Closed Weingarten Hypersurfaces in Space Forms

Claus Gerhardt

Dedicated to Stefan Hildebrandt on the occasion of his sixtieth birthday

0. Introduction.

In a complete $(n+1)$-dimensional manifold N we want to find closed hypersurfaces M of *prescribed curvature*, so-called *Weingarten* hypersurfaces. To be more precise, let Ω be a connected open subset of N, $f \in C^{2,\alpha}(\overline{\Omega})$, F a smooth, symmetric function defined in the positive cone $\Gamma_+ \subset \mathbf{R}^n$, then we look for a convex hypersurface $M \subset \Omega$ such that

$$F|_M = f(x) \quad \forall x \in M, \tag{0.1}$$

where $F|_M$ means that F is evaluated at the vector $(\kappa_i(x))$ the components of which are the principal curvatures of M.

This is in general a fully nonlinear partial differential equation problem, which is elliptic if we assume F to satisfy

$$\frac{\partial F}{\partial \kappa_i} > 0 \quad \text{in} \quad \Gamma_+. \tag{0.2}$$

Classical examples of curvature functions F are the elementary symmetric polynomials of order k, H_k, defined by

$$H_k = \sum_{i_1<\ldots<i_k} \kappa_{i_1} \ldots \kappa_{i_k}, \quad 1 \le k \le n. \tag{0.3}$$

H_1 is the mean curvature H, H_2 is the scalar curvature—for hypersurfaces in Euclidean space—, and H_n is the Gaussian curvature K.

For technical reasons it is convenient to consider the homogeneous polynomials of degree 1

$$\sigma_k = H_k^{1/k} \tag{0.4}$$

instead of H_k. Then, the σ_k's are not only monotone increasing but also *concave*. Their *inverses* $\tilde{\sigma}_k$, defined through

$$\tilde{\sigma}(\kappa_i) = \frac{1}{\sigma_k(\kappa_i^{-1})} \tag{0.5}$$

share these properties; a proof of this non-trivial result can be found in [12]. $\tilde{\sigma}_1$ is the so-called *harmonic curvature* G, and, evidently, we have $\tilde{\sigma}_n = \sigma_n$.

To describe the general curvature functions we have in mind, let us define

Definition 0.1. Let $F \in C^0(\overline{\Gamma}_+) \cap C^{2,\alpha}(\Gamma_+)$ be a symmetric function, (positively) homogeneous of degree 1 satisfying

$$F_i = \frac{\partial F}{\partial \kappa_i} > 0 \quad \text{on} \quad \Gamma_+ \tag{0.6}$$

and

$$F \quad \textit{is concave.} \tag{0.7}$$

Then we say

(i) F is of class $(\tilde{K})$, if

$$F_{|\partial\Gamma_+} = 0; \tag{0.8}$$

(ii) F is of class $(\tilde{H})$, if

$$\textit{its inverse} \quad \tilde{F} \quad \textit{is also concave,} \tag{0.9}$$

and

$$F \in C^{2,\alpha}(\overline{\Lambda}_{\varepsilon,c}) \quad \text{and} \quad 0 < F_i \quad \text{in} \quad \overline{\Lambda}_{\varepsilon,c}, \tag{0.10}$$

where $\Lambda_{\varepsilon,c} \subset \Gamma_+$ is defined through

$$\Lambda_{\varepsilon,c} = \{(\kappa_i) \,:\, 0 < \varepsilon \le F,\ 0 < \kappa_i \le c\}. \tag{0.11}$$

(iii) F is of class (H), if

$$\tilde{F} \quad \textit{is concave.} \tag{0.12}$$

and

$$F \in C^{2,\alpha}(\overline{\Gamma}_+) \quad \text{and} \quad 0 < F_i \quad \text{in} \quad \overline{\Gamma}_+. \tag{0.13}$$

Remark 0.2. Since F_i are homogeneous of degree 0, the condition (0.13) implies that the F_i are also uniformly bounded in $\overline{\Gamma}_+$.

Remark 0.3. Here are some classical curvature functions which satisfy the above definitions.

(i) The $\widetilde{\sigma}_k$'s are of class $(\widetilde{K})$, and also the inverse of the length of the second fundamental form

$$F(\kappa_i) = \frac{1}{\left(\sum_i k_i^{-2}\right)^{1/2}} \tag{0.14}$$

(ii) The σ_k's (and their inverses) are of class $(\widetilde{H})$.

(iii) The mean curvature is of class (H).

Our main assumption in the existence proof is a barrier assumption.

Definition 0.4. Let M_1, M_2 be strictly convex, closed hypersurfaces in N, homeomorphic to S^n and of class $C^{4,\alpha}$ which bound a connected open subset Ω, such that the mean curvature vector of M_1 points outside of Ω and the mean curvature vector of M_2 points inside of Ω. M_1, M_2 are barriers for (F, f) if

$$F|_{M_1} \leq f \tag{0.15}$$

and

$$F|_{M_2} \geq f. \tag{0.16}$$

Remark 0.5. In view of the Harnack inequality we deduce from the properties of the barriers that they do not touch, unless both coincide and are solutions of our problem. In this case Ω would be empty.

Then we can prove

Theorem 0.6. *Let N be a space form with curvature $K_N = 0$, let F be of class $(\widetilde{K})$, $0 < f \in C^{2,\alpha}(\overline{\Omega})$ and assume that M_1, M_2 are barriers for (F, f), then the problem*

$$F|_M = f \tag{0.17}$$

has a strictly convex solution $M \subset \overline{\Omega}$ of class $C^{4,\alpha}$.

Theorem 0.7. *Let N be a space form with curvature $K_N = 0$ and $F \in (\widetilde{H})$; let $0 < f \in C^{2,\alpha}(\overline{\Omega})$ be such that $\log f$ is concave and assume that M_1, M_2 are barriers for (F, f), then the problem*

$$F|_M = f \tag{0.18}$$

has a convex solution $M \subset \overline{\Omega}$ of class $C^{4,\alpha}$.

Theorem 0.8. *Let N be a space form with curvature K_N and $F \in (H)$; let $f \in C^{2,\alpha}(\overline{\Omega})$ satisfy*

$$-K_N f \bar{g}_{\alpha\beta} + f_{\alpha\beta} \leq 0 \quad in \quad \Omega \tag{0.19}$$

and assume that M_1, M_2 are barriers for (F, f), then the problem

$$F|_M = f \tag{0.20}$$

has a convex solution $M \subset \overline{\Omega}$ of class $C^{4,\alpha}$ if $K_N \leq 0$, or—in the case $K_N > 0$—if in addition f is strictly positive in $\overline{\Omega}$.

Remark 0.9. In the first part of Theorem 0.8 ($K_N \leq 0$) f is not supposed to be strictly positive in $\overline{\Omega}$. Though, in view of the barrier condition, f has to be positive in a neighbourhood of M_1. The solution M will be contained in the support of $\max(f, 0)$ and also the assumption (0.19) should only be valid there.

In a separate paper we considered closed Weingarten hypersurfaces in arbitrary Riemannian manifolds with non-positive sectional curvature, cf. [9]. In that paper we have also proved that we can isometrically lift the geometric setting Ω, M_1, M_2 and f to the universal cover $\widetilde{N}$ even in the case of a space form N with $K_N > 0$. Thus, we may—and shall—assume in the following that N is simply connected.

