Journal of Algebra 529 (2019) 124–144

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

On strong vanishing property and plurigenera of isolated singularities

Huaiqing Zuo [1]

Department of Mathematical Sciences, Tsinghua University, Beijing, 100084, PR China

ARTICLE INFO

Article history:
Received 29 January 2018
Available online 9 April 2019
Communicated by Steven Dale Cutkosky

MSC:
14B05
32S05

Keywords:
Strong vanishing property
Plurigenera
Isolated singularities

ABSTRACT

We introduce a new concept of strong vanishing property and investigate this property for minimally elliptic singularities. A comparison theorem about strong vanishing property is also obtained. We use Zariski decomposition to give a formula of plurigenera for Gorenstein singularities and use it to test whether a divisor is f-nef.

1. Introduction

Let (X, x) be an isolated surface singularity and $f : (M, A) \to (X, x)$ be a resolution, where $A = f^{-1}(x)$ is the exceptional set, and K is the canonical divisor on M. Let $A = \cup_{i=1}^{n} A_i$ be the decomposition of A into irreducible components. It is known that the resolution dual graph Γ and the link ∂X of the singularity determine each other

E-mail address: hqzuo@mail.tsinghua.edu.cn.

[1] The author is supported by NSFC Grants (11771231 and 11531007), Tsinghua University Initiative Scientific Research Program (523010006).

https://doi.org/10.1016/j.jalgebra.2019.04.002

[16]. So an invariant is topological if it is determined by Γ. The typical topological invariants are the Euler characteristic $\chi(Z_{min})$ and the self-intersection number Z_{min}^2, where Z_{min} is Artin's fundamental cycle. However, one invariant is topological or not is generally difficult to see, for example the plurigenera $\delta_m(X,x)$ (see Definition 2.2) are very fundamental analytic invariants. However, in general, it is difficult to see whether they are topological or not. The main reason is that these invariants are difficult to compute, even if (X,x) is Gorenstein. One of the main contributions of this paper is the following result.

Theorem 1.1. *Suppose that (X,x) is an isolated Gorenstein surface singularity but not a log-canonical singularity. Let $f:(M,A)\to(X,x)$ be the minimal good resolution. Let $P+N$ be the Zariski decomposition of the divisor $K+A$. $\{\delta_{m+1}(X,x)\}_{m\in\mathbb{N}}$ are plurigenera. Then*

$$\delta_{m+1}(X,x) = -m^2P^2/2 - mK\cdot P/2 + p_g(X,x)$$

if and only if $K+A$ is f-nef.

Remark 1.1. Under the same conditions in Theorem 1.1, it follows that $\delta_{m+1}(X,x)$ are topological invariants when $p_g(X,x)$ is a topological invariant and $K+A$ is f-nef.

One direction in Theorem 1.1 can be improved further.

Corollary 1.1. *Suppose that (X,x) is an isolated Gorenstein surface singularity but not a quotient singularity. Let $f:(M,A)\to(X,x)$ be the minimal good resolution. Let $P+N$ be the Zariski decomposition of the divisor $K+A$. If $K+A$ is f-nef, then we have the formula*

$$\delta_{m+1}(X,x) = -m^2P^2/2 - mK\cdot P/2 + p_g(X,x).$$

Remark 1.2. Let $\mathbb{N}^*$ denote the set of positive integers. If (X,x) is an isolated quotient singularity, then $\delta_m(X,x)=0$ for any $m\in\mathbb{N}^*$(cf. [24], Theorem 3.9) and $P=0$ by Theorem 2.4. Thus the equation in Corollary 1.1 still holds.

Let $f:(M,A)\to(X,x)$ be a resolution of an isolated surface singularity (X,x) and Div(M) denotes the group of divisors on M. Suppose $D\in$Div(M), $\chi_M(D)$ is given by

$$\chi_M(D) := \dim_{\mathbb{C}} H^0(M\setminus A, \mathcal{O}_M(D))/H^0(M,\mathcal{O}_M(D)) + h^1(M,\mathcal{O}_M(D)).$$

Let D be a cycle on M. Then it is not hard to get a formula for $\chi_M(D)$, i.e., $\chi_M(D) = -D\cdot(D-K)/2 + p_g(X,x)$ (cf. Theorem 2.7). However, when D is a divisor on M, the formula for $\chi_M(D)$, which is obtained by Morales (cf. Theorem 2.8), is very complicated.

A main result used in his proof is the following: for any $D \in \mathrm{Div}(M)$, Giraud [2] (see subsection 2.4) proved that there exists a unique cycle $\langle D\rangle \in \oplus_{i=1}^{n}\mathbb{Z}A_i$ such that

(1) $D - \langle D\rangle$ is f-nef,

(2) if D' is a cycle and $D - D'$ is f-nef, then $\langle D\rangle \le D'$.

In this article, we shall prove the following result.

Theorem 1.2. *Let $D \in Div(M)$ such that $\langle D\rangle > 0$. Then $H^0(\mathcal{O}_{\langle D\rangle}(D)) = 0$ (i.e. $D\cdot\langle D\rangle < 0$), thus*

$$H^0(M, \mathcal{O}_M(D - \langle D\rangle)) = H^0(M, \mathcal{O}_M(D)).$$

As a corollary of Theorem 1.2, we recover a theorem in Giraud [2].

Corollary 1.2. *Let $D \in Div(M)$ such that $\langle D\rangle \ge 0$. Then $f_*\mathcal{O}_M(D)$ is reflexive, i.e.,*

$$H^0(M \setminus A, \mathcal{O}_M(D)) = H^0(M, \mathcal{O}_M(D)).$$

In a beautiful paper [21], four vanishing theorems are proved. Specifically, let $f : (M, A) \to (X, x)$ be the minimal good resolution of an isolated surface singularity (X, x), with exceptional divisor $A = \sum A_i$. Let S denote the rank 2 subbundle of the tangent bundle θ_M of M of derivations vanishing along A. Wahl proved that

(A) $H^1_A(\mathcal{O}_M) = 0$;

(B) if (X, x) is rational, then $H^1_A(\mathcal{O}_M(A)) = 0$;

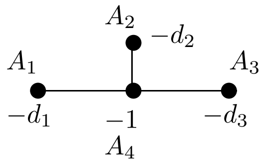
(C) in characteristic $0, H^1_A(S) = 0$;

(D) if (X, x) is rational and the characteristic is 0, then $H^1_A(S(A)) = 0$.

These vanishing results play important roles in the deformation theory of singularities, especially rational ones. In this paper, we pay exceptional attention on the result (B) which can be used to test whether a reduced divisor on a deformation of M can specialize to a non-reduced divisor on M (cf. [21], Corollary 2.8 and Remark 2.9). It is natural to ask whether result (B) can be generalized to more general singularities. In this article, we shall introduce two new concepts: strong and weak vanishing property (see Definition 4.1). In fact, the vanishing result (B) coincides with the weak vanishing property in this paper and it is implied by strong vanishing property. It follows from a result of Wahl (cf. Proposition 4.2) that all rational singularities have strong vanishing property. We investigate the strong vanishing property for minimally elliptic singularities and obtain the following results.

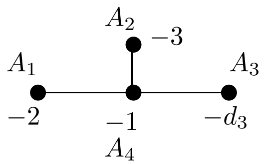
Theorem 1.3. *Suppose (X, x) is a minimally elliptic singularity. Let $f : M \to X$ be the minimal good resolution of the singularity, whose reduced exceptional divisor is $A = \sum_i A_i$, and Z is a positive cycle on M. If $Z + A \ne Z_{min}$, then there exists an $A_i \subset |Z|$ with $(Z + A)\cdot A_i < 0$ except in the following cases:*

1) (X, x) has the following dual resolution graph,



where $d_3 \geq d_2 \geq d_1 \geq 2$, and $\frac{1}{d_1} + \frac{1}{d_2} + \frac{1}{d_3} < 1$, and with $Z = A_4$ or $2A_4$ (notice that in the case $Z = 2A_4$, then $d_1 = 2$, otherwise $Z + A = Z_{min}$); and

2) (X, x) has the following dual resolution graph,



with $d_3 \geq 7$ and $Z = A_1 + 3A_4$.

