0}x_3^{a_3-2}x_4^{a_4-1}x_5^{a_5-1}x_6^{a_6-1}\cdots x_{n-1}^{a_{n-1}-1}x_n^{a_n-1})\\
&\geq wt(x_1^{a_1-1}x_2^{0}x_3^{a_3-1}x_4^{0}x_5^{a_5-2}x_6^{a_6-1}\cdots x_{n-1}^{a_{n-1}-1}x_n^{a_n-1})\\
&\vdots\\
&\geq wt(x_1^{a_1-1}x_2^{0}x_3^{a_3-1}x_4^{0}x_5^{a_5-1}x_6^{0}\cdots x_{n-3}^{a_{n-3}-1}x_{n-2}^{0}x_{n-1}^{a_{n-1}-2}x_n^{a_n-1})\\
&\geq wt(x_1^{a_1-1}x_2^{0}x_3^{a_3-1}x_4^{0}x_5^{a_5-1}x_6^{0}\cdots x_{n-3}^{a_{n-3}-1}x_{n-2}^{0}x_{n-1}^{a_{n-1}-1}x_n^{0}).
\end{aligned}$$

Remark 10.3. All the monomials above are monomial bases in $A(f)$.

Proof. We just check the first inequality; the other inequalities are similar.

$$\begin{aligned}
&wt(x_1^{a_1-2}x_2^{a_2-1}x_3^{a_3-1}x_4^{a_4-1}x_5^{a_5-1}x_6^{a_6-1}\cdots x_{n-1}^{a_{n-1}-1}x_n^{a_n-1})\\
&\qquad - wt(x_1^{a_1-1}x_2^{0}x_3^{a_3-2}x_4^{a_4-1}x_5^{a_5-1}x_6^{a_6-1}\cdots x_{n-1}^{a_{n-1}-1}x_n^{a_n-1})\\
&\quad = -w_1 + (a_2-1)w_2 + w_3 = d - w_1 - w_2.
\end{aligned}$$

Since $mult(f) \geq 3$, $w_i < \frac{d}{2}, 1 \leq i \leq n$, then $d - w_1 - w_2 \geq 0$. Thus the first inequality is satisfied. □

Lemma 10.5. *Let $f = x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}$, $n \geq 2$, be a weighted homogeneous fewnomial isolated singularity of type with positive weights $w_1, \ldots, w_n$. Suppose $mult(f) \geq 4$ and n is even. Then for any $1 \leq i \leq n$,*

$$\begin{aligned}
&w_i + wt(x_1^{a_1-1}x_2^{0}x_3^{a_3-2}x_4^{a_4-1}x_5^{a_5-1}x_6^{a_6-1}\cdots x_{n-1}^{a_{n-1}-1}x_n^{a_n-1}) \leq nd - 2(w_1 + \cdots + w_n),\\
&w_i + wt(x_1^{a_1-1}x_2^{0}x_3^{a_3-1}x_4^{0}x_5^{a_5-2}x_6^{a_6-1}\cdots x_{n-1}^{a_{n-1}-1}x_n^{a_n-1}) \leq nd - 2(w_1 + \cdots + w_n),\\
&\vdots\\
&w_i + wt(x_1^{a_1-1}x_2^{0}x_3^{a_3-1}x_4^{0}x_5^{a_5-1}x_6^{0}\cdots x_{n-3}^{a_{n-3}-1}x_{n-2}^{0}x_{n-1}^{a_{n-1}-2}x_n^{a_n-1})\\
&\qquad \leq nd - 2(w_1 + \cdots + w_n),\\
&w_i + wt(x_1^{a_1-1}x_2^{0}x_3^{a_3-1}x_4^{0}x_5^{a_5-1}x_6^{0}\cdots x_{n-3}^{a_{n-3}-1}x_{n-2}^{0}x_{n-1}^{a_{n-1}-1}x_n^{0})\\
&\qquad \leq nd - 2(w_1 + \cdots + w_n).
\end{aligned}$$

Proof. Notice that $wt(x_1^{a_1-2}x_2^{a_2-1}x_3^{a_3-1}x_4^{a_4-1}x_5^{a_5-1}x_6^{a_6-1}\cdots x_{n-1}^{a_{n-1}-1}x_n^{a_n-1}) = nd - 2(w_1 + \cdots + 2w_n)$. By Lemma 10.4, it suffices to show that, for any $1 \leq i \leq n$,

$$\begin{aligned}
&w_i + wt(x_1^{a_1-1}x_2^{0}x_3^{a_3-2}x_4^{a_4-1}x_5^{a_5-1}x_6^{a_6-1}\cdots x_{n-1}^{a_{n-1}-1}x_n^{a_n-1})\\
&\leq wt(x_1^{a_1-2}x_2^{a_2-1}x_3^{a_3-1}x_4^{a_4-1}\cdots x_{n-1}^{a_{n-1}-1}x_n^{a_n-1}) = nd - 2(w_1 + \cdots + 2w_n), \qquad (10.5)
\end{aligned}$$

i.e., $d - (w_i + w_1 + w_2) \leq 0$. Since $mult(f) \geq 4$, $a_i \geq 3$, then $w_i < \frac{d}{3}$. Thus $d - (w_i + w_1 + w_2) \leq 0$. □

Remark 10.4. Lemma 10.5 will help us to prove Proposition 10.5. Explicitly they will be used to determine the candidates of maximal graded pieces in $A(f)^n/\text{Hess}(f)A(f)^n$. Lemma 10.5 tells us we don't need to consider the monomial bases of $A(f)$ of the above forms.

