INVERSE CURVATURE FLOWS IN HYPERBOLIC SPACE

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Dedicated to Stefan Hildebrandt on the occasion of his 75th birthday.

ABSTRACT. We consider inverse curvature flows in $\mathbb{H}^{n+1}$ with star-shaped initial hypersurfaces and prove that the flows exist for all time, and that the leaves converge to infinity, become strongly convex exponentially fast and also more and more totally umbilic. After an appropriate rescaling the leaves converge in C^∞ to a sphere.

CONTENTS

1. INTRODUCTION

Curvature flows (driven by extrinsic curvatures) of compact hypersurfaces in a Riemannian space generally exist only for a finite time and then develop a singularity provided the flow is a pure curvature flow without an additional force term. This phenomenon occurs in the case of direct flows, which can also be characterized as *contracting* flows, cf. [9], as well as for inverse flows, which can also be characterized as *expanding* flows, see [10].

In non-compact spaces of constant curvature we can expect that the inverse flows behave differently than the direct flows, since the inverse flows of geodesic spheres exist for all time. In [3] we proved that inverse curvature flows of star-shaped hypersurfaces in Euclidean space exist for all time, converge to infinity and, after rescaling, converge to spheres.

In this paper we want to prove a similar result in hyperbolic space $\mathbb{H}^{n+1}$, $n \geq 2$. The initial hypersurface M_0 is supposed to be star-shaped with respect

to a given point $p \in \mathbb{H}^{n+1}$, i.e., after introducing geodesic polar coordinates with center p, M_0 can be written as a graph over a geodesic sphere with center p which we identify topologically with the standard sphere $\mathbb{S}^n$. Let F be a smooth curvature function, homogeneous of degree 1, monotone, and concave, defined in a symmetric, convex, open cone $\Gamma \subset \mathbb{R}^n$, such that

$$(1.1) \qquad\qquad F_{|\Gamma} > 0 \quad \wedge \quad F_{|\partial\Gamma} = 0.$$

Then we consider the curvature flow

$$(1.2) \qquad\qquad \dot{x} = -\Phi\nu$$

with initial hypersurface M_0, the principal curvatures of which are supposed to lie in the cone Γ; such a hypersurface is called *admissible*. Here, the function Φ is defined by

$$(1.3) \qquad\qquad \Phi = \Phi(r) = -r^{-1}, \qquad r > 0,$$

and $\Phi\nu$ stands for

$$(1.4) \qquad\qquad \Phi(F)\nu,$$

i.e., the flow equation is

$$(1.5) \qquad\qquad \dot{x} = \frac{1}{F}\nu,$$

where ν is outward normal.

However, to simplify comparisons with former results and formulas, and also to make a generalization to more general flows easier, most results in this paper are formulated, and some are also proved, for a general smooth real valued function Φ defined on the positive real axis satisfying

$$(1.6) \qquad\qquad \dot{\Phi} > 0 \quad \wedge \quad \ddot{\Phi} \le 0.$$

We shall normalize F such that

$$(1.7) \qquad\qquad F(1,\dots,1) = n$$

and shall also use the same notation F when we assume F to depend on the second fundamental form h_j^i instead of the principal curvatures. Sometimes we also use the notation $\breve{h}_j^i$ for the second fundamental form of a hypersurface embedded in $\mathbb{H}^{n+1}$ to distinguish it from the second fundamental form h_j^i of the same hypersurface viewed as being embedded in $\mathbb{R}^{n+1}$, which will happen when we parameterize $\mathbb{H}^{n+1}$ over the open ball $B_2(0) \subset \mathbb{R}^{n+1}$.

We can now state our first result.

1.1. **Theorem.** *The flow* (1.5) *with a smooth and admissible initial hypersurface M_0 exists for all time. The flow hypersurfaces in hyperbolic space converge to infinity, become strongly convex exponentially fast and also more and more totally umbilic. In fact there holds*

$$(1.8) \qquad\qquad |\breve{h}_j^i - \delta_j^i| \le ce^{-\frac{t}{n}},$$

i.e., the principal curvatures are uniformly bounded and converge exponentially fast to 1.

For a more detailed analysis of the asymptotic behaviour we have to parameterize $\mathbb{H}^{n+1}$ over $B_2(0) \subset \mathbb{R}^{n+1}$ such that the metric can be expressed in the form

$$
\begin{aligned}
(1.9) \qquad d\breve{s}^2 &= \frac{1}{(1 - \frac{1}{4}|x|^2)^2} dx^2 \\
&= \frac{1}{(1 - \frac{1}{4}r^2)^2} \{dr^2 + r^2 \sigma_{ij} dx^i dx^j\}.
\end{aligned}
$$

The flow hypersurfaces $M(t)$ can now also be viewed as graphs

$$
(1.10) \qquad M(t) = \operatorname{graph} u(t, \cdot)
$$

over $\mathbb{S}^n$ in Euclidean space, such that $0 < u < 2$, and convergence to infinity is tantamount to $u \to 2$. The second fundamental form in $\mathbb{R}^{n+1}$ is denoted by h_j^i, or simply by A, where we omit the tensor indices. Then, we can prove:

1.2. **Theorem.** *Let $M(t) = \operatorname{graph} u(t)$ be the leaves of the inverse curvature flow, where F and the initial hypersurface are smooth, then the estimate*

$$
(1.11) \qquad \|D^m A\| \le c_m e^{-\frac{t}{n}} \qquad \forall m \ge 1
$$

is valid and the function

$$
(1.12) \qquad (u - 2) e^{\frac{t}{n}}
$$

converges in $C^\infty(\mathbb{S}^n)$ to a strictly negative function.

1.3. **Remark.** After publishing a first version of the paper in the arXiv we learnt that Qi Ding in [1] published a similar result for the inverse mean curvature flow in $\mathbb{H}^{n+1}$ even claiming that the rescaled flow hypersurfaces would converge to a sphere. However, he used a somewhat crude rescaling, namely,

$$
(1.13) \qquad \frac{\breve{u}}{t}
$$

and not the finer

$$
(1.14) \qquad \breve{u} - \frac{t}{n}
$$

which we consider.

The fact that the functions in (1.13) converge to $\frac{1}{n}$ follows immediately from the estimate (3.19) on page 7.

2. Definitions and Conventions

The main objective of this section is to state the equations of Gauß, Codazzi, and Weingarten for hypersurfaces. For greater generality we shall formulate the governing equations of a hypersurface M in a semi-riemannian (n+1)-dimensional manifold N, which is either Riemannian or Lorentzian. Geometric quantities in N will be denoted by $(\bar{g}_{\alpha\beta}), (\bar{R}_{\alpha\beta\gamma\delta})$, etc., and those in M by $(g_{ij}), (R_{ijkl})$, etc. Greek indices range from 0 to n and Latin from 1 to n; the summation convention is always used. Generic coordinate systems in N resp. M will be denoted by (x^α) resp. (ξ^i). Covariant differentiation

will simply be indicated by indices, only in case of possible ambiguity they will be preceded by a semicolon, i.e., for a function u in N, (u_α) will be the gradient and $(u_{\alpha\beta})$ the Hessian, but e.g., the covariant derivative of the curvature tensor will be abbreviated by $\bar{R}_{\alpha\beta\gamma\delta;\epsilon}$. We also point out that

$$(2.1) \qquad \bar{R}_{\alpha\beta\gamma\delta;i} = \bar{R}_{\alpha\beta\gamma\delta;\epsilon} x_i^\epsilon$$

with obvious generalizations to other quantities.

Let M be a *spacelike* hypersurface, i.e., the induced metric is Riemannian, with a differentiable normal ν. We define the signature of ν, $\sigma = \sigma(\nu)$, by

$$(2.2) \qquad \sigma = \bar{g}_{\alpha\beta}\nu^\alpha\nu^\beta = \langle \nu, \nu \rangle.$$

In case N is Lorentzian, $\sigma = -1$, and ν is time-like.

In local coordinates, (x^α) and (ξ^i), the geometric quantities of the spacelike hypersurface M are connected through the following equations

$$(2.3) \qquad x_{ij}^\alpha = -\sigma h_{ij}\nu^\alpha$$

the so-called *Gauß formula*. Here, and also in the sequel, a covariant derivative is always a *full* tensor, i.e.,

$$(2.4) \qquad x_{ij}^\alpha = x_{,ij}^\alpha - \Gamma_{ij}^k x_k^\alpha + \bar{\Gamma}_{\beta\gamma}^\alpha x_i^\beta x_j^\gamma.$$

The comma indicates ordinary partial derivatives.

In this implicit definition the *second fundamental form* (h_{ij}) is taken with respect to $-\sigma\nu$.

The second equation is the *Weingarten equation*

$$(2.5) \qquad \nu_i^\alpha = h_i^k x_k^\alpha,$$

where we remember that ν_i^α is a full tensor.

Finally, we have the *Codazzi equation*

$$(2.6) \qquad h_{ij;k} - h_{ik;j} = \bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta x_j^\gamma x_k^\delta$$

and the *Gauß equation*

$$(2.7) \qquad R_{ijkl} = \sigma\{h_{ik}h_{jl} - h_{il}h_{jk}\} + \bar{R}_{\alpha\beta\gamma\delta}x_i^\alpha x_j^\beta x_k^\gamma x_l^\delta.$$

Here, the signature of ν comes into play.

Now, let us assume that N is a topological product $\mathbb{R} \times \mathcal{S}_0$, where $\mathcal{S}_0$ is a compact Riemannian manifold, and that there exists a Gaussian coordinate system (x^α), such that the metric in N has the form

$$(2.8) \qquad d\bar{s}_N^2 = e^{2\psi}\{\sigma dx^{0^2} + \sigma_{ij}(x^0, x)dx^i dx^j\},$$

where σ_{ij} is a Riemannian metric, ψ a function on N, and x an abbreviation for the spacelike components (x^i),

We also assume that the coordinate system is *future oriented*, i.e., the time coordinate x^0 increases on future directed curves. Hence, the *contravariant* time-like vector $(\xi^\alpha) = (1, 0, \ldots, 0)$ is future directed as is its *covariant* version $(\xi_\alpha) = e^{2\psi}(\sigma, 0, \ldots, 0)$.

Let $M = \operatorname{graph} u_{|\mathcal{S}_0}$ be a spacelike hypersurface

$$(2.9) \qquad M = \{ (x^0, x) : x^0 = u(x), \ x \in \mathcal{S}_0 \},$$

then the induced metric has the form

$$(2.10) \qquad g_{ij} = e^{2\psi}\{\sigma u_i u_j + \sigma_{ij}\}$$

where σ_{ij} is evaluated at (u, x), and its inverse $(g^{ij}) = (g_{ij})^{-1}$ can be expressed as

$$(2.11) \qquad g^{ij} = e^{-2\psi}\{\sigma^{ij} - \sigma \frac{u^i}{v} \frac{u^j}{v}\},$$

where $(\sigma^{ij}) = (\sigma_{ij})^{-1}$ and

$$(2.12) \qquad \begin{aligned} u^i &= \sigma^{ij} u_j \\ v^2 &= 1 + \sigma \sigma^{ij} u_i u_j \equiv 1 + \sigma |Du|^2. \end{aligned}$$

The covariant form of a normal vector of a graph looks like

$$(2.13) \qquad (\nu_\alpha) = \pm v^{-1} e^{\psi}(1, -u_i).$$

and the contravariant version is

$$(2.14) \qquad (\nu^\alpha) = \pm v^{-1} e^{-\psi}(\sigma, -u^i).$$

In the Gauß formula (2.3) we are free to choose any of two normals, but we stipulate that in general we use

$$(2.15) \qquad (\nu^\alpha) = v^{-1} e^{-\psi}(\sigma, -u^i).$$

as normal vector.

Look at the component $\alpha = 0$ in (2.3), then we obtain

$$(2.16) \qquad e^{-\psi} v^{-1} h_{ij} = -u_{ij} - \bar{\Gamma}^0_{00} u_i u_j - \bar{\Gamma}^0_{0i} u_j - \bar{\Gamma}^0_{0j} u_i - \bar{\Gamma}^0_{ij}.$$

Here, the covariant derivatives a taken with respect to the induced metric of M, and

$$(2.17) \qquad -\bar{\Gamma}^0_{ij} = e^{-\psi} \bar{h}_{ij},$$

where $(\bar{h}_{ij})$ is the second fundamental form of the hypersurfaces $\{x^0 = \text{const}\}$.

3. First Estimates

Let $F \in C^{m,\alpha}(\Gamma)$, $m \geq 4$, be a monotone and concave curvature function, homogeneous of degree 1, and normalized such that

$$(3.1) \qquad F(1, \ldots, 1) = n.$$

We first look at the flow of a geodesic sphere S_{r_0}. Fix a point $p_0 \in \mathbb{H}^{n+1}$ and consider geodesic polar coordinates centered at p_0. Then the hyperbolic metric can be expressed as

$$(3.2) \qquad d\bar{s}^2 = dr^2 + \sinh^2 r \sigma_{ij} \, dx^i dx^j,$$

where σ_{ij} is the canonical metric of $\mathbb{S}^n$.