Corollary 1.3. *The following singularities have strong vanishing property:*

(1) Cusp singularities,

(2) Simple elliptic singularities,

(3) Gorenstein Du Bois singularities.

Given a singularity (X, x), there are ways to get new singularities (X', x') in Theorem 1.4. It is interesting to compare invariants and properties of (X, x) and (X', x'). For example, Watanabe [24] proved that $\delta_m(X, x) \geq \delta_m(X', x')$ where $\delta_m(X, x)$ is the plurigenera. In this article, we shall prove the following comparison theorem about the strong vanishing property between two related singularities.

Theorem 1.4. *Let $f : (M, A) \to (X, x)$ be a minimal good resolution of an isolated surface singularity (X, x). Let A' be a connected cycle such that $0 < A' \leq A$. Let (X', x') be the singularity obtained by contracting A'. If (X, x) has strong vanishing property, then so does (X', x').*

The paper is organized as follows: we introduce some preliminary knowledge in section 2. In section 3 we prove the Theorems 1.1, 1.2, and their corollaries. We introduce a new concept in section 4. Theorems 1.3, 1.4 and Corollary 1.3 are proved in section 5.

I would like to thank the anonymous referee for careful reading and providing valuable comments on improving the presentation of this paper.

2. Preliminaries

Let M be a nonsingular surface, not necessarily compact. Let $D = \sum a_i D_i$ be a $\mathbb{Q}$-divisor on M, where D_i's are mutually distinct prime divisors. We put $D_{red} =$

$\sum_{a_i \neq 0} D_i$. The divisor D is said to be connected if the support of D is connected, and said to be positive if D is effective and $D \neq 0$. If the support of D is compact, then we call D a $\mathbb{Q}$-cycle; furthermore if each a_i is an integer, then we call D a $\mathbb{Z}$-cycle, or a cycle for short.

Let (X, x) be an isolated surface singularity and $f : (M, A) \to (X, x)$ a resolution, i.e., M is smooth, and f is a biholomorphic isomorphism above $X \setminus \{x\}$. We say that f is a good resolution if the exceptional divisor $A := f^{-1}(x)$ is a normal crossing divisor. A good resolution always exists (and it is not unique). f is called a minimal resolution if for any resolution $f' : M' \to X$ there exists a unique morphism $g : M' \to M$ such that $f' = f \circ g$.

A good resolution $f : (M, A) \to (X, x)$ is called a minimal good resolution if for any good resolution $f' : M' \to X$ there exists a unique morphism $g : M' \to M$ such that $f' = f \circ g$. For any surface singularity, there exists a unique minimal good resolution (cf. [11], Theorem 5.12).

Let $f : (M, A) \to (X, x)$ be a resolution of a normal surface singularity (X, x). Then any $\mathbb{Q}$-cycle on M is supported in A. Let $A = \cup_{i=1}^n A_i$ be the decomposition of A into irreducible components. We denote by K the canonical divisor on M.

Let $D = \sum_{i=1}^n d_i A_i$ be a positive cycle on M. $|D| = \cup A_i$, $d_i \neq 0$, denotes the support of D. Recall that $\chi(D) := \dim H^0(M, \mathcal{O}_D) - \dim H^1(M, \mathcal{O}_D)$.

Let E be a divisor on M and D a positive cycle on M. Then we have

$$\chi(D) = -D \cdot (D + K)/2 \text{ and } \chi(\mathcal{O}_D(E)) = D \cdot E + \chi(D).$$

For any positive cycles D and F, we have $\chi(D + F) = \chi(D) + \chi(F) - D \cdot F$.

Definition 2.1. A $\mathbb{Q}$-divisor D on M is said to be f-numerically effective, or f-nef for short, if $D \cdot A_i \geq 0$ for all A_i. $\mathbb{Q}$-divisors D and E on M are said to be f-numerically equivalent, written $D \equiv E$, if $(D - E) \cdot A_i = 0$ for all A_i.

Let (X, x) be a $\mathbb{Q}$-Gorenstein singularity. (X, x) is said to be terminal (resp. canonical, log-terminal, log-canonical) if there exists a good resolution $f : (M, A) \to (X, x)$ such that

$$K_M = f^*(K_X) + \sum a_i A_i,$$

with $a_i > 0$ (resp. $a_i \geq 0$, $a_i > -1$, $a_i \geq -1$) for all i, where A_i are mutually distinct prime divisors supported in A.

We recall a beautiful result of Kawamata.

Theorem 2.1. *(Kawamata, [[9], Corollary 1.9]) A surface singularity (X, x) is a log-terminal singularity if and only if (X, x) is a quotient singularity. In fact, any log-terminal surface singularity is the quotient with respect to a cyclic group of $(\mathbb{C}^2, 0)$ or a rational double point.*

2.1. Plurigenera

Let (X, x) be a normal isolated singularity of dimension $n \geq 2$ and $f : (M, A) \to (X, x)$ a good resolution. Let $U = X \setminus \{x\}$. For any positive integer m, a section $\omega \in \Gamma(U, \mathcal{O}_X(mK_X))$ is regarded as an m-ple holomorphic n-form.

An m-ple holomorphic n-form $\omega \in \Gamma(U, \mathcal{O}_X(mK_X))$ is said to be $L^{2/m}$-integrable if

$$\int_{W\setminus\{x\}} (\omega \wedge \bar{\omega})^{1/m} < \infty$$

for any sufficiently small relatively compact neighborhood W of $x \in X$. We denote by $L^{2/m}(U)$ the set of $L^{2/m}$-integrable m-ple holomorphic n-forms on U, which is a subspace of $\Gamma(U, \mathcal{O}_X(mK_X))$.

Definition 2.2. (Watanabe [24]) For each $m \in \mathbb{N}^*$, the m-th L^2-plurigenus $\delta_m(X, x)$ of a normal isolated singularity (X, x) is defined by

$$\delta_m(X, x) = \dim_{\mathbb{C}} \frac{\Gamma(X \setminus \{x\}, \mathcal{O}_X(mK_X))}{L^{2/m}(X \setminus \{x\})}.$$

It is known that $\Gamma(M, \mathcal{O}_M(mK_M + (m-1)A)) \cong L^{2/m}(U)$ (cf. Sakai [18]). The next proposition is obvious and useful for computing the plurigenera.

Proposition 2.1. *For any good resolution $f : (M, A) \to (X, x)$, we have*

$$\delta_m(X, x) = dim_{\mathbb{C}} \frac{\Gamma(M \setminus A, \mathcal{O}_M(mK_M))}{\Gamma(M, \mathcal{O}_M(mK_M + (m-1)A)}.$$

The $p_g(X, x) = \delta_1(X, x)$ was proved by Laufer [12] for surface singularity and Yau [25] for higher-dimensional singularity. The plurigenera was systematic studied by Ishii (cf. [3–7]) and also studied by the author in [30], [14].

2.2. Minimally elliptic singularities

Let $f : (M, A) \to (X, x)$ be a resolution of an isolated surface singularity (X, x) and $A = \cup_{i=1}^{n} A_i$. Associated to f is a unique fundamental cycle Z_{min} [1, pp. 131-132] such that $Z_{min} > 0$, $A_i \cdot Z_{min} \leq 0$ for all A_i and such that Z_{min} is minimal with respect to those two properties. Z_{min} may be computed from the intersection as follows [12, Proposition 4.1, p. 607].

$$\begin{aligned} Z_0 = 0, Z_1 = A_{i_1}, Z_2 = Z_1 + A_{i_2}, \ldots, Z_j = Z_{j-1} + A_{i_j}, \ldots, \\ Z_\ell = Z_{\ell-1} + A_{i_\ell} = Z_{min} \end{aligned}$$

where A_{i_1} is arbitrary and $A_{i_j} \cdot Z_{j-1} > 0$, $1 < j \leq \ell$.