Lemma 10.6. *Let $f = x_1^{a_1}x_2+x_2^{a_2}x_3+\cdots+x_{n-1}^{a_{n-1}}x_n+x_n^{a_n}$, $n \geq 2$, be a weighted homogeneous fewnomial isolated singularity of type C with positive weights $w_1, \ldots, w_n$ satisfying $mult(f) \geq 5$. $A(f) = \mathbb{C}\{x_1, \ldots, x_n\}/(f_{x_1}, f_{x_2}, \ldots, f_{x_n})$ is the moduli algebra of f. Then $w_i < \frac{d}{4}$ for all $1 \leq i \leq n-1$ and $w_n \leq \frac{d}{5}$.*

Proof. From a similar argument as in Lemma 10.1, it follows from the definition of the weights. □

Proposition 10.5. *Let $f = x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}$, $n \geq 2$, be a weighted homogeneous fewnomial isolated singularity of type C with positive weights $w_1, \ldots, w_n$. Suppose $mult(f) \geq 5$ and n is even.*

$$A(f) = \mathbb{C}\{x_1, \ldots, x_n\}/(f_{x_1}, f_{x_2}, \ldots, f_{x_n})$$

is the moduli algebra of f. Then for any $1 \leq i \leq n$, w_i, + maximal weight of monomial bases of $A(f)/I_i \leq nd - 2(w_1 + w_2 + \cdots + w_n)$.

Before giving the proof of Proposition 10.5, we need to show several lemmas.

Similarly as before, we can use the above relations (10.3) and (10.4) to simplify

$$A(f)^n/\mathrm{Hess}(f)A(f)^n.$$

We obtain (notice that we assume $n \geq 5$)

$$A(f)^n/\mathrm{Hess}(f)A(f)^n \subseteq A(f)/I_1 \oplus A(f)/I_2 \oplus \cdots \oplus A(f)/I_n,$$

where $I_1 = \langle x_1^{a_1-2}x_2^2\rangle$, $I_2 = \langle x_2^{a_2-1}x_3\rangle$, $I_3 = \langle x_2x_3^{a_3-1}x_4, x_3^{a_3-1}x_4^2x_5\rangle$, $I_i = \langle x_{i-1}x_i^{a_i-1}x_{i+1}^2,\ x_{i-2}x_{i-1}x_i^{a_i-1}x_{i+1},\ x_i^{a_i-1}x_{i+1}^2x_{i+2}\rangle$ where $4 \leq i \leq n-2$, $I_{n-1} = \langle x_{n-1}^{a_{n-1}-1}x_n^2,\ x_{n-3}x_{n-2}x_{n-1}^{a_{n-1}-1}x_n\rangle$, and $\langle I_n =< x_{n-1}x_n^{a_n-1}\rangle$.

For $I_1 = \langle x_1^{a_1-2}x_2^2\rangle$, by Lemmas 10.4 and 10.5, the candidates for maximal weight of bases of $A(f)/I_1$ are as follows:

case(C) 1.1. $x_1^{a_1-3}x_2^{a_2-1}\cdots x_n^{a_n-1}$,

case(C) 1.2. $x_1^{a_1-2}x_2x_3^{a_3-1}\cdots x_n^{a_n-1}$,

case(C) 1.3. $x_1^{a_1-1}x_3^{a_3-2}x_4^{a_4-1}\cdots x_n^{a_n-1}$.

Lemma 10.7. *Under the same assumption as in Proposition 10.5, for case(C) 1.1–case(C) 1.3 above, the weights of the monomial bases are less than or equal to $nd - 2(w_1 + w_2 + \cdots + w_n) - wt(x_1)$.*

Proof. Case(C) 1.1. It is obvious that $wt(x_1^{a_1-3}x_2^{a_2-1}\cdots x_n^{a_n-1}) + wt(x_1) = nd - 2(w_1 + \cdots + w_n)$. So the conclusion is true in this case.

Case(C) 1.2. Notice that in this case, we want to show that

$$wt(x_1^{a_1-2}x_2x_3^{a_3-1}\cdots x_n^{a_n-1}) + wt(x_1) \leq nd - 2(w_1 + \cdots + w_n).$$

It follows that $w_1 + 2w_2 + w_3 \leq d$ from $a_iw_i + w_{i+1} = d$, for $1 \leq i \leq n-1$ and $a_nw_n = d$. We need to show that $w_1 + 2w_2 + w_3 \leq d$. This follows from Lemma 10.6.