Geodesic spheres S_r with center in p_0 are umbilic and their second fundamental form is given by

$$(3.3) \qquad \bar{h}_{ij} = \coth r \, \bar{g}_{ij},$$

where

$$(3.4) \qquad \bar{g}_{ij} = \sinh^2 r \sigma_{ij}.$$

Hence, if we consider an inverse curvature flow (ICF) with initial hypersurface S_{r_0}, then the flow hypersurfaces $M(t)$ will be spheres with radii $r(t)$ satisfying the scalar curvature flow equation

$$(3.5) \qquad \dot r = \frac{1}{F} = \frac{1}{n \coth r},$$

and we deduce further, from

$$(3.6) \qquad \coth r \, dr = \frac{1}{n} dt,$$

$$(3.7) \qquad \log \sinh r - \log \sinh r_0 = \frac{t}{n},$$

or equivalently,

$$(3.8) \qquad \sinh r = \sinh r_0 e^{\frac{t}{n}}.$$

Let us now consider the inverse curvature flow of a star-shaped hypersurface M_0 which is given as a graph over $\mathbb{S}^n$

$$(3.9) \qquad M_0 = \operatorname{graph} u_{0|\mathbb{S}^n}.$$

The flow exists on a maximal time interval $[0, T^*)$, $0 < T^* \le \infty$, and its leaves are also graphs

$$(3.10) \qquad M(t) = \operatorname{graph} u(t)_{|\mathbb{S}^n},$$

which satisfy, besides the original flow equation,

$$(3.11) \qquad \dot x = -\Phi\nu = \frac{1}{F}\nu,$$

the scalar flow equation

$$(3.12) \qquad \dot u = \frac{\tilde v}{F},$$

where

$$(3.13) \qquad \tilde v = v^{-1}, \quad v^2 = 1 + |Du|^2 = 1 + \frac{1}{\sinh^2 u}\sigma^{ij}u_i u_j,$$

where the dot indicates a total time derivative. If we instead consider a partial time derivate, then we get

$$(3.14) \qquad \dot u \equiv \frac{\partial u}{\partial t} = \frac{v}{F},$$

cf. [6, p. 98].

Let S_{r_i}, $i = 1, 2$, be geodesic spheres satisfying

$$(3.15) \qquad r_1 < u_0 < r_2,$$

and let u_i, $i = 1, 2$, be the solutions to the corresponding inverse curvature flows, then this inequality will also be valid for $t > 0$, i.e.,

$$(3.16) \qquad u_1(t) < u(t) < u_2(t) \qquad \forall t \in [0, T^*),$$

in view of the maximum principle, and we conclude:

3.1. **Lemma.** *The solutions* $M(t) = \operatorname{graph} u(t)$ *of the ICF satisfy the estimates*

$$(3.17) \qquad \sinh r_1 < \sinh u(t) e^{-\frac{t}{n}} < \sinh r_2 \qquad \forall\, t \in [0, T^*),$$

and there exist constants c_i, $i = 1, 2$, *such that the function*

$$(3.18) \qquad \tilde{u} = u - \frac{t}{n}$$

is uniformly bounded by

$$(3.19) \qquad c_1 < \tilde{u}(t) < c_2 \qquad \forall\, t \in [0, T^*).$$

Proof. The inequality (3.17) follows from (3.8) and the parabolic maximum principle, while (3.19) is due to the trivial estimate

$$(3.20) \qquad 0 < \tilde{c}_1 \le \sinh r\, e^{-r} \le \tilde{c}_2 \qquad \forall\, 0 < r_0 \le r$$

with appropriate constants $\tilde{c}_i$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Next, we want to derive an a priori estimate for v, or equivalently, for

$$(3.21) \qquad |Du|^2 = \frac{1}{\sinh^2 u} \sigma^{ij} u_i u_j.$$

Let us write the metric (3.2) in a more general form

$$(3.22) \qquad d\bar{s}^2 = dr^2 + \vartheta^2(r)\sigma_{ij}dx^i dx^j.$$

The second fundamental form of $\operatorname{graph} u$ can then be expressed as

$$(3.23) \qquad \begin{aligned} h_{ij}v^{-1} &= -u_{ij} + \bar{h}_{ij} \\ &= -u_{ij} + \dot{\vartheta}\vartheta\sigma_{ij}. \end{aligned}$$

Define the metric

$$(3.24) \qquad \tilde{\sigma}_{ij} = \vartheta^2(u)\sigma_{ij},$$

and denote covariant differentiation with respect to this metric by a semicolon, then

$$(3.25) \qquad h_{ij}v^{-1} = -v^{-2}u_{;ij} + \dot{\vartheta}\vartheta\sigma_{ij},$$

cf. [6, Lemma 2.7.6], and we conclude further

$$(3.26) \qquad \begin{aligned} h^i_j &= g^{ik}h_{kj} \\ &= v^{-1}\vartheta^{-1}\{-(\sigma^{ik} - v^{-2}\varphi^i\varphi^k)\varphi_{jk} + \dot{\vartheta}\delta^i_j\}, \end{aligned}$$

where σ^{ij} is the inverse of σ_{ij},

$$(3.27) \qquad \varphi = \int_{r_0}^{u} \vartheta^{-1},$$

$$(3.28) \qquad \varphi^i = \sigma^{ik}\varphi_k,$$

and φ_{jk} are the second covariant derivatives of φ with respect to the metric σ_{ij}.

Thus, the scalar curvature equation (3.14) can now be expressed as

$$(3.29) \qquad \dot{u} = \frac{v}{F(h^i_j)},$$

or equivalently,

$$(3.30) \qquad \dot\varphi = \vartheta^{-1}\dot u = \frac{1}{F(\vartheta v^{-1}h^i_j)} \equiv \frac{1}{F(\tilde h^i_j)},$$

where

$$(3.31) \qquad \tilde h^i_j = v^{-2}\{-(\sigma^{ik} - v^{-2}\varphi^i\varphi^k)\varphi_{jk} + \dot\vartheta\delta^i_j\}.$$

Let

$$(3.32) \qquad \tilde g_{ij} = \varphi_i\varphi_j + \sigma_{ij},$$

then we consider the eigenvalues of

$$(3.33) \qquad \tilde h_{ij} = \tilde g_{ik}\tilde h^k_j$$

with respect to this metric and we define F^{ij} resp. F^i_j accordingly

$$(3.34) \qquad F^{ij} = \frac{\partial F}{\partial \tilde h_{ij}}$$

and

$$(3.35) \qquad F^i_j = \frac{\partial F}{\partial \tilde h^j_i} = \tilde g_{jk}F^{ik}.$$

Note that $\tilde h_{ij}$ is symmetric, since h_{ij} and $\tilde g_{ij}$ can be diagonalized simultaneously. We also emphasize that

$$(3.36) \qquad |Du|^2 = \sigma^{ij}\varphi_i\varphi_j \equiv |D\varphi|^2.$$

3.2. **Lemma.** *Let u be solution of the scalar curvature equation*

$$(3.37) \qquad \dot u = \frac{v}{F(h_{ij})},$$

then

$$(3.38) \qquad |Du|^2 \le c.$$

Moreover, if F is bounded from above

$$(3.39) \qquad F \le c_0,$$

then there exists $0 < \lambda = \lambda(c_0)$ such that

$$(3.40) \qquad |Du|^2 \le ce^{-\lambda t} \qquad \forall\, t \in [0, T^*).$$

Proof. „(3.38)"   In view of (3.36), we may estimate

$$(3.41) \qquad w = \tfrac{1}{2}|D\varphi|^2.$$

Differentiating equation (3.30) covariantly with respect to

$$(3.42) \qquad \varphi^k D_k$$

we deduce

$$(3.43) \qquad \begin{aligned} \dot w = F^{-2}\{ &2v^{-2}Fw_i\varphi^i + v^{-2}F^k_l\tilde g^{lr}w_{kr} - v^{-2}F^k_l\tilde g^{lr}\varphi_{ik}\varphi^i_r \\ &+ v^{-2}F^k_l\tilde g^{lr}{}_{;i}\varphi^i\varphi_{kr} + v^{-2}F^k_l\tilde g^{lr}\varphi_r\varphi_k - v^{-2}F^k_l\tilde g^{lr}\sigma_{kr}|D\varphi|^2 \\ &- 2v^{-2}F^k_k\ddot\vartheta\vartheta w\}, \end{aligned}$$

where covariant derivatives with respect to the metric σ_{ij} are simply denoted by indices, if no ambiguities are possible, and by a semi-colon otherwise. In deriving the previous equation we also used the Ricci identities and the properties of the Riemann curvature tensor of $\mathbb{S}^n$.

Now, let $0 < T < T^*$ and suppose that

$$(3.44) \qquad \sup_{Q_T} w, \qquad Q_T = [0, T] \times \mathbb{S}^n,$$

is attained at (t_0, x_0) with $t_0 > 0$. Then the maximum principle implies

$$(3.45) \qquad \begin{aligned} 0 \leq v^{-2}\{&-F_l^k \tilde{g}^{lr} \varphi_{ik} \varphi_r^i + (F_l^k \tilde{g}^{lr} \varphi_r \varphi_k - F_l^k \tilde{g}^{lr} \sigma_{kr} |D\varphi|^2) \\ &- 2F_k^k \sinh^2 u \, w\}. \end{aligned}$$

The right-hand side, however, is strictly negative, if $w > 0$, hence $t_0 > 0$ is not possible, since we didn't assume M_0 to be a sphere, and we conclude

$$(3.46) \qquad w \leq \sup_{\mathbb{S}^n} w(0).$$

„(3.40)" Now, assume that the original curvature function is uniformly bounded

$$(3.47) \qquad F(h_{ij}) \leq c_0,$$

and let $0 < \lambda$ be a constant, then

$$(3.48) \qquad \tilde{w} = we^{\lambda t}$$

satisfies the same equation as w with an additional term

$$(3.49) \qquad \lambda \tilde{w}$$

at the right-hand side.

Applying the maximum principle as before, we deduce, that at a point (t_0, x_0), $t_0 > 0$, where $\tilde{w}$ attains a positive maximum, there holds instead of (3.45)

$$(3.50) \qquad 0 < -2F_k^k \sinh^2 u \, \tilde{w} + \lambda v^2 F^2(\tilde{h}_j^i)\tilde{w},$$

but

$$(3.51) \qquad vF(\tilde{h}_j^i) = \sinh u F(h_{ij}) \leq \sinh u \, c_0,$$

and hence

$$(3.52) \qquad we^{\lambda t} \leq \sup_{\mathbb{S}^n} w(0)$$

for all

$$(3.53) \qquad 0 < \lambda \leq 2nc_0^{-2},$$

since

$$(3.54) \qquad F_k^k \geq n.$$

$\square$

4. C^2-estimates and existence for all time

To prove estimates for h_{ij}, we first need an a priori bound for F.

4.1. **Lemma.** *Let $M(t)$ be the leaves of the ICF*

$$\dot{x} = -\Phi\nu, \tag{4.1}$$

then there exists a positive constant c_1 such that

$$0 < c_1 \le F \qquad \forall\, t \in [0, T^*). \tag{4.2}$$

Proof. The function Φ, or equivalently $-\Phi$, satisfies the linear parabolic equation

$$\Phi' - \dot{\Phi}F^{ij}\Phi_{ij} = \dot{\Phi}F^{ij}h_{ik}h_j^k\Phi + K_N\dot{\Phi}F^{ij}g_{ij}\Phi, \tag{4.3}$$

when the ambient Riemannian space N is a space of constant curvature K_N, cf. [4, Corollary 3.5].

Another very useful equation is satisfied by a quantity χ which is defined by

$$\chi = v\eta, \tag{4.4}$$

where $0 < \eta = \eta(r)$ is a solution of

$$\dot{\eta} = -\frac{\bar{H}}{n}\eta; \tag{4.5}$$

here r is the radial distance to the center of geodesic polar coordinates in a spaceform N, and $\bar{H} = \bar{H}(r)$ is the mean curvature of S_r.

When $N = \mathbb{H}^{n+1}$, η is given by

$$\eta = \frac{1}{\sinh r}, \tag{4.6}$$

and

$$\chi = v\eta(u) \tag{4.7}$$

then satisfies

$$\dot{\chi} - \dot{\Phi}F^{ij}\chi_{ij} = -\dot{\Phi}F^{ij}h_{ik}h_j^k - 2\chi^{-1}\dot{\Phi}F^{ij}\chi_i\chi_j + \{\dot{\Phi}F + \Phi\}\frac{\bar{H}}{n}v\chi \tag{4.8}$$

for a general function Φ. In case of the inverse curvature, the term in the braces on the right-hand side vanishes.

In view of Lemma 3.1, the function

$$\tilde{\chi} = \chi e^{\frac{t}{n}} \tag{4.9}$$

is uniformly bounded,

$$0 < c_1 \le \tilde{\chi} \le c_2 \qquad \forall\, t \in [0, T^*), \tag{4.10}$$

and

$$\dot{\tilde{\chi}} - \dot{\Phi}F^{ij}\tilde{\chi}_{ij} = -\dot{\Phi}F^{ij}h_{ik}h_j^k\tilde{\chi} - 2\tilde{\chi}^{-1}\dot{\Phi}F^{ij}\tilde{\chi}_i\tilde{\chi}_j + \tfrac{1}{n}\tilde{\chi}, \tag{4.11}$$

when we consider an ICF.

We claim that

$$w = \log(-\Phi) + \log\tilde{\chi} \le \text{const} \tag{4.12}$$

during the evolution, which in turn would prove (4.2).