Recall that a cycle $E > 0$ is *minimally elliptic* if $\chi(E) = 0$ and $\chi(D) > 0$ for all cycles D such that $0 < D < E$.

Theorem 2.2 *([13]). Let $f: M \to X$ be the minimal resolution of the normal two dimensional variety X with one singular point x. Let Z_{min} be the fundamental cycle on the exceptional set $A = f^{-1}(x)$. Then the following are equivalent:*

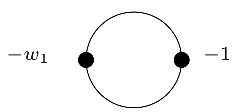
(1) *Z_{min} is a minimally elliptic cycle,*
(2) *$A_i \cdot Z_{min} = -A_i \cdot K$ for all irreducible components A_i in A,*
(3) *$\chi(Z_{min}) = 0$ and any connected proper subvariety of A is the exceptional set for a rational singularity.*

In [13], a singularity is called minimally elliptic singularity if it satisfies one of the conditions of Theorem 2.2. Minimally elliptic singularity and its generalization were intensively studied by Wagreich [23], Laufer [13], Yau ([26–29]) and others. We recall the following beautiful result will be used later.

Proposition 2.2 *([13]). Let $f: M \to X$ and $f': M' \to X$ be the minimal resolution and minimal good resolution respectively for a minimally elliptic singularity (X, x). Then $f = f'$ and all the A_i are rational curves except for the following cases:*

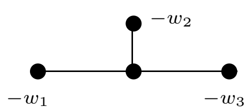
(i) *A is an elliptic curve. f is a minimal good resolution.*
(ii) *A is a rational curve with a node singularity.*
(iii) *A is a rational curve with a cusp singularity.*
(iv) *A is two smooth rational curves which have first order tangential contact at one point.*
(v) *A is three smooth rational curves all meeting transversely at the same point.*

In case (ii), the weighted dual graph of the minimal good resolution is



with $w_1 \geq 5$.

In cases (iii)–(v), f' has the following weighted dual graph



with $w_i \geq 2$, $1 \leq i \leq 3$.

2.3. Zariski decomposition

Let (X, x) be a normal surface singularity, $f : (M, A) \to (X, x)$ a resolution with $A = \cup_{i=1}^n A_i$, and K the canonical divisor on M. Let $D = \sum_{i=1}^n d_i A_i$ be a $\mathbb{Q}$-cycle. $[D] := \sum [d_i] A_i$ where $[d]$ denotes the greatest integer not more than d.

Since the intersection matrix $(A_i \cdot A_j)$ is negative definite, so for any $D \in \mathrm{Div}(M)$, there exists a unique $\mathbb{Q}$-cycle $\sum_{i=1}^n d_i A_i$ such that $D \equiv \sum_{i=1}^n d_i A_i$. Then the map $D \mapsto \sum_{i=1}^n d_i A_i$ induces a homomorphism of groups

$$\pi : \mathrm{Div}(M) \to \oplus_{i=1}^n \mathbb{Q} A_i.$$

The intersection number of divisors $D_1, D_2 \in \mathrm{Div}(M)$ is defined by

$$D_1 \cdot D_2 = \pi(D_1) \cdot \pi(D_2).$$

Definition 2.3. Let D be a divisor on M. Then a decomposition $\pi(D) = P + N$ is called a Zariski decomposition of D if the following conditions are satisfied:

(1) P is an f-nef $\mathbb{Q}$-cycle;
(2) N is an effective $\mathbb{Q}$-cycle;
(3) $P \cdot N = 0$, i.e., $P \cdot A_i = 0$ for all $A_i \subset |N|$.

For any divisor D, there exists a unique Zariski decomposition of D (Sakai [19], Theorem A.1).

Theorem 2.3. *(Sakai [19], Theorem A.2; [17], Theorem 4.6) Let $\pi(D) = P + N$ the Zariski decomposition of $D \in Div(M)$. Then*

$$H^1(\mathcal{O}_M(K + D - [N])) = 0.$$

In particular, if D is an f-nef divisor on M, then $H^1(\mathcal{O}_M(K + D)) = 0$.

We assume that $f : (M, A) \to (X, x)$ is the minimal good resolution, and that $P + N$ is the Zariski decomposition of $K + A$. Then P and N are computed from the weighted dual graph. Note that $P^2 = 0$ if and only if $P = 0$.

The following theorem implies that log-canonical surface singularities are characterized by their weighted dual graphs.

Theorem 2.4. *(Kawamata [10] section 9; Wahl [22] 2.4) A normal surface singularity (X, x) is a log-canonical singularity if and only if $P = 0$. Furthermore we have the following:*

(1) (X, x) is a log-terminal singularity if and only if $P = 0$ and $Supp(N) = A$;
(2) (X, x) is a simple elliptic or cusp singularity if and only if $P = N = 0$.

Theorem 2.5. *(Watanabe [24], Ishii [4]) Suppose that the singularity (X, x) is $\mathbb{Q}$-Gorenstein. If (X, x) is log-canonical but not log-terminal, then*

$$\delta_m(X, x) = \begin{cases} 1 & m \equiv 0 \ (mod\, r) \\ 0 & otherwise \end{cases}$$

where r is the index of (X, x);

Theorem 2.6. *(Ishii [3]) Let (X, x) be a minimally elliptic singularity. Then the following are equivalent:*

(1) (X, x) is a simple elliptic or cusp singularity;
(2) (X, x) is a log-canonical singularity;
(3) (X, x) is Du Bois;
(4) $\delta_m(X, x) = 1$ for all $m \in \mathbb{N}^*$.

2.4. Riemann-Roch for resolution spaces

We use the notation of the preceding subsection, but do not assume that $f : (M, A) \to (X, x)$ is the minimal good resolution. Let D be a divisor on M, the Euler characterization $\chi_M(D)$ is given by

$$\chi_M(D) = \dim_{\mathbb{C}} H^0(M \setminus A, \mathcal{O}_M(D))/H^0(M, \mathcal{O}_M(D)) + h^1(M, \mathcal{O}_M(D)).$$

Theorem 2.7. *(Kato [8]) Let D be a cycle on M. Then*

$$\chi_M(D) = -D \cdot (D - K)/2 + p_g(X, x).$$

Morales [15] generalized the result of Theorem 2.7. Let us review these results.

For any $D \in \operatorname{Div}(M)$, Giraud [2] show that there exists a unique cycle $\langle D \rangle \in \oplus_{i=1}^n \mathbb{Z}A_i$ such that

(1) $D - \langle D \rangle$ is f-nef, and
(2) if D' is a cycle and if $D - D'$ is f-nef, then $\langle D \rangle \leq D'$.

Let $D \in \operatorname{Div}(M)$, we define $\mathcal{C}(D) = \{F \in \oplus_{i=1}^n \mathbb{Z}A_i \mid D - F \text{ is } f\text{-nef}\}$.

Lemma 2.1. *Let D and D' be divisors on M.*

(1) If $\pi(D') \in \oplus_{i=1}^n \mathbb{Z}A_i$, then $\langle D + D' \rangle = \langle D \rangle + \pi(D')$.
(2) $\langle D - \langle D \rangle \rangle = 0$.

Proof. If $\pi(D')$ is a cycle, then $\mathcal{C}(D + D') = \{G + \pi(D') | \ G \in \mathcal{C}(D)\}$. Hence (1) holds. (2) follows from (1). Q.E.D.

Let $\mathcal{E} = \{D \in \operatorname{Div}(M) | \langle D \rangle = 0\}$. We define a map $\alpha : \mathcal{E} \to \mathbb{Z}$ by $\alpha(D) = h^1(\mathcal{O}_M(D)) - p_g(X, x)$, and a map $\beta : \mathcal{E} \to \mathbb{Q}$ by $\beta(D) = D \cdot (D - K)$.

