Torsion pairs in finite 2-Calabi-Yau triangulated categories with maximal rigid objects[*]

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Abstract

We give a complete classification of (co)torsion pairs in finite 2-Calabi-Yau triangulated categories with maximal rigid objects which are not cluster tilting. These finite 2-Calabi-Yau triangulated categories are divided into two main classes: one denoted by $\mathcal{A}_{n,t}$ called of type A, and the other denoted by $D_{n,t}$ called of type D. By using the geometric model of cluster categories of type A, or type D, we give a geometric description of (co)torsion pairs in $\mathcal{A}_{n,t}$ or $D_{n,t}$ respectively, via defining the periodic Ptolemy diagrams. This allows to count the number of (co)torsion pairs in these categories. Finally, we determine the hearts of (co)torsion pairs in all finite 2-Calabi-Yau triangulated categories with maximal rigid objects which are not cluster tilting via quivers and relations.

Key words: Finite 2-Calabi-Yau triangulated category; Periodic Ptolemy diagram; Torsion pair, Heart of torsion pair.

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1 Introduction

The notion of torsion theory in abelian categories was first introduced by Dickson [D], and triangulated version goes back to Iyama and Yoshino [IY]. Torsion pairs play an important role in the study of the algebraic structure and geometric structure of triangulated categories, and unify the notion of t-structures, co-t-structures, cluster tilting subcategories and maximal rigid subcategories. They are used to construct concrete abelian structures inside triangulated categories (as the hearts of torsion pairs) after Nakaoka's work [N1]. For a given triangulated category, one may ask a question: How many abelian subquotient categories can be constructed from the triangulated category? In general, this is a difficult question, one may think the hearts of t-structures in a given triangulated category in the sense of Beilinson-Bernstein-Deligne[BBD]. To attack this question, one plan is to classify torsion pairs in the given triangulated category, and then to determine the hearts of torsion pairs. The aim of the paper is to give a classification of torsion pairs and to determine their hearts for certain finite 2-Calabi-Yau triangulated categories.

Classification of torsion pairs (equivalently cotorsion pairs) has been studied by many people recently. Ng [Ng] gave a classification of torsion pairs in the cluster category of type A_∞ by Ptolemy diagrams of an ∞-gon. Holm, Jørgensen and Rubey gave a classification of torsion pairs in the cluster category of type A_n via Ptolemy diagrams of a regular $(n+3)$-gon [HJR1], they also did the same work for

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the cluster category of type D_n by Ptolemy diagrams of a regular $2n$-gon [HJR3] and for cluster tubes [HJR2]. Zhang, Zhou and Zhu [ZZZ] gave a classification of torsion pairs in the cluster category of a marked surface. Zhou and Zhu [ZZ2] gave a construction and a classification of torsion pairs in any 2-Calabi-Yau triangulated category with cluster tilting objects.

Cluster categories associated with finite dimensional hereditary algebras [BMRRT] (see also [CCS] for type A) and the stable category of the preprojective algebra Λ of a Dynkin quiver [GLS] have been used for the categorification of cluster algebras. These categories are 2-Calabi-Yau triangulated categories. They have an important class of objects called cluster-tilting objects, which are the analogues of clusters in cluster algebras. The cluster-tilting objects are closely related to a class of objects called maximal rigid objets. Indeed, cluster-tilting objects are maximal rigid objects, but the converse is not true in general [BIKR, BMV, KZ]. For a 2-Calabi-Yau triangulated category, either all maximal rigid objects are cluster tilting, or none of them are [ZZ1].

Triangulated categories with finitely many indecomposable objects (which we call finite triangulated categories) are a special class of locally finite triangulated categories, which are deserved a classification. By Amiot [A] and Burban-Iyama-Keller-Reiten [BIKR] (see also [BPR]), finite 2-Calabi-Yau triangulated categories with non-zero maximal rigid objects have a classification which depends on wether the maximal rigid objects are cluster tilting or not. Standard finite 2-Calabi-Yau triangulated categories with non-zero maximal rigid objects which are not cluster tilting are exactly the following orbit categories:

- *(Type A)* $\mathcal{A}_{n,t} = D^b(\mathbb{K}A_{(2t+1)(n+1)-3})/\tau^{t(n+1)-1}[1]$, where $n \geq 1$ and $t > 1$;

- *(Type D)* $\mathcal{D}_{n,t} = D^b(\mathbb{K}D_{2t(n+1)})/\tau^{(n+1)}\varphi^n$, where $n, t \geq 1$, and where φ is induced by an automorphism of $D_{2t(n+1)}$ of order 2;

- *(Type E)* $D^b(\mathbb{K}E_7)/\tau^2$ and $D^b(\mathbb{K}E_7)/\tau^5$.

Recently Buan-Palu-Reiten [BPR] classified the algebras arising from these triangulated categories as the endomorphism algebras of maximal rigid objects via mutations of quivers with relations.

In this paper, we use the geometric models of torsion pairs in cluster categories of type A_n or type D_n in [HJR1] or [HJR3] respectively to define a notion of periodic Ptolemy diagrams. This allows us to give a complete classification of torsion pairs in the categories $\mathcal{A}_{n,t}$, $\mathcal{D}_{n,t}$. From this classification, we count the number of torsion pairs in these categories. We also determine the hearts of these torsion pairs. These results, combining with results in [ZZ2] give a complete picture on torsion pairs and their hearts in finite 2-Calabi-Yau triangulated categories with non-zero maximal rigid objects.

The paper is organized as follows: In Section 2, some basic definitions and related results are recalled and some conclusions are achieved on torsion pairs. In Section 3, we give a geometric description of torsion pairs in $\mathcal{A}_{n,t}$, where $n \geq 1$ and $t > 1$, in the first subsection. In the second subsection, we count the number of torsion pairs in these categories. In the final subsection, we use the same approach to count the number of torsion pairs in $\mathcal{A}_{n,1}$, which is a 2-Calabi-Yau finite triangulated category of type A with cluster tilting objects. In Section 4, we give a geometric description of torsion pairs in $\mathcal{D}_{n,t}$, and count the number of torsion pairs in these categories. In the last section, we determine the hearts

of torsion pairs in finite 2-Calabi-Yau triangulated categories with maximal rigid objects which are not cluster tilting.