The existence of closed Weingarten hypersurfaces in $\mathbf{R}^{n+1}$ has been studied extensively in previous papers: the case $F = H$ by Bakelman and Kantor [2], Treibergs and Wei [14], the case $F = K$ by Oliker [13], Delanoë [5], and for general curvature functions by Caffarelli, Nirenberg and Spruck [4]. In all papers—except in [5]—the authors imposed a sign condition for the radial derivative of the right-hand side to prove the existence. This condition was necessary for two reasons, first to derive the a priori estimates for the C^1-norm and secondly to apply the inverse function theorem, i.e. the kernel of the linearized operator had to be trivial.

Without this condition the kernel is no longer trivial and the inverse function theorem of Leray–Schauder type arguments fail.

We therefore use the evolution method to approximate stationary solutions. But there is still the difficulty of obtaining the C^1-estimates: either one has to impose some artificial condition on the right-hand side, i.e. the condition depends on the choice of a special coordinate system, or one has to stay in the class of convex hypersurfaces where the C^1-estimates are a trivial consequence of the convexity, but then the preservation of the convexity has to be proved and this can only be achieved for special curvature functions like the Gaussian curvature, or by assuming f to satisfy the condition (0.19).

The paper is organized as follows: In Section 1 we consider general curvature functions and state some basic properties.

In Section 2 we formulate the evolution problem and prove short-time existence.

In Section 3 we derive the evolution equation for some geometric quantities like the metric and the second fundamental form.

In Section 4 we prove that the flow stays in $\overline{\Omega}$.

In Section 5 we state the parabolic equations satisfied by h_{ij} resp. $v = \sqrt{1+|Du|^2}$.

In Section 6 the C^2-estimates are derived, while in Section 7 the convergence to a smooth stationary solution is proved.

1. Curvature Functions.

Let $F \in C^{2,\alpha}(\Gamma_+) \cap C^0(\overline{\Gamma}_+)$ be a symmetric function satisfying the conditions (0.6) and (0.7); then, F can also be viewed as a function defined on the space of symmetric, positive definite matrices $\mathcal{S}_+$, or to be more precise, at least in this section, let $(h_{ij}) \in \mathcal{S}_+$ with eigenvalues κ_i, $1 \le i \le n$, then define $\widehat{F}$ on $\mathcal{S}_+$ by

$$\widehat{F}(h_{ij}) = F(\kappa_i). \tag{1.1}$$

It is well known, see e.g. [3], that $\widehat{F}$ is as smooth as F and that $\widehat{F}^{ij} = \dfrac{\partial F}{\partial h_{ij}}$ satisfies

$$\widehat{F}^{ij}\xi_i\xi_j = \frac{\partial F}{\partial \kappa_i}|\xi_i|^2, \tag{1.2}$$

where we use the summation convention throughout this paper unless otherwise stated.

Moreover, if F is concave then $\widehat{F}$ is also concave, i.e.

$$\widehat{F}^{ij,kl}\eta_{ij}\eta_{kl} \le 0 \tag{1.3}$$

for any symmetric (η_{ij}), where

$$\widehat{F}^{ij,kl} = \frac{\partial^2}{\partial h_{ij}\partial h_{kl}}\widehat{F}. \tag{1.4}$$

An even sharper estimate is valid, namely,

Lemma 1.1. *Let F, $\widehat{F}$ be defined as above, then*

$$\widehat{F}^{ij,kl}\eta_{ij}\eta_{kl} = \frac{\partial^2 F}{\partial\kappa_i\partial\kappa_j}\eta_{ii}\eta_{jj} + \sum_{i\neq j}\frac{F_i - F_j}{\kappa_i - \kappa_j}(\eta_{ij})^2, \tag{1.5}$$

for any $(\eta_{ij}) \in \mathcal{S}$, where $\mathcal{S}$ is the space of all symmetric matrices and where $F_i = \frac{\partial F}{\partial\kappa_i}$. The second term on the right-hand side of (1.5) is non-positive and has to be interpreted as a limit if $\kappa_i = \kappa_j$.

Proof. In [7, Lemma 2] it is shown that

$$\left(\frac{\partial F}{\partial\kappa_i} - \frac{\partial F}{\partial\kappa_j}\right)(\kappa_i - \kappa_j) \leq 0 \tag{1.6}$$

if F is concave, hence the second term of the right-hand side in (1.5) is non-positive.

A proof of inequality (1.5) can be found in [9, Lemma 1.1].

We also want to mention that F need not to be defined on the positive cone, any open, convex cone will do.

For the rest of the paper we shall no longer distinguish between F and $\widehat{F}$; instead we shall consider F to be defined both on $\mathcal{S}_+$ and Γ_+.

For $(h_{ij}) \in \mathcal{S}_+$ let $(\tilde{h}^{ij}) = (h_{ij})^{-1}$, then we have

Lemma 1.2. *Let F be a curvature function on Γ_+ and $\tilde{F}$ be its inverse, and assume that both F and $\tilde{F}$ are concave, then*

$$F^{ij,kl}\eta_{ij}\eta_{kl} + 2F^{ik}\tilde{h}^{jl}\eta_{ij}\eta_{kl} \geq 2F^{-1}(F^{ij}\eta_{ij})^2 \tag{1.7}$$

for all $(\eta_{ij}) \in \mathcal{J}$.

A proof of the lemma is given in [15, p. 112].

The preceding considerations are also applicable if the κ_i are the principal curvatures of a hypersurface M with metric (g_{ij}). F can then be looked at as being defined on the space of all symmetric tensors (h_{ij}) with eigenvalues κ_i with respect to the metric.

$$F^{ij} = \frac{\partial F}{\partial h_{ij}} \tag{1.8}$$

is then a contravariant tensor of second order. Sometimes, it will be convenient to circumvent the dependence on the metric by considering F to depend on the mixed tensor

$$h^i_j = g^{ik}h_{kj}. \tag{1.9}$$

Then

$$F_i^j = \frac{\partial F}{\partial h_j^i} \tag{1.10}$$

is also a mixed tensor with contravariant index j and covariant index i.

2. The evolution problem.

Let N be a complete $(n+1)$-dimensional Riemannian manifold and M a closed hypersurface. Geometric quantities in N will be denoted by $(\bar{g}_{\alpha\beta})$, $(\bar{R}_{\alpha\beta\gamma\delta})$, etc., and those in M by (g_{ij}), (R_{ijkl}), etc. Greek indices range from 0 to n and Latin from 1 to n; the summation convention is always used. Generic coordinate systems in N resp. M will be denoted by (x^α) resp. (ξ^i). Covariant differentiation will simply be indicated by indices, only in case of possible ambiguity they will be preceded by a semicolon, i.e. for a function u on N, (u_α) will be the gradient and $(u_{\alpha\beta})$ the Hessian, but, e.g. the covariant derivative of the curvature tensor will be abbreviated by $\bar{R}_{\alpha\beta\gamma\delta;\varepsilon}$. We also point out that

$$\bar{R}_{\alpha\beta\gamma\delta;i} = \bar{R}_{\alpha\beta\gamma\delta;\varepsilon} x_i^\varepsilon \tag{2.1}$$

with obvious generalizations to other quantities.

If N is a space of constant curvature, then

$$\bar{R}_{\alpha\beta\gamma\delta} = K_N(\bar{g}_{\alpha\gamma}\bar{g}_{\beta\delta} - \bar{g}_{\alpha\delta}\bar{g}_{\beta\gamma}). \tag{2.2}$$

In local coordinates x^α and ξ^i the geometric quantities of the hypersurface M are connected through the following equations

$$x_{ij}^\alpha = -h_{ij}\nu^\alpha \tag{2.3}$$

the so-called *Gauß formula.* Here, and also in the sequel, a covariant derivative is always a *full* tensor, i.e.

$$x_{ij}^\alpha = x_{,ij}^\alpha - \Gamma_{ij}^k x_k^\alpha + \bar{\Gamma}_{\beta\gamma}^\alpha x_i^\beta x_j^\gamma. \tag{2.4}$$

The comma indicates ordinary partial derivatives.