Theorem 2.8. *(Morales [15]) Let D be a divisor on M. Then we have*

$$\chi_M(D) = -D \cdot (D - K)/2 + p_g(X, x) + \epsilon(D),$$

where $\epsilon(D) = \alpha(D - \langle D \rangle) + \beta(D - \langle D \rangle)/2$.

3. Proofs of Theorems 1.1 and 1.2

We assume that $f : (M, A) \to (X, x)$ is the minimal good resolution. Let $P + N$ be the Zariski decomposition of the divisor $K + A$, where P is an f-nef cycle. We set $L_m = m(K + A)$. With the same notations as in previous subsections.

Lemma 3.1. *([17], Lemma 4.36) Suppose that (X, x) is not a quotient singularity. Let r be the minimal positive integer such that rN belongs to $\oplus \mathbb{Z} A_i$. We put $q = m - [m/r]r$ and $B_q = qN - [qN]$. Then*

$$h^1(\mathcal{O}_M(K + L_m)) = -m^2 N^2/2 - mK \cdot N/2 + \rho(m),$$

where $\rho(m) = B_q \cdot B_q/2 + K \cdot B_q/2 + h^0(\mathcal{O}_{[qN]}(K + L_q))$.

Theorem 3.1. *([17], Theorem 4.37) We have the formula*

$$\delta_{m+1}(X, x) = -m^2 P^2/2 - mK \cdot P/2 + \nu(m),$$

where ν is a bounded function. If (X, x) is not a quotient singularity, then

$$\nu(m) = p_g(X, x) + \epsilon(K + L_m) - \rho(m).$$

Lemma 3.2. *([17], Lemma 4.47) Suppose that (X, x) is a $\mathbb{Q}$-Gorenstein but not a log-canonical singularity. Then we have $\rho(r - 1) < 0$ if $N \neq 0$.*

Now we prove the theorems in section 1.

Proof of Theorem 1.1. "⇒". Since (X, x) is not a log-canonical singularity, so (X, x) is not a quotient singularity. By Theorem 3.1, we have

$$\delta_{m+1}(X, x) = -m^2 P^2/2 - mK \cdot P/2 + \nu(m),$$

where $\nu(m) = p_g(X, x) + \epsilon(K + L_m) - \rho(m)$. By assumption, $\delta_{m+1}(X, x) = -m^2 P^2/2 - mK \cdot P/2 + p_g(X, x)$. Thus $\epsilon(K + L_m) = \rho(m)$ for any $m \geq 0$. Let r be as in Lemma 3.1. Since (X, x) is a Gorenstein singularity, so $K + L_{r-1}$ is linearly equivalent to a cycle, $\epsilon(K + L_{r-1}) = 0$. Hence $\rho(r - 1) = 0$. By assumption, (X, x) is not a log-canonical singularity, thus it follows that $N = 0$ from Lemma 3.2. This means that $K + A$ is f-nef.

"$\Leftarrow$". Let $L = K + A$, then

$$\chi_M(K+L_m) = \dim_{\mathbb{C}} H^0(M \setminus A, \mathcal{O}_M(K+L_m))/H^0(M, \mathcal{O}_M(K+L_m)) \\ + h^1(M, \mathcal{O}_M(K+L_m)).$$

Since $K + A$ is f-nef, then $h^1(\mathcal{O}_M(K+L_m)) = 0$ by Theorem 2.3. Therefore, it follows from Proposition 2.1 that

$$\begin{aligned} \delta_{m+1} &= \dim_{\mathbb{C}} H^0(M \setminus A, \mathcal{O}_M((m+1)K))/H^0(M, \mathcal{O}_M((m+1)K + mA) \\ &= \dim_{\mathbb{C}} H^0(M \setminus A, \mathcal{O}_M(K+L_m))/H^0(M, \mathcal{O}_M(K+L_m)) \\ &= \chi_M(K+L_m). \end{aligned}$$

Since (X, x) is Gorenstein, so $K + L_m$ is a cycle. $K + A$ is f-nef, this implies $K + A = P$. On the one hand, by Theorem 2.7, we have $\chi_M(K + L_m) = -(K + L_m) \cdot L_m/2 + p_g(X, x) = -m^2P^2/2 - mK \cdot P/2 + p_g(X, x)$. On the other hand, by Theorem 3.1, we have $\delta_{m+1}(X, x) = -m^2P^2/2 - mK \cdot P/2 + \nu(m)$. Therefore $\nu(m) = p_g(X, x)$ for $m \geq 0$. Hence $\delta_{m+1}(X, x) = -m^2P^2/2 - mK \cdot P/2 + p_g(X, x)$. Q.E.D.

Proof of Corollary 1.1. Notice that if (X, x) is a simple elliptic or cusp singularity, then $K + A = 0$, $P = 0$ and $\delta_{m+1}(X, x) = 1, m \in \mathbb{N}$ (see Theorem 2.6), the corollary is obviously true. By Theorem 2.1, Theorem 2.5 and Theorem 2.6, if the (X, x) is a Gorenstein singularity and not a quotient singularity, simple elliptic or cusp singularity, then (X, x) is not a log-canonical singularity. Thus the corollary follows from Theorem 1.1 immediately. Q.E.D.

Proof of Theorem 1.2. Consider

$$0 \to \mathcal{O}_M(D - \langle D \rangle) \to \mathcal{O}_M(D) \to \mathcal{O}_{\langle D \rangle}(D) \to 0.$$

We have the exact sequence

$$0 \to H^0(\mathcal{O}_M(D - \langle D \rangle)) \to H^0(\mathcal{O}_M(D)) \to H^0(\mathcal{O}_{\langle D \rangle}(D)) \to \cdots .$$

Thus it suffices to show $H^0(\mathcal{O}_{\langle D \rangle}(D)) = 0$. Since $\langle D \rangle > 0$, so D is not f-nef. Hence there exists a component A_i such that $D \cdot A_i < 0$. Notice that $D - \langle D \rangle$ is f-nef, this implies $(D - \langle D \rangle) \cdot A_i \geq 0$. Hence $\langle D \rangle \cdot A_i < 0$ and thus $A_i \leq \langle D \rangle$. We have the exact sequence

$$0 \to H^0(\mathcal{O}_{\langle D \rangle - A_i}(D - A_i)) \to H^0(\mathcal{O}_{\langle D \rangle}(D)) \to H^0(\mathcal{O}_{A_i}(D)) = 0.$$

Notice that Lemma 2.1 implies that $\langle D - A_i \rangle = \langle D \rangle - A_i$. By induction, we obtain $H^0(\mathcal{O}_{\langle D \rangle}(D)) = 0$. Q.E.D.

Proof of Corollary 1.2. It is clear that $H^0(M \setminus A, \mathcal{O}_M(D)) = H^0(M \setminus A, \mathcal{O}_M(D - \langle D\rangle))$. If $\langle D\rangle > 0$, it follows from Theorem 1.2 that $H^0(M, \mathcal{O}_M(D - \langle D\rangle)) = H^0(M, \mathcal{O}_M(D))$. Hence we may assume that $\langle D\rangle = 0$, by Lemma 2.1, (2). Since the vector space

$$H^0(M \setminus A, \mathcal{O}_M(D))/H^0(M, \mathcal{O}_M(D))$$

is finite-dimensional, there exists a positive cycle V such that

$$H^0(M \setminus A, \mathcal{O}_M(D)) = H^0(M, \mathcal{O}_M(D + V)).$$

Since there exists the exact sequence

$$0 \to \mathcal{O}_M(D) \to \mathcal{O}_M(D + V) \to \mathcal{O}_V(D + V) \to 0,$$

it suffices to show that $H^0(\mathcal{O}_V(D + V)) = 0$. If $D + V$ is f-nef, then $\langle D\rangle \leq -V$ by the definition of $\langle D\rangle$: it contradicts that $V > 0$. Hence there exists a component A_i such that $(D + V) \cdot A_i < 0$. Since D is f-nef, $V \cdot A_i < 0$, and thus $A_i \leq V$. We have the exact sequence

$$0 \to H^0(\mathcal{O}_{V-A_i}(D + V - A_i)) \to H^0(\mathcal{O}_V(D + V)) \to H^0(\mathcal{O}_{A_i}(D + V)) = 0.$$

By induction, we obtain $H^0(\mathcal{O}_V(D + V)) = 0$. Q.E.D.