Notation. Unless stated otherwise, $\mathbb{K}$ will be an algebraically closed field of characteristic zero. Our categories will be assumed $\mathbb{K}$-linear, Hom-finite, Krull-Remark-Schmidt additive categories. $\operatorname{add} T$ denotes the additive closure of T. Let $\mathscr{X}$ and $\mathscr{Y}$ be subcategories of a triangulated category $\mathcal{C}$. $\mathscr{X} * \mathscr{Y}$ denotes the extension subcategory of $\mathscr{X}$ by $\mathscr{Y}$, whose objects are by definition the objects M with the triangle $X \to M \to Y \to X[1]$, where $X \in \mathscr{X}$ and $Y \in \mathscr{Y}$. A subcategory $\mathscr{X}$ is called an extension closed subcategory provided that $\mathscr{X} * \mathscr{X} \subseteq \mathscr{X}$. For a subcategory $\mathcal{D}$ of $\mathcal{C}$, we denote by $\mathcal{D}^{\perp}$ (resp. ${}^{\perp}\mathcal{D}$) the subcategory whose objects are $M \in \mathcal{C}$ satisfying $\operatorname{Hom}_{\mathcal{C}}(D, M) = 0$ (resp. $\operatorname{Hom}_{\mathcal{C}}(M, D) = 0$) for any $D \in \mathcal{D}$. For the sake of convenience, we write [1] for the shift functor in any triangulated category unless other stated.

2 Preliminaries

Firstly, we recall some basic notions based on [BMRRT, IY, N2].

Definition 2.1. Let $\mathscr{X}$ and $\mathscr{Y}$ be subcategories of a triangulated category $\mathcal{C}$.

(1) The pair $(\mathscr{X}, \mathscr{Y})$ is a *torsion pair* if

$$\operatorname{Hom}_{\mathcal{C}}(\mathscr{X}, \mathscr{Y}) = 0 \text{ and } \mathcal{C} = \mathscr{X} * \mathscr{Y}.$$

The subcategory $\mathcal{I} = \mathscr{X} \cap \mathscr{Y}[-1]$ is called the *core* of the torsion pair.

(2) The pair $(\mathscr{X}, \mathscr{Y})$ is a *cotorsion pair* if

$$\operatorname{Ext}^{1}_{\mathcal{C}}(\mathscr{X}, \mathscr{Y}) = 0 \text{ and } \mathcal{C} = \mathscr{X} * \mathscr{Y}[1].$$

Moreover, we call the subcategory $\mathcal{I} = \mathscr{X} \cap \mathscr{Y}$ the *core* of the cotorsion pair.

(3) A *t-structure* $(\mathscr{X}, \mathscr{Y})$ in $\mathcal{C}$ is a torsion pair such that $\mathscr{X}$ is closed under [1] (equivalently $\mathscr{Y}$ is closed under $[-1]$).

(4) A subcategory $\mathcal{T}$ is called *rigid* if $\operatorname{Ext}^{1}_{\mathcal{C}}(\mathcal{T}, \mathcal{T}) = 0$. $\mathcal{T}$ is called *maximal rigid* if $\mathcal{T}$ is maximal with respect to this property, i.e., if $\operatorname{Ext}^{1}_{\mathcal{C}}(\mathcal{T} \oplus addM, \mathcal{T} \oplus addM) = 0$, then $M \in \operatorname{add} T$.

T is called a *rigid object* if $\operatorname{add} T$ is rigid. T is *maximal rigid* if $\operatorname{add} T$ is maximal rigid.

(5) A functorially finite subcategory $\mathcal{T}$ is called cluster tilting if it is rigid and satisfies the property that $\operatorname{Ext}^{1}_{\mathcal{C}}(M, \mathcal{T}) = 0$ implies $M \in \mathcal{T}$. An object T is a *cluster tilting* object if $\operatorname{add} T$ is a cluster tilting subcategory.

Remark 2.2. By Definition 2.1, we know that a pair $(\mathscr{X}, \mathscr{Y})$ is a cotorsion pair if and only if $(\mathscr{X}, \mathscr{Y}[1])$ is a torsion pair.

Remark 2.3. For a cotorsion pair $(\mathscr{X}, \mathscr{Y})$ with core $\mathcal{I} = \mathscr{X} \cap \mathscr{Y}$, it is easy to see that $(\mathscr{X}, \mathscr{Y})$ is a t-structure if and only if $\mathcal{I} = \{0\}$ [ZZ2]; $\mathscr{X}$ is a cluster tilting subcategory if and only if $\mathcal{I} = \mathscr{X} = \mathscr{Y}$.

The following result can be found in [IY, Proposition 2.3].

Lemma 2.4. *(1) Let $\mathscr{X}$ be a contravariantly finite and extension closed subcategory of a triangulated category $\mathcal{C}$. Then $(\mathscr{X}, \mathscr{X}^{\perp})$ is a torsion pair.*

(2) Let $\mathscr{X}$ be a covariantly finite and extension closed subcategory of a triangulated category $\mathcal{C}$. Then $(^{\perp}\mathscr{X}, \mathscr{X})$ is a torsion pair.

Definition 2.5. 1. A triangulated category $\mathcal{C}$ is called 2-*Calabi-Yau* (shortly 2−CY) provided there is a functorially isomorphism $\mathrm{Hom}_{\mathcal{C}}(X, Y) \simeq D\mathrm{Hom}_{\mathcal{C}}(Y, X[2])$, for all $X, Y \in \mathcal{C}$, where $D = \mathrm{Hom}_{\mathbb{K}}(-, \mathbb{K})$.

2. [XZ] A triangulated category $\mathcal{C}$ is *locally finite* if for any indecomposable object X, there exists only a finite number of isomorphism classes of indecomposable objects Y such that $\mathrm{Hom}_{\mathcal{C}}(X, Y) = 0$. $\mathcal{C}$ is called a *finite triangulated category* if it contains only finitely many indecomposable objects up to isomorphisms.

Remark 2.6. Any finite 2-CY triangulated category $\mathcal{C}$ contains a maximal rigid object (may be zero). If the maximal rigid objects of a connected finite 2-CY triangulated category $\mathcal{C}$ are zero , then any torsion pair $(\mathscr{X}, \mathscr{Y})$ is a t-structure. It follows that $\mathscr{X}, \mathscr{Y}$ are triangulated subcategories and $\mathcal{C} = \mathscr{X} \oplus \mathscr{Y}$. This implies that $(\mathscr{X}, \mathscr{Y}) = (\mathcal{C}, 0)$ or $(0, \mathcal{C})$. So the finite 2-CY triangulated category $\mathcal{C}$ which we consider in this paper is assumed to contain a nonzero maximal rigid object. If $\mathcal{C}$ contains a cluster tilting object, then any maximal rigid object is cluster tilting [ZZ1]. Finite 2-CY triangulated categories with non-zero maximal rigid objects are divided into two classes: one with cluster tilting objects, one without cluster tilting objects. For the first class, we can apply results in [ZZ2] to obtain a classification of torsion pairs. So we are interested in triangulated categories in the second class in this paper. For these triangulated categories, Amiot gave a classification (see [A, BPR])

Lemma 2.7. [BPR, Proposition 2.2] *The standard, finite 2-Calabi-Yau, triangulated categories with non-zero maximal rigid objects which are not cluster tilting are exactly the orbit categories:*

- *(Type A)* $\mathcal{A}_{n,t} = D^b(\mathbb{K}A_{(2t+1)(n+1)-3})/\tau^{t(n+1)-1}[1]$, *where $n \geq 1$ and $t > 1$;*

- *(Type D)* $\mathcal{D}_{n,t} = D^b(\mathbb{K}D_{2t(n+1)})/\tau^{(n+1)}\varphi^n$, *where $n, t \geq 1$, and where φ is induced by an automorphism of $D_{2t(n+1)}$ of order 2;*

- *(Type E)* $D^b(\mathbb{K}E_7)/\tau^2$ *and* $D^b(\mathbb{K}E_7)/\tau^5$.