Case(C) 1.3. We want to show that $wt(x_1^{a_1-1}x_3^{a_3-2}x_4^{a_4-1}\cdots x_n^{a_n-1})+wt(x_1) \leq nd - 2(w_1 + \cdots + w_n)$. Simplifying it as in case 1.2 we get $2w_1 + w_2 \leq d$. This is obviously true. □

For $I_2 = \langle x_2^{a_2-1}x_3\rangle$, by Lemmas 10.4 and 10.5, the candidates for maximal weight of basis of $A(f)/I_2$ are as follows:

$$case(C)\ 2.1.\ x_1^{a_1-2}x_2^{a_2-2}x_3^{a_3-1}\cdots x_n^{a_n-1},$$
$$case(C)\ 2.2.\ x_1^{a_1-2}x_2^{a_2-1}x_4^{a_4-1}\cdots x_n^{a_n-1},$$
$$case(C)\ 2.3.\ x_1^{a_1-1}x_3^{a_3-2}x_4^{a_4-1}.\cdots x_n^{a_n-1}.$$

Lemma 10.8. *Under the same assumption as in Proposition* 10.5*, for case*(C) 2.1*–case*(C) 2.3 *above, the weights of the monomial bases are less than or equal to* $nd - 2(w_1 + w_2 + \cdots + w_n) - wt(x_2)$*.*

Proof. Case(C) 2.1. It is obvious that $wt(x_1^{a_1-2}x_2^{a_2-2}x_3^{a_3-1}\cdots x_n^{a_n-1}) + wt(x_2) = nd - 2(w_1 + \cdots + w_n)$. So the conclusion is true in this case.

Case(C) 2.2. Notice that in this case, we want to show that

$$wt(x_1^{a_1-2}x_2^{a_2-1}x_4^{a_4-1}\cdots x_n^{a_n-1}) + wt(x_2) \leq nd - 2(w_1 + \cdots + w_n).$$

We need to show that $w_2 + w_3 + w_4 \leq d$. This follows from Lemma 10.6.

Case(C) 2.3. We want to show that $wt(x_1^{a_1-1}x_3^{a_3-2}x_4^{a_4-1}\cdots x_n^{a_n-1})+wt(x_2) \leq nd - 2(w_1 + \cdots + w_n)$. Simplifying it as in case(C) 2.2 we get $w_1 + 2w_2 \leq d$. This follows from Lemma 10.6. □

For $I_3 = \langle x_2x_3^{a_3-1}x_4,\ x_3^{a_3-1}x_4^2x_5\rangle$, the candidates for maximal weight of basis of $A(f)/I_3$ are as follows:

$$case(C)\ 3.1.\ x_1^{a_1-2}x_2^{a_2-1}x_3^{a_3-2}x_4^{a_4-1}\cdots x_n^{a_n-1},$$
$$case(C)\ 3.2.\ x_1^{a_1-2}x_2^{a_2-1}x_3^{a_3-1}x_5^{a_5-1}\cdots x_n^{a_n-1},$$
$$case(C)\ 3.3.\ x_1^{a_1-2}x_3^{a_3-1}x_4^{a_4-1}x_6^{a_6-1}x_7^{a_7-1}\cdots x_n^{a_n-1}.$$

Lemma 10.9. *Under the same assumption as in Proposition* 10.5*, for case*(C)3.1*–case*(C) 3.3 *above, the weights of the monomial bases are less than or equal to* $nd - 2(w_1 + w_2 + \cdots + w_n) - wt(x_3)$*.*

Proof. Case(C) 3.1. It is obvious that $wt(x_1^{a_1-2}x_2^{a_2-1}x_3^{a_3-2}x_4^{a_4-1}\cdots x_n^{a_n-1})+wt(x_3) = nd - 2(w_1 + \cdots + w_n)$. So the conclusion is true in this case.

Case(C) 3.2. Note that in this case, we want to show that

$$wt(x_1^{a_1-2}x_2^{a_2-1}x_3^{a_3-1}x_5^{a_5-1}\cdots x_n^{a_n-1}) + wt(x_3) \leq nd - 2(w_1 + \cdots + w_n).$$

It follows that $w_1 + 2w_2 + w_3 \leq d$, which is correct following from Lemma 10.6.

Case(C) 3.3. We want to show that $wt(x_1^{a_1-2}x_3^{a_3-1}x_4^{a_4-1}x_6^{a_6-1}x_7^{a_7-1}\cdots x_n^{a_n-1}) + wt(x_3) \leq nd - 2(w_1 + \cdots + w_n)$. Simplifying it as in case 3.2 we get $w_2 + 2w_3 + w_5 + w_6 \leq 2d$, which is correct following from Lemma 10.6. □