4. Local Cohomology and vanishing properties

Let $f : (M, A) \to (X, x)$ be the minimal good resolution of an isolated surface singularity (X, x), with exceptional divisor $A = \sum_i A_i$. For a positive cycle D on M, we write $N_D = \mathcal{O}_D(D)$.

Proposition 4.1 *([21]). Let $\mathcal{F}$ be locally free. Suppose for every cycle D there is an $A_i \subset |D|$ so that the natural map*

$$H^0(\mathcal{F} \otimes N_D) \to H^0(\mathcal{F} \otimes \mathcal{O}_{A_i}(D)) \tag{4.1}$$

is zero. Then $H^1_A(\mathcal{F}) = 0$.

Proposition 4.1 plays crucial role in the proof of the following Theorem 4.1. The strategy is that given D, if one can find $A_i \subset |D|$ such that $H^0(\mathcal{F} \otimes \mathcal{O}_{A_i}(D)) = 0$, then (4.1) is zero map. One gets a vanishing theorem.

Theorem 4.1 *([21]). If (X, x) is an isolated rational surface singularity, and $f : M \to X$ is the minimal good resolution with exceptional divisor $A = \sum_i A_i$, then*

$$H^1_A(M, \mathcal{O}_M(A)) = 0.$$

Theorem 4.1 follows from Proposition 4.1 and the following Proposition 4.2 first observed by Wahl [21].

Proposition 4.2. *If (X, x) is an isolated rational surface singularity, and $f : M \to X$ is the minimal good resolution with exceptional divisor $A = \sum_i A_i$, then for every positive cycle Z on M, there is an $A_i \subset |Z|$ with $(Z+A) \cdot A_i < 0$, i.e., $H^0(\mathcal{O}_M(A) \otimes \mathcal{O}_{A_i}(Z)) = 0$.*

Proof. Let $Z' = Z + A = \sum r_i A_i$, then all $r_i \geq 1$, and at least one $r_i > 1$. Let $d_i = -A_i \cdot A_i$. We want an A_i so that $r_i > 1$ and $Z' \cdot A_i < 0$. We proceed by induction on s, the number of components of A, the case $s = 1$ being obvious. Since the singularity is rational, so $\chi(Z') \geq 1$, thus $Z' \cdot (Z' + K) \leq -2$. Thus we have

$$\sum r_i(Z' \cdot A_i + d_i - 2) \leq -2.$$

It implies that there is a component (say A_1) with $Z' \cdot A_1 + d_1 - 2 \leq -1$, or $Z' \cdot A_1 \leq 1 - d_1 \leq -1$ (since all $d_i \geq 2$ for a rational singularity). If $r_1 > 1$, we are done; if $r_1 = 1$, then $Z' \cdot A_1 = (\sum r_j A_j) \cdot A_1 = -d_1 + \sum r_k$, where $\sum r_k$ is over those components intersecting A_1. Since $\sum r_k = Z' \cdot A_1 + d_1 \leq 1$, we have that A_1 intersects only one component (say A_2), with $r_2 = 1$. Thus, $Z' - A_1$ is supported on a divisor (corresponding to another singularity, still rational) with a smaller number of components; by induction, there is an $A_3 \subset |Z' - A_1|$, with $r_3 > 1$ (hence $A_3 \neq A_2$) and $(Z' - A_1) \cdot A_3 < 0$. Since $A_1 \cdot A_3 = 0$, we have $Z' \cdot A_3 < 0$, as desired. Q.E.D.

From the proof above, it is natural to introduce the following new concepts.

Definition 4.1. If (X, x) is an isolated normal surface singularity, and $f : M \to X$ is the minimal good resolution with exceptional divisor $A = \sum_i A_i$.

(1) If $H^1_A(M, \mathcal{O}_M(A)) = 0$, then we call (X, x) has weak vanishing property.

(2) If for every positive cycle Z on M, there is an $A_i \subset |Z|$ with $(Z + A) \cdot A_i < 0$, then we call (X, x) has strong vanishing property.

5. Proofs of Theorems 1.3 and 1.4

It follows from Proposition 4.1 that strong vanishing property implies weak vanishing property. Proposition 4.2 implies a rational singularity has strong vanishing property. Since minimally elliptic singularities are the most simple singularities after rational singularities, so a natural question is whether minimally elliptic singularities have strong vanishing property. This is answered in Theorem 1.3 and its corollary.

Proof of Theorem 1.3. Let $d_i = -A_i \cdot A_i$. Since X has a minimally elliptic singularity, so for all $Z > 0$, $\chi(Z) \geq 0$, thus

$$Z \cdot (K + Z) \leq 0.$$

Let $Z' = Z + A = \sum_{i=1}^{s} r_i A_i$, where $r_i \geq 1$ and with at least one i such that $r_i > 1$.

We claim that if $Z+A \neq Z_{min}$, then there exists an $A_i \subset |Z|$ such that $(Z+A) \cdot A_i < 0$.

Thus we want to show that there exists an A_i so that $r_i > 1$ and $Z' \cdot A_i < 0$. We proceed the proof by induction on s, the number of components of A. The case $s = 1$ is obvious.

By Proposition 2.2, there are two cases to be considered:

(a) f is also a minimal resolution and A_i's are rational.

In this case, we have $Z_{min} = -K$ (see Theorem 2.2), where K is the canonical cycle. Then we have

$$K \cdot A_i + A_i^2 = 2g - 2 = -2.$$

Since $Z' \cdot (K + Z') \leq 0$, we have

$$\sum r_i(-2 + d_i + A_i \cdot Z') \leq 0.$$

Firstly, we assume there exists a component (say A_1) such that $-2+d_1+A_1 \cdot Z' \leq -1$. Thus $A_1 \cdot Z' = (\sum r_i A_i) \cdot A_1 \leq 1 - d_1 \leq -1$ since f is a minimal resolution, all $d_i \geq 2$. If $r_1 > 1$, then we are done; otherwise, $(\sum r_i A_i) \cdot A_1 = -d_1 + \sum r_k \leq 1 - d_1$, where $\sum r_k$ are over all components intersecting with A_1. Thus $\sum r_k = 1$. Therefore A_1 intersects only one component (say A_2), with $r_2 = 1$. Thus $|Z' - A_1|$ has its number of components smaller than s and thus, by induction, there is an $A_3 \subset |Z' - A_1|$, with $r_3 > 1$ and $(Z' - A_1) \cdot A_3 < 0$. Since $A_1 \cdot A_3 = 0$, we have $Z' \cdot A_3 < 0$, as desired.

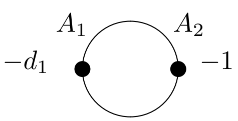
Secondly, for any $1 \leq i \leq s$, $(K + Z') \cdot A_i = -2 + d_i + A_i \cdot Z' = 0$.

We write $K + Z' = \sum t_i A_i$. On the one hand, $Z + A \neq Z_{min}$, so $Z + A \neq -K$, $K + Z' \neq 0$, thus $(K + Z')^2 < 0$. On the other hand, since $(K + Z') \cdot A_i = 0$, we have $(K + Z')^2 = (K + Z') \cdot \sum t_i A_i = 0$. This is a contradiction.

(b) When the minimal resolution does not equal to the minimal good resolution, there are five different cases (i)-(v) in Proposition 2.2. Notice that in this case Z_{min} is not necessarily to be the same as $-K$, since we are not assuming minimal resolution.

In case (i), there is only one component A_1 which is an elliptic curve such that $A_1 \cdot A_1 \leq -1$ and $-K = Z_{min} = A_1$. It is easy to see the validation of the claim.