To derive (4.12) we first fix $0 < T < T^*$ and let

(4.13) $$(t_0, \xi_0) \in Q_T = [0, T] \times \mathbb{S}^n, \quad t_0 > 0,$$

be such that

(4.14) $$w(t_0, \xi_0) = \sup_{Q_T} w.$$

The equations (4.3), (4.11) and the maximum principle then yield in (t_0, ξ_0)

(4.15) $$0 \leq -\dot{\Phi} F^{ij} g_{ij} + \tfrac{1}{n},$$

which can only hold, if

(4.16) $$n \leq F(t_0, \xi_0);$$

hence w is uniformly bounded from above. $\qquad\square$

4.2. **Lemma.** *During the evolution F is uniformly bounded from above.*

Proof. The function u satisfies the parabolic equation

(4.17) $$\begin{aligned}\dot{u} - \dot{\Phi} F^{ij} u_{ij} &= -\Phi v^{-1} + \dot{\Phi} F v^{-1} - \dot{\Phi} F^{ij} \bar{h}_{ij} \\ &= 2\dot{\Phi} F v^{-1} - \dot{\Phi} F^{ij} \bar{h}_{ij},\end{aligned}$$

cf. [6, Lemma 3.3.2], and the rescaled function

(4.18) $$\tilde{u} = u - \frac{t}{n}$$

is uniformly bounded, cf. Lemma 3.1, and there holds

(4.19) $$\dot{\tilde{u}} - \dot{\Phi} F^{ij} \tilde{u}_{ij} = 2\dot{\Phi} F v^{-1} - \dot{\Phi} F^{ij} \bar{h}_{ij} - \tfrac{1}{n}.$$

The lemma will be proved, if we can show

(4.20) $$w = -\log(-\Phi) + \tilde{u} = \log F + \tilde{u} \leq \text{const}$$

during the evolution.

Applying the maximum principle as before, we conclude

(4.21) $$0 \leq 2F^{-1} v^{-1} - \tfrac{1}{n},$$

hence, F has to be bounded, proving the claim. $\qquad\square$

As an immediate corollary we deduce, in view of Lemma 3.2:

4.3. **Remark.** $|Du|^2$ satisfies the estimate (3.40), i.e., it decays exponentially, if $T^* = \infty$.

We are now ready to prove a priori estimates for the principal curvatures κ_i. The proof will be similar to a corresponding proof in [7, Theorem 1.4] valid in arbitrary Riemannian spaces. Our former result cannot be applied directly, since we assumed that the flow stays in a compact subset and also considered a contracting flow not an expanding one as we do now.

4.4. **Lemma.** *The principal curvatures of the flow hypersurfaces are uniformly bounded from above*

$$(4.22) \qquad \kappa_i \leq \text{const} \qquad \forall\, 1 \leq i \leq n,$$

and hence, are compactly contained in Γ, in view of the estimate (4.2).

Proof. In a Riemannian space of constant curvature the second fundamental forms h_j^i of the flow hypersurfaces $M(t)$ satisfy the evolution equation

$$(4.23) \quad \begin{aligned} \dot{h}_j^i - \dot{\Phi} F^{kl} h_{j;kl}^i &= \dot{\Phi} F^{kl} h_{kr} h_k^r h_j^i + (\Phi - \dot{\Phi} F) h_i^k h_{kj} + \ddot{\Phi} F_j F^i \\ &\quad \dot{\Phi} F^{kl,rs} h_{kl;i} h_{rs;}{}^i + K_N \{ (\Phi + \dot{\Phi} F) \delta_j^i - \dot{\Phi} F^{kl} g_{kl} h_j^i \} \end{aligned}$$

cf. [6, Lemma 2.4.3].

Here, the flow is given as an embedding

$$(4.24) \qquad x = x(t,\xi), \qquad (t,\xi) \in [0,T^*) \times \mathbb{S}^n,$$

and

$$(4.25) \qquad F_i = \frac{\partial F}{\partial \xi^i} = F^{kl} h_{kl;i}.$$

By assumption, F is monotone and concave. Thus, choosing, in a given point, coordinates (ξ^i) such that

$$(4.26) \qquad g_{ij} = \delta_{ij} \quad \wedge \quad h_{ij} = \kappa_i \delta_{ij},$$

and labelling the κ_i such that

$$(4.27) \qquad \kappa_1 \leq \cdots \leq \kappa_n,$$

then

$$(4.28) \quad \begin{aligned} F^{kl,rs} \eta_{kl} \eta_{rs} &\leq \sum_{k \neq l} \frac{F^{kk} - F^{ll}}{\kappa_k - \kappa_l} (\eta_{kl})^2 \\ &\leq \frac{2}{\kappa_n - \kappa_1} \sum_{k=1}^{n} (F^{nn} - F^{kk})(\eta_{nk})^2, \end{aligned}$$

and

$$(4.29) \qquad F^{nn} \leq \cdots \leq F^{11}.$$

For a proof of (4.28) see [7, Lemma 1.1] and of (4.29) [2, Lemma 2].

Let $\tilde{\chi}$ be the rescaled function in (4.9) and define

$$(4.30) \qquad \hat{\chi} = \tilde{\chi}^{-1},$$

then there exists a constant $\theta > 0$ such that

$$(4.31) \qquad 2\theta \leq \hat{\chi}.$$

Next, let ζ, φ and w be defined by

$$(4.32) \qquad \zeta = \sup\{ h_{ij} \eta^i \eta^j : \|\eta\| = 1 \},$$

$$(4.33) \qquad \varphi = -\log(\hat{\chi} - \theta)$$

and

$$(4.34) \qquad w = \log \zeta + \varphi + \lambda \tilde{u},$$

where $\tilde{u}$ is the function in (3.18) on page 7 and $\lambda > 0$ is supposed to be large. We claim that w is bounded, if λ is chosen sufficiently large.

Let $0 < T < T^*$, and $x_0 = x_0(t_0, \xi_0)$, with $0 < t_0 \leq T$, be a point in $M(t_0)$ such that

$$
(4.35) \qquad \sup_{M_0} w < \sup\{ \sup_{M(t)} w \colon 0 < t \leq T \} = w(x_0).
$$

We then introduce a Riemannian normal coordinate system (ξ^i) at $x_0 \in M(t_0)$ such that at $x_0 = x(t_0, \xi_0)$ we have

$$
(4.36) \qquad g_{ij} = \delta_{ij} \quad \text{and} \quad \zeta = h_n^n.
$$

Let $\tilde{\eta} = (\tilde{\eta}^i)$ be the contravariant vector field defined by

$$
(4.37) \qquad \tilde{\eta} = (0, \ldots, 0, 1),
$$

and set

$$
(4.38) \qquad \tilde{\zeta} = \frac{h_{ij}\tilde{\eta}^i\tilde{\eta}^j}{g_{ij}\tilde{\eta}^i\tilde{\eta}^j}.
$$

$\tilde{\zeta}$ is well defined in neighbourhood of (t_0, ξ_0).

Now, define $\tilde{w}$ by replacing ζ by $\tilde{\zeta}$ in (4.32); then, $\tilde{w}$ assumes its maximum at (t_0, ξ_0). Moreover, at (t_0, ξ_0) we have

$$
(4.39) \qquad \dot{\tilde{\zeta}} = \dot{h}_n^n,
$$

and the spatial derivatives do also coincide; in short, at (t_0, ξ_0) $\tilde{\zeta}$ satisfies the same differential equation (4.23) as h_n^n. For the sake of greater clarity, let us therefore treat h_n^n like a scalar and pretend that w is defined by

$$
(4.40) \qquad w = \log h_n^n + \varphi + \lambda \tilde{u}.
$$

From equations (4.23), (4.28), (4.11) and (4.19) we infer that in (t_0, ξ_0)

$$
\begin{aligned}
0 \leq\ & -\dot{\Phi}F^{ij}h_{ik}h_j^k \frac{\theta}{\hat{\chi} - \theta} + (\Phi - \dot{\Phi}F)h_n^n - (\Phi + \dot{\Phi}F)(h_n^n)^{-1} \\
& + \dot{\Phi}FF^{kl}g_{kl} + (\Phi + \dot{\Phi}F)\frac{\bar{H}}{n}v\frac{\hat{\chi}}{\hat{\chi} - \theta} + \frac{1}{n}\frac{\hat{\chi}}{\hat{\chi} - \theta} \\
& + \lambda(-\Phi + \dot{\Phi}F)v^{-1} - \lambda\dot{\Phi}F^{ij}\bar{h}_{ij} - \frac{\lambda}{n} \\
& + \dot{\Phi}F^{ij}(\log h_n^n)_i(\log h_n^n)_j - \dot{\Phi}F^{ij}\varphi_i\varphi_j \\
& + \frac{2}{\kappa_n - \kappa_1}\dot{\Phi}\sum_{i=1}^{n}(F^{nn} - F^{ii})(h_{ni;}{}^n)^2(h_n^n)^{-1}.
\end{aligned}
$$

(4.41)

There holds

$$
(4.42) \qquad F^{ij}\bar{h}_{ij} \geq c_0 F^{ij}g_{ij}, \quad c_0 > 0;
$$

moreover,

$$
(4.43) \qquad h_{ni;n} = h_{nn;i},
$$

and

$$
(4.44) \qquad \Phi + \dot{\Phi}F = 0,
$$

though

$$(4.45) \qquad \qquad \Phi \leq 0 \quad \wedge \quad |\Phi| \leq c\dot{\Phi}F$$

would suffice.

We then distinguish two cases.

Case 1. Suppose that

$$(4.46) \qquad \qquad \kappa_1 < -\epsilon_1 \kappa_n,$$

where $\epsilon_1 > 0$ is small, note that the principal curvatures are labelled according to (4.27). Then, we infer from [5, Lemma 8.3]

$$(4.47) \qquad \qquad F^{ij} h_{ki} h_j^k \geq \tfrac{1}{n} F^{ij} g_{ij} \epsilon_1^2 \kappa_n^2,$$

and

$$(4.48) \qquad \qquad F^{ij} g_{ij} \geq F(1,\dots,1),$$

for a proof see e.g., [6, Lemma 2.2.19].

Since $Dw = 0$,

$$(4.49) \qquad \qquad D \log h_n^n = -D\varphi - \lambda D\tilde{u},$$

we obtain

$$(4.50) \quad \dot{\Phi} F^{ij} (\log h_n^n)_i (\log h_n^n)_j = \dot{\Phi} F^{ij} \varphi_i \varphi_j + 2\lambda \dot{\Phi} F^{ij} \varphi_i \tilde{u}_j + \lambda^2 \dot{\Phi} F^{ij} \tilde{u}_i \tilde{u}_j,$$

where

$$(4.51) \qquad \qquad |\varphi_i| \leq c|\kappa_i| \|Du\| + c\|Du\|,$$

as one easily checks.

Hence, we conclude that κ_n is a priori bounded in this case for any choice of $\lambda > 0$, if we use

$$(4.52) \qquad \qquad F \leq \text{const},$$

or for $\lambda > 2$ otherwise.

Let us remark that

$$(4.53) \qquad \qquad \frac{\hat{\chi}}{\hat{\chi} - \theta} \leq 2$$

and

$$(4.54) \qquad \qquad F \leq F(1,\dots,1)\kappa_n = n\kappa_n.$$

Case 2. Suppose that

$$(4.55) \qquad \qquad \kappa_1 \geq -\epsilon_1 \kappa_n,$$

then the last term in inequality (4.41) can be estimated from above by

$$(4.56) \qquad \qquad \frac{2}{1+\epsilon_1} \dot{\Phi} \sum_{i=1}^{n} (F^{nn} - F^{ii})(\log h_{n;i}^n)^2.$$

The terms in (4.41) containing derivatives of h_n^n can therefore be estimated from above by

$$- \frac{1 - \epsilon_1}{1 + \epsilon_1} \dot{\Phi} F^{ij} (\log h_n^n)_i (\log h_n^n)_j + \frac{1}{1 + \epsilon_1} \dot{\Phi} F^{nn} \sum_{i=1}^{n} (\log h_{n;i}^n)^2$$

$$
\begin{aligned}
(4.57) \qquad & \leq \dot{\Phi} F^{nn} \sum_{i=1}^{n} (\log h_{n;i}^n)^2 \\
& = \dot{\Phi} F^{nn} \| D\varphi + \lambda D\tilde{u} \|^2 \\
& = \dot{\Phi} F^{nn} \{ \| D\varphi \|^2 + \lambda^2 \| D\tilde{u} \|^2 + 2\lambda \langle D\varphi, D\tilde{u} \rangle \}.
\end{aligned}
$$

Hence, we finally deduce

$$
\begin{aligned}
(4.58) \qquad 0 \leq & -\dot{\Phi} F^{nn} \kappa_n^2 \frac{\theta}{\hat{\chi} - \theta} - \dot{\Phi} F \kappa_n + \dot{\Phi} F^{kl} g_{kl} (1 - \lambda c_0) + c \\
& + \lambda c \dot{\Phi} F - \frac{\lambda}{n} + \lambda^2 c \dot{\Phi} F^{nn} (1 + \kappa_n).
\end{aligned}
$$

Thus, we obtain an a priori estimate

$$(4.59) \qquad \kappa_n \leq \text{const},$$

if λ is chosen large enough. Note that ϵ_1 is only subject to the requirement

$$(4.60) \qquad 0 < \epsilon_1 < 1.$$

$\square$

As a corollary we can state:

4.5. **Corollary.** *Let the initial hypersurface $M_0 \in C^{m+2,\alpha}$, $4 \leq m \leq \infty$, $0 < \alpha < 1$, then the solution of the curvature flow*

$$(4.61) \qquad \dot{x} = -\Phi \nu$$

exists for all time and belongs to the parabolic Hölder space

$$(4.62) \qquad H^{m+\alpha, \frac{m+\alpha}{2}}(Q),$$

while the solution u of the scalar flow belongs to

$$(4.63) \qquad H^{m+2+\alpha, \frac{m+2+\alpha}{2}}(Q),$$

where

$$(4.64) \qquad Q = [0, \infty) \times \mathbb{S}^n.$$

The norm will still depend on t however due to the present coordinate system.