These categories depend on parameters n, t. We note that when $t = 1$, $\mathcal{A}_{n,1}$ is also a finite 2-CY triangulated category, it has cluster tilting objects [BPR] which we are also interested in.

In the following, we have some conclusions for torsion pairs in orbit triangulated categories [K, G]. Firstly, we recall the definition of orbit categories.

Definition 2.8. [K] Let $\mathcal{T}$ be a triangulated category and $F \colon \mathcal{T} \to \mathcal{T}$ be an autoequivalence. The *orbit category* $\mathcal{O}_F := \mathcal{T}/F$ has the same objects as $\mathcal{T}$ and its morphisms from X to Y are in bijection with $\bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(X, F^i Y)$.

Lemma 2.9. *Let $\mathcal{T}$ be a locally finite triangulated category and $F \colon \mathcal{T} \to \mathcal{T}$ be an autoequivalence such that the orbit category $\mathcal{O}_F = \mathcal{T}/F$ is a triangulated category and the projection functor $\pi : \mathcal{T} \to \mathcal{O}_F$ is a triangle functor. If $(\mathscr{X}, \mathscr{Y})$ is a torsion pair in $\mathcal{O}_F$, then $(\pi^{-1}(\mathscr{X}), \pi^{-1}(\mathscr{Y}))$ is a torsion pair in $\mathcal{T}$.*

Proof. We first show $\pi^{-1}(\mathscr{X})$ is closed under extensions. For any $Z \in \pi^{-1}(\mathscr{X}) * \pi^{-1}(\mathscr{X})$, there exists a triangle $X_1 \to Z \to X_2 \to X_1[1]$ with $X_1, X_2 \in \pi^{-1}(\mathscr{X})$ in $\mathcal{T}$. Since π is a triangle functor, we have that $\pi(X_1) \to \pi(Z) \to \pi(X_2) \to \pi(X_1[1])$ is a triangle in $\mathcal{O}_F$. Thus $\pi(Z) \in \mathscr{X} * \mathscr{X} \subseteq \mathscr{X}$, that is, $Z \in \pi^{-1}(\mathscr{X})$.

Since $\mathcal{T}$ is locally finite, any subcategory of $\mathcal{T}$ is functorially finite. Since $(\mathscr{X}, \mathscr{Y})$ is a torsion pair in $\mathcal{O}_F$, we have $\pi^{-1}(\mathscr{Y}) = \pi^{-1}(\mathscr{X}^\perp) = \pi^{-1}(\mathscr{X})^\perp$. $\qquad\square$

Definition 2.10. Let $\mathscr{X}$ and $\mathscr{Y}$ be subcategories of a triangulated category $\mathcal{C}$, and $F \colon \mathcal{C} \to \mathcal{C}$ be an autoequivalence. The pair $(\mathscr{X}, \mathscr{Y})$ is called an *F-periodic* torsion pair if $(\mathscr{X}, \mathscr{Y})$ is a torsion pair and $\mathscr{X}$ is F-periodic (or equivalently, $\mathscr{Y}$ is F-periodic),i.e., $F\mathscr{X} \subseteq \mathscr{X}$ (resp. $F\mathscr{Y} \subseteq \mathscr{Y}$).

Lemma 2.11. *Let $\mathcal{T}$ be a locally finite triangulated category and $F \colon \mathcal{T} \to \mathcal{T}$ be an autoequivalence such that $\mathcal{O}_F = \mathcal{T}/F$ is a triangulated category and the projection functor $\pi \colon \mathcal{T} \to \mathcal{O}_F$ is a triangle functor. If $(\mathscr{X}, \mathscr{Y})$ is an F-periodic torsion pair in $\mathcal{T}$, then $(\pi(\mathscr{X}), \pi(\mathscr{Y}))$ is a torsion pair in $\mathcal{O}_F$.*

Proof. Since $\mathscr{X}$ is F-periodic, we have

$$\operatorname{Hom}_{\mathcal{O}_F}(\pi(X), \pi(Y)) = \bigoplus_{i \in \mathbb{Z}} \operatorname{Hom}_{\mathcal{T}}(F^i X, Y) = 0,$$

where $X \in \mathscr{X}$ and $Y \in \mathscr{Y}$. For any object $Z \in \mathcal{O}_F$, let Z' be an object in its preimage in $\mathcal{T}$, that is, $Z' \in \mathcal{T} = \mathscr{X} * \mathscr{Y}$. Then there exists a triangle $X \to Z' \to Y \to X[1]$ in $\mathcal{T}$ with $X \in \mathscr{X}$ and $Y \in \mathscr{Y}$. Since π is a triangle functor, we have that $\pi(X) \to Z \to \pi(Y) \to \pi(X[1])$ is a triangle in $\mathcal{O}_F$. This proves that $(\pi(\mathscr{X}), \pi(\mathscr{Y}))$ is a torsion pair in $\mathcal{O}_F$. $\qquad\square$

The following theorem gives a one-to-one correspondence between F-periodic torsion pairs in $\mathcal{T}$ and torsion pairs in $\mathcal{O}_F$.

Theorem 2.12. *Let $\mathcal{T}$ be a locally finite triangulated category and $F \colon \mathcal{T} \to \mathcal{T}$ be an autoequivalence such that $\mathcal{O}_F := \mathcal{T}/F$ is a triangulated category and the projection functor $\pi \colon \mathcal{T} \to \mathcal{O}_F$ is a triangle functor. Then there is a bijection between the following sets:*

(1) *The set of F-periodic torsion pairs in $\mathcal{T}$;*

(2) *The set of torsion pairs in $\mathcal{O}_F$.*

Proof. This follows directly from Lemma 2.9 and Lemma 2.11. $\qquad\square$

Proposition 2.13. *Let $\mathcal{C}$ be a connected finite 2-CY triangulated category. Then the t-structures of $\mathcal{C}$ are trivial,i.e. $(\mathcal{C}, 0)$ or $(0, \mathcal{C})$.*

Proof. Suppose $(\mathscr{X}, \mathscr{Y})$ is a t-structure in $\mathcal{C}$. From definition we have $\mathscr{X}[1] \subseteq \mathscr{X}, \mathscr{Y}[-1] \subseteq \mathscr{Y}$. It follows that $\mathscr{X}[1] = \mathscr{X}, \mathscr{Y}[1] = \mathscr{Y}$, i.e. $\mathscr{X}, \mathscr{Y}$ are triangulated subcategories of $\mathcal{C}$, and $\mathcal{C} = \mathscr{X} \oplus \mathscr{Y}$. Thus $\mathscr{X} = \mathcal{C}$ and $\mathscr{Y} = 0$ or $\mathscr{X} = 0$ and $\mathscr{Y} = \mathcal{C}$. $\qquad\square$

3 Classification of torsion pairs in $\mathcal{A}_{n,t}$

In this section, we give a classification of torsion pairs in finite 2-CY triangulated categories of type A with non-zero maximal rigid objects. These categories are denoted by $\mathcal{A}_{n,t}$ [BPR]. When $t = 1$, the category $\mathcal{A}_{n,1}$ has cluster tilting objects; when $t > 1$, the category $\mathcal{A}_{n,t}$ has non-zero maximal rigid objects which are not cluster tilting.