In case (ii), the weighted dual graph of the minimal good resolution is



where $d_1 \geq 5$ and $Z_{min} = A_1 + 2A_2$. If $Z + A \neq Z_{min}$, then $Z \neq A_2$. Suppose that $Z = r_1 A_1 + r_2 A_2$, with $r_1 > 0$ or $r_2 > 1$. Consider firstly $r_1 > 0$ and $r_2 > 0$. We have

$$A_1 \cdot (Z + A) = A_1 \cdot ((r_1 + 1)A_1 + (r_2 + 1)A_2) = -(r_1 + 1)d_1 + 2(r_2 + 1),$$

and

$$A_2 \cdot (Z + A) = A_2 \cdot ((r_1 + 1)A_1 + (r_2 + 1)A_2) = 2(r_1 + 1) - (r_2 + 1).$$

We claim that either $A_1 \cdot (Z + A) \leq -1$ or $A_2 \cdot (Z + A) \leq -1$. Otherwise, if both $A_1 \cdot (Z + A) \geq 0$ and $A_2 \cdot (Z + A) \geq 0$, then $-(r_1 + 1)d_1 + 2(r_2 + 1) \geq 0$ and $2(r_1 + 1) - (r_2 + 1) \geq 0$, thus $(4 - d_1)(r_1 + 1) \geq 0$ which contradicts with $d_1 \geq 5$.

If $r_1 > 0$ and $r_2 = 0$, then we have

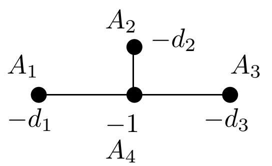
$$A_1 \cdot (Z + A) = A_1 \cdot ((r_1 + 1)A_1 + A_2) = -(r_1 + 1)d_1 + 2 < 0.$$

If $r_1 = 0$ and $r_2 > 1$, then we have

$$A_2 \cdot (Z + A) = A_2 \cdot (A_1 + (r_2 + 1)A_2) = 2 - (r_2 + 1) < 0.$$

Thus the claim is true in this case.

In cases (iii)–(v), the weighted dual graph of the minimal good resolution is



where $d_i \geq 2$, $1 \leq i \leq 3$, and $\frac{1}{d_1} + \frac{1}{d_2} + \frac{1}{d_3} < 1$ (cf. [17], Remark 2.45).

Let $Z = r_1A_1 + r_2A_2 + r_3A_3 + r_4A_4$ be a cycle, where $r_1, r_2, r_3, r_4 \geq 0$ but not all 0. We claim that there is an $A_i \subset |Z|$ with $A_i \cdot (Z + A) < 0$. We only need to consider the following cases:

(1) $r_1, r_2, r_3 > 0$,	(2) $r_1 = r_2 = r_3 = 0$,
(3) $r_1 = 0, r_2, r_3 > 0$,	(4) $r_2 = 0, r_1, r_3 > 0$,
(5) $r_3 = 0, r_1, r_2 > 0$,	(6) $r_1 = r_2 = 0, r_3 > 0$,
(7) $r_2 = r_3 = 0, r_1 > 0$,	(8) $r_3 = r_1 = 0, r_2 > 0$.

We first prove the following lemma.

Lemma 5.1. *Let $Z = r_1A_1 + r_2A_2 + r_3A_3 + r_4A_4$ be a cycle, with $r_1, r_2, r_3, r_4 \geq 0$ but not all 0. Then each one of the following three conditions*

1) $r_i > 0$ for $i = 1, 2, 3, 4$;

2) $r_4 = 0$;

3) $r_4 > (r_1 + r_2 + r_3) + 2$;

implies that there exists an $A_i \subset |Z|$ with $A_i \cdot (Z + A) < 0$.

Proof. Since

$$A_1 \cdot (Z + A) = -(r_1 + 1)d_1 + r_4 + 1, \tag{5.1}$$

$$A_2 \cdot (Z + A) = -(r_2 + 1)d_2 + r_4 + 1, \tag{5.2}$$

$$A_3 \cdot (Z + A) = -(r_3 + 1)d_3 + r_4 + 1, \tag{5.3}$$

$$A_4 \cdot (Z + A) = -(r_4 + 1) + (r_1 + 1) + (r_2 + 1) + (r_3 + 1). \tag{5.4}$$

For case 1), we claim that there exists i, such that $A_i \cdot (Z + A) < 0$. If not, then $A_i \cdot (Z + A) \geq 0$ for $i = 1, 2, 3, 4$. It follows from (5.1)-(5.4) that

$$\frac{1}{d_1} + \frac{1}{d_2} + \frac{1}{d_3} \geq 1,$$

which contradicts with $\frac{1}{d_1} + \frac{1}{d_2} + \frac{1}{d_3} < 1$.

The case 2) follows from (5.1)-(5.3) easily.

The case 3) follows from (5.4) directly. Q.E.D.

Now we continue with the proof of Theorem 1.3. Without loss of generality, we assume that $d_3 \geq d_2 \geq d_1 \geq 2$. Since $\frac{1}{d_1} + \frac{1}{d_2} + \frac{1}{d_3} < 1$, so we divided it into two cases:

1). $d_3 \geq d_2 \geq d_1 \geq 3$,

2). $d_3 \geq d_2 \geq 3, d_1 = 2$.

It follows from 2), 3) in Lemma 5.1 that we only need to consider the case when $0 < r_4 \leq (r_1 + r_2 + r_3) + 2$.

In case (1), we have $r_1, r_2, r_3 > 0$, and $r_4 > 0$. It follows from Lemma 5.1 that the claim is true.

In case (2), combining $r_1 = r_2 = r_3 = 0$ with $0 < r_4 \leq (r_1 + r_2 + r_3) + 2$, we have $r_4 = 1$ or 2, i.e., $Z = A_4$ or $2A_4$, $A_4 \cdot (Z + A) = 1 + 1 + 1 - (r_4 + 1) \geq 0$, so the claim can't hold. Thus we get a exceptional case 1) in Theorem 1.3.

In case (3), we have $r_1 = 0, r_2, r_3 > 0$, and $r_4 > 0$. By contradiction we assume that

$$A_2 \cdot (Z + A) = -(r_2 + 1)d_2 + r_4 + 1 \geq 0, \tag{5.5}$$

$$A_3 \cdot (Z + A) = -(r_3 + 1)d_3 + r_4 + 1 \geq 0, \tag{5.6}$$

$$A_4 \cdot (Z + A) = -(r_4 + 1) + 1 + (r_2 + 1) + (r_3 + 1) \geq 0. \tag{5.7}$$

It follows from (5.5)-(5.7) that $\frac{1}{d_2} + \frac{1}{d_3} \geq \frac{r_4}{r_4+1}$. Since $d_3 \geq d_2 \geq 3$, so $r_4 = 1$ or 2. However it follows from (5.5) that $r_4 \geq 5$. This is a contradiction. Thus the claim is true.

In case (4), we have $r_2 = 0, r_1, r_3 > 0$, and $r_4 > 0$. By contradiction we assume that

$$A_1 \cdot (Z + A) = -(r_1 + 1)d_1 + r_4 + 1 \geq 0, \tag{5.8}$$

$$A_3 \cdot (Z + A) = -(r_3 + 1)d_3 + r_4 + 1 \geq 0, \tag{5.9}$$

$$A_4 \cdot (Z + A) = -(r_4 + 1) + (r_1 + 1) + 1 + (r_3 + 1) \geq 0. \tag{5.10}$$

It follows from (5.8)-(5.10) that $\frac{1}{d_1} + \frac{1}{d_3} \geq \frac{r_4}{r_4+1}$. If $d_3 \geq d_2 \geq d_1 \geq 3$, then the claim is true as in Case (3). If $d_3 \geq d_2 \geq 3, d_1 = 2$, notice that $\frac{1}{d_1} + \frac{1}{d_2} + \frac{1}{d_3} < 1$, so $d_3 \geq 7$, then $r_4 = 1$. However it follows from (5.8) that $r_4 \geq 3$. This is a contradiction. Thus this validates the claim.