For $I_4 = \langle x_3x_4^{a_4-1}x_5^2,\ x_2x_3x_4^{a_4-1}x_5,\ x_4^{a_4-1}x_5^2x_6\rangle$, the candidates for maximal weight of basis of $A(f)/I_4$ are as follows:

$$case(C)\ 4.1.\ x_1^{a_1-2}x_2^{a_2-1}x_3^{a_3-1}x_4^{a_4-2}x_5^{a_5-1}\cdots x_n^{a_n-1},$$
$$case(C)\ 4.2.\ x_1^{a_1-2}x_3^{a_3-1}x_4^{a_4-1}x_5x_6^{a_6-1}\cdots x_n^{a_n-1},$$
$$case(C)\ 4.3.\ x_1^{a_1-2}x_2^{a_2-1}x_4^{a_4-1}x_5x_6^{a_6-1}\cdots x_n^{a_n-1},$$
$$case(C)\ 4.4.\ x_1^{a_1-2}x_2^{a_2-1}x_3^{a_3-1}x_4^{a_4-1}x_6^{a_6-1}\cdots x_n^{a_n-1},$$
$$case(C)\ 4.5.\ x_1^{a_1-2}x_2^{a_2-1}x_4^{a_4-1}x_5^{a_5-1}x_7^{a_7-1}\cdots x_n^{a_n}.$$

Lemma 10.10. *Under the same assumption as in Proposition* 10.5*, for case*(C) 4.1–*case*(C) 4.3 *above, the weights of the monomial bases are less than or equal to* $nd-2(w_1+w_2+\cdots+w_n)-wt(x_4)$.

Proof. Case(C) 4.1. It is obvious that $wt(x_1^{a_1-2}x_2^{a_2-1}x_3^{a_3-1}x_4^{a_4-2}x_5^{a_5-1}\cdots x_n^{a_n-1})+wt(x_4)=nd-2(w_1+\cdots+w_n)$. So the conclusion is true in this case.

Case(C) 4.2. Note that in this case, we want to show that

$$wt(x_1^{a_1-2}x_3^{a_3-1}x_4^{a_4-1}x_5x_6^{a_6-1}\cdots x_n^{a_n-1})+wt(x_4)\leq nd-2(w_1+\cdots+w_n).$$

It follows that we need to show that $w_2+w_3+w_4+2w_5+w_6\leq 2d$, which is correct following from Lemma 10.6.

Case(C) 4.3. We want to show that

$$wt(x_1^{a_1-2}x_2^{a_2-1}x_4^{a_4-1}x_5x_6^{a_6-1}\cdots x_n^{a_n-1})+wt(x_4)\leq nd-2(w_1+\cdots+w_n).$$

Simplifying it as in case 4.2 we get $w_3+2w_4+2w_5+w_6\leq d$, which is correct following from Lemma 10.6.

Case(C) 4.4. We want to show that $wt(x_1^{a_1-2}x_2^{a_2-1}x_3^{a_3-1}x_4^{a_4-1}x_6^{a_6-1}\cdots x_n^{a_n-1})$ $+wt(x_4)\leq nd-2(w_1+\cdots+w_n)$. Simplifying it as in case 4.2 we get $w_4+w_5+w_6\leq d$, which is correct following from Lemma 10.6.

Case(C) 4.5. We want to show that $wt(x_1^{a_1-2}x_2^{a_2-1}x_4^{a_4-1}x_5^{a_5-1}x_7^{a_7-1}\cdots x_n^{a_n-1})$ $+wt(x_4)\leq nd-2(w_1+\cdots+w_n)$. Simplifying it as in case 4.2 we get $w_3+2w_4+w_6+w_7\leq 2d$, which is correct following from Lemma 10.6. □

For $I_i=\langle x_{i-1}x_i^{a_i-1}x_{i+1}^2,\, x_{i-2}x_{i-1}x_i^{a_i-1}x_{i+1},\, x_i^{a_i-1}x_{i+1}^2x_{i+2}\rangle$, where $4\leq i\leq n-2$, the candidates for maximal weight of bases of $A(f)/I_i$ are as follows:

case$(C)\,i.1.$ $x_1^{a_1-2}x_2^{a_2-1}\cdots x_{i-1}^{a_{i-1}-1}x_i^{a_i-2}x_{i+1}^{a_{i+1}-1}\cdots x_n^{a_n-1}$,

case$(C)\,i.2.$ $x_1^{a_1-2}x_2^{a_2-1}\cdots x_{i-3}^{a_{i-3}-1}x_{i-2}x_{i-1}^{a_{i-1}-1}x_i^{a_i-1}x_{i+1}x_{i+2}^{a_{i+2}-1}\cdots x_n^{a_n-1}$,

case$(C)\,i.3.$ $x_1^{a_1-2}x_2^{a_2-1}\cdots x_{i-2}^{a_{i-2}-1}x_i^{a_i-1}x_{i+1}x_{i+2}^{a_{i+2}-1}\cdots x_n^{a_n-1}$,

case$(C)\,i.4.$ $x_1^{a_1-2}x_2^{a_2-1}\cdots x_{i-1}^{a_{i-1}-1}x_i^{a_i-1}x_{i+2}^{a_{i+2}-1}\cdots x_n^{a_n-1}$,

case$(C)\,i.5.$ $x_1^{a_1-2}x_2^{a_2-1}\cdots x_{i-2}^{a_{i-2}-1}x_i^{a_i-1}x_{i+1}^{a_{i+1}-1}x_{i+3}^{a_{i+3}-1}\cdots x_n^{a_n-1}$.