3.1 A geometric description of torsion pairs in $\mathcal{A}_{n,t}$

Let $\mathcal{C}_{A_{N-3}}$ be the cluster category of type A_{N-3}, where $N = (2t+1)(n+1)$. By the universal property of orbit categories [K], also by the proof of Lemma 2.4 in [BPR], we know that there exists a covering functor $\pi \colon \mathcal{C}_{A_{N-3}} \to \mathcal{A}_{n,t}$. Write $F = \tau^{t(n+1)}$, then $F \colon \mathcal{C}_{A_{N-3}} \to \mathcal{C}_{A_{N-3}}$ is an autoequivalence. Since $\tau^{N-2} = [-2]$ in $D^b(\mathbb{K}\vec{A}_{N-3})$ by [K] and $\tau = [1]$ in $\mathcal{C}_{A_{N-3}}$, τ is of order N and τ^{n+1} is of order $2t+1$ in $\mathcal{C}_{A_{N-3}}$. Moreover, $\gcd(t, 2t+1) = 1$ implies that $F = \tau^{t(n+1)}$ is also of order $2t+1$, so the groups generated by F and by τ^{n+1} are the same, i.e. $< F >=< \tau^{n+1} >$. Therefore $\mathcal{A}_{n,t}$ can be seen as the orbit category $\mathcal{C}_{A_{N-3}}/\tau^{n+1}$, and π is a $(2t+1)$-covering functor (see [BPR]), and a triangle functor. By Theorem 2.12, we have the following consequence.

Corollary 3.1. *There is a bijection between the set of τ^{n+1}-periodic torsion pairs in $\mathcal{C}_{A_{N-3}}$ and torsion pairs in $\mathcal{A}_{n,t}$.*

In the following, we recall the description of Ptolemy diagrams based on [HJR1], and give a correspondence between subcategories of $\mathcal{A}_{n,t}$ and collections of diagonals of N-gon.

Let P be an N-gon, we label the vertices of P clockwise by $1, 2, \ldots N$ consecutively, where $N = (2t+1)(n+1)$. A *diagonal* is a set of two non-neighbouring vertices $\{\alpha, \beta\}$. Two diagonals $\{\alpha_1, \alpha_2\}$ and $\{\beta_1, \beta_2\}$ *cross* if their end points are all distinct and come in the order α_1, β_1, α_2, β_2 when moving around the polygon in one direction or the other.

Definition 3.2. Let $\mathfrak{U}$ be a set of diagonals in the N-gon P.

1[HJR1]. $\mathfrak{U}$ is called a *Ptolemy diagram* if for any two crossing diagonals $\alpha = \{\alpha_1, \alpha_2\}$ and $\beta = \{\beta_1, \beta_2\}$ in $\mathfrak{U}$, those of $\{\alpha_1, \beta_1\}$, $\{\alpha_1, \beta_2\}$, $\{\alpha_2, \beta_1\}$, $\{\alpha_2, \beta_2\}$ which are diagonals are in $\mathfrak{U}$ (see figure 1 for an example).

2. Fix a positive integer $k|N$, and $N = k\ell$ for some integer ℓ. $\mathfrak{U}$ is called an *k-periodic* collection of diagonals of P if for each diagonal $(i, j) \in \mathfrak{U}$, all diagonals $(i + kr, j + kr)$ *(modulo N)* for $1 \leq r \leq \ell$ are in $\mathfrak{U}$ (see figure 2 for an example).

3. $\mathfrak{U}$ is a *k-periodic Ptolemy diagram* if it is a Ptolemy diagram and is k-periodic.

There is a bijection between indecomposable object of the cluster category $\mathcal{C}_{A_{N-3}}$ and diagonal of N-gon P [CCS]. The Auslander-Reiten translation τ acts on diagonal is rotation by one vertex in counterclockwise.

$$\dim \operatorname{Ext}^1_{\mathcal{C}_{A_{N-3}}}(a, b) = \begin{cases} 1 & \text{if a and b cross,} \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathcal{C}_{A_{N-3}}$ has only finitely many indecomposable objects, any subcategory of $\mathcal{C}_{A_{N-3}}$ closed under direct sums and direct summands is completely determined by the set of indecomposable objects it contains. Then the bijection between indecomposable objects of $\mathcal{C}_{A_{N-3}}$ and diagonals of P extends to
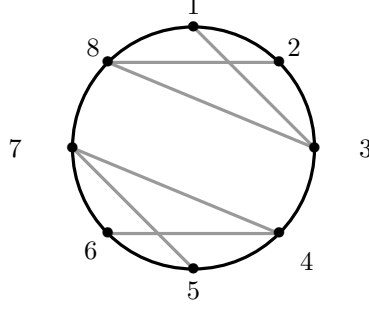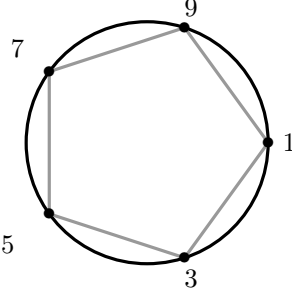
Figure 1: $N = 8$

Figure 2: $n = 1, t = 2$

a bijection between subcategories of $\mathcal{C}_{A_{N-3}}$ closed under direct sums and direct summands and sets of diagonals of P.

For any subcategory $\mathscr{X}$ in $\mathcal{A}_{n,t}$, the preimage under the covering functor π is an τ^{n+1}-periodic subcategory $\widetilde{\mathscr{X}} = \pi^{-1}(\mathscr{X})$ in $\mathcal{C}_{A_{N-3}}$. Moreover, the subcategory $\widetilde{\mathscr{X}}$ corresponds to the set of diagonals of N-gon P by the discussion above, we still denote the corresponding set of diagonals by $\widetilde{\mathscr{X}}$. The corresponding set of diagonals is $(n+1)$-periodic. In the rest of this section, we always use a coordinate (i, j) to represent an indecomposable object of $\mathcal{C}_{A_{N-3}}$ and $[(i, j)]$ to represent the image under the functor π.