In this implicit definition (2.3) the *second fundamental form* (h_{ij}) is taken with respect to $-\nu$.

The second equation is the *Weingarten equation*

$$\nu_i^\alpha = h_i^k x_k^\alpha, \tag{2.5}$$

where we remember that ν_i^α is full tensor.

Finally, we have the *Codazzi equation*

$$h_{ij;k} - h_{ik;j} = \overline{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta x_j^\gamma x_k^\delta = 0, \tag{2.6}$$

if N is a space of constant curvature, and the *Gauß equation*

$$R_{ijkl} = h_{ik}h_{jl} - h_{il}h_{jk} + \overline{R}_{\alpha\beta\gamma\delta}x_i^\alpha x_j^\beta x_k^\gamma x_l^\delta. \tag{2.7}$$

We want to prove that the equation

$$F = f \tag{2.8}$$

has a solution. For technical reasons it is convenient to solve instead of (2.8) the equivalent equation

$$\Phi(F) = \Phi(f) \tag{2.9}$$

where Φ is real function defined on $\mathbf{R}_+$ such that

$$\dot\Phi > 0 \quad \text{and} \quad \ddot\Phi \le 0. \tag{2.10}$$

For notational reasons let us abbreviate

$$\tilde f = \Phi(f). \tag{2.11}$$

To solve (2.9) we look at the evolution problem

$$\begin{aligned} \dot x &= -(\Phi - \tilde f)\nu \\ x(0) &= x_0 \end{aligned} \tag{2.12}$$

where x_0 is an embedding of an initial strictly convex hypersurface M_0 diffeomorphic to S^n, $\Phi = \Phi(F)$, and F is evaluated for the principal curvatures of the flow hypersurfaces $M(t)$, or, equivalently, we may assume that F depends on the second fundamental form (h_{ij}) and the metric (g_{ij}) of $M(t)$; $x(t)$ is the embedding for $M(t)$.

This is a parabolic problem, so short-time existence is guaranteed—an exact proof is given below—, and under suitable assumptions we shall be able to prove that the solution exists for all time and that the velocity tends to zero if t goes to infinity.

Consider now a tubular neighbourhood $\mathcal{U}$ of the initial hypersurface M_0, then we can introduce so-called *normal Gaussian coordinates* x^α, such that the metric in $\mathcal{U}$ has the form

$$d\bar{s}^2 = dr^2 + \bar{g}_{ij}\, dx^i dx^j \tag{2.13}$$

where $r = x^0$, $\bar{g}_{ij} = \bar{g}_{ij}(r, x)$; here we use slightly ambiguous notation.

A point $p \in \mathcal{U}$ can be represented by its signed distance from M_0 and its base point $x \in M_0$, thus $p = p(r, x)$.

Let $M \subset \mathcal{U}$ be a hypersurface which is a graph over M_0, i.e.

$$M = \{(r, x) \,:\, r = u(x), x \in M_0\}. \tag{2.14}$$

The induced metric of M, g_{ij}, can then be expressed as

$$g_{ij} = \bar{g}_{ij} + u_i u_j \tag{2.15}$$

with inverse

$$g^{ij} = \bar{g}^{ij} - \frac{u^i}{v}\frac{u^j}{v}, \tag{2.16}$$

where $(\bar{g}^{ij}) = (\bar{g}_{ij})^{-1}$ and

$$\begin{aligned} u^i &= \bar{g}^{ij} u_j \\ v^2 &= 1 + \bar{g}^{ij} u_i u_j \end{aligned} \tag{2.17}$$

The normal vector ν of M then takes the form

$$(\nu^\alpha) = v^{-1}(1, -u^i) \tag{2.18}$$

if x^0 is chosen appropriately.

From the Gauß formula we immediately deduce that the second fundamental form of M is given by

$$v^{-1} h_{ij} = -u_{ij} + \bar{h}_{ij}, \tag{2.19}$$

where

$$\bar{h}_{ij} = \frac{1}{2}\dot{\bar{g}}_{ij} = \frac{1}{2}\frac{\partial \bar{g}_{ij}}{\partial r} \tag{2.20}$$

is the second fundamental form of the level surfaces $\{r = \text{const}\}$, and where the second covariant derivatives of u are defined with respect to the induced metric.

At least for small t the hypersurfaces $M(t)$ are graphs over M_0 and the embedding vector looks like

$$\begin{aligned} x^0(t) &= u(t, x^i(t)) \\ x^i(t) &= x^i(t, \xi^i) \end{aligned} \tag{2.21}$$

where the ξ^i are local coordinates for $M(t)$ independent of t.

Furthermore,

$$\dot{x}^0 = \dot{u} = \frac{\partial u}{\partial t} + \dot{x}^i u_i \tag{2.22}$$

and from (2.12) we conclude

$$\begin{aligned} \dot{x}^0 &= -(\Phi - \tilde{f}) v^{-1} \\ \dot{x}^i &= v^{-1} u^i (\Phi - \tilde{f}) \end{aligned} \tag{2.23}$$

hence, we obtain

$$\frac{\partial u}{\partial t} = -(\Phi - \tilde{f}) v. \tag{2.24}$$

This is a scalar equation, which can be solved on a cylinder $[0, \varepsilon] \times M_0$ for small ε, if the principal curvatures of the initial hypersurface M_0 are strictly positive. The equation (2.23) for the embedding vector is then a classical ordinary differential equation of the form

$$\dot{x} = \varphi(t, x). \tag{2.25}$$

We have therefore proved

Theorem 2.1. *The evolution problem* (2.12) *has a solution on a small time interval* $[0, \varepsilon]$.

3. The evolution equations of some geometric quantities.

In this section we want to show how the metric, the second fundamental form, and the normal vector of the hypersurfaces $M(t)$ evolve. All time derivatives are *total* derivatives.

Lemma 3.1 (Evolution of the metric). *The metric g_{ij} of $M(t)$ satisfies the evolution equation*

$$\dot{g}_{ij} = -2(\Phi - \tilde{f})h_{ij}. \tag{3.1}$$

Proof. Let ξ^i be local coordinates for $M(t)$, then

$$g_{ij} = \overline{g}_{\alpha\beta} x_i^\alpha x_j^\beta \tag{3.2}$$

and thus

$$\dot{g}_{ij} = 2\overline{g}_{\alpha\beta} \dot{x}_i^\alpha x_j^\beta. \tag{3.3}$$

On the other hand, differentiating

$$\dot{x}^\alpha = -(\Phi - \tilde{f})\nu^\alpha \tag{3.4}$$

with respect to ξ^i yields

$$\dot{x}_i^\alpha = -(\Phi - \tilde{f})_i \nu^\alpha - (\Phi - \tilde{f})\nu_i^\alpha \tag{3.5}$$

and the desired result follows from the Weingarten equation.