Lemma 10.11. *Under the same assumption as in Proposition* 10.5*, for case*(C) $i.1$–*case*(C) $i.5$ *above, the weight of the monomial bases is less than or equal to* $nd-2(w_1+w_2+\cdots+w_n)-wt(x_i)$*, where* $4\leq i\leq n-2$.

Proof. The proof is the same as I_4, with just a little adjustment of the indices. □

For $I_{n-1}=\langle x_{n-1}^{a_{n-1}-1}x_n^2,\, x_{n-3}x_{n-2}x_{n-1}^{a_{n-1}-1}x_n\rangle$, the candidates for maximal weight of basis of $A(f)/I_{n-1}$ are as follows:

case$(C)\,(n-1).1.$ $x_1^{a_1-2}x_2^{a_2-1}\cdots x_{n-2}^{a_{n-2}-1}x_{n-1}^{a_{n-1}-2}x_n^{a_n-1}$,

case$(C)\,(n-1).2.$ $x_1^{a_1-2}x_2^{a_2-1}\cdots x_{n-3}^{a_{n-3}-1}x_{n-1}^{a_{n-1}-1}x_n$,

case$(C)\,(n-1).3.$ $x_1^{a_1-2}x_2^{a_2-1}\cdots x_{n-4}^{a_{n-4}-1}x_{n-2}^{a_{n-2}-1}x_{n-1}^{a_{n-1}-1}x_n$,

case$(C)\,(n-1).4.$ $x_1^{a_1-2}x_2^{a_2-1}\cdots x_{n-2}^{a_{n-2}-1}x_{n-1}^{a_{n-1}-1}$.

Lemma 10.12. *Under the same assumption as in Proposition* 10.5*, for case*(C) $(n-1).1$–*case*(C) $(n-1).3$ *above, the weights of the monomial bases are less than or equal to* $nd-2(w_1+w_2+\cdots+w_n)-wt(x_{n-1})$.

Proof. Case(C) $(n-1)$.1. It is obvious that

$$wt(x_1^{a_1-2}x_2^{a_2-1}\cdots x_{n-2}^{a_{n-2}-1}x_{n-1}^{a_{n-1}-2}x_n^{a_n-1})+wt(x_{n-1})=nd-2(w_1+\cdots+w_n).$$

So the conclusion is true in this case.

Case(C) $(n-1)$.2. Note that in this case, we want to show that

$$wt(x_1^{a_1-2}x_2^{a_2-1}\cdots x_{n-3}^{a_{n-3}-1}x_{n-1}^{a_{n-1}-1}x_n)+wt(x_{n-1})\le nd-2(w_1+\cdots+w_n).$$

It follows that $w_{n-2}+2w_{n-1}+2w_n\le 2d$, which is correct following from Lemma 10.6.

Case(C) $(n-1)$.3. We want to show that

$$wtx_1^{a_1-2}x_2^{a_2-1}\cdots x_{n-4}^{a_{n-4}-1}x_{n-2}^{a_{n-2}-1}x_{n-1}^{a_{n-1}-1}x_n)+wt(x_{n-1})\le nd-2(w_1+\cdots+w_n).$$

Simplifying it as in case(C) $(n-2)$.2 we get $w_{n-3}+w_{n-2}+w_{n-1}+2w_n\le 2d$, which is correct following from Lemma 10.6.

Case(C) $(n-1)$.4. We want to show that $wt(x_1^{a_1-2}x_2^{a_2-1}\cdots x_{n-2}^{a_{n-2}-1}x_{n-1}^{a_{n-1}-1})$ $+wt(x_{n-1})\le nd-2(w_1+\cdots+w_n)$. Simplifying it as in case(C) $(n-1)$.2 we get $w_{n-1}+w_n\le d$, which is correct following from Lemma 10.6. □

For $I_n=\langle x_{n-1}x_n^{a_n-1}\rangle$, the candidates for maximal weight of bases of $A(f)/I_n$ are as follows:

$$\begin{aligned}&case(C)\,n.1.\ x_1^{a_1-2}x_2^{a_2-1}\cdots x_{n-1}^{a_{n-1}-1}x_n^{a_n-2},\\&case(C)\,n.2.\ x_1^{a_1-2}x_2^{a_2-1}\cdots x_{n-2}^{a_{n-2}-1}x_n^{a_n-1}.\end{aligned}$$

Lemma 10.13. *Under the same assumption as in Proposition* 10.5*, for case*(C) $n.1$*–case*(C) $n.2$ *above, the weights of the monomial bases are less than or equal to* $nd-2(w_1+w_2+\cdots+w_n)-wt(x_n)$

Proof. Case(C) $n.1$. It is obvious that $wt(x_1^{a_1-2}x_2^{a_2-1}\cdots x_{n-1}^{a_{n-1}-1}x_n^{a_n-2})+wt(x_n)=$ $nd-2(w_1+\cdots+w_n)$. So the conclusion is true in this case.