Proof. Let us look at the scalar flow equation (3.14) on page 6. In view of the previous estimates the nonlinear operator is uniformly elliptic and by assumption also concave, hence we may apply the Krylov-Safonov estimates yielding uniform Hölder estimates for $\dot{u}$ and $D^2 u$ estimates. Now, the linear theory and the parabolic Schauder estimates can be applied; for details see e.g. [6, Chapter 2.6] and [8, Section 6]. $\square$

5. The conformally flat parametrization

Hyperbolic space is conformally flat such that

(5.1)
$$\begin{aligned}
d\bar{s}^2 &= \frac{1}{(1 - \frac{1}{4}|x|^2)^2} dx^2 \\
&= \frac{1}{(1 - \frac{1}{4}r^2)^2} \{dr^2 + r^2 \sigma_{ij} dx^i dx^j\} \\
&\equiv e^{2\psi} \{dr^2 + r^2 \sigma_{ij} dx^i dx^j\}
\end{aligned}$$

after introducing polar coordinates.

Define the variable τ by

(5.2)
$$d\tau = \frac{1}{1 - \frac{1}{4}r^2} dr$$

such that

(5.3)
$$\tau = \log(2 + r) - \log(2 - r),$$

then

(5.4)
$$\sinh^2 \tau = \frac{r^2}{(1 - \frac{1}{4}r^2)^2},$$

and we see that τ is the radial distance in hyperbolic space from the origin of the euclidean ball $B_2(0)$.

A star-shaped hypersurface $M \subset \mathbb{H}^{n+1}$ is also star-shaped in $\mathbb{R}^{n+1}$ under this correspondence.

Let us distinguish geometric quantities in $\mathbb{H}^{n+1}$ by an additional breve from the corresponding quantities in $\mathbb{R}^{n+1}$, e.g., $\breve{g}_{\alpha\beta}$, $\breve{g}_{ij}$, $M = \operatorname{graph} \breve{u}$, $\breve{h}_{ij}$, $\breve{\nu}$, etc.

Consider a hypersurface

(5.5)
$$M = \operatorname{graph} \breve{u} = \operatorname{graph} u,$$

then

(5.6)
$$\breve{u} = \log(2 + u) - \log(2 - u)$$

and

(5.7)
$$\breve{u}_i = \frac{1}{1 - \frac{1}{4}u^2} u_i$$

and $|D\breve{u}|^2$ as defined in (3.21) on page 7 can be expressed as

(5.8)
$$|D\breve{u}|^2 = u^{-2} \sigma^{ij} u_i u_j \equiv |Du|^2,$$

hence the term v is identical in both coordinate systems which is also evident from the invariant definition of v by

(5.9)
$$v^{-1} = \langle \eta, \nu \rangle,$$

where

(5.10)
$$\eta = Dd$$

and d is the distance function in hyperbolic space from the origin.

The second fundamental forms are connected through the relation

$$(5.11) \qquad \begin{aligned} e^{\psi} \breve{h}^i_j &= h^i_j + \psi_{\alpha} \nu^{\alpha} \delta^i_j \\ &\equiv h^i_j + v^{-1} \tilde{\vartheta} \delta^i_j, \end{aligned}$$

where

$$(5.12) \qquad \tilde{\vartheta} = \tfrac{1}{2} \frac{r}{1 - \tfrac{1}{4} r^2}.$$

Let

$$(5.13) \qquad \breve{h}_{ij} = h_{ij} + v^{-1} \tilde{\vartheta} g_{ij},$$

$$(5.14) \qquad g_{ij} = u_i u_j + u^2 \sigma_{ij},$$

then the curvature flow in $\mathbb{H}^{n+1}$

$$(5.15) \qquad \dot{x} = F^{-1} \breve{\nu}$$

can also be viewed as a curvature flow in $\mathbb{R}^{n+1}$

$$(5.16) \qquad \dot{x} = F^{-1} \nu,$$

where now F depends on the eigenvalues of $\breve{h}_{ij}$ with respect to the metric g_{ij}

$$(5.17) \qquad F = F(\breve{h}_{ij}) = F(\breve{h}^i_j).$$

For the rest of this paper we shall mainly consider the curvature flow (5.16).

Let us quickly summarize the most important flow equations.

Writing (5.16) slightly more general

$$(5.18) \qquad \dot{x} = -\Phi(F)\nu \equiv -\Phi\nu$$

there holds

$$(5.19) \qquad \dot{h}^j_i = \Phi^j_i + \Phi h^k_i h^j_k,$$

cf. [6, Lemma 2.3.3], which will be the main ingredient to derive the subsequent modified flow equations:

$$(5.20) \qquad \begin{aligned} \Phi' - \dot{\Phi} F^{ij} \Phi_{ij} &= \dot{\Phi} F^{ij} h_{ki} h^k_j \Phi - \dot{\Phi} F^{ij} g_{ij} r_{\alpha\beta} \nu^{\alpha} \nu^{\beta} \tilde{\vartheta} \Phi \\ &\quad - \dot{\Phi} F^{ij} g_{ij} \dot{\tilde{\vartheta}} v^{-2} \Phi + \dot{\Phi} F^{ij} g_{ij} \tilde{\vartheta} \Phi_k u^k, \end{aligned}$$

$$(5.21) \qquad \dot{u} - \dot{\Phi} F^{ij} u_{ij} = 2v^{-1} F^{-1} - \dot{\Phi} F^{ij} g_{ij} \tilde{\vartheta} v^{-2} - \dot{\Phi} F^{ij} \bar{h}_{ij},$$

where we used that

$$(5.22) \qquad \Phi(t) = -t^{-1}, \qquad t > 0,$$

here t is just a symbol for a real variable, and where

$$(5.23) \qquad \bar{h}_{ij} = u^{-1} \bar{g}_{ij} = u \sigma_{ij}$$

is the second fundamental form of the slices $\{x^0 = u\}$, i.e., of spheres in $\mathbb{R}^{n+1}$ with center in the origin and radius $r = u$.

The evolution equation for the second fundamental form looks like

$$\dot{h}_i^j - \dot{\Phi}F^{kl}h_{i;kl}^j = \dot{\Phi}F^{kl}h_{kr}h_l^r h_i^j + (\Phi - \dot{\Phi}F)h_{ik}h^{kj}$$
$$+ \ddot{\Phi}F_i F^j + \dot{\Phi}F^{kl,rs}\check{h}_{kl;i}\check{h}_{rs;}^{\ j}$$
$$+ \dot{\Phi}F^{kl}g_{kl}\{-\tilde{\vartheta}v^{-1}h_i^r h_r^j - v^{-2}\dot{\tilde{\vartheta}}h_i^j + v^{-1}\dot{\tilde{\vartheta}}\bar{h}_{ik}g^{kj}$$
$$(5.24)\qquad\qquad - r_{\alpha\beta}\nu^\alpha\nu^\beta\tilde{\vartheta}h_i^j + \tilde{\vartheta}u_r h_{i;}^{r,\ j} + \dot{\tilde{\vartheta}}h_i^r u_r u^j + \dot{\tilde{\vartheta}}h^{rj}u_r u_i$$
$$+ r_{\alpha\beta}x_k^\alpha x^{\beta j}h_i^k\tilde{\vartheta} + v^{-1}\ddot{\tilde{\vartheta}}u_i u^j + r_{\alpha\beta\gamma}\nu^\alpha x_i^\beta x_j^\gamma\tilde{\vartheta}$$
$$+ r_{\alpha\beta}\nu^\alpha x_i^\beta\dot{\tilde{\vartheta}}u^j + r_{\alpha\beta}\nu^\alpha x^{\beta j}\dot{\tilde{\vartheta}}u_i\}.$$

The function χ is now defined by

$$(5.25)\qquad\qquad\qquad \chi = vu^{-1}$$

and there holds:

5.1. **Lemma.** χ *satisfies the evolution equation*

$$(5.26)\qquad \dot{\chi} - \dot{\Phi}F^{ij}\chi_{ij} = -\dot{\Phi}F^{ij}h_i^k h_{kj}\chi - 2\chi^{-1}\dot{\Phi}F^{ij}\chi_i\chi_j + \{\dot{\Phi}F + \Phi\}\chi^2$$
$$+ \dot{\Phi}F^{ij}g_{ij}\{-\tilde{\vartheta}\chi^2 + \chi_k u^k u\theta - \chi\dot{\theta}\|Du\|^2 u\},$$

where

$$(5.27)\qquad\qquad \theta(r) = r^{-1}\tilde{\vartheta} = \tfrac{1}{2}\frac{1}{1 - \frac{1}{4}r^2}.$$

Proof. We consider a general Φ in the curvature flow in $\mathbb{R}^{n+1}$

$$(5.28)\qquad\qquad\qquad \dot{x} = -\Phi\nu,$$

where

$$(5.29)\qquad\qquad\qquad F = F(\check{h}_{ij})$$

and $\check{h}_{ij}$ is defined by (5.13).

For the above flow the normal evolves according to

$$(5.30)\qquad\qquad\qquad \dot{\nu} = \Phi^k x_k,$$

cf. [6, Lemma 2.3.2].

Using an euclidean coordinate system (x^α) in $\mathbb{R}^{n+1}$ it follows immediately that χ can be expressed as

$$(5.31)\qquad\qquad\qquad \chi = \langle x, \nu\rangle^{-1},$$

and hence

$$\dot{\chi} = -\chi^2\langle\dot{x}, \nu\rangle - \chi^2\langle x, \dot{\nu}\rangle$$
$$= \Phi\chi^2 - \chi^2\Phi^k u_k u$$
$$= \Phi\chi^2 - \chi^2\dot{\Phi}F^{ij}\check{h}_{ij;}^{\ k}u_k u$$
$$(5.32)\qquad\qquad = \Phi\chi^2 - \chi^2\dot{\Phi}F^{ij}\{h_{ij;}^{\ k}u_k u + (\chi^{-1}\theta)_k u^k u g_{ij}\}$$
$$= \Phi\chi^2 - \chi^2\dot{\Phi}F^{ij}h_{ij;}^{\ k}u_k u$$
$$+ \dot{\Phi}F^{ij}g_{ij}\{\chi_k u^k u\theta - \chi\dot{\theta}\|Du\|^2 u\}.$$

Differentiating χ covariantly with respect to $\xi = (\xi^i)$ we obtain

$$(5.33) \qquad \chi_i = -\chi^2 h_i^k \langle x_k, x \rangle,$$

$$(5.34) \qquad \chi_{ij} = 2\chi^{-1}\chi_i\chi_j - \chi^2 h_{i;j}^k \langle x_k, x \rangle + h_i^k h_{kj}\chi - \chi^2 h_{ij}.$$

Combining (5.32) and (5.34) the result follows immediately due to the homogeneity of F. $\qquad\square$

We want to prove that h_{ij} is uniformly bounded. However, this result can only be achieved in several steps.

We observe that in view of the relation (5.11) and the boundedness of $\breve{h}_j^i$

$$(5.35) \qquad h_j^i(1 - \tfrac{1}{4}u^2)$$

is uniformly bounded, or equivalently,

$$(5.36) \qquad |h_j^i|e^{-\frac{t}{n}} \leq \text{const},$$

because of (5.4) and (3.17) on page 7.

As a first new step we shall improve (5.36) slightly:

5.2. **Lemma.** *Define* λ_ϵ *by*

$$(5.37) \qquad \lambda_\epsilon = \tfrac{1}{n} - \epsilon, \qquad \epsilon > 0,$$

where ϵ *is small. Then the principal curvatures* κ_i *of the flow hypersurfaces can be estimated from above by*

$$(5.38) \qquad \kappa_i \leq ce^{\lambda_\epsilon t},$$

if $\epsilon > 0$ *is small*

$$(5.39) \qquad 0 < \epsilon < \epsilon_0.$$

Proof. Define ζ as in (4.32) on page 12 and let

$$(5.40) \qquad \tilde{\zeta} = \zeta e^{-\lambda_\epsilon t},$$

then we claim that

$$(5.41) \qquad w = \log\tilde{\zeta} + \log\chi$$

is uniformly bounded from above, if ϵ is sufficiently small.