Lemma 3.2 (Evolution of the normal). *The normal vector ν evolves according to*

$$\dot{\nu} = \nabla_M(\Phi - \tilde{f}) = g^{ij}(\Phi - \tilde{f})_i x_j. \tag{3.6}$$

Proof. Since ν is a unit normal vector we have $\dot{\nu} \in T(M)$. Furthermore, differentiating

$$0 = \langle \nu, x_i \rangle \tag{3.7}$$

with respect to t, we deduce

$$\langle \dot{\nu}, x_i \rangle = -\langle \nu, \dot{x}_i \rangle = (\Phi - \tilde{f})_i. \tag{3.8}$$

Lemma 3.3 (Evolution of the second fundamental form). *The second fundamental form evolves according to*

$$\dot{h}_i^j = (\Phi - \tilde{f})_i^j + (\Phi - \tilde{f})h_i^k h_k^j + (\Phi - \tilde{f})\overline{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta \nu^\gamma x_k^\delta g^{kj} \tag{3.9}$$

and

$$\dot{h}_{ij} = (\Phi - \tilde{f})_{ij} - (\Phi - \tilde{f})h_i^k h_{kj} + (\Phi - \tilde{f})\overline{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta \nu^\gamma x_j^\delta. \tag{3.10}$$

Proof. We use the *Ricci identities* to interchange the covariant derivatives of ν with respect to t and ξ^i

$$
\begin{aligned}
\frac{d}{dt}(\nu_i^\alpha) &= (\dot\nu^\alpha)_i - \overline{R}^\alpha{}_{\beta\gamma\delta}\nu^\beta x_i^\gamma \dot x^\delta \\
&= g^{kl}(\Phi-\tilde f)_{ki}x_l^\alpha + g^{kl}(\Phi-\tilde f)_k x_{li}^\alpha - \overline{R}^\alpha{}_{\beta\gamma\delta}\nu^\beta x_i^\gamma \dot x^\delta
\end{aligned}
\tag{3.11}
$$

For the second equality we used (3.6).

On the other hand, in view of the Weingarten equation

$$
\frac{d}{dt}(\nu_i^\alpha) = \frac{d}{dt}\left(h_i^k x_k^\alpha\right) = \dot h_i^k x_k^\alpha + h_i^k \dot x_k^\alpha.
\tag{3.12}
$$

Multiplying the resulting equation with $\overline{g}_{\alpha\beta}x_j^\beta$ we conclude

$$
\dot h_i^k g_{kj} - (\Phi-\tilde f)h_i^k h_{kj} = (\Phi-\tilde f)_{ij} + (\Phi-\tilde f)\overline{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta \nu^\gamma x_j^\delta
\tag{3.13}
$$

or equivalently (3.9).

To derive (3.10), we differentiate

$$
h_{ij} = h_i^k g_{kj}
\tag{3.14}
$$

with respect to t and use (3.3).

Lemma 3.4 (Evolution of $(\Phi-\tilde f)$). *The term $(\Phi-\tilde f)$ evolves according to the equation*

$$
\begin{aligned}
(\Phi-\tilde f)' - \dot\Phi F^{ij}(\Phi-\tilde f)_{ij} &= \dot\Phi F^{ij}h_{ik}h_j^k(\Phi-\tilde f) + \tilde f_\alpha \nu^\alpha(\Phi-\tilde f) \\
&\quad + \dot\Phi F^{ij}\overline{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta \nu^\gamma x_j^\delta(\Phi-\tilde f)
\end{aligned}
\tag{3.15}
$$

where

$$
(\Phi-\tilde f)' = \frac{d}{dt}(\Phi-\tilde f)
\tag{3.16}
$$

and

$$
\dot\Phi = \frac{d}{dr}\Phi(r).
\tag{3.17}
$$

Proof. When we differentiate F with respect to t it is advisable to consider F as a function of the mixed tensor h_j^i; then we obtain

$$
(\Phi-\tilde f)' = \dot\Phi F_j^i \dot h_i^j - \tilde f_\alpha \dot x^\alpha.
\tag{3.18}
$$

The result now follows from (3.9) and (3.4).

Corollary 3.5. *Let N be a space form then the equation* (3.15) *takes the form*

$$
\begin{aligned}
(\Phi-\tilde f)' - \dot\Phi F^{ij}(\Phi-\tilde f)_{ij} &= \dot\Phi F^{ij}h_{ik}h_j^k(\Phi-\tilde f) + \tilde f_\alpha \nu^\alpha(\Phi-\tilde f)\\
&\quad + K_N\dot\Phi F^{ij}g_{ij}(\Phi-\tilde f)
\end{aligned}
\tag{3.19}
$$

4. Barriers and a priori estimates in the C^0-norm.

In [9, Section 4] we have shown that, if the sectional curvature of the ambient space N is non-positive or if N is a space form with positive curvature, then, the geometric setting of our problem, i.e. Ω, M_1, M_2 and f can be isometrically lifted to the universal cover. If N is space form with $K_N > 0$, then $\overline{\Omega} \subset \tilde N$ is contained in an open hemisphere. The barriers M_i are boundaries of convex bodies $\langle M_i\rangle$ and, if we introduce geodesic polar coordinates $(x^\alpha) = (r, x^i) = (r, x)$ around a point in $\langle M_1\rangle$ such that

$$
d\bar s^2 = dr^2 + \bar g_{ij}\, dx^i dx^j \tag{4.1}
$$

then the second fundamental form $\bar h_{ij}$ of a geodesic sphere $\{r = \text{const}\}$ that intersects $\overline{\Omega}$ is uniformly positive definite. The M_i are graphs over a fixed geodesic sphere S_0, $M_i = \operatorname{graph} u_i|_{S_0}$.

Moreover, let $M(t)$ be a solution of the evolution problem (2.12) in a maximal time interval $I = [0, T^*)$ such that the hypersurfaces are strictly convex. Then, each $M(t)$ can be represented as a graph over S_0

$$
M(t) = \{(r, x) \,:\, r = u(t, x), x \in S_0\}. \tag{4.2}
$$

In [9, Section 5] we also proved the following lemmata

Lemma 4.1. *Choose as initial hypersurface M_0 either M_1 or M_2, then we have for the embedding vector $x = x(t)$*

$$
x(t) \in \overline{\Omega} \quad \forall t \in I. \tag{4.3}
$$

and

Lemma 4.2. *Let $M(t)$ be a solution of the evolution problem* (2.12) *defined on a maximal interval $[0, T^*)$. As initial hypersurface M_0 we choose either M_1 or M_2; then we obtain*

$$
\Phi - \tilde f \le 0 \quad \forall t \tag{4.4}
$$

if $M_0 = M_1$, *and*

$$\Phi - \tilde{f} \geq 0 \quad \forall t \tag{4.5}$$

if $M_0 = M_2$.

5. The evolution equations for h_{ij} and v.

Let $M(t)$ be a solution of problem (2.12); in [9, Section 7] we derived the following evolution equations for h_{ij} resp. h^i_j

Lemma 5.1. *Let* $M(t)$ *be a solution of the problem* (2.12), *then the second fundamental form satisfies*

$$\begin{aligned}
\dot{h}_{ij} - \dot\Phi F^{kl} h_{ij;kl} = {} & \dot\Phi F^{kl} h_{kr} h^r_l h_{ij} - (\Phi - \tilde{f}) h^k_i h_{kj} - \dot\Phi F h^k_i h_{kj} \\
& - \tilde{f}_{\alpha\beta} x^\alpha_i x^\beta_j + \tilde{f}_\alpha \nu^\alpha h_{ij} + \ddot\Phi F_i F_j \\
& + \dot\Phi F^{kl,rs} h_{kl;i} h_{rs;j} + (\Phi - \tilde{f}) \overline{R}_{\alpha\beta\gamma\delta} \nu^\alpha x^\beta_i \nu^\gamma x^\delta_j \\
& + 2\dot\Phi F^{kl} \overline{R}_{\alpha\beta\gamma\delta} x^\alpha_r x^\beta_i x^\gamma_k x^\delta_j h^r_l \\
& - \dot\Phi F^{kl} \overline{R}_{\alpha\beta\gamma\delta} x^\alpha_r x^\beta_k x^\gamma_i x^\delta_l h^r_j \\
& - \dot\Phi F^{kl} \overline{R}_{\alpha\beta\gamma\delta} x^\alpha_r x^\beta_k x^\gamma_j x^\delta_l h^r_i \\
& + \dot\Phi F^{kl} \overline{R}_{\alpha\beta\gamma\delta} \nu^\alpha x^\beta_k \nu^\gamma x^\delta_l h_{ij} \\
& - \dot\Phi F \overline{R}_{\alpha\beta\gamma\delta} \nu^\alpha x^\beta_i \nu^\gamma x^\delta_j \\
& + \dot\Phi F^{kl} \overline{R}_{\alpha\beta\gamma\delta;\epsilon} \{\nu^\alpha x^\beta_k x^\gamma_l x^\delta_i x^\epsilon_j + \nu^\alpha x^\beta_i x^\gamma_k x^\delta_j x^\epsilon_l\}
\end{aligned} \tag{5.1}$$