Case(C) $n.2$. Note that in this case, we want to show that

$$wt(x_1^{a_1-2}x_2^{a_2-1}\cdots x_{n-2}^{a_{n-2}-1}x_n^{a_n-1})+wt(x_n)\le nd-2(w_1+\cdots+w_n).$$

It follows that $w_{n-1}+2w_n\le 2d$, which is correct following from Lemma 10.6. □

Now we can give the proof of Proposition 10.5 easily.

Proof. Proposition 10.5 follows from Lemmas 10.7, 10.8, 10.9, 10.11, 10.12, and 10.13 immediately. □

Now we can give the proof of Proposition 10.4.

Proof. Let $D\in \mathrm{Der}(A(f))$ be a negative weight derivation. We can write $D=\sum_{i=1}^n g_i\partial x_i$, where $g_i\in A(f)$ and $wt(g_i)<w_i$. By Proposition 10.5, we have $wt(g_i)$+maximal weight of monomial bases of

$$A(f)/I_i<nd-2(w_1+w_2+\cdots+w_n),\ 1\le i\le n.$$

By Theorem 8.1(3), the weight of socle is $nd-2(w_1+w_2+\cdots+w_n)$. Thus wt(g_i)+maximal weight of monomial bases of $A(f)/I_i<wt(socle)$, which means the projection map (7.2) is zero. From Theorem 7.1 we have $D=0$. Thus $\mathrm{Der}(A(f))$ has no negative weight derivations. □

Proposition 10.6. *Let $f = x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}$ $n \geq 2$ be a weighted homogeneous fewnomial isolated singularity of type C. Suppose mult$(f) \geq 5$ and n is odd. $A(f) = \mathbb{C}\{x_1, \ldots, x_n\}/(f_{x_1}, f_{x_2}, \ldots, f_{x_n})$ is the moduli algebra of f. Then there are no non-zero negative weight derivations on $A(f)$.*

Proof. The only difference between Proposition 10.4 and Proposition 10.6 is whether n is odd or even, which will affect the monomial bases of $A(f)$. In the case of Proposition 10.4, $A(f) = \mathbb{C}\{x_1, x_2, \ldots, x_n\}/(f_{x_1}, f_{x_2}, \ldots, f_{x_n})$ is a vector space spanned by monomial basis $\{x_1^{k_1}x_2^{k_2}\cdots x_{n-1}^{k_{n-1}}x_n^{k_n}\}$. Here $(k_1, k_2, \ldots, k_n) \in \{0 \leq k_1 \leq a_1 - 2, 0 \leq k_j \leq a_j - 1 \text{ for } 2 \leq j \leq n\} \cup \{k_1 = a_1 - 1, k_2 = 0, 0 \leq k_3 \leq a_3 - 2, 0 \leq k_j \leq a_j - 1 \text{ for } 4 \leq j \leq n\} \cup \cdots \cup \{k_{2j-1} = a_{2j-1} - 1, k_{2j} = 0, \text{ for } 1 \leq j \leq i, 0 \leq k_{2i+1} \leq a_{2i+1} - 2, 0 \leq a_j \leq a_j - 1 \text{ for } 1 \leq 2i+1 < j \leq n\} \cup \cdots \cup \{k_{2j-1} = a_{2j-1} - 1 \text{ and } k_{2j} = 0 \text{ for } 1 \leq j \leq \frac{n-1}{2} \text{ and } 0 \leq k_n \leq a_n - 2\}$. However, the difference doesn't affect the proof. A similar argument to Proposition 10.4 will prove Proposition 10.6. ∎

The following proposition will be used in the proof of Main Theorem C.

Proposition 10.7. *Let $f(x_1, \ldots, x_n)$ and $g(x_{n+1}, \ldots, x_m)$ be holomorphic functions with isolated singularities at the origins of $\mathbb{C}^n$ and $\mathbb{C}^{m-n}$, respectively. Let $A(f)$, $A(g)$, and $A(f+g)$ be the moduli algebras of $f, g, f+g$, respectively. If $f(x_1, \ldots, x_n)$ is a weighted-homogeneous holomorphic function with an isolated singularity at origin, then $A(f+g) \cong A(f) \otimes A(g)$.*

Proof.

$$\begin{aligned} A(f+g) &:= \mathbb{C}\{x_1, \ldots, x_m\}/(f+g, f_{x_1}, \ldots, f_{x_n}, g_{x_{n+1}}, \ldots, g_{x_m}) \\ &= \mathbb{C}\{x_1, \ldots, x_m\}/(g, f_{x_1}, \ldots, f_{x_n}, g_{x_{n+1}}, \ldots, g_{x_m}) \\ &\cong \mathbb{C}\{x_1, \ldots, x_n\}/(f_{x_1}, \ldots, f_{x_n}) \otimes \mathbb{C}\{x_{n+1}, \ldots, x_m\}/(g, g_{x_{n+1}}, \ldots, g_{x_m}) \\ &= A(f) \otimes A(g). \end{aligned}$$

The second and last equalities come from f being weighted homogeneous. ∎

Main Theorem C. Let $f \in \mathbb{C}\{x_1, \ldots, x_n\}$ be a weighted homogeneous fewnomial isolated singularity with positive weights $w_1, w_2, \ldots, w_n$ and multiplicity at least 5.

$$A(f) = \mathbb{C}\{x_1, \ldots, x_n\}/(f_{x_1}, f_{x_2}, \ldots, f_{x_n})$$

is the moduli algebra of f. Then there are no non-zero negative weight derivations on $A(f)$.