Lemma 3.3. *There is a bijection between the following sets:*

(1) *Subcategory $\mathscr{X}$ of $\mathcal{A}_{n,t}$;*

(2) *Collection of diagonals $\widetilde{\mathscr{X}}$ of the N-gon P which are $(n+1)$-periodic.*

Lemma 3.4. [HJR1, Theorem A] *There is a bijection between Ptolemy diagrams of the $(n+3)$-gon and torsion pairs in the cluster category of type A_n.*

The following Lemma gives an equivalent description of torsion pairs in $\mathcal{A}_{n,t}$.

Lemma 3.5. *Let $\mathscr{X}$ be a subcategory of $\mathcal{A}_{n,t}$, and let $\widetilde{\mathscr{X}}$ be the corresponding $(n+1)$-periodic collections of diagonals of N-gon P. Then the following conditions are equivalent:*

(1) *$(\mathscr{X}, \mathscr{X}^{\perp})$ is a torsion pair in $\mathcal{A}_{n,t}$;*

(2) *$\mathscr{X} = {}^{\perp}(\mathscr{X}^{\perp})$;*

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(3) $\widetilde{\mathscr{X}}$ is an $(n+1)$-periodic Ptolemy diagram of N-gon.

Proof. $(1) \Longleftrightarrow (2)$ is clear.

"$(1) \Rightarrow (3)$". If $(\mathscr{X}, \mathscr{X}^\perp)$ is a torsion pair in $\mathcal{A}_{n,t}$, then $\mathscr{X}$ corresponds to an $(n+1)$-periodic collections of diagonals $\widetilde{\mathscr{X}}$ of the N-gon P by Lemma 3.3. Moreover, $(\pi^{-1}(\mathscr{X}), \pi^{-1}(\mathscr{X}^\perp))$ is a torsion pair in $\mathcal{C}_{A_{N-3}}$ by Lemma 2.9, and $\pi^{-1}(\mathscr{X})$ corresponds to a Ptolemy diagram of N-gon, so $\widetilde{\mathscr{X}}$ is an $(n+1)$-periodic Ptolemy diagram of N-gon.

"$(3) \Rightarrow (1)$". If $\widetilde{\mathscr{X}}$ is an $(n+1)$-periodic Ptolemy diagram of N-gon, then it corresponds to a torsion pair in $\mathcal{C}_{A_{N-3}}$ by Lemma 3.4. Moreover, $\widetilde{\mathscr{X}}$ is $(n+1)$-periodic, this implies the corresponding subcategory $\mathscr{X}$ and $\mathscr{X}^\perp$ can be a torsion pair in $\mathcal{A}_{n,t}$ by Lemma 2.11. $\qquad \square$

We will frequently use in this section the coordinate system for the category $\mathcal{A}_{n,t}$ corresponding to the N-gon P, see [BPR] for more details.

Definition 3.6. For a coordinate (i,j) (modulo N) with $j > i$ corresponding to an indecomposable object in the AR-quiver of $\mathcal{C}_{A_{N-3}}$, we call $j - i - 1$ the *level* of the vertex. Denote the vertex in the AR-quiver of $\mathcal{A}_{n,t}$ by $[(i,j)]$ such that all $(i + rn, j + rn)$ for $1 \leq r \leq 2t+1$ has to be identified. We also call $j - i - 1$ the *level* of the vertex $[(i,j)]$. The *length* of the vertex $[(i,j)]$ is $j - i$. Buan, Palu and Reiten determined all the indecomposable rigid objects in $\mathcal{A}_{n,t}$: for an indecomposable object $[(i,j)]$ in $\mathcal{A}_{n,t}$ with level $j - i - 1$, it is rigid if and only if $j - i - 1 \leq n$ [BPR].

3.2 Torsion pairs in $\mathcal{A}_{n,t}$ with $t > 1$

Proposition 3.7. *Let $(\mathscr{X}, \mathscr{X}^\perp)$ be a torsion pair in $\mathcal{A}_{n,t}$, $n \geq 1$, $t > 1$, and let $\widetilde{\mathscr{X}}$ be the corresponding $(n+1)$-periodic collections of diagonals of the N-gon P. Then precisely one of the following situations occurs:*

(1) *The level of all the indecomposable objects in $\mathscr{X} \leq n$.*

(2) *The level of all the indecomposable objects in $\mathscr{X}^\perp \leq n$.*

Proof. By the AR-quiver of $\mathcal{A}_{n,t}$, we can always choose an element $(i,j) \in [(i,j)]$ such that $1 \leq i \leq n+1$, $3 \leq j \leq (t+1)(n+1)$. For example, we have $[(1, (t+1)(n+1)+1)] = [(1, 1 + t(n+1))]$.

I. If the level of all the indecomposable objects of $\mathscr{X} \leq n$, we claim that $\mathscr{X}^\perp$ must contain an element with level$> n$. Indeed, since $\mathscr{X}$ contains only (finitely many) indecomposable rigid objects, we pick an indecomposable object from $\mathscr{X}$ with maximal length. Suppose that its coordinate is $(1, \ell)$. Since $(1, \ell)$ corresponds to a rigid object, $3 \leq \ell \leq n+2$. If we can show $(1, \ell + n + 1) \in \mathscr{X}^\perp[-1]$, then $\mathscr{X}^\perp[-1]$ contains an element with level$> n$, and so does $\mathscr{X}^\perp$. Since $t > 1$, $\ell + n + 1 \leq n + 2 + n + 1 = 2n + 3 < (t+1)(n+1)$, i.e., $(1, \ell + n + 1)$ represents a different element from $(1, \ell)$ in the AR-quiver of $\mathcal{A}_{n,t}$, and the level of $(1, \ell + n + 1)$ is $\ell + n - 1 \geq 3 + n - 1 = n + 2$. This will complete the proof of this case. Now we prove that $(1, \ell + n + 1) \in \mathscr{X}^\perp[-1]$. Since $(1, \ell)$ is in $\mathscr{X}$ with maximal length, there is no diagonal $(i, j) \in \widetilde{\mathscr{X}}$ with $i < \ell < j$, otherwise $(1, \ell)$ has to cross (i, j), but the Ptolemy condition yields a diagonal $(1, j)$ whose length is longer than $(1, \ell)$, a contradiction. If $(1, \ell + n + 1)$ crosses a diagonal (a, b) in $\widetilde{\mathscr{X}}$, then $b - a > \ell - 1$, a contradiction. This means $(1, \ell + n + 1)$ does not cross any diagonal from $\widetilde{\mathscr{X}}$, i.e., $(1, \ell + n + 1) \in \mathscr{X}^\perp[-1]$.
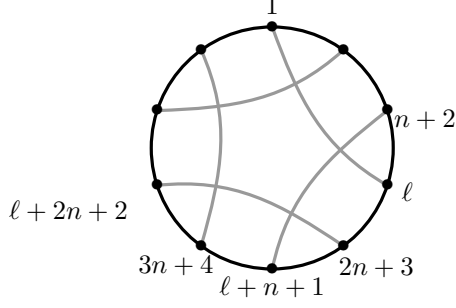
8