and

Lemma 5.2. *The evolution equation for* h^i_i (*no summation over* i) *has the form*

$$\begin{aligned}
\dot{h}^i_i - \dot\Phi F^{kl} h^i_{i;kl} = {} & \dot\Phi F^{kl} h_{kr} h^r_l h^i_i + (\Phi - \tilde{f}) h^k_i h^i_k - \dot\Phi F h^k_i h^i_k \\
& - \tilde{f}_{\alpha\beta} x^\alpha_i x^\beta_k g^{ki} + \tilde{f}_\alpha \nu^\alpha h^i_i + \ddot\Phi F_i F^i \\
& + \dot\Phi F^{kl,rs} h_{kl;i} h_{rs;m} g^{mi} \\
& + (\Phi - \tilde{f}) \overline{R}_{\alpha\beta\gamma\delta} \nu^\alpha x^\beta_i \nu^\gamma x^\delta_m g^{mi} \\
& + 2\dot\Phi F^{kl} \overline{R}_{\alpha\beta\gamma\delta} x^\alpha_r x^\beta_i x^\gamma_k x^\delta_m g^{mi} h^r_l \\
& - 2\dot\Phi F^{kl} \overline{R}_{\alpha\beta\gamma\delta} x^\alpha_r x^\beta_k x^\gamma_i x^\delta_l h^{ri} \\
& + \dot\Phi F^{kl} \overline{R}_{\alpha\beta\gamma\delta} \nu^\alpha x^\beta_k \nu^\gamma x^\delta_l h^i_i \\
& - \dot\Phi F \overline{R}_{\alpha\beta\gamma\delta} \nu^\alpha x^\beta_i \nu^\gamma x^\delta_m g^{mi} \\
& + \dot\Phi F^{kl} \overline{R}_{\alpha\beta\gamma\delta;\epsilon} \{\nu^\alpha x^\beta_k x^\gamma_l x^\delta_i x^\epsilon_m + \nu^\alpha x^\beta_i x^\gamma_k x^\delta_m x^\epsilon_l\} g^{mi}
\end{aligned} \tag{5.2}$$

In case N is a space form with curvature K_N, the preceding evolution equations simplify to

Corollary 5.3. *Let N be a space form with curvature K_N, then equation (5.1) takes the form*

$$
\begin{aligned}
\dot{h}_{ij} - \dot{\Phi} F^{kl} h_{ij;kl} = \;& \dot{\Phi} F^{kl} h_{kr} h_l^r h_{ij} - (\Phi - \widetilde{f}) h_i^k h_{kj} - \dot{\Phi} F h_i^k h_{kj} \\
& - \widetilde{f}_{\alpha\beta} x_i^\alpha x_j^\beta + \widetilde{f}_\alpha \nu^\alpha h_{ij} + \ddot{\Phi} F_i F_j \\
& + \dot{\Phi} F^{kl,rs} h_{kl;i} h_{rs;j} \\
& + K_N \{(\Phi - \widetilde{f}) + \dot{\Phi} F\} g_{ij} - K_N \dot{\Phi} F^{kl} g_{kl} h_{ij}
\end{aligned}
\tag{5.3}
$$

and

Corollary 5.4. *Let N be a space form with curvature K_N, then equation (5.2) takes the form*

$$
\begin{aligned}
\dot{h}_i^i - \dot{\Phi} F^{kl} h_{i;kl}^i = \;& \dot{\Phi} F^{kl} h_{kr} h_l^r h_i^i + (\Phi - \widetilde{f}) h_i^k h_k^i - \dot{\Phi} F h_i^k h_k^i \\
& - \widetilde{f}_{\alpha\beta} x_i^\alpha x_k^\beta g^{ki} + \widetilde{f}_\alpha \nu^\alpha h_i^i + \ddot{\Phi} F_i F^i \\
& + \dot{\Phi} F^{kl,rs} h_{kl;i} h_{rs;m} g^{mi} \\
& + K_N \{(\Phi - \widetilde{f}) + \dot{\Phi} F\} \delta_i^i - K_N \dot{\Phi} F^{kl} g_{kl} h_i^i
\end{aligned}
\tag{5.4}
$$

The proof is straightforward, if one observes that F^{ij} and h_{ij} can be diagonalized simultaneously, cf. [9, equ. (1.12)].

Suppose now, that we have introduced geodesic polar coordinates $(x^\alpha) = (r, x^i)$ such that the hypersurfaces $M(t)$ are graphs over a geodesic sphere S_0. From the relation (2.18) we conclude

$$
v = \sqrt{1 + |Du|^2} = (r_\alpha \nu^\alpha)^{-1}.
\tag{5.5}
$$

We know, that as long as the hypersurfaces are convex, the quantity v is uniformly bounded, or more precisely, cf. [9, Lemma 6.1]

Lemma 5.5. *Let $M = \operatorname{graph} u|_{S_0}$ be a closed convex hypersurface represented in normal Gaussian coordinates then the quantity $v = \sqrt{1 + |Du|^2}$ can be estimated by*

$$
v \le c(|u|, S_0, \bar{g}_{ij}).
\tag{5.6}
$$

Furthermore, the function u and the quantity v satisfy the following evolution equations

Lemma 5.6. *Consider the flow in a normal Gaussian coordinate system where the $M(t)$ can be written as graphs of a function $u(t)$. Then u resp. v satisfy the evolution equations*

$$\dot{u} - \dot{\Phi}F^{ij}u_{ij} = -(\Phi - \tilde{f})v^{-1} + \dot{\Phi}Fv^{-1} - \dot{\Phi}F^{ij}\bar{h}_{ij} \tag{5.7}$$

resp.

$$\begin{aligned} \dot{v} - \dot{\Phi}F^{ij}v_{ij} &= -\dot{\Phi}F^{ij}h_{ik}h^k_j v - 2v^{-1}\dot{\Phi}F^{ij}v_i v_j \\ &\quad + r_{\alpha\beta}\nu^\alpha\nu^\beta\big[(\Phi - \tilde{f}) - \dot{\Phi}F\big]v^2 \\ &\quad + \dot{\Phi}F^{ij}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_i x^\gamma_j x^\delta_k r_\varepsilon x^\varepsilon_m g^{mk} v^2 \\ &\quad + 2\dot{\Phi}F^{ij}r_{\alpha\beta}h^k_i x^\alpha_k x^\beta_j v^2 \\ &\quad + \dot{\Phi}F^{ij}r_{\alpha\beta\gamma}\nu^\alpha x^\beta_i x^\gamma_j v^2 + \tilde{f}_\alpha x^\alpha_m g^{mk} r_\beta x^\beta_k v^2 \end{aligned} \tag{5.8}$$

cf. [9, equ. (8.2) resp. Lemma 7.3].

In a simply connected space form we can deduce a considerable simpler and more aesthetic form of (5.8).