In the case (5), we have $r_3 = 0, r_1, r_2 > 0$, and $r_4 > 0$. By contradiction we assume that

$$A_1 \cdot (Z + A) = -(r_1 + 1)d_1 + r_4 + 1 \geq 0, \tag{5.11}$$

$$A_2 \cdot (Z + A) = -(r_2 + 1)d_2 + r_4 + 1 \geq 0, \tag{5.12}$$

$$A_4 \cdot (Z + A) = -(r_4 + 1) + (r_1 + 1) + (r_2 + 1) + 1 \geq 0. \tag{5.13}$$

It follows from (5.11)-(5.13) that $\frac{1}{d_1} + \frac{1}{d_2} \geq \frac{r_4}{r_4+1}$. If $d_3 \geq d_2 \geq d_1 \geq 3$, then the claim is true. If $d_3 \geq d_2 \geq 3, d_1 = 2$, then $r_4 = 1, 2, 3, 4$, or 5. However it follows from (5.12) that $r_4 \geq 5$. Thus we have $r_4 = 5, r_2 = 1, r_1 = 2, d_2 = 3, d_1 = 2, d_3 \geq 7, d_4 = -1$. And $Z = 2A_1 + A_2 + 5A_4$. It is easy to check that $Z_{min} = 3A_1 + 2A_2 + A_3 + 6A_4$. Thus $Z + A = Z_{min}$. This contracts with our assumption $Z + A \neq Z_{min}$. Therefore this validates the claim.

In the case (6), we have $r_1 = r_2 = 0$, $r_3 > 0$, and $r_4 > 0$. By contradiction we assume that

$$A_3 \cdot (Z + A) = -(r_3 + 1)d_3 + r_4 + 1 \geq 0, \tag{5.14}$$

$$A_4 \cdot (Z + A) = -(r_4 + 1) + 1 + 1 + (r_3 + 1) \geq 0. \tag{5.15}$$

It follows from (5.14)-(5.15) that $r_3 \leq \frac{3-d_3}{d_3-1}$. Since $d_3 \geq 3$, so $r_3 \leq 0$ which contradicts with the assumption $r_3 > 0$. This validates the claim.

In the case (7), we have $r_2 = r_3 = 0, r_1 > 0$, and $r_4 > 0$. By contradiction we assume that

$$A_1 \cdot (Z + A) = -(r_1 + 1)d_1 + r_4 + 1 \geq 0, \tag{5.16}$$

$$A_4 \cdot (Z + A) = -(r_4 + 1) + (r_1 + 1) + 1 + 1 \geq 0. \tag{5.17}$$

It follows from (5.16)-(5.17) that $r_1 \leq \frac{3-d_1}{d_1-1}$. If $d_3 \geq d_2 \geq d_1 \geq 3$, then the claim is true. If $d_3 \geq d_2 \geq 3, d_1 = 2$, then $r_1 = 1, r_4 = 3$. Notice that if $d_2 \geq 4$, then $Z_{min} = 2A_1 + A_2 + A_3 + 4A_4$. Combining the condition $Z + A \neq Z_{min}$, we conclude that $d_2 = 3$. Thus we get a exceptional case 2) in Theorem 1.3.

In the case (8) we have $r_1 = r_3 = 0, r_2 > 0$, and $r_4 > 0$. We assume by contradiction that

$$A_2 \cdot (Z + A) = -(r_2 + 1)d_2 + r_4 + 1 \geq 0, \tag{5.18}$$

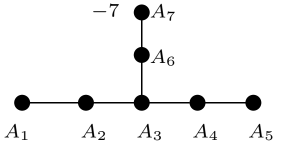
$$A_4 \cdot (Z + A) = -(r_4 + 1) + 1 + (r_2 + 1) + 1 \geq 0. \tag{5.19}$$

It follows from (5.18)-(5.19) that $r_2 \leq \frac{3-d_2}{d_2-1}$. Then the claim is true. Q.E.D.

Remark 5.1. In Theorem 1.3, if $Z + A = Z_{min}$, then in general we can not find $A_i \subset |Z|$, such that $(Z + A) \cdot A_i < 0$. We have only $(Z + A) \cdot A_i \leq 0$ for any $A_i \subset |Z|$. We give three examples as follows.

In the following weighted dual graphs, the weights $-d_i$'s are omitted if $d_i = 2$.

Example 1.

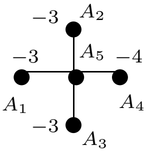


For this example,

$$\begin{array}{rccccc} & & & 1 & & \\ & & & 2 & & \\ Z_{min} = & 1 & 2 & 3 & 2 & 1 \end{array}$$

i.e., $Z_{min} = A_1 + 2A_2 + 3A_3 + 2A_4 + A_5 + 2A_6 + A_7$. Suppose $Z + A = Z_{min}$, i.e., $Z = A_2 + 2A_3 + A_4 + A_6$ and for any $A_i \subset |Z|$, we have $(Z + A) \cdot A_i = 0$.

Example 2.

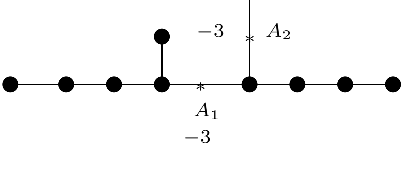


For this example,

$$\begin{array}{rccc} & & 1 & \\ Z_{min} = 1 & & 2 & 1 \\ & & 1 & \end{array}$$

i.e., $Z_{min} = A_1 + A_2 + A_3 + A_4 + 2A_5$. Suppose that $Z + A = Z_{min}$, thus $Z = A_5$ and $(Z + A) \cdot A_5 = 0$.

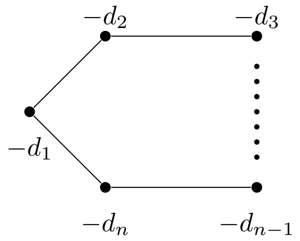
Example 3.



$$\begin{array}{lccccccccc} & & & & & & 1 & & & \\ & & & & 2 & & 2 & & & \\ Z_{min} = & 1 & 2 & 3 & 4 & 3 & 4 & 3 & 2 & 1 \end{array}$$

For this example, suppose that $Z + A = Z_{min}$, then we know that $A_1 \subset |Z|$ and $A_1 \cdot (Z + A) = A_1 \cdot Z_{min} = -1$.

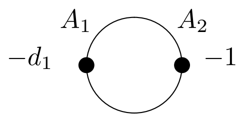
Before we give the proof of Corollary 1.3, we first recall the definition of cusp singularities and simple elliptic singularities. Suppose $f : (M, A) \to (X, x)$ is the minimal resolution. Then (X, x) is called a simple elliptic singularity if A is a nonsingular elliptic curve. (X, x) is called a cusp singularity if A is a cycle of nonsingular rational curves (see the dual resolution graph below), or a rational curve with a node. It is clear that simple elliptic or cusp singularity are minimally elliptic.



where $d_i \geq 2$ for all $i = 1, \cdots, n$ $(n \geq 3)$ and negative definite intersection matrix $(A_i \cdot A_j)_{i,j}$ (or equivalently, $\sum_i (d_i - 2) > 0$).

Corollary 5.1. *Cusp singularities have strong vanishing property.*

Proof. Let $f : (M, A) \to (X, x)$ be the minimal good resolution of the cusp singularity (X, x), with exceptional divisor $A = \sum_i A_i$. If A is a cycle of nonsingular rational curves, then by Proposition 2.2, f is also a minimal resolution. So the dual resolution graph is a cycle as above. It is easy to calculate that $Z_{min} = A$, so for any positive cycle Z supported on A. We have $Z + A \neq Z_{min}$. The condition of the Theorem 1.3 is satisfied. The corollary follows immediately from Theorem 1.3. If the minimal resolution of (X, x) is a rational curve with a node, then the dual resolution graph of minimal good resolution is as follows:



where $d_1 \geq 5$. Then the corollary also follows from the proof the Theorem 1.3. Q.E.D.