Let $0 < T < \infty$ be large and assume that

$$(5.42) \qquad \sup_{Q_T} w = w(t_0, \xi_0)$$

with

$$(5.43) \qquad 0 < t_0 \leq T.$$

Arguing as in the proof of Lemma 4.4 on page 12, we may assume that κ_n is the largest principal curvature and that w is defined by

$$(5.44) \qquad w = \log\tilde{h}_n^n + \log\chi,$$

where

$$(5.45) \qquad \tilde{h}_n^n = h_n^n e^{-\lambda_\epsilon t}.$$

We shall suppose that

(5.46) $$\tilde{h}_n^n(t_0, \xi_0) >> 1.$$

Applying the maximum principle we then infer from (5.24) and (5.26)

$$
\begin{aligned}
(5.47) \quad 0 \leq \dot{\Phi} F^{kl} g_{kl} \{ & -\tilde{\vartheta} v^{-1} e^{\lambda_\epsilon t} \tilde{h}_n^n - v^{-2} \dot{\tilde{\vartheta}} + c \dot{\tilde{\vartheta}} e^{-\lambda_\epsilon t} (\tilde{h}_n^n)^{-1} + c\tilde{\vartheta} \\
& + c\dot{\tilde{\vartheta}} \|Du\|^2 + c\ddot{\tilde{\vartheta}} \|Du\|^2 e^{-\lambda_\epsilon t} (\tilde{h}_n^n)^{-1} \\
& + \tilde{\vartheta} u^k (\log \tilde{h}_n^n)_k + u\theta u^k (\log \chi)_k \},
\end{aligned}
$$

where we used the concavity of F, the properties of Φ and at one point the vanishing of Dw in (t_0, ξ_0).

Our assumption that $\tilde{h}_n^n$ is very large implies that $t = t_0$ is very large and, hence, powers of e^t will be the dominating terms.

In view of (3.17) on page 7 and (5.4) we have

(5.48) $$\tilde{\vartheta} \sim ce^{\frac{t}{n}} \quad \wedge \quad \dot{\tilde{\vartheta}} \sim ce^{\frac{2t}{n}} \quad \wedge \quad \ddot{\tilde{\vartheta}} \sim ce^{\frac{3t}{n}},$$

while

(5.49) $$\|Du\| \leq ce^{-\lambda_0 t}$$

for some $0 < \lambda_0$, cf. the estimate (3.40) on page 8.

The best term inside the braces on the right-hand side of (5.47) is

(5.50) $$- v^{-2} \dot{\tilde{\vartheta}} \sim -ce^{\frac{2t}{n}}$$

and the worst is

(5.51) $$c\ddot{\tilde{\vartheta}} \|Du\|^2 e^{-\lambda_\epsilon t} (\tilde{h}_n^n)^{-1} \sim c(\tilde{h}_n^n)^{-1} e^{\frac{t}{n}(2+n\epsilon-2n\lambda_0)},$$

hence, choosing

(5.52) $$\epsilon = 2\lambda_0$$

we obtain an a priori estimate for $\tilde{h}_n^n$, since the terms in (5.47) involving the derivatives of $\log \tilde{h}_n^n$ and $\log \chi$ vanish, for

(5.53) $$u\theta = \tilde{\vartheta} \quad \wedge \quad Dw = 0.$$

$\square$

As a corollary we deduce:

5.3. **Corollary.** *The quantity $F = F(\check{h}_j^i)$ can be estimated from above by*

(5.54) $$F \leq nv^{-1}\tilde{\vartheta}(1 + ce^{-2\lambda_0 t}) \qquad \forall\, 0 \leq t < \infty.$$

Proof. This follows at once from (5.13), (5.38), (5.52) and the normalization (3.1) on page 5. $\square$

We are now able to improve the decay rate of $|Du|$.

5.4. **Lemma.** *$|Du|$ satisfies the estimate*

(5.55) $$|Du| \leq ce^{-\frac{t}{n}} \qquad \forall\, 0 \leq t < \infty.$$

Proof. We look at the scalar flow equation

$$\dot{u} = \frac{\partial u}{\partial t} = \frac{v}{F}, \tag{5.56}$$

where $F = F(\check{h}^i_j)$. Let

$$\varphi = \log u, \tag{5.57}$$

then

$$h^i_j = g^{ik}h_{kj} = v^{-1}u^{-1}\{-(\sigma^{ik} - v^{-2}\varphi^i\varphi^k)\varphi_{jk} + \delta^i_j\}, \tag{5.58}$$

where all space derivatives are covariant derivatives with respect to σ_{ij}, cf. (3.26) on page 7, but now we are in euclidean space, i.e., the factor ϑ in (3.22) is equal to

$$\vartheta(r) = r. \tag{5.59}$$

Hence, we infer from (5.56)

$$\dot{\varphi} = \frac{1}{F(\tilde{h}^i_j)}, \tag{5.60}$$

where

$$\tilde{h}^i_j = v^{-2}\{-(\sigma^{ik} - v^{-2}\varphi^i\varphi^k)\varphi_{jk} + \vartheta\delta^i_j\}, \tag{5.61}$$

and ϑ is defined by

$$\vartheta = \vartheta(r) = r\tilde{\vartheta}. \tag{5.62}$$

The term

$$w = \tfrac{1}{2}|D\varphi|^2 \tag{5.63}$$

then satisfies

$$
\begin{aligned}
\dot{w} = F^{-2}\{&2v^{-2}Fw_i\varphi^i + v^{-2}F^k_l\tilde{g}^{lr}w_{kr} - v^{-2}F^k_l\tilde{g}^{lr}\varphi_{ik}\varphi^i_r \\
&+ v^{-2}F^k_l\tilde{g}^{lr}{}_{;i}\varphi^i\varphi_{kr} + v^{-2}F^k_l\tilde{g}^{lr}\varphi_r\varphi_k - v^{-2}F^k_l\tilde{g}^{lr}\sigma_{kr}|D\varphi|^2 \\
&- 2v^{-2}F^k_k\dot{\vartheta}e^\varphi w\},
\end{aligned}
\tag{5.64}
$$

cf. (3.43) on page 8 observing that now ϑ is defined differently and

$$\vartheta_i\varphi^i = \dot{\vartheta}e^\varphi|D\varphi|^2 = 2\dot{\vartheta}e^\varphi w. \tag{5.65}$$

The metric $\tilde{g}_{ij}$ is defined by

$$\tilde{g}_{ij} = \varphi_i\varphi_j + \sigma_{ij}, \tag{5.66}$$

and $\tilde{g}^{ij}$ is its inverse.

Let

$$0 < \lambda \le \tfrac{2}{n} \tag{5.67}$$

be arbitrary and define

$$\tilde{w} = we^{\lambda t} \quad \wedge \quad \tilde{\varphi}_i = \varphi_i e^{\frac{\lambda t}{2}}. \tag{5.68}$$

Now choose $0 < T < \infty$ and suppose that

$$\sup_{Q_T} w, \qquad Q_T = [0, T] \times \mathbb{S}^n, \tag{5.69}$$

is attained at (t_0, x_0) with $t_0 > 0$. Then the maximum principle implies

$$(5.70) \quad 0 \le F^{-2}\{F_l^k \tilde{g}^{lr} \tilde{\varphi}_r \tilde{\varphi}_k - F_l^k \tilde{g}^{lr} \sigma_{kr} |D\tilde{\varphi}|^2 - 2F_k^k \dot{\vartheta} e^\varphi \tilde{w} + \lambda v^2 F^2 \tilde{w}\}.$$

By definition

$$(5.71) \qquad\qquad\qquad \tilde{h}_j^i = v^{-1} u^{-1} \check{h}_j^i,$$

and thus

$$(5.72) \qquad v^2 F(\tilde{h}_j^i) \le u^2 F^2(\check{h}_j^i) \le u^2 n^2 v^{-2} \tilde{\vartheta}^2 (1 + ce^{-2\lambda_0 t})^2.$$

On the other hand,

$$(5.73) \qquad\qquad\qquad 2n\dot{\vartheta}u = 8n\tilde{\vartheta}^2$$

hence, we obtain an a priori estimate for $\tilde{w}$ provided

$$(5.74) \qquad\qquad\qquad 0 < \lambda < \tfrac{2}{n}.$$

To derive an a priori estimate in the limit case

$$(5.75) \qquad\qquad\qquad \lambda = \tfrac{2}{n}$$

we define, with a slight abuse of notation,

$$(5.76) \qquad\qquad w = w(t) = \sup_{\mathbb{S}^n} w(t, \cdot) \quad \wedge \quad \tilde{w} = w e^{\lambda t}$$

with $\lambda = \frac{2t}{n}$; w is Lipschitz continuous and the maximum principle then yields, instead of (5.70),

$$(5.77) \qquad\qquad \dot{\tilde{w}} \le F^{-2}\{-2F_k^k \dot{\vartheta} e^\varphi \tilde{w} + \lambda v^2 F^2 \tilde{w}\}$$

for almost every $t > 0$. Because of the relations (5.72), (5.73) and the previous estimates for w we then conclude

$$(5.78) \qquad\qquad\qquad \dot{\tilde{w}} \le ce^{-\delta t}$$

for a.e. $t > 0$ with some $\delta > 0$, completing the proof of the lemma. $\qquad\square$

As a corollary we deduce:

5.5. **Corollary.** *The principal curvatures κ_i of the flow hypersurfaces are uniformly bounded from above*

$$(5.79) \qquad\qquad \kappa_i \le c \qquad \forall\, 0 \le t < \infty.$$

Proof. Choosing in the proof of Lemma 5.2 $\lambda_\epsilon = 0$ and applying the maximum principle we obtain the inequality (5.47) with λ_ϵ replaced by 0. In view of the estimate (5.55) we then infer an a priori estimate for κ_n. $\qquad\square$

An estimate from below for the κ_i is much more difficult and requires two steps.

5.6. **Lemma.** *Let κ_i, $1 \leq i \leq n$, be the principal curvatures of the flow hypersurfaces*

$$(5.80) \qquad \kappa_1 \leq \cdots \leq \kappa_n,$$

and let

$$(5.81) \qquad \tilde{\kappa}_i(t) = \inf_{\mathbb{S}^n} \kappa_i(t,\xi)(2 - u(t,\xi)),$$

then

$$(5.82) \qquad \liminf_{t \to \infty} \tilde{\kappa}_1(t) = 0.$$

Proof. We argue by contradiction. Suppose that

$$(5.83) \qquad \liminf_{t \to \infty} \tilde{\kappa}_1(t) < 0.$$

Let φ be defined by

$$(5.84) \qquad \begin{aligned} \varphi &= F(2-u) = F(h^i_j + v^{-1}\tilde{\vartheta}\delta^i_j)(2-u) \\ &= F(h^i_j v\tilde{\vartheta}^{-1} + \delta^i_j)v^{-1}\tilde{\vartheta}(2-u), \end{aligned}$$

then

$$(5.85) \qquad \lim_{t \to \infty} |v^{-1}\tilde{\vartheta}(2-u)|_{0,\mathbb{S}^n} = 1$$

and (5.83) is equivalent to

$$(5.86) \qquad \liminf_{t \to \infty} \inf_{\mathbb{S}^n} F(2-u) = F(1+\tilde{\kappa}_1, \ldots, 1+\tilde{\kappa}_n) < F(1, \ldots, 1) = n,$$

since the non-negative κ_i are uniformly bounded and F is strictly monotone. Thus, (5.83) implies

$$(5.87) \qquad \liminf_{t \to \infty} \inf_{\mathbb{S}^n} w < \log n,$$

where

$$(5.88) \qquad \begin{aligned} w &= \log \varphi - \log \chi - \log 2 \\ &= -\log(-\Phi) + \log(2-u) - \log \chi - \log 2, \end{aligned}$$

since

$$(5.89) \qquad \lim_{t \to \infty} |-\log \chi - \log 2|_{0,\mathbb{S}^n} = 0.$$

Let $\epsilon > 0$ be so small such that

$$(5.90) \qquad \liminf_{t \to \infty} \inf_{\mathbb{S}^n} w < (1 - 2\epsilon) \log n$$

and let τ be so large such that

$$(5.91) \qquad t \geq \tau \quad \wedge \quad \inf_{\mathbb{S}^n} w(t,\xi) = w(t,\xi_0) < (1-\epsilon) \log n$$

implies

$$(5.92) \qquad \tilde{\kappa}_1(t) < -\epsilon_0 \quad \wedge \quad \kappa_1 \tilde{\vartheta}^{-1}(t,\xi_0) < -\epsilon_0$$

for a fixed $0 < \epsilon_0 = \epsilon_0(\epsilon)$. The existence of τ follows from the relations (5.89) and (5.90).

Define the set

$$(5.93) \qquad \Lambda = \{\, t : t \geq \tau \quad \wedge \quad \inf_{\mathbb{S}^n} w(t,\cdot) < (1-\epsilon) \log n \,\},$$

then $\Lambda \neq \emptyset$, since it contains a sequence $t_k \to \infty$.