First, we observe that by symmetry

$$\bar{g}_{ij} = h(r)\sigma_{ij}, \tag{5.9}$$

where σ_{ij} is the metric of a geodesic sphere of radius 1. Then, we fix a point in N and choose the coordinates (x^i) such that in that point

$$\bar{g}_{ij,k} = 0. \tag{5.10}$$

Let us calculate the corresponding Christoffel symbols in N. We have

$$\bar{\Gamma}^0_{ij} = -\frac{1}{2}\dot{\bar{g}}_{ij} = -\bar{h}_{ij}, \tag{5.11}$$

$$\bar{\Gamma}^0_{00} = \bar{\Gamma}^0_{0i} = \bar{\Gamma}^i_{jk} = 0, \tag{5.12}$$

and

$$\bar{\Gamma}^i_{0j} = \bar{h}^i_j, \tag{5.13}$$

from which we conclude

Lemma 5.7. *In the above coordinate system the covariant derivatives of r can be expressed as follows*

$$\begin{cases} r_{0\alpha} = r_{\alpha 0} = 0 \\ r_{ij} = \bar{h}_{ij} \end{cases} \tag{5.14}$$

$$\text{(5.15)} \qquad \begin{cases} r_{0ij} = -\dfrac{\overline{H}^2}{n^2}\bar{g}_{ij} \\ r_{ij0} = \dfrac{\dot{\overline{H}}}{n}\bar{g}_{ij} \\ r_{i0j} = -\dfrac{\overline{H}^2}{n^2}\bar{g}_{ij} \end{cases}$$

and

$$\text{(5.16)} \qquad r_{00j} = r_{0i0} = r_{ijk} = 0.$$

Proof. To prove (5.14), we use $|Dr| = 1$ to obtain

$$\text{(5.17)} \qquad 0 = r_{\alpha\beta} r^\beta = r_{\alpha 0} = r_{0\alpha}$$

and

$$\text{(5.18)} \qquad r_{ij} = -\overline{\Gamma}^\alpha_{ij} r_\alpha = -\overline{\Gamma}^0_{ij} = \bar{h}_{ij}.$$

The covariant derivatives of the third order are defined by

$$\text{(5.19)} \qquad r_{\alpha\beta\gamma} = r_{\alpha\beta,\gamma} - \overline{\Gamma}^m_{\alpha\gamma} r_{\beta m} - \overline{\Gamma}^m_{\beta\gamma} r_{m\alpha},$$

where we already used (5.17). The relation (5.16) now follows immediately and also

$$\text{(5.20)} \qquad r_{0ij} = r_{i0j} = \bar{h}^m_i \bar{h}_{mj}$$

and

$$\text{(5.21)} \qquad r_{ij0} = \dot{\bar{h}}_{ij} - 2\bar{h}^m_i \bar{h}_{mj}.$$

To complete the proof, we observe that the geodesic spheres are totally umbilical, i.e.

$$\text{(5.22)} \qquad \bar{h}_{ij} = \frac{\overline{H}}{n}\bar{g}_{ij}$$

and hence

$$\text{(5.23)} \qquad \dot{\bar{h}}_{ij} = \frac{\dot{\overline{H}}}{n}\bar{g}_{ij} + 2\frac{\overline{H}^2}{n^2}\bar{g}_{ij}.$$

To derive a simpler version of equation (5.8), let $\eta = \eta(r)$ be a positive solution of

$$\dot\eta = -\frac{\overline{H}}{n}\eta, \quad r > 0, \tag{5.24}$$

wherever it is defined and set

$$\chi = v\eta(u). \tag{5.25}$$

Then, we can prove

Lemma 5.8. *The function χ satisfies the evolution equation*

$$\begin{aligned} \dot\chi - \dot\Phi F^{ij}\chi_{ij} &= -\dot\Phi F^{ij} h_{ik} h^k_j \chi - 2\chi^{-1}\dot\Phi F^{ij}\chi_i\chi_j \\ &\quad + \{\dot\Phi F + (\Phi - \tilde f)\}\frac{\overline{H}}{n} v\chi + \tilde f_\alpha x^\alpha_k g^{ik} u_i v\chi \end{aligned} \tag{5.26}$$

Proof. Using the same notation as before, we obtain

$$\begin{aligned} \dot\chi - \dot\Phi F^{ij}\chi_{ij} &= \{\dot v - \dot\Phi F^{ij} v_{ij}\}\eta + \{\dot u - \dot\Phi F^{ij} u_{ij}\} v\dot\eta \\ &\quad - 2\dot\eta\dot\Phi F^{ij} v_i u_j - v\ddot\eta\dot\Phi F^{ij} u_i u_j \end{aligned} \tag{5.27}$$

We then rewrite the equation (5.8) using the expressions in (5.14) to (5.16) to deduce

$$\begin{aligned} \dot v - \dot\Phi F^{ij} v_{ij} &= -\dot\Phi F^{ij} h_{ik} h^k_j v - 2v^{-1}\dot\Phi F^{ij} v_i v_j \\ &\quad + 2\dot\Phi F^{ij} v_i u_j \frac{\overline{H}}{n} \\ &\quad - \dot\Phi F^{ij} g_{ij} \frac{\overline{H}^2}{n^2} v - \dot\Phi F^{ij} u_i u_j \frac{\dot{\overline{H}}}{n} v \\ &\quad + \frac{\overline{H}}{n}|Du|^2[(\Phi - \tilde f) - \dot\Phi F] \\ &\quad + 2\dot\Phi F \frac{\overline{H}}{n} v^2 + \tilde f_\alpha x^\alpha_k u^k v^2 \end{aligned} \tag{5.28}$$

where we also took into account that

$$\begin{aligned} v_i &= -v^2 h^k_i u_k + v^3 \overline{h}_{ik} u^k \\ &= -v^2 h^k_i u_k + v\frac{\overline{H}}{n} u_i \end{aligned} \tag{5.29}$$

Inserting (5.28) and (5.7) in (5.27) and observing that in view of (5.24)

$$\ddot\eta = -\frac{\dot{\overline{H}}}{n}\eta - \frac{\overline{H}}{n}\dot\eta, \tag{5.30}$$

the equation (5.26) can be easily deduced.

As we have already remarked before, the mean curvature $\overline{H}$ of the geodesic spheres in question is uniformly strictly positive.

6. A priori estimates in the C^2-norm.

Let $M(t)$ be a solution of the evolution problem (2.12) with initial hypersurface $M_0 = M_1$ or $M_0 = M_2$ defined in a maximal time interval $I = [0, T^*)$. We assume $M(t)$ to be represented as the graph of a function u in geodesic polar coordinates. We know that during the evolution the flow stays in the compact set $\overline{\Omega}$ and that the hypersurfaces are strictly convex—this is contained in the definition of the maximal time-interval—, and, hence, Du is uniformly bounded.

We want to show that the second derivatives of u are uniformly bounded or equivalently that the principal curvatures of the flow hypersurfaces are uniformly bounded and positive.

1. The Case of Theorem 0.6.

We first prove

Lemma 6.1. *Let $F \in (\widetilde{K})$, $M_0 = M_1$, $\Phi(t) = \log t$ and $K_N = 0$, then the principal curvatures of the evolution hypersurfaces are uniformly bounded from above.*

Proof. First, we observe, that

$$\Phi \leq \tilde{f} \quad \text{or} \quad F \leq f \tag{6.1}$$

in view of the results in Lemma 4.2.