Proof. Since $f \in \mathbb{C}\{x_1, \ldots, x_n\}$ is a weighted homogeneous fewnomial isolated singularity, f is a summation of the following three types:

Type A. $x_1^{a_1} + x_2^{a_2} + \cdots + x_{n-1}^{a_{n-1}} + x_n^{a_n}$, $n \geq 1$,
Type B. $x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}x_1$, $n \geq 2$,
Type C. $x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}$, $n \geq 2$.

The moduli algebra $A(f)$ is a tensor product of those moduli algebras of the above three types from Proposition 10.7. By Theorem 8.2, it suffices to prove that the moduli algebras of the above three types have no non-zero negative weight derivations. That is exactly what we have proved in Propositions 10.1, 10.2, 10.4, and 10.6. Therefore there are no non-zero negative weight derivations on $A(f)$. ∎

References

[A1] A. G. Aleksandrov, *Cohomology of a quasihomogeneous complete intersection* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **49** (1985), no. 3, 467–510, 672. MR794953

[A2] A. G. Aleksandrov, *Duality and deformations of Artinian algebras*, Proceedings of the International Conference on the Theory of Rings, Algebras and Modules in Memory of A. I. Shirshov, Barnaul, USSR, 1991.

[A3] A. G. Aleksandrov, *Duality, derivations and deformations of zero-dimensional singularities*, Zero-dimensional schemes (Ravello, 1992), de Gruyter, Berlin, 1994, pp. 11–31. MR1292472

[AM] Aleksandr G. Aleksandrov and Bernd Martin, *Derivations and deformations of Artinian algebras*, Beiträge Algebra Geom. **33** (1992), 115–130. MR1163646

[AVZ] V. Arnold, A. Varchenko, and S. Gusein-Zade, *Singularities of differentiable mappings I, II*, 2nd ed. (Russian) MCNMO, Moskva, 2004. MR0685918, MR0755329.

[BK1] Igor Belegradek and Vitali Kapovitch, *Topological obstructions to nonnegative curvature*, Math. Ann. **320** (2001), no. 1, 167–190, DOI 10.1007/PL00004467. MR1835067

[BK2] Igor Belegradek and Vitali Kapovitch, *Obstructions to nonnegative curvature and rational homotopy theory*, J. Amer. Math. Soc. **16** (2003), no. 2, 259–284, DOI 10.1090/S0894-0347-02-00418-6. MR1949160

[Bo] Armand Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts* (French), Ann. of Math. (2) **57** (1953), 115–207, DOI 10.2307/1969728. MR0051508

[Bl] Richard E. Block, *Determination of the differentiably simple rings with a minimal ideal*, Ann. of Math. (2) **90** (1969), 433–459, DOI 10.2307/1970745. MR0251088

[CG] Jeff Cheeger and Detlef Gromoll, *On the structure of complete manifolds of nonnegative curvature*, Ann. of Math. (2) **96** (1972), 413–443, DOI 10.2307/1970819. MR0309010

[CXY] Hao Chen, Yi-Jing Xu, and Stephen S.-T. Yau, *Nonexistence of negative weight derivation of moduli algebras of weighted homogeneous singularities*, J. Algebra **172** (1995), no. 2, 243–254, DOI 10.1016/S0021-8693(05)80001-8. MR1322403

[Ch1] Hao Chen, *On negative weight derivations of the moduli algebras of weighted homogeneous hypersurface singularities*, Math. Ann. **303** (1995), no. 1, 95–107, DOI 10.1007/BF01460981. MR1348357

[Ch2] Hao Chen, *Nonexistence of negative weight derivations on graded Artin algebras: a conjecture of Halperin*, J. Algebra **216** (1999), no. 1, 1–12, DOI 10.1006/jabr.1998.7750. MR1694610

[ET] Wolfgang Ebeling and Atsushi Takahashi, *Strange duality of weighted homogeneous polynomials*, Compos. Math. **147** (2011), no. 5, 1413–1433, DOI 10.1112/S0010437X11005288. MR2834726

[FHT] Yves Félix, Stephen Halperin, and Jean-Claude Thomas, *Rational homotopy theory*, Graduate Texts in Mathematics, vol. 205, Springer-Verlag, New York, 2001. MR1802847

[Ia] A. Iarrobino, *Hilbert scheme of points: overview of last ten years*, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), Proc. Sympos. Pure Math., vol. 46, Amer. Math. Soc., Providence, RI, 1987, pp. 297–320. MR927986

[Kho] A. G. Khovanskiĭ, *Fewnomials*, translated from the Russian by Smilka Zdravkovska, Translations of Mathematical Monographs, vol. 88, American Mathematical Society, Providence, RI, 1991. MR1108621

[Khi] G. Khimshiashvili, *Yau algebras of fewnomial singularities*, preprint, `http://www.math.uu.nl/publications/preprints/1352.pdf`.