We shall now prove

$$(5.94) \qquad \Lambda = [\tau, \infty),$$

and

$$(5.95) \qquad \tilde{w}(t) = \inf_{\mathbb{S}^n} w(t, \cdot)$$

is (weakly) monotone increasing in $[\tau, \infty)$, i.e.,

$$(5.96) \qquad \tilde{w}(t_1) \leq \tilde{w}(t_2) \qquad \forall \, \tau \leq t_1 < t_2 < \infty.$$

Let T, $\tau < T < \infty$, be arbitrary but so large such that

$$(5.97) \qquad \Lambda \cap [\tau, T] \neq \emptyset,$$

and suppose that

$$(5.98) \qquad \inf\{\, w(t, \xi) \colon \tau \leq t \leq T, \ \xi \in \mathbb{S}^n \,\} = w(t_0, \xi_0)$$

with $t_0 > \tau$.

w satisfies the evolution equation

$$\dot{w} - \dot{\Phi} F^{ij} w_{ij} =$$

$$(5.99) \quad \begin{aligned} &\dot{\Phi} F^{ij} g_{ij} r_{\alpha\beta} \nu^\alpha \nu^\beta \tilde{\vartheta} + \dot{\Phi} F^{ij} g_{ij} \dot{\tilde{\vartheta}} v^{-2} - \dot{\Phi} F^{ij} g_{ij} (\log(-\Phi))_k u^k \tilde{\vartheta} \\ &- \dot{\Phi} F^{ij} (\log(-\Phi))_i (\log(-\Phi))_j + \dot{\Phi} F^{ij} (\log(2-u))_i (\log(2-u))_j \\ &+ \dot{\Phi} F^{ij} (\log \chi)_i (\log \chi)_j \\ &+ \dot{\Phi} F^{ij} g_{ij} \tilde{\vartheta} v^{-2} (2-u)^{-1} + \dot{\Phi} F^{ij} \bar{h}_{ij} (2-u)^{-1} - 2 v^{-1} \varphi^{-1} \\ &+ \dot{\Phi} F^{ij} g_{ij} \{ \tilde{\vartheta} \chi - (\log \chi)_k u^k u \theta + \dot{\theta} \| Du \|^2 u \}. \end{aligned}$$

At the point (t_0, ξ_0) $Dw = 0$, hence the terms in line two and three on the right-hand side of (5.99) add up to

$$(5.100) \qquad 2 \dot{\Phi} F^{ij} (\log \chi)_i (\log(2-u))_j,$$

which in turn is equal to

$$(5.101) \qquad 2 \dot{\Phi} F^{ij} h_{ik} u^k u_j (2-u)^{-1} \chi^{-1} \geq -c \dot{\Phi},$$

due to the estimates (5.55), (5.36), (5.4) and (3.17) on page 7.

Analogously, we conclude

$$(5.102) \quad \begin{aligned} &\dot{\Phi} F^{ij} g_{ij} \{ -(\log(-\Phi))_k u^k u \theta - (\log \chi)_k u^k \tilde{\vartheta} \} = \\ &\dot{\Phi} F^{ij} g_{ij} (-\log(2-u))_k u^k) \tilde{\vartheta} \geq 0, \end{aligned}$$

where we also used

$$(5.103) \qquad u\theta = \tilde{\vartheta}.$$

Hence, applying the maximum principle we infer from (5.99)

$$(5.104) \quad \begin{aligned} 0 &\geq \dot{\Phi} F^{ij} g_{ij} \dot{\tilde{\vartheta}} v^{-2} + \dot{\Phi} F^{ij} g_{ij} \tilde{\vartheta} v^{-2} (2-u)^{-1} - 2 v^{-1} \varphi^{-1} - c \dot{\Phi} \\ &\geq 2\delta - c e^{-\frac{2t}{n}} \geq \delta, \end{aligned}$$

if τ is large enough, with some uniform $\delta = \delta(\epsilon_0) > 0$; a contradiction.

Thus, we have proved that $t_0 = \tau$ and therefore

$$(5.105) \qquad \tilde{w}(\tau) \leq \tilde{w}(t) \qquad \forall \tau \leq t \leq T.$$

Since we can replace τ by any $t_1 \in [\tau, T)$ we conclude

$$(5.106) \qquad \tilde{w}(t_1) \leq \tilde{w}(t_2) \qquad \forall \tau \leq t_1 \leq t_2 \leq T,$$

and we have proved (5.94) as well as (5.96), since $\tau < T < \infty$ is arbitrary.

However, the arguments we used to derive the contradiction in inequality (5.104) yield

$$(5.107) \qquad \dot{w}(t, \xi_t) \geq \delta > 0 \qquad \forall \tau \leq t < \infty,$$

where

$$(5.108) \qquad \inf\{\, w(t, \xi) \colon \xi \in \mathbb{S}^n \,\} = w(t, \xi_t),$$

in view of (5.94) and the definition of Λ. The left-hand side of the preceding equation is the definition of $\tilde{w}(t)$, which is Lipschitz continuous and satisfies for a.e. $t > \tau$

$$(5.109) \qquad \dot{\tilde{w}}(t) = \dot{w}(t, \xi_t).$$

Hence, we deduce

$$(5.110) \qquad \dot{\tilde{w}}(t) \geq \delta$$

for a.e. $t > \tau$, which is a contradiction, since $\tilde{w}$ is uniformly bounded, completing the proof of the lemma. $\qquad \square$

Now, we can prove that the principal curvatures are uniformly bounded.

5.7. **Lemma.** *The principal curvatures* κ_i, $1 \leq i \leq n$, *are uniformly bounded during the evolution*

$$(5.111) \qquad |\kappa_i| \leq c.$$

Proof. We shall estimate

$$(5.112) \qquad \varphi = \tfrac{1}{2}|A|^2 = \tfrac{1}{2} h_{ij} h^{ij},$$

which satisfies the evolution equation

$$(5.113) \qquad \dot{\varphi} - \dot{\Phi} F^{ij} \varphi_{ij} = -\dot{\Phi} F^{kl} h_{ij;k} h^{ij}{}_{;l} + \{\dot{h}_i^j - \dot{\Phi} F^{kl} h_{i;kl}^j\} h_j^i.$$

Looking at (5.24) and observing that, in view of the previous estimates,

$$(5.114) \qquad \lim_{t \to \infty} |A| e^{-\frac{t}{n}} = 0$$

uniformly in $\xi \in \mathbb{S}^n$, and, because of the homogeneity of F,

$$(5.115) \qquad F^{-1} + |F^{kl,rs}| \leq c e^{-\frac{t}{n}},$$

we deduce that the terms

$$(5.116) \qquad - \dot{\Phi} F^{kl} h_{ij;k} h^{ij}{}_{;l} - \dot{\Phi} F^{kl} g_{kl} v^{-2} \dot{\tilde{\vartheta}} 2\varphi,$$

which are either explicitly or implicitly contained in the right-hand side of (5.113), are dominating; they can absorb any bad term such that an application of the maximum principle gives an a priori estimate for φ. $\qquad \square$

As a corollary we obtain:

5.8. **Theorem.** *The flow hypersurfaces in hyperbolic space become strongly convex exponentially fast and also more and more totally umbilic. In fact there holds*

(5.117) $$|\breve{h}^i_j - \delta^i_j| \le ce^{-\frac{t}{n}}.$$

Proof. We infer from (5.11) on page 17

(5.118) $$\begin{aligned} \breve{h}^i_j - \delta^i_j &= \breve{h}^i_j - \frac{u}{2v}\delta^i_j + (\frac{u}{2v} - 1)\delta^i_j \\ &= h^i_j \frac{u}{2\tilde{\vartheta}} + (\frac{u}{2v} - 1)\delta^i_j, \end{aligned}$$

from which the estimate (5.117) immediately follows, in view of (5.111), (5.55), (5.4), and (3.17) on page 7. □

6. Higher order estimate

Assuming the curvature function F to be smooth, we want to prove higher order estimates for h_{ij}, or equivalently, for u. Since we already know that h_{ij} is uniformly bounded,

(6.1) $$\dot{g}_{ij} = -2\Phi h_{ij}$$

as well as the Riemannian curvature tensor of the induced metric then are also uniformly bounded.

Let A represent the second fundamental form, where we omit the tensor indices, then we want to prove

(6.2) $$\|D^m A\| \le c_m e^{-\frac{t}{n}} \qquad \forall\, m \ge 1.$$

This estimate will immediately imply a corresponding estimate

(6.3) $$\|D^m u\| \le c_m e^{-\frac{t}{n}} \qquad \forall\, m \ge 1,$$

in view of the relation

(6.4) $$h_{ij} v^{-1} = -u_{ij} + \bar{h}_{ij}$$

and the estimate (5.55) on page 20.

To obtain an estimate for $D^2 u$ we have to apply the following interpolation lemma:

6.1. **Lemma.** *Let $M = M^n$ be a compact Riemannian manifold of class C^m, $m \ge 2$, and $u \in C^2(M)$, then*

(6.5) $$\|Du\|_{0,M} \le c|u|_{2,M}^{\frac{1}{2}} |u|_{0,M}^{\frac{1}{2}},$$

where $c = c(M)$ and

(6.6) $$|u|_{2,M} = |u|_{0,M} + \|Du\|_{0,M} + \|D^2 u\|_{0,M}$$

and the norms on the right-hand side are supremum norms.

Proof. Using a partition of unity we may assume that u has support in a coordinate chart and hence we may assume that

$$u \in C_c^2(\mathbb{R}^n), \tag{6.7}$$

where $\mathbb{R}^n$ is equipped with the Euclidean metric. Moreover, we may assume $n = 1$.

Let $x \in \mathbb{R}$, $\epsilon > 0$ be arbitrary, and choose $x_1, x_2 \in \mathbb{R}$ such that

$$x_2 - x_1 = \epsilon \quad \wedge \quad x \in (x_1, x_2). \tag{6.8}$$

Then we deduce

$$u(x_2) - u(x_1) = Du(\xi)(x_2 - x_1), \quad \xi \in (x_1, x_2), \tag{6.9}$$

$$Du(x) = Du(\xi) + \int_\xi^x D^2 u, \tag{6.10}$$

and hence,

$$|Du(x)| \le 2\epsilon^{-1}|u|_0 + \epsilon |D^2 u|_0 \equiv \varphi(\epsilon), \tag{6.11}$$

where we assume without loss of generality that

$$|D^2 u|_0 > 0, \tag{6.12}$$

otherwise, we replace $|D^2 u|_0$ by $|D^2 u|_0 + \delta$, $\delta > 0$.

Minimizing φ by solving

$$\dot{\varphi}(\epsilon) = -2\epsilon^{-2}|u|_0 + |D^2 u|_0 = 0, \tag{6.13}$$

we conclude

$$\epsilon = \sqrt{2}|u|_0^{\frac{1}{2}}|D^2 u|_0^{-\frac{1}{2}}, \tag{6.14}$$

and thus,

$$|Du| \le 2\sqrt{2}|u|_0^{\frac{1}{2}}|D^2 u|_0^{\frac{1}{2}}. \tag{6.15}$$

$\square$

6.2. **Corollary.** *Let $M = M^n$ be a compact Riemannian manifold of class C^m, $m \ge 2$, and $u \in C^m(M)$, then*

$$\|D^{m-1}u\|_{0,M} \le c|u|_{m-2,M}^{\frac{1}{2}}|u|_{m,M}^{\frac{1}{2}}, \tag{6.16}$$

where $c = c(m, M)$.

We shall apply the corollary to the function $(u - 2)$ using either $M = \mathbb{S}^n$ or $M = \text{graph}\, u$.

The starting point for deriving the estimate (6.2) is equation (5.24) on page 18 which will be differentiated covariantly. However, we first have to derive some preparatory lemmata.

6.3. **Lemma.** *Let φ be defined by*

$$\varphi = (u - 2)^{-1} \tag{6.17}$$

and assume

$$m \geq 2 \quad \wedge \quad \|D^k A\| \leq c_\lambda e^{-\lambda t} \quad \forall\, 1 \leq k \leq m - 1, \tag{6.18}$$

and for all $0 < \lambda < \frac{1}{n}$, then

$$\|D^{m+1}\varphi\| \leq c_\epsilon e^{(\frac{1}{n}+\epsilon)t} \qquad \forall\, 0 < \epsilon. \tag{6.19}$$

The estimate

$$\|D\varphi\| \leq c e^{\frac{t}{n}} \tag{6.20}$$

has already been proved.