Next, let φ be defined by

$$\varphi = \sup\{h_{ij}\eta^i\eta^j \;:\; \|\eta\| = 1\} \tag{6.2}$$

and w by

$$w = \log\varphi + \log\chi. \tag{6.3}$$

We claim that w is bounded. Let $0 < T < T^*$, and $x_0 = x(t_0)$, $0 < t_0 \leq T$, be a point in $M(t_0)$ such that

$$\sup_{M_0} w < \sup\{\sup_{M(t)} w \;:\; 0 < t \leq T\} = w(x_0). \tag{6.4}$$

We then can introduce a Riemannian normal coordinate system ξ^i at $x_0 \in M(t_0)$ such that at $x_0 = x(t_0, \xi_0)$ we have

$$g_{ij} = \delta_{ij} \quad \text{and} \quad \varphi = h^n_n. \tag{6.5}$$

Let $\eta = (\eta^i)$ be the contravariant vector defined by

$$\eta = (0, \dots, 0, 1) \tag{6.6}$$

and set

$$\widetilde{\varphi} = \frac{h_{ij}\eta^i\eta^j}{g_{ij}\eta^i\eta^j}. \tag{6.7}$$

$\widetilde{\varphi}$ is well defined in a neighbourhood of (t_0, ξ_0).

Now, define $\widetilde{w}$ by replacing φ by $\widetilde{\varphi}$ in (6.3); then $\widetilde{w}$ assumes its maximum at (t_0, ξ_0). Moreover, at (t_0, ξ_0) we have

$$\dot{\widetilde{\varphi}} = \dot{h}^n_n \tag{6.8}$$

and the spacial derivatives do also coincide; in short, $\widetilde{\varphi}$ satisfies at (t_0, ξ_0) the same differential equation (5.2) as h^n_n. For the sake of greater clarity, let us therefore treat h^n_n like a scalar and pretend that w is defined by

$$w = \log h^n_n + \log \chi. \tag{6.9}$$

At (t_0, ξ_0) we have $\dot{w} \geq 0$, and, in view of the maximum principle, we deduce from (5.4) and (5.26)

$$0 \leq -h^n_n + c, \tag{6.10}$$

where we have estimated bounded terms by a constant c.

Thus, the principal curvatures are bounded from above.

We further claim that the principal curvatures are uniformly strictly positive, or equivalently—because of the condition (0.8)—

Lemma 6.2. *Under the assumptions of the preceding lemma, we have*

$$0 < \varepsilon_0 \leq F \quad \forall t \tag{6.11}$$

with a given ε_0.

Proof. Consider the function

$$w = \log(-(\Phi - \tilde{f})) + \log \chi. \tag{6.12}$$

Let $0 < T < T^*$ and suppose

$$\sup_{M_0} w < \sup\{\sup_{M(t)} w \,:\, 0 \le t \le T\}. \tag{6.13}$$

Then, there is $x_0 = x(t_0)$, $0 < t_0 \le T$, such that

$$w(x_0) = \sup\{\sup_{M(t)} w \,:\, 0 \le t \le T\}. \tag{6.14}$$

From (3.19), (5.26) and the maximum principle we then infer

$$0 \le (\Phi - \tilde{f})\frac{\overline{H}}{n} v + c, \tag{6.15}$$

i.e. w is a priori bounded.

2. The Case of Theorem 0.7.

First, we obtain by the same arguments as before

Lemma 6.3. *Let $K_N = 0$, $F \in (\widetilde{H})$, $M_0 = M_1$, $\Phi(t) = \log t$, and $0 < f \in C^{2,\alpha}(\overline{\Omega})$, then the principal curvatures of the evolution hypersurfaces are bounded from above as long as they are non-negative.*

Lemma 6.4. *Let $K_N = 0$, $F \in (\widetilde{H})$, $M_0 = M_1$, $\Phi(t) = \log t$, and $0 < f \in C^{2,\alpha}(\overline{\Omega})$, then there exists ε_0 such that*

$$0 < \varepsilon_0 \le F \quad \forall t \tag{6.16}$$

as long as the evolution hypersurfaces are convex.

It remains to prove that the principal curvatures stay positive during the evolution. For this achievement we need to know the evolution equation for the inverse of the second fundamental form.

Lemma 6.5. *Let $(\tilde{h}^{ij}) = (h_{ij})^{-1}$ in contravariant form, then the mixed tensor $(\tilde{h}^i_j)$ satisfies the evolution equation (no summation over i)*

$$\begin{aligned}
\dot{\tilde{h}}^i_i - \dot{\Phi} F^{kl} \tilde{h}^i_{i;kl} = {} & -\dot{\Phi} F^{kl} h_{kr} h^r_l \tilde{h}^i_i + \{\dot{\Phi} F - (\Phi - \tilde{f})\}\delta^i_i \\
& - K_N\{\dot{\Phi} F + (\Phi - \tilde{f})\}\tilde{h}_{ki}\tilde{h}^{ki} + K_N \dot{\Phi} F^{kl} g_{kl} \tilde{h}^i_i \\
& + \tilde{f}_{\alpha\beta} x^\alpha_k x^\beta_l \tilde{h}^{ki} \tilde{h}^l_i - \tilde{f}_\alpha \nu^\alpha \tilde{h}^i_i \\
& - \{\dot{\Phi} F^{rs,kl} h_{rs;p} h_{kl;q} + 2\dot{\Phi} F^{rl} \tilde{h}^{ks} h_{rs;p} h_{kl;q} \\
& + \ddot{\Phi} F_p F_q\} \tilde{h}^{pi} \tilde{h}^q_i
\end{aligned} \tag{6.17}$$

Proof. We write

$$\tilde{h}_i^i = g_{ij}\tilde{h}^{ij} \tag{6.18}$$

and use the rule for differentiation of the inverse of a second order tensor to obtain the desired result in view of Corollary 5.3 and the evolution equation of the metric, cf. equation (3.1).

We can then prove

Lemma 6.6. *Let $K_N = 0$, $F \in (\tilde{H})$, $M_0 = M_1$, $\Phi(t) = \log t$, and $0 < f \in C^{2,\alpha}(\bar{\Omega})$ be such that $\log f$ is concave, then there exists a constant λ such that the principal curvatures κ_i of the evolution hypersurfaces $M(t)$ are bounded below by*

$$0 < e^{-\lambda t} \le \kappa_i. \tag{6.19}$$

Proof. Since M_0 is strictly convex the inverse $\tilde{h}^{ij}$ is well-defined during the evolution. We shall show that the eigenvalues of $\tilde{h}^{ij}$ (with respect to g_{ij}) grow at most exponential in t.

Define

$$\varphi = \sup\{\tilde{h}_{ij}\eta^i\eta^j \,:\, \|\eta\| = 1\} \tag{6.20}$$

and w by

$$w = \varphi e^{-\lambda t}, \quad \lambda > 0. \tag{6.21}$$

We claim that w is bounded. Let $0 < T < T^*$, and $x_0 = x(t_0)$, $0 < t_0 \le T$, be a point in $M(t_0)$ such that

$$\sup_{M_0} w < \sup\{\sup_{M(t)} w \,:\, 0 < t \le T\} = w(x_0). \tag{6.22}$$

Arguing as in the proof of Lemma 6.1, we introduce Riemannian normal coordinates ξ^i in $x_0 \in M(t_0)$ such that

$$\varphi(x_0) = \tilde{h}_n^n \tag{6.23}$$

and we may pretend as before that w is defined by

$$w = \tilde{h}_n^n e^{-\lambda t}. \tag{6.24}$$

Applying the maximum principle we deduce from (6.17)

$$0 \le -\lambda w + c + cw, \tag{6.25}$$

where we used that $\tilde{f}$ is concave, the estimates in Lemma 6.3 and 6.4, and also the inequality (1.7) to estimate the terms involving the derivatives of the second fundamental form; we should also point out that, because of the homogeneity of F,

$$F_i = F^{kl} h_{kl;i}. \tag{6.26}$$

Thus, the lemma is proved if λ is chosen large enough.