[KL] Steven L. Kleiman and John Landolfi, *Geometry and deformation of special Schubert varieties*, Compositio Math. **23** (1971), 407–434. MR0314855

[KS] Maximilian Kreuzer and Harald Skarke, *On the classification of quasihomogeneous functions*, Comm. Math. Phys. **150** (1992), no. 1, 137–147. MR1188500

[Lu] G. Lupton, Private communication, December 8, 2015.

[MY] John N. Mather and Stephen S. T. Yau, *Classification of isolated hypersurface singularities by their moduli algebras*, Invent. Math. **69** (1982), no. 2, 243–251, DOI 10.1007/BF01399504. MR674404

[Me] W. Meier, *Rational universal fibrations and flag manifolds*, Math. Ann. **258** (1981/82), no. 3, 329–340, DOI 10.1007/BF01450686. MR649203

[Pa] V. P. Palamodov, *Deformations of complex spaces* (Russian), Uspehi Mat. Nauk **31** (1976), no. 3(189), 129–194. MR0508121

[Pi1] Henry Pinkham, *Groupe de monodromie des singularités unimodulaires exceptionnelles*, C. R. Acad. Sci. Paris Sér. A-B **284** (1977), no. 23, A1515–A1518. MR0439840

[Pi2] Henry Pinkham, *Deformations of normal surface singularities with C^* action*, Math. Ann. **232** (1978), no. 1, 65–84, DOI 10.1007/BF01420623. MR0498543

[PP1] Stefan Papadima and Laurentiu Paunescu, *Reduced weighted complete intersection and derivations*, J. Algebra **183** (1996), no. 2, 595–604, DOI 10.1006/jabr.1996.0234. MR1399041

[PP2] Ştefan Papadima and Laurenţiu Păunescu, *Isometry-invariant geodesics and nonpositive derivations of the cohomology*, J. Differential Geom. **71** (2005), no. 1, 159–176. MR2191771

[Sa1] Kyoji Saito, *Quasihomogene isolierte Singularitäten von Hyperflächen* (German), Invent. Math. **14** (1971), 123–142, DOI 10.1007/BF01405360. MR0294699

[Sa2] Kyoji Saito, *Einfach-elliptische Singularitäten* (German), Invent. Math. **23** (1974), 289–325, DOI 10.1007/BF01389749. MR0354669

[SW] Günter Scheja and Hartmut Wiebe, *Über Derivationen von lokalen analytischen Algebren* (German), Symposia Mathematica, Vol. XI (Convegno di Algebra Commutativa, INDAM, Rome, 1971), Academic Press, London, 1973, pp. 161–192. MR0338461

[Th1] Jean-Claude Thomas, *Sur les fibrations de Serre pures* (French, with English summary), C. R. Acad. Sci. Paris Sér. A-B **290** (1980), no. 17, A811–A813. MR580572

[Th2] Jean-Claude Thomas, *Rational homotopy of Serre fibrations* (English, with French summary), Ann. Inst. Fourier (Grenoble) **31** (1981), no. 3, v, 71–90. MR638617

[Wie] Hartmut Wiebe, *Über homologische Invarianten lokaler Ringe* (German), Math. Ann. **179** (1969), 257–274, DOI 10.1007/BF01350771. MR0255531

[Wil] Burkhard Wilking, *On fundamental groups of manifolds of nonnegative curvature*, Differential Geom. Appl. **13** (2000), no. 2, 129–165, DOI 10.1016/S0926-2245(00)00030-9. MR1783960

[XY] Yi-Jing Xu and Stephen S.-T. Yau, *Micro-local characterization of quasi-homogeneous singularities*, Amer. J. Math. **118** (1996), no. 2, 389–399. MR1385285

[Ya1] Stephen S.-T. Yau, *Solvability of Lie algebras arising from isolated singularities and non-isolatedness of singularities defined by* $\mathrm{sl}(2, \mathbf{C})$ *invariant polynomials*, Amer. J. Math. **113** (1991), no. 5, 773–778, DOI 10.2307/2374785. MR1129292

[Ya2] Stephen S.-T. Yau, *Solvable Lie algebras and generalized Cartan matrices arising from isolated singularities*, Math. Z. **191** (1986), no. 4, 489–506, DOI 10.1007/BF01162338. MR832806

[YZ1] Stephen S.-T. Yau and Huai Qing Zuo, *A sharp upper estimate conjecture for the Yau number of a weighted homogeneous isolated hypersurface singularity*, Pure Appl. Math. Q. **12** (2016), no. 1, 165–181, DOI 10.4310/PAMQ.2016.v12.n1.a6. MR3613969

[YZ2] Stephen S.-T. Yau and Huaiqing Zuo, *Derivations of the moduli algebras of weighted homogeneous hypersurface singularities*, J. Algebra **457** (2016), 18–25, DOI 10.1016/j.jalgebra.2016.03.003. MR3490075

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