Proof. Set

$$\tilde{u} = (u - 2)e^{\frac{t}{n}}, \tag{6.21}$$

then $\tilde{u}$ satisfies

$$-c_1 \leq \tilde{u} \leq -c_2 \qquad \forall\, 0 \leq t < \infty, \tag{6.22}$$

and

$$\varphi = \tilde{u}^{-1} e^{\frac{t}{n}}. \tag{6.23}$$

Let $\alpha \in \mathbb{N}^n$ be a multi-index of order $m+1$, $m \geq 2$, then $D^\alpha \varphi$ can be written as

$$D^\alpha \varphi = \sum_{|\beta_1|+\cdots+|\beta_{m+1}|=m+1} c_{\beta_1,\ldots,\beta_{m+1}} D^{\beta_1}\tilde{u} \cdots D^{\beta_{m+1}}\tilde{u}\, e^{\frac{t}{n}}, \tag{6.24}$$

where the coefficients $c_{\beta_1,\ldots,\beta_{m+1}}$ depend smoothly on $\tilde{u}$, and, if we allow some of the coefficients to vanish, the sum is taken over all multi-indices β_i, $1 \leq i \leq m + 1$, satisfying

$$\sum_{i=1}^{m+1} |\beta_i| = m + 1. \tag{6.25}$$

The estimate (6.20) is trivial in view of (5.55) on page 20. $\qquad\square$

6.4. **Lemma.** *Let $f = f(u, Du, \tilde{u}, D\tilde{u})$ be any smooth function and assume that the conditions (6.18) are valid, then, for any $\epsilon > 0$, there holds*

$$\|D^m(f\dot{\tilde{\vartheta}})\| \leq c_\epsilon e^{(\frac{1}{n}+\epsilon)t}, \tag{6.26}$$

$$\|D^m(f\dot{\tilde{\vartheta}}u_i)\| \leq c_\epsilon e^{(\frac{1}{n}+\epsilon)t}, \quad \forall\, 1 \leq i \leq n, \tag{6.27}$$

$$\|D^m(f\ddot{\tilde{\vartheta}}u_i u_j)\| \leq c_\epsilon e^{(\frac{1}{n}+\epsilon)t}, \quad \forall\, 1 \leq i, j \leq n. \tag{6.28}$$

Proof. Let us only prove (6.26), since we can write

$$\begin{aligned}
\dot{\tilde{\vartheta}}u_i &= \dot{\tilde{\vartheta}}(u-2)(u-2)^{-1}u_i \\
(6.29) \qquad &= \dot{\tilde{\vartheta}}\tilde{u}^{-1}\tilde{u}_i \\
&\equiv f\tilde{\vartheta}
\end{aligned}$$

with some smooth function $f = f(u, Du, \tilde{u}, D\tilde{u})$, and similarly

$$(6.30) \qquad \ddot{\tilde{\vartheta}}u_i u_j \equiv f\tilde{\vartheta}.$$

„(6.26)" Define $\tilde{\theta}$ by

$$(6.31) \qquad \tilde{\theta} = -\tilde{\vartheta}(u-2),$$

then $\tilde{\theta}$ is smooth and

$$(6.32) \qquad f\tilde{\vartheta} = -f\tilde{\theta}\varphi.$$

The estimate then follows by applying the general Leibniz rule and (6.19) observing that $\epsilon > 0$ is assumed to be arbitrary. $\qquad\square$

Let $\check{A}$ be a symbol for $\check{h}^i_j$, then

$$(6.33) \qquad \check{A} = A + v^{-1}\tilde{\vartheta}\delta^i_j$$

and we deduce from (6.26), if the assumptions (6.18) are satisfied,

$$(6.34) \qquad \|D^m\check{A}\| \le \|D^m A\| + c_\epsilon e^{(\frac{1}{n}+\epsilon)t}$$

for any $\epsilon > 0$, where c_ϵ also depends on m. Here, we also used the relation

$$(6.35) \qquad v^{-2} = 1 - \|Du\|^2.$$

We also note that in case $m = 1$ the relation (6.34) is valid for $\epsilon = 0$.

Inside the braces of the right-hand side of equation (5.24) on page 18 there is the crucial term

$$(6.36) \qquad -v^{-2}\dot{\tilde{\vartheta}}h^j_i + v^{-1}\dot{\tilde{\vartheta}}\bar{h}_{ik}g^{kj},$$

which is equal to

$$(6.37) \qquad v^{-1}\dot{\tilde{\vartheta}}\{-v^{-1}h^j_i + \bar{h}_{ik}g^{kj}\} = v^{-1}\dot{\tilde{\vartheta}}u^j_i,$$

in view of (6.4).

Differentiating (6.36) covariantly with respect to a multi-index α, $|\alpha| = m$, $m \ge 1$, we therefore obtain

$$\begin{aligned}
(6.38) \qquad &\sum_{\beta\le\alpha}\binom{\alpha}{\beta}D^{\alpha-\beta}(v^{-1}\dot{\tilde{\vartheta}})D^\beta\{-v^{-1}h^j_i + \bar{h}_{ik}g^{kj}\} \\
&= v^{-1}\dot{\tilde{\vartheta}}D^\alpha\{-v^{-1}h^j_i + \bar{h}_{ik}g^{kj}\} + \sum_{\beta<\alpha}\binom{\alpha}{\beta}D^{\alpha-\beta}(v^{-1}\dot{\tilde{\vartheta}})D^\beta u^j_i.
\end{aligned}$$

Furthermore, there holds

$$(6.39) \qquad v^{-1}\dot{\tilde{\vartheta}} = f(u, Du, \tilde{u})\tilde{\vartheta}e^{\frac{t}{n}},$$

and hence,

$$(6.40) \qquad D^{\alpha-\beta}(v^{-1}\dot{\tilde{\vartheta}})D^\beta u_i^j = D^{\alpha-\beta}(f\tilde{\vartheta})D^\beta \tilde{u}_i^j$$

and we conclude

$$(6.41) \qquad \|\sum_{\beta<\alpha}\binom{\alpha}{\beta}D^{\alpha-\beta}(v^{-1}\dot{\tilde{\vartheta}})D^\beta u_i^j\| \le c_\epsilon e^{(\frac{1}{n}+\epsilon)t} \quad \forall \epsilon > 0,$$

provided (6.18) is valid, in view of (6.26), (6.24) and (6.19), since

$$(6.42) \qquad \beta < \alpha \quad \implies \quad |\beta| + 2 \le m + 1.$$

In case $m = 1$ we have

$$(6.43) \qquad \|D(v^{-1}\dot{\tilde{\vartheta}})u_i^j\| \le c e^{\frac{2}{n}t}\|D^2 u\|.$$

Hence we obtain:

6.5. **Lemma.** *Let α be a multi-index of order $m \ge 2$ and suppose that* (6.18) *is valid, then*

$$(6.44) \qquad \begin{aligned} D^\alpha\{-v^{-2}\dot{\tilde{\vartheta}}h_i^j + v^{-1}\dot{\tilde{\vartheta}}\bar{h}_{ik}g^{kj}\} &= v^{-1}\dot{\tilde{\vartheta}}D^\alpha\{-v^{-1}h_i^j + \bar{h}_{ik}g^{kj}\} + \mathcal{O}_\epsilon^1 \\ &= -v^{-2}\dot{\tilde{\vartheta}}D^\alpha h_i^j + \mathcal{O}_\epsilon^1, \end{aligned}$$

where $\mathcal{O}_\epsilon^1$ represents a tensor that can be estimated like

$$(6.45) \qquad \|\mathcal{O}_\epsilon^1\| \le c_\epsilon e^{(\frac{1}{n}+\epsilon)t} \quad \forall \epsilon > 0.$$

In case $m = 1$ we have

$$(6.46) \qquad \begin{aligned} D\{-v^{-2}\dot{\tilde{\vartheta}}h_i^j + v^{-1}\dot{\tilde{\vartheta}}\bar{h}_{ik}g^{kj}\} &= v^{-1}\dot{\tilde{\vartheta}}D\{-v^{-1}h_i^j + \bar{h}_{ik}g^{kj}\} + \mathcal{O}_0^2 u_i^j \\ &= -v^{-2}\dot{\tilde{\vartheta}}Dh_i^j + \mathcal{O}_0^1 + \mathcal{O}_0^2 u_i^j, \end{aligned}$$

where $\mathcal{O}_0^1$ resp. $\mathcal{O}_0^2$ represent tensors that can be estimated like

$$(6.47) \qquad \|\mathcal{O}_0^1\| \le c e^{\frac{t}{n}}$$

resp.

$$(6.48) \qquad \|\mathcal{O}_0^2\| \le c e^{\frac{2}{n}t}.$$

Proof. Observing that

$$(6.49) \qquad \bar{h}_{ik} = u^{-1}\bar{g}_{ik} = u^{-1}g_{ik} - u^{-1}u_i u_k$$

the relation (6.44) follows from (6.38), (6.41) and (6.18), while (6.46) can be deduced from (6.43). $\qquad\square$

6.6. **Definition.** Let $k \in \mathbb{Z}$, then the symbol $\mathcal{O}_\epsilon^k$ represents any tensor that can be estimated by

$$(6.50) \qquad \|\mathcal{O}_\epsilon^k\| \le c_\epsilon e^{(\frac{k}{n}+\epsilon)t} \qquad \forall \epsilon > 0,$$

and the symbol $\mathcal{O}_0^k$ represents any tensor that can be estimated by

$$(6.51) \qquad \|\mathcal{O}_0^k\| \le c e^{\frac{k}{n}t}.$$

Thus $\mathcal{O}_0^0$ represents a uniformly bounded tensor.

We also denote by $\mathcal{D}^m F$ the derivatives of order m of F with respect to the argument $\check{h}^i_j$, and when S, T are arbitrary tensors then $S \star T$ will symbolize any linear combination of tensors formed by contracting over S and T. The result can be a tensor or a function. Note that we do not distinguish between $S \star T$ and $cS \star T$, c a constant.

From (6.34), the homogeneity of F and the definition of Φ we then deduce

6.7. **Lemma.** *Let $m = 1$ or assume that* (6.18) *is valid, then we have*

$$(6.52) \qquad D^m F = \mathcal{D}F \star D^m A + \mathcal{D}F \star \mathcal{O}^1_\epsilon,$$

$$(6.53) \qquad D^m \Phi^{(k)} = \Phi^{(k+1)} D^m F + \Phi^{(k+1)} \star \mathcal{O}^1_\epsilon,$$

and similarly

$$(6.54) \qquad D^m \mathcal{D}^k F = \mathcal{D}^{k+1} F \star D^m F + \mathcal{D}^{k+1} F \star \mathcal{O}^1_\epsilon,$$

where $\Phi^{(k)}$ is the k-th derivative of Φ. In case $m = 1$ $\mathcal{O}^1_\epsilon$ can be replaced by $\mathcal{O}^1_0$.

We are now ready to differentiate (5.24) on page 18 covariantly.

6.8. **Lemma.** *The tensor DA satisfies the evolution equation*

$$\frac{D}{dt}(DA) - \dot{\Phi}F^{kl}(DA)_{;kl} =$$

$$\ddot{\Phi}\,\mathcal{O}^0_0 \star (DA + \mathcal{O}^1_\epsilon) \star \mathcal{D}^2 A + \dot{\Phi}\,\mathcal{D}^2 F \star (DA + \mathcal{O}^1_\epsilon) \star D^2 A$$

$$+ \dot{\Phi}\,\mathcal{O}^0_0 \star DA + \ddot{\Phi}\,\mathcal{O}^0_0 \star (DA + \mathcal{O}^1_0) + \dot{\Phi}\,\mathcal{O}^0_0 \star (DA + \mathcal{O}^1_0)$$

$$+ \Phi\,\mathcal{O}^0_0 \star DA + \dddot{\Phi}\,\mathcal{O}^0_0 \star (DA + \mathcal{O}^1_0) \star (DA + \mathcal{O}^1_0) \star (DA + \mathcal{O}^1_0)$$

$$+ \ddot{\Phi}\,\mathcal{O}^0_0 \star (D^2 A + \mathcal{O}^1_0 + \mathcal{O}^2_0 \star D^2 u) \star (DA + \mathcal{O}^1_0)$$

$$(6.55) \qquad + \ddot{\Phi}\,\mathcal{D}^2 F \star \mathcal{D}F \star (DA + \mathcal{O}^1_0) \star (DA + \mathcal{O}^1_0) \star (DA + \mathcal{O}^1_0)$$

$$+ \dot{\Phi}\,\mathcal{D}^3 F \star (DA + \mathcal{O}^1_0) \star (DA + \mathcal{O}^1_0) \star (DA + \mathcal{O}^1_0)$$

$$+ \dot{\Phi}\,\mathcal{D}^2 F \star (D^2 A + \mathcal{O}^1_0 + \mathcal{O}^2_0 \star D^2 u) \star (DA + \mathcal{O}^1_0)$$

$$+ \ddot{\Phi}\,\mathcal{O}^0_0 \star (DA + \mathcal{O}^1_0) \star (\dot{\tilde{\vartheta}} D^2 u \star \mathcal{O}^0_0 + \tilde{\vartheta}\,\mathcal{O}^0_0 + DA \star \mathcal{O}^0_0)$$

$$+ \dot{\Phi}\,\mathcal{O}^0_0 \star (\mathcal{O}^1_0 + \mathcal{O}^1_0 \star DA + \mathcal{O}^0_0 \star D^2 A + \mathcal{O}^2_0 \star D^2 u)$$

$$+ \dot{\Phi}F^{kl} g_{kl}(-v^{-2}\dot{\tilde{\vartheta}} DA).$$

Proof. Differentiate (5.24) on page 18 covariantly with respect to a spatial variable and apply Lemma 5.4 on page 20, Lemma 6.5, Definition 6.6, and Lemma 6.7. $\qquad\square$

An almost identical proof—where we also have to rely on Lemma 6.4—yields the evolution equation for higher derivatives of A.