Corollary 5.2. *Simple elliptic singularities have strong vanishing property.*

Proof. Let $f : (M, A) \to (X, x)$ be the minimal good resolution of the simple elliptic singularity (X, x), with exceptional divisor $A = A_1$. A is a nonsingular elliptic curve. The corollary follows from the proof the Theorem 1.3. Q.E.D.

Recall that isolated Du Bois singularities are characterized as follows.

Theorem 5.1. *(Steenbrink, [20], (3.6)). Let (X, x) be an isolated singularities and f : $(M, A) \to (X, x)$ a good resolution. Then (X, x) is Du Bois if and only if the natural map*

$$H^i(M, \mathcal{O}_M) \to H^i(A, \mathcal{O}_A)$$

is isomorphism for $i > 0$.

If (X, x) is an isolated Gorenstein surface singularity, then (X, x) is a Du Bois singularity if and only if (X, x) is either rational, simple elliptic or cusp (cf. [3]).

Corollary 5.3. *Gorenstein Du Bois singularities have strong vanishing property.*

Proof. It follows from the Proposition 4.2, Corollary 5.1 and Corollary 5.2. Q.E.D.

Proof of Corollary 1.3. It follows from the Corollary 5.1, Corollary 5.2 and Corollary 5.3. Q.E.D.

Lemma 5.2. *Let $f : (M, A) \to (X, x)$ be any resolution of a surface singularity (X, x). Let A' be a connected cycle such that $0 < A' \leq A$. Then A' is contractible to a surface singularity.*

Proof. Suppose that $A' = \sum_{i=1}^{k} A_i$. Let Z be the fundamental cycle on M. Consider the sequence of positive cycles:

$$Z_1 = A_1, \cdots, Z_{i+1} = Z_i + A_{j_i}, \cdots$$

where $j_i \in \{1, \cdots, k\}$ and $Z_i \cdot A_{j_i} > 0$. Since $Z_i \leq Z$ for each Z_i, the sequence will end. Hence we obtain the positive cycle Z' such that $\mathrm{Supp}(Z') = A'$, and $Z' \cdot A_i \leq 0$ for $i = 1, \cdots, k$. Since the intersection matrix of A is negative definite, $Z' \cdot Z' < 0$. By Artin's result (cf. [1], Proposition 2), the intersection matrix of A' is also negative definite. Thus the assertion follows. Q.E.D.

Proof of Theorem 1.4. Let M' be a neighborhood of A' and $f' : (M', A') \to (X', x')$ a morphism, which contracts A'. We assume that X' is Stein. If $A' = A$, then it is done. Therefore we assume that $A' < A$. For every positive cycle Z supported on A', due to the (X, x) has strong vanishing property, there is an $A_i \subset |Z|$ with $(Z + A) \cdot A_i < 0$.

We claim that $(Z + A') \cdot A_i < 0$. Let $A - A' = A_{j_1} + \cdots + A_{j_s} (s \geq 1)$. $(Z + A) \cdot A_i = (Z + A' + A_{j_1} + \cdots + A_{j_s}) \cdot A_i = (Z + A') \cdot A_i + (A_{j_1} + \cdots + A_{j_s}) \cdot A_i < 0$. Since $A_i \subset |Z| \subset A'$, so $(A_{j_1} + \cdots + A_{j_s}) \cdot A_i \geq 0$, then $(Z + A') \cdot A_i < 0$. Thus the assertion follows. Q.E.D.

References

[1] M. Artin, On isolated rational singularities of surfaces, Amer. J. Math. 88 (1966) 129–136.
[2] J. Giraud, Improvement of Grauert-Riemenschneider's theorem for a normal surface, Ann. Inst. Fourier (Grenoble) 32 (4) (1982) 13–23.
[3] S. Ishii, On isolated Gorenstein singularities, Math. Ann. 270 (1985) 541–554.
[4] S. Ishii, Isolated $\mathbb{Q}$-Gorenstein singularities of dimension three, in: T. Suwa, P. Wagreich (Eds.), Complex Analytic Singularities, in: Advanced Studies in Pure Math., vol. 8, Kinokuniya, Tokyo, North-Holland, Amsterdam, 1986, pp. 165–198.
[5] S. Ishii, Small deformation of normal singularities, Math. Ann. 275 (1986) 139–148.
[6] S. Ishii, Two dimensional singularities with bounded plurigenera δ_m are $\mathbb{Q}$-Gorenstein singularities, Contemp. Math. 90 (1989) 135–145.
[7] S. Ishii, The asymptotic behavior of plurigenera for a normal isolated singularity, Math. Ann. 286 (1990) 803–812.
[8] M. Kato, Riemann-Roch theorem for strongly pseudoconvex manifolds of dimension 2, Math. Ann. 222 (1976) 243–250.
[9] Y. Kawamata, The cone of curves of algebraic varieties, Ann. of Math. 119 (1984) 603–633.
[10] Y. Kawamata, Crepant blowing-up of 3-dimensional canonical singularities and its application to degenerations of surfaces, Ann. of Math. 127 (1988) 93–163.
[11] H. Laufer, Normal Two-Dimensional Singularities, Ann. of. Math. Studies, vol. 71, Princeton University Press, 1971.
[12] H. Laufer, On rational singularities, Amer. J. Math. 94 (1972) 597–608.
[13] H. Laufer, On minimally elliptic singularities, Amer. J. Math. 99 (1977) 1257–1295.
[14] K.-P. Lin, S.S.-T. Yau, H. Zuo, Plurigenera of compact strongly pseudoconvex CR manifolds, Sci. China Math. 58 (2015) 525–530.
[15] M. Morales, Calcul de quelques invariants des singularités de surface normale, Enseign. Math. 31 (1983) 191–203.
[16] W.D. Neumann, A calculus for plumbing applied to the topology of complex surface singularities and degenerating complex curves, Trans. Amer. Math. Soc. 268 (1981) 299–344.
[17] T. Okuma, Plurigenera of Surface Singularities, Nova Science Publishers, Inc., Huntington, NY.
[18] F. Sakai, Kodaira dimension of complements of divisors, in: W.L. Baily, T. Shioda (Eds.), Complex Analysis and Algebraic Geometry, Iwanami Shoten, Cambridge University Press, 1977, pp. 239–257.
[19] F. Sakai, Anticanonical models of rational surfaces, Math. Ann. 269 (1984) 389–410.
[20] J. Steenbrink, Mixed Hodge structures associated with isolated singularities, in: P. Orlik, et al. (Eds.), Singularities, in: Proc. Sympos. Pure. Math., vol. 40, Amer. Math. Soc., 1983, pp. 513–536, Part 2.
[21] J. Wahl, Vanishing theorems for resolutions of surface singularities, Invent. Math. 31 (1975) 17–41.
[22] J. Wahl, A characteristic number for links of surface singularities, J. Amer. Math. Soc. 3 (1990) 625–637.
[23] P. Wagreich, Elliptic singularities of surfaces, Amer. J. Math. 92 (1970) 419–454.
[24] K. Watanabe, On plurigenera of normal isolated singularities I, Math. Ann. 250 (1980) 65–94.
[25] S.S.-T. Yau, Two theorems on higher dimensional singularities, Math. Ann. 231 (1977) 55–59.
[26] S.S.-T. Yau, On almost minimally elliptic singularities, Bull. Amer. Math. Soc. 83 (1977) 362–364.
[27] S.S.-T. Yau, Normal two-dimensional elliptic singularities, Trans. Amer. Math. Soc. 254 (1979) 117–134.
[28] S.S.-T. Yau, On strongly elliptic singularities, Amer. J. Math. 101 (1979) 855–884.
[29] S.S.-T. Yau, On maximally elliptic singularities, Trans. Amer. Math. Soc. 257 (1980) 269–329.
[30] S.S.-T. Yau, H. Zuo, A sharp lower bound for the geometric genus and Zariski multiplicity question, Math. Z. 289 (2018) 1299–1310.