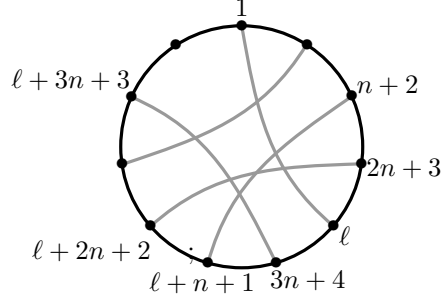Figure 3: The maximal length of non-crossing objects is $n + 1$

Figure 4: The maximal length of non-crossing objects is $n + 1$

II. If $\mathscr{X}$ contains an element with level$> n$, we claim that $\mathscr{X}^{\perp}$ contains only rigid indecomposable objects. Indeed, without losing generality, we suppose $(1, \ell) \in \mathscr{X}$ with level $\ell - 1 - 1 \geq n + 1$, that is, $(t + 1)(n + 1) \geq \ell \geq n + 3$. We choose ℓ for different intervals $[n + 3, 2n + 3]$, $[2n + 3, 3n + 4], \ldots, [2t(n + 1) + 1, (2t + 1)(n + 1) + 1]$.

   (a). If $\ell \leq 2n + 3$, then the corresponding diagonals in $\widetilde{\mathscr{X}}$ are shown in figure 3.

   (b). If $2n + 3 \leq \ell \leq 3n + 4$, then the corresponding diagonals in $\widetilde{\mathscr{X}}$ are shown in figure 4.

   (c). The other cases are similar.

   This shows the level of indecomposable objects in $[(1, \ell)]^{\perp} \leq n$, so does $\mathscr{X}^{\perp}$, since $\mathscr{X}^{\perp} \subseteq [(1, \ell)]^{\perp}$.

As a consequence, for a torsion pair $(\mathscr{X}, \mathscr{X}^{\perp})$ in $\mathcal{A}_{n,t}$, precisely one of (1) and (2) occurs. If neither (1) nor (2) occurs, this is a contradiction with II. Contrarily, if both (1) and (2) occur, this will be a contradiction with I. $\qquad\qquad\square$

   This Proposition immediately yields the following important conclusion.

Corollary 3.8. [BPR] $\mathcal{A}_{n,t}$ *does not contain any cluster tilting object.*

   By Proposition 3.7, the classification of torsion pairs $(\mathscr{X}, \mathscr{X}^{\perp})$ in $\mathcal{A}_{n,t}$ reduces to the classification of the possible halves $\mathscr{X}$ (or $\mathscr{X}^{\perp}$) of a torsion pair, whose all indecomposable objects are strictly below level $(n + 1)$ in the AR-quiver of $\mathcal{A}_{n,t}$.

Definition 3.9. Let (i, j) be a diagonal of N-gon P. The *wing* $W(i, j)$ of (i, j) consists of all diagonals (r, s) of the N-gon such that $i \leq r \leq s \leq j$, that is all diagonals which are overarched by (i, j). $[(i, j)]$ represents a vertex in the AR-quiver of $\mathcal{A}_{n,t}$, the corresponding wing is denoted by $W[(i, j)]$.

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Theorem 3.10. *There are bijections between the following sets:*

(1) *Torsion pairs $(\mathscr{X}, \mathscr{X}^\perp)$ in $\mathcal{A}_{n,t}$ such that the level of all the indecomposable objects in $\mathscr{X} \leq n$;*

(2) *$(n+1)$-periodic Ptolemy diagrams $\widetilde{\mathscr{X}}$ of N-gon P such that all diagonals in $\widetilde{\mathscr{X}}$ have length at most $n+1$;*

(3) *Collections $\{([(i_1, j_1)], [W_1]), \ldots, ([(i_r, j_r)], [W_r])\}$ of pairs consisting of vertices $[(i_\ell, j_\ell)]$ of level$\leq$ n in the AR-quiver of $\mathcal{A}_{n,t}$ and subset $[W_\ell] \subset W[(i_\ell, j_\ell)]$ of their wings such that for any different $k, \ell \in \{1, 2, \ldots, r\}$, we have*

$$W[(i_k, j_k)][1] \cap W[(i_\ell, j_\ell)] = \emptyset,$$

and the $(n+1)$-periodic collection W_ℓ corresponding to $[W_\ell]$ is a Ptolemy diagram.

Proof. The proof is similar as in the case of cluster tubes [HJR2]. ∎

Note that the number of indecomposable rigid objects in $\mathcal{A}_{n,t}$ is independent of t, we have the following.

Corollary 3.11. *The number of torsion pairs in $\mathcal{A}_{n,t}$ with $n \geq 1$, $t > 1$ is independent of t.*

Therefore counting the number of torsion pairs in $\mathcal{A}_{n,t}$ reduces to counting the possible sets of pairs in the AR-quiver of $\mathcal{A}_{n,t}$: $\{([(i_1, j_1)], [W_1]), \ldots, ([(i_r, j_r)], [W_r])\}$. This is the same as in the process of counting torsion pairs in the cluster tube of rank $n+1$, see [HJR2] for details.

Theorem 3.12. *The number of torsion pairs in $\mathcal{A}_{n,t}$ with $n \geq 1$, $t > 1$ is the same as the cluster tube of rank $n+1$, that is*

$$T_{n+1} = \sum_{\ell \geq 0} 2^{\ell+1} \binom{n+\ell}{\ell} \binom{2n+1}{n-2\ell},$$

where T_{n+1} represents the number of torsion pairs in the cluster tube of rank $n+1$.

Example 3.13. When $n = 2, t = 2$, $\mathcal{A}_{2,2} = D^b(\mathbb{K}A_{12})/\tau^5[1]$ is 2-CY with non-zero maximal rigid objects, whose Auslander-Reiten quiver is shown in figure 5. From Theorem 3.12, we have $T_{2+1} = 32$. Now we construct all torsion pairs by Theorem 3.10.