3. The Case of Theorem 0.8.

We first consider the case $K_N \le 0$.

Lemma 6.7. *Let $F \in (H)$, $K_N \le 0$, $\Phi(t) = t$, $M_0 = M_1$, and suppose that $f \in C^{2,\alpha}(\overline{\Omega})$ satisfies* (0.19). *Let $M(t)$ be strictly convex solutions of the evolution problem in a maximal time-interval $[0, T^*)$, then there are constants λ and c such that the principal curvatures can be estimated by*

$$e^{-\lambda t} \le \kappa_i \le c. \tag{6.27}$$

Proof. First, we observe that in view of Lemma 4.2

$$F \le f \tag{6.28}$$

and hence

$$0 < \kappa_i \le c, \tag{6.29}$$

because

$$F = F^{ij} h_{ij} \tag{6.30}$$

and F^{ij} is by assumption uniformly positive definite in $\overline{\Gamma}_+$.

Thus, it remains to prove the lower estimate in (6.27). The proof is identical to that of Lemma 6.6 with the only exception, that, when we apply the maximum principle, we have to use the assumption (0.19) in order to neglect the quadratic terms in w.

Consider now the second part of Theorem 0.8, $K_N > 0$.

Lemma 6.8. *Let $F \in (H)$, $K_N > 0$, $\Phi(t) = \log t$, $M_0 = M_2$, and suppose that $0 < f \in C^{2,\alpha}(\overline{\Omega})$. Let $M(t)$ be strictly convex solutions in a maximal time-interval $[0, T^*)$, then there are constants ε_0 and c such that*

$$0 < \varepsilon_0 \le F \le c \quad \forall t. \tag{6.31}$$

Proof. First, we obtain from Lemma 4.2

$$\Phi \ge \tilde{f} \quad \forall t, \tag{6.32}$$

hence, the lower estimate in (6.31). To prove the upper estimate, we define

$$w = \log(\Phi - \tilde{f}) + \log \chi + \lambda u, \tag{6.33}$$

where λ is supposed to be large and χ is the function in Lemma 5.8. We claim that w is bounded from above. Let $0 < T < T^*$, and $x_0 = x(t_0)$, $0 < t_0 \le T$, be a point in $M(t_0)$ such that

$$\sup_{M_0} w < \sup\{\sup_{M(t)} w \,:\, 0 < t \le T\} = w(x_0). \tag{6.34}$$

Combining the equations (3.19), (5.26) and (5.7) we conclude from the maximum principle

$$\begin{aligned} 0 \le \dot{w} - \dot{\Phi} F^{ij} w_{ij} \;\le\; & \dot{\Phi} F^{ij} \log(\Phi - \tilde{f})_i \log(\Phi - \tilde{f})_j \\ & - \dot{\Phi} F^{ij} \log \chi_i \log \chi_j \\ & + (\Phi - \tilde{f}) \frac{\overline{H}}{n} v + K_N \dot{\Phi} F^{ij} g_{ij} \\ & - \lambda (\Phi - \tilde{f}) v^{-1} - \lambda \dot{\Phi} F^{ij} \overline{h}_{ij} + c\lambda \end{aligned} \tag{6.35}$$

Let us first consider the terms involving the derivatives; since $Dw = 0$ they are equal to

$$2\lambda \dot{\Phi} F^{ij} \chi_i u_j \chi^{-1} + \lambda^2 \dot{\Phi} F^{ij} u_i u_j. \tag{6.36}$$

The first term is non-positive, since

$$\begin{aligned} F^{ij} \chi_i u_j &= F^{ij} v_i u_j \eta + \dot{\eta} F^{ij} u_i u_j v \\ &= -F^{ij} h_i^k u_k u_j \eta v^2 + \frac{\overline{H}}{n} F^{ij} u_i u_j v \eta + \dot{\eta} F^{ij} u_i u_j v \\ &= -F^{ij} h_i^k u_k u_j \eta v^2 \le 0 \end{aligned} \tag{6.37}$$

where we used (5.29) and (5.24).

Thus the right-hand side of inequality (6.35) can be estimated from above by

$$(\Phi - \tilde{f})\frac{\overline{H}}{n}v - \lambda(\Phi - \tilde{f})v^{-1} + c(1 + \lambda^2) \tag{6.38}$$

which yields the desired estimate if λ is chosen large enough. Here, we also used the assumption that F^{ij} is uniformly bounded in $\overline{\Gamma}_+$.

Next, let us prove the a priori estimates for the principal curvatures.

Lemma 6.9. *Suppose that the assumptions of the preceding lemma are valid and that in addition f satisfies* (0.19)*, then the principal curvatures of the evolution hypersurfaces can be estimated by*

$$e^{-\lambda t} \leq \kappa_i \leq c \quad \forall t \tag{6.39}$$

for suitable constants λ and c.

The proof is identical to that of Lemma 6.7 since we know already upper and lower bounds for F.

7. Convergence to a stationary solution.

We are ready to prove the Theorems. Let $M(t)$ be a flow with initial hypersurface $M_0 = M_1$ or $M_0 = M_2$. Let us look at the scalar version of the flow, cf. (2.24),

$$\frac{\partial u}{\partial t} = -(\Phi - \tilde{f})v. \tag{7.1}$$

This is a scalar parabolic differential equation defined on the cylinder

$$Q_{T^*} = [0, T^*) \times S_0 \tag{7.2}$$

with initial value $u_0 \in C^{4,\alpha}(S_0)$, where $u_0 = u_i$, $i \in \{1, 2\}$. S_0 is a geodesic sphere equipped with the induced metric. In view of the a priori estimates we have proved in the preceding sections, we know that

$$|u|_{2,0,S_0} \leq c \tag{7.3}$$

and

$$F \quad \textit{is uniformly elliptic in} \quad u \tag{7.4}$$

independent of t. Furthermore, F is concave and thus, we can apply the regularity results in Krylov [11, Chapter 5.5] to conclude that uniform $C^{2,\alpha}$-estimates are valid, leading further to uniform $C^{4,\alpha}$-estimates in view of the regularity results for linear operators.

Therefore, the maximal time interval is unbounded, i.e. $T^* = \infty$.

Now, integrate (7.1) and observe that the right-hand side has a sign to obtain

$$|u(t,x) - u(0,x)| = \int_0^t |\Phi - \tilde{f}| v \geq \int_0^t |\Phi - \tilde{f}|, \tag{7.5}$$

i.e.

$$\int_0^\infty |\Phi - \tilde{f}| < \infty \quad \forall x \in S_0. \tag{7.6}$$

Thus, for any $x \in S_0$ there is a sequence $t_k \to \infty$ such that $(\Phi - \tilde{f}) \to 0$.
On the other hand, $u(\cdot, x)$ is monotone and therefore

$$\lim_{t\to\infty} u(t,x) = \tilde{u}(x) \tag{7.7}$$

exists and is of class $C^{4,\alpha}(S_0)$ in view of the a priori estimates. We finally deduce that $\tilde{u}$ is a stationary solution of our problem and that

$$\lim_{t\to\infty} (\Phi - \tilde{f}) = 0. \tag{7.8}$$

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RUPRECHT–KARLS–UNIVERSITÄT HEIDELBERG
INSTITUT FÜR ANGEWANDTE MATHEMATIK
IM NEUENHEIMER FELD 294
D–69120 HEIDELBERG
E-MAIL ADDRESS: GERHARDT@MATH.UNI-HEIDELBERG.DE