6.9. **Lemma.** *Let $m \geq 2$ and assume that assumptions* (6.18) *are valid, then the tensor $D^m A$, where $D^m A$ represents any covariant derivative $D^\alpha A$, $|\alpha| = m$, satisfies the evolution equation*

$$\frac{D}{dt}(D^m A) - \dot\Phi F^{kl}(D^m A)_{;kl} =$$

$$\ddot\Phi \mathcal{D}F \star (D^m A + \mathcal{O}^1_\epsilon) \star D^2 A + \dot\Phi \mathcal{D}^2 F \star (D^m A + \mathcal{O}^1_\epsilon) \star D^2 A$$

$$+ \Phi \mathcal{O}^0_0 \star D^m A + \dot\Phi \mathcal{O}^1_\epsilon \star D^{m+1} A + \dot\Phi \mathcal{D}^2 F \star \mathcal{O}^1_\epsilon \star (D^{m+1}A + \mathcal{O}^1_\epsilon)$$

$$+ \ddot\Phi \mathcal{D}^2 F \star (D^2 F + \mathcal{O}^1_\epsilon) \star \mathcal{D}F \star (D^m A + \mathcal{O}^1_\epsilon)$$

$$+ \dot\Phi \mathcal{D}^2 F \star (D^2 A + \mathcal{O}^1_\epsilon) \star (D^m A + \mathcal{O}^1_\epsilon) \star \mathcal{D}F$$

$$+ \ddot\Phi (\mathcal{D}F \star D^m A + \mathcal{O}^1_\epsilon) \star \mathcal{O}^0_0 + \dot\Phi \mathcal{D}^2 F \star (D^m A + \mathcal{O}^1_\epsilon) \star \mathcal{O}^0_0$$

$$+ \dot\Phi (D^m A + \mathcal{O}^1_\epsilon) \star \mathcal{O}^0_0 + \dddot\Phi (D^m A + \mathcal{O}^1_\epsilon) \star D\check A \star D\check A \star \mathcal{O}^0_0$$

(6.56)
$$+ \dddot\Phi \mathcal{D}^2 F \star D^m A \star D\check A + \dddot\Phi \mathcal{D}^2 F \star \mathcal{O}^1_\epsilon \star D\check A$$

$$+ \dddot\Phi \mathcal{D}F \star (D^m A + \mathcal{O}^1_\epsilon) \star \mathcal{D}^2 F \star D\check A \star D\check A \star \mathcal{O}^0_0$$

$$+ \dot\Phi \mathcal{D}^3 F \star (D^m A + \mathcal{O}^1_\epsilon) \star D\check A \star D\check A$$

$$+ \dot\Phi \mathcal{D}^2 F \star (D^{m+1} A + \mathcal{O}^1_\epsilon) \star (DA + \mathcal{O}^1_\epsilon)$$

$$+ \ddot\Phi (D^m A + \mathcal{O}^1_\epsilon) \star (\dot{\tilde\vartheta} D^2 u \star \mathcal{O}^0_0 + \tilde\vartheta \mathcal{O}^0_0)$$

$$+ \dot\Phi \mathcal{D}^2 F \star (D^m A + \mathcal{O}^1_\epsilon) \star (\dot{\tilde\vartheta} D^2 u + \tilde\vartheta \mathcal{O}^0_0) \star \mathcal{O}^0_0$$

$$+ \dot\Phi \mathcal{D}F \star (\mathcal{O}^1_\epsilon + \tilde\vartheta D^m A \star \mathcal{O}^0_0 + \mathcal{O}^0_0 \star D^{m+1} A)$$

$$- \dot\Phi F^{kl} g_{kl} v^{-2} \dot{\tilde\vartheta} D^m A$$

We are now going to prove uniform bounds for

(6.57)
$$\tfrac{1}{2}\|D^m \tilde A\|^2 = \tfrac{1}{2} \sum_{|\alpha|=m} \|D^\alpha A\|^2 e^{2\lambda t}$$

for all $m \geq 1$ and

(6.58)
$$0 \leq \lambda < \tfrac{1}{n}.$$

First, we observe that

$$\frac{D}{dt}(\tfrac{1}{2}\|D^m \tilde A\|^2) - \dot\Phi F^{kl}(\tfrac{1}{2}\|D^m \tilde A\|^2)_{;kl} =$$

(6.59)
$$\left\{\frac{D}{dt}(D^m A) - \dot\Phi F^{kl}(D^m A)_{;kl}\right\} e^{\lambda t} D^m \tilde A - \dot\Phi F^{kl}(D^m \tilde A)_{;k}(D^m \tilde A)_{;l}$$

$$+ \lambda \|D^m \tilde A\|^2$$

6.10. **Lemma.** *The quantities $\tfrac{1}{2}\|D^m \tilde A\|^2$ are uniformly bounded during the evolution for any $m \geq 1$ and $0 \leq \lambda < \tfrac{1}{n}$,*

Proof. We prove the lemma recursively by estimating

(6.60)
$$\varphi = \log(\tfrac{1}{2}\|D^m \tilde A\|^2) + \mu \tfrac{1}{2}\|D^{m-1} A\|^2,$$

where

$$(6.61) \qquad 0 < \mu = \mu(m) << 1,$$

cf. the proof of [6, Lemma 7.6.3].

We shall only treat the case $m = 1$, since the proof for $m \geq 2$ is almost identical by considering the evolution equation (6.56) instead of (6.55).

Thus, let

$$(6.62) \qquad \varphi = \log(\tfrac{1}{2}\|D\tilde{A}\|^2) + \mu \tfrac{1}{2}\|A\|^2.$$

Fix $0 < T < \infty$, T very large, and suppose that

$$(6.63) \qquad \sup_{[0,T]} \sup_{M(t)} \varphi = \varphi(t_0, \xi_0)$$

is large, and hence, $0 < t_0 \leq T$, is sufficiently large, such that the previous decay estimates for $\|Du\|$, etc. can be employed.

Applying the maximum principle we deduce from (6.55), (6.59) and the evolution equation for $\tfrac{1}{2}\|A\|^2$, see (5.113) on page 25,

$$
\begin{aligned}
(6.64) \qquad 0 \leq \{&-\dot{\Phi} F^{kl} g_{kl} v^{-2} \dot{\tilde{\vartheta}} + \lambda\} - 2\dot{\Phi} F^{kl} (D\tilde{A})_{;k} (D\tilde{A})_{;l} \\
&+ \dot{\Phi} F^{kl} \log(\tfrac{1}{2}\|\tilde{A}\|^2)_k \log(\tfrac{1}{2}\|D\tilde{A}\|^2)_l \\
&- \mu \dot{\Phi} F^{kl} h_{ij;k} h^{ij}{}_{;l} + \text{rest}.
\end{aligned}
$$

When t_0 is large then terms in the braces can be estimated from above by

$$(6.65) \qquad -2\delta,$$

where

$$(6.66) \qquad \delta = \delta(\lambda) \approx \tfrac{1}{2}(\tfrac{1}{n} - \lambda).$$

To estimate

$$(6.67) \qquad \dot{\Phi} F^{kl} \log(\tfrac{1}{2}\|\tilde{A}\|^2)_k \log(\tfrac{1}{2}\|D\tilde{A}\|^2)_l$$

we use $D\varphi = 0$ and conclude that this term can be estimated from above by

$$(6.68) \qquad \mu^2 \dot{\Phi} F^{kl} h_{ij;k} h^{ij}{}_{;l} \|A\|^2 \leq \tfrac{\mu}{2} \dot{\Phi} F^{kl} h_{ij;k} h^{ij}{}_{;l},$$

if $0 < \mu$ is small.

Most terms in the „rest" can be easily absorbed; a few are a bit more delicate. These can be estimated from above by

$$(6.69) \qquad c\frac{\|D^2 u\| e^{\lambda t} \|D\tilde{A}\|}{\|D\tilde{A}\|^2} < c\|D\tilde{A}\|^{-\frac{1}{2}},$$

where the last inequality is due to the interpolation lemma, cf. Corollary 6.2, applied to $(u - 2)$. Here, we also used the assumption that $\|D\tilde{A}\| \geq 1$.

A thorough inspection of the right-hand side of (6.64) then yields

$$(6.70) \qquad 0 \leq -\delta + ce^{-(\frac{1}{n} - \lambda)t_0} + c\|D\tilde{A}\|^{-\frac{1}{2}}$$

and hence an a priori estimate for $\|D\tilde{A}\|$, if t_0 is large. $\qquad \square$

It remains to prove the optimal decay (6.2) and the convergence to a constant. The optimal decay will be achieved by deriving the equivalent estimate

$$(6.71) \qquad \|D^m u\| \le c_m e^{-\frac{t}{n}} \qquad \forall\, m \ge 1.$$

6.11. **Theorem.** *Let $M(t) = \operatorname{graph} u(t)$ be the leaves of the inverse curvature flow, where F and the initial hypersurface are smooth, then the estimate* (6.71) *is valid and the function*

$$(6.72) \qquad (u-2)e^{\frac{t}{n}}$$

converges in $C^\infty(\mathbb{S}^n)$ to a strictly negative function.

Proof. It suffices to prove (6.71), in view of the relations (3.17) on page 7 and (5.4) on page 16, and to show that the limit exists.

(i) Our starting point is equation (5.21) on page 17 satisfied by u as well as by $(u-2)$.

Let φ, $\tilde\varphi$, $\tilde F$, and $\tilde\Phi$ be defined by

$$(6.73) \qquad \varphi = (2-u)^{-1} \quad \wedge \quad \tilde\varphi = \varphi e^{-\frac{t}{n}},$$

$$(6.74) \qquad \tilde F = F(\check h_l^k (2-u)),$$

and

$$(6.75) \qquad \tilde\Phi = \Phi(\tilde F),$$

then we deduce from (5.21)

$$(6.76) \quad \begin{aligned}
\dot{\tilde\varphi} - \dot{\tilde\Phi}\tilde\varphi^{-2} e^{-\frac{2}{n}t} \tilde F^{ij}\tilde\varphi_{ij} &= -2\dot{\tilde\Phi}\tilde\varphi^{-2} e^{-\frac{2}{n}t} \tilde F^{ij}\tilde\varphi_i\tilde\varphi_j\tilde\varphi^{-1} + 2v^{-1}\tilde F^{-1}\tilde\varphi \\
&\quad - \tilde F^{-2}F^{ij}g_{ij}\tilde\theta\tilde\varphi v^{-2} - \tilde F^{-2}F^{ij}\bar h_{ij}e^{-\frac{t}{n}} - \tfrac{1}{n}\tilde\varphi,
\end{aligned}$$

where

$$(6.77) \qquad \tilde\theta = \tilde\vartheta(2-u).$$

$\tilde\theta$ depends smoothly on u and is strictly positive. The derivatives of arbitrary order of $\tilde\theta$, $\tilde F$, $\tilde\Phi$, F^{ij}, $\tilde F^{ij}$, v, and $\bar h_{ij}$ are uniformly bounded and decay exponentially fast, if t goes to infinity, while the C^m-norms of $\tilde\varphi$ can be estimated by

$$(6.78) \qquad \|D^m \tilde\varphi\| \le c_{m,\epsilon} e^{\epsilon t} \qquad \forall\, \epsilon > 0,$$

in view of our previous estimates.

Differentiating then (6.76) covariantly we obtain the following differential inequality for

$$(6.79) \qquad w = \tfrac{1}{2}\|D^m\tilde\varphi\|^2$$

$$(6.80)$$
$$\dot w - \dot{\tilde\Phi}\tilde\varphi^{-2}e^{-\frac{2}{n}t}\tilde F^{ij}w_{ij} \le \mathcal{O}^0_{-2\delta} + 2\{2v^{-1}\tilde F^{-1} - \tilde F^{-2}F^{ij}g_{ij}\tilde\theta v^{-2} - \tfrac{1}{n}\}w,$$

where $\mathcal{O}^0_r$, $r \in \mathbb{R}$, represents a term that can be estimated by

$$(6.81) \qquad |\mathcal{O}^0_r| \le c e^{rt} \qquad \forall\, 0 \le t < \infty.$$

In inequality (6.80) we may choose $\delta > 0$ independently of $m \geq 1$. The terms inside the braces of that inequality are also an $\mathcal{O}^0_{-2\delta}$ for an appropriate $\delta > 0$, and, because of (6.78),

$$\mathcal{O}^0_{-2\delta} w = \mathcal{O}^0_{-\delta}. \tag{6.82}$$

Hence, applying the maximum principle to the function

$$w + \mu e^{-\delta t} \tag{6.83}$$

we derive an a priori estimate for w by choosing μ large enough.

(ii) It remains to prove that the pointwise limit

$$\lim_{t \to \infty} (u(t, \xi) - 2) e^{\frac{t}{n}} \tag{6.84}$$

exists for any $\xi \in \mathbb{S}^n$.

Using the scalar flow equation (5.56) on page 21 we deduce

$$\dot{\tilde{u}} = \frac{v}{F} e^{\frac{t}{n}} + \tfrac{1}{n} \tilde{u}, \tag{6.85}$$

where

$$\tilde{u} = (u - 2) e^{\frac{t}{n}} \tag{6.86}$$

and F depends on

$$\check{h}^i_j = h^i_j + v^{-1} \tilde{\vartheta} \delta^i_j. \tag{6.87}$$

In view of the homogeneity of F we further conclude

$$
\begin{aligned}
\dot{\tilde{u}} &= \frac{v}{F(\check{h}^i_j e^{-\frac{t}{n}})} + \tfrac{1}{n} \tilde{u} \\
&= (-\tilde{u}) \left\{ \frac{v}{F(h^i_j e^{-\frac{t}{n}}(-\tilde{u}) + v^{-1} \frac{u}{(1+\frac{1}{2}u)} \delta^i_j)} - \tfrac{1}{n} \right\} \\
&\geq -c e^{-\frac{t}{n}}
\end{aligned}
\tag{6.88}
$$

in view of our previous estimates, and we finally obtain

$$(\tilde{u} - nc e^{-\frac{t}{n}})' \geq 0, \tag{6.89}$$

from which the convergence result immediately follows. $\qquad\square$

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