$$
\begin{aligned}
\mathscr{X}_1 &= \{0\} & \mathscr{X}_1^\perp[-1] &= \mathcal{A}_{2,2} \\
\mathscr{X}_2 &= \{13\} & \mathscr{X}_2^\perp[-1] &= \{13, 14, 16, 17, 19, 36, 37, 39\} \\
\mathscr{X}_3 &= \{24\} & \mathscr{X}_3^\perp[-1] &= \{14, 15, 17, 18, 24, 25, 27, 28\} \\
\mathscr{X}_4 &= \{35\} & \mathscr{X}_4^\perp[-1] &= \{25, 26, 28, 29, 35, 36, 38, 39\} \\
\mathscr{X}_5 &= \{14\} & \mathscr{X}_5^\perp[-1] &= \{13, 14, 17, 24\} \\
\mathscr{X}_6 &= \{25\} & \mathscr{X}_6^\perp[-1] &= \{24, 25, 28, 35\} \\
\mathscr{X}_7 &= \{36\} & \mathscr{X}_7^\perp[-1] &= \{13, 35, 36, 39\} \\
\mathscr{X}_8 &= \{13, 14\} & \mathscr{X}_8^\perp[-1] &= \{13, 14, 17\} \\
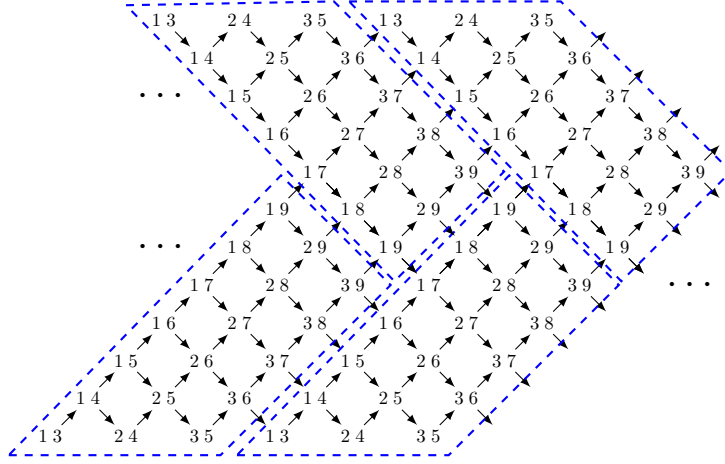\mathscr{X}_9 &= \{24, 25\} & \mathscr{X}_9^\perp[-1] &= \{24, 25, 28\}
\end{aligned}
$$

Figure 5: The AR-quiver of $\mathcal{A}_{2,2}$

$$\mathscr{X}_{10} = \{35, 36\} \qquad \mathscr{X}_{10}^{\perp}[-1] = \{35, 36, 39\}$$

$$\mathscr{X}_{11} = \{14, 24\} \qquad \mathscr{X}_{11}^{\perp}[-1] = \{14, 17, 24\}$$

$$\mathscr{X}_{12} = \{25, 35\} \qquad \mathscr{X}_{12}^{\perp}[-1] = \{25, 28, 35\}$$

$$\mathscr{X}_{13} = \{13, 36\} \qquad \mathscr{X}_{13}^{\perp}[-1] = \{13, 36, 39\}$$

$$\mathscr{X}_{14} = \{13, 14, 24\} \qquad \mathscr{X}_{14}^{\perp}[-1] = \{14, 17\}$$

$$\mathscr{X}_{15} = \{24, 25, 35\} \qquad \mathscr{X}_{15}^{\perp}[-1] = \{25, 28\}$$

$$\mathscr{X}_{16} = \{13, 35, 36\} \qquad \mathscr{X}_{16}^{\perp}[-1] = \{36, 39\}$$

In this example, the collection of pairs in Theorem 3.10 (3) has one element,i.e., $r = 1$. The pair $\{([(i_1, j_1)], [W_1])\}$ with $[W_1]$ containing zero object is the subcategory $\mathscr{X}_1$; The pairs $\{([(i_1, j_1)], [W_1])\}$ with $[W_1]$ containing one object are the following subcategories: $\mathscr{X}_2$ which corresponds to the set $([(1, 3)]; \{[(1, 3)]\})$, $\mathscr{X}_3$ which corresponds to the set $([(2, 4)]; \{[(2, 4)]\})$, $\mathscr{X}_4$ which corresponds to the set $([(3, 5)]; \{[(3, 5)]\})$, $\mathscr{X}_5$ which corresponds to the set $([(1, 4)]; \{[(1, 4)]\})$, $\mathscr{X}_6$ which corresponds to the set $([(2, 5)]; \{[(2, 5)]\})$, $\mathscr{X}_7$ which corresponds to the set $([(3, 6)]; \{[(3, 6)]\})$. The pairs $\{([(i_1, j_1)], [W_1])\}$ with $[W_1]$ containing two objects are the subcategories: $\mathscr{X}_8$ which corresponds to the set $([(1, 4)]; \{[(1, 3)], [(1, 4)]\})$, $\mathscr{X}_9$ which corresponds to the set $([(2, 5)]; \{[(2, 4)], [(2, 5)]\})$, $\mathscr{X}_{10}$ which corresponds to the set $([(3, 6)]; \{[(3, 5)], [(3, 6)]\})$, $\mathscr{X}_{11}$ which corresponds to the set $([(1, 4)]; \{[(2, 4)], [(1, 4)]\})$, $\mathscr{X}_{12}$ which corresponds to the set $([(2, 5)]; \{[(2, 5)], [(3, 5)]\})$, $\mathscr{X}_{13}$ which corresponds to the set $([(3, 6)]; \{[(1, 3)], [(3, 6)]\})$. The pairs $\{([(i_1, j_1)], [W_1])\}$ with $[W_1]$ containing three objects are the subcategories: $\mathscr{X}_{14}$ which corresponds to the set $([(1, 4)]; \{[(1, 3)], [(1, 4)], [(2, 4)]\})$, $\mathscr{X}_{15}$ which corresponds to the set $([(2, 5)]; \{[(2, 5)], [(3, 5)], [(2, 4)]\})$, $\mathscr{X}_{16}$ which corresponds to the set $([(3, 6)]; \{[(1, 3)], [(3, 5)], [(3, 6)]\})$.

Then $(\mathscr{X}_i, \mathscr{X}_i^{\perp}[-1])$ is a cotorsion pair in $\mathcal{A}_{2,2}$, where $i = 1, 2, 3, \cdots, 16$. It follows that $(\mathscr{X}_i, \mathscr{X}_i^{\perp})$ is a torsion pair in $\mathcal{A}_{2,2}$, for any i. Similarly, we can know that $(^{\perp}\mathscr{X}_i, \mathscr{X}_i)$ is a torsion pair in $\mathcal{A}_{2,2}$, for $i = 1, 2, 3, \cdots, 16$.

3.3 Torsion pairs in $\mathcal{A}_{n,1}$

By Amiot [A], Burban-Iyama-Keller-Reiten [BIKR], the standard, finite 2-CY triangulated categories of type A with cluster tilting objects are cluster categories of type A and the orbit categories $D^b(A_{3n})/\tau^n[1]$ with $n \geq 1$, which is denoted by $\mathcal{A}_{n,1}$ (see Proposition 2.1 in [BPR]). The later are just the orbit categories $\mathcal{A}_{n,t}$ with $t = 1$. There is also a covering functor from the cluster category $\mathcal{C}_{A_{3n}}$ to $\mathcal{A}_{n,1}$, and the same discussions above on torsion pairs work in the category $\mathcal{A}_{n,1}$. We have the following.

Theorem 3.14. *The number of torsion pairs in $\mathcal{A}_{n,1}$ is*

$$N_{n,1} = T_{n+1} - t_{n,1} = \sum_{\ell \geq 0} 2^{\ell+1} \binom{n+\ell}{\ell}\binom{2n+1}{n-2\ell} - \sum_{\ell \geq 0} 2^{\ell}\binom{n+\ell}{\ell}\binom{2n}{n-2\ell},$$

where T_{n+1} represents the number of torsion pairs in the cluster tube of rank $n+1$, and $t_{n,1} = (n+1)s_{n+2}$, where s_{n+2} represents the number of torsion pairs in the cluster category of type A_{n-1}.

Proof. For a torsion pair $(\mathcal{X}, \mathcal{Y})$ in $\mathcal{A}_{n,1}$, if $\mathcal{X}$ (resp. $\mathcal{Y}$) contains a diagonal whose length is longer than $n+1$, then $\mathcal{Y}$ (resp. $\mathcal{X}$) will contain only indecomposable rigid objects. The proof is the same as II in the proof of Proposition 3.7. We consider the case that both $\mathcal{X}$ and $\mathcal{Y}$ contain only indecomposable rigid objects in the following.

If $\mathcal{X}$ is a cluster tilting subcategory in $\mathcal{A}_{n,1}$, then $\mathcal{A}_{n,1}$ has a cotorsion pair $(\mathcal{X}, \mathcal{X})$. Besides these, there are some other torsion pairs $(\mathcal{X}, \mathcal{Y})$ with both $\mathcal{X}$ and $\mathcal{Y}$ containing only indecomposable rigid objects. For a torsion pair $(\mathcal{X}, \mathcal{Y})$, both $\mathcal{X}$ and $\mathcal{Y}$ contain only indecomposable rigid objects if and only if $\mathcal{X}$ and $\mathcal{Y}$ contain one of the rigid objects $[(i, (i+n+1))]$ with $i \in \{1, 2, \ldots, n+1\}$, where these objects are indecomposable rigid objects with maximal length in $\mathcal{A}_{n,1}$. We prove the statement in the following.

Let $\widetilde{\mathcal{X}}$ be the corresponding $(n+1)$-periodic collections of diagonals of the N-gon with $N = (2t+1)(n+1) = 3(n+1)$. If $\mathcal{X}$ contains the rigid object $[(i, (i+n+1))]$ with $i \in \{1, 2, \ldots, n+1\}$. Suppose $(1, n+2) \in \mathcal{X}$ (up to shifting). Then $(1, n+2), (n+2, 2n+3)$ and $(2n+3, 3n+4) = (1, 2n+3)$ are in $\widetilde{\mathcal{X}}$, so the maximal length of diagonals that do not cross with diagonals in $\widetilde{\mathcal{X}}$ is $n+1$, that is $\mathcal{Y}[1]$ contains only indecomposable rigid objects, so does $\mathcal{Y}$. Same proof for $\mathcal{X}$ contains only rigid indecomposable objects. Conversely, if $(\mathcal{X}, \mathcal{Y})$ is a torsion pair with both $\mathcal{X}$ and $\mathcal{Y}$ contain only indecomposable rigid objects, we pick a diagonal from $\mathcal{X}$ with maximal length, suppose its coordinate is $(1, \ell)$ with $3 \leq \ell \leq n+2$ since $(1, \ell)$ corresponds to an indecomposable rigid object. If $\ell < n+2$, then $(1, \ell+n+1)$ is in $\mathcal{X}^{\perp}[-1]$, which is similar to the proof of Proposition 3.7, i.e., $\mathcal{Y}$ contains an indecomposable object with level$> n$, which is non-rigid, a contradiction. This completes the proof of the statement above. It is easy to prove that for a torsion pair $(\mathcal{X}, \mathcal{Y})$, $\mathcal{X}$ (or $\mathcal{Y}$) contains at most one of indecomposable rigid objects with maximal length in $\mathcal{A}_{n,1}$.

Thus the torsion pairs in $\mathcal{A}_{n,1}$ divide into two subclasses: one (denoted by I) consists of torsion pairs $(\mathcal{X}, \mathcal{Y})$ with $\mathcal{X}$ or $\mathcal{Y}$ (not both) containing non-rigid indecomposable objects; another one (denoted by II) consists of torsion pairs $(\mathcal{X}, \mathcal{Y})$ with $\mathcal{X}$ and $\mathcal{Y}$ containing one and only one indecomposable rigid objects with maximal length in $\mathcal{A}_{n,1}$. The counting of torsion pairs in $\mathcal{A}_{n,1}$ is the same with $\mathcal{A}_{n,t}, t > 1$ (compare to Theorem 3.12). $(\mathcal{X}, \mathcal{Y})$ belongs to the subclass II if and only if so does $(\mathcal{Y}, \mathcal{X})$. Thus

the number $N_{n,1}$ of torsion pairs in $\mathcal{A}_{n,1}$ is the number T_{n+1} of torsion pairs in $\mathcal{A}_{n,t}$ with $t > 1$ minus the number of repetitive ones. That means we should remove from T_{n+1} the number $t_{n,1}$ of extension closed subcategory $\mathscr{X}$, which consists of indecomposable rigid objects and contains one and only one of indecomposable rigid objects with maximal length; that is $[(i, (i + n + 1))]$ with $i \in \{1, 2, \ldots, n + 1\}$. Therefore we have $t_{n,1} = (n + 1)s_{n+2}$, where s_{n+2} is the number of Ptolemy diagrams of the $(n + 2)$-gon, $s_{n+2} = \frac{1}{n+1} \sum_{\ell \geq 0} 2^\ell \binom{n+\ell}{\ell} \binom{2n}{n-2\ell}$ by [HJR1]. Then we get the conclusion. $\qquad\square$

Example 3.15. When $n = 2, t = 1$, $\mathcal{A}_{2,1} = D^b(\mathbb{K}A_6)/\tau^2[1]$ is 2-CY with cluster tilting objects, whose Auslander-Reiten quiver is shown in figure 6. By Theorem 3.14, we have that the number of torsion pairs in $\mathcal{A}_{2,1} : N_{2,1} = 20$. We can construct them as follows:

$$\begin{aligned}
\mathscr{X}_1 &= \{0\} & \mathscr{X}_1^\perp[-1] &= \mathcal{A}_{2,1} \\
\mathscr{X}_2 &= \{13\} & \mathscr{X}_2^\perp[-1] &= \{13, 14, 16, 36\} \\
\mathscr{X}_3 &= \{24\} & \mathscr{X}_3^\perp[-1] &= \{14, 15, 24, 25\} \\
\mathscr{X}_4 &= \{35\} & \mathscr{X}_4^\perp[-1] &= \{25, 26, 35, 36\} \\
\mathscr{X}_5 &= \{14\} & \mathscr{X}_5^\perp[-1] &= \{13, 14, 24\} \\
\mathscr{X}_6 &= \{25\} & \mathscr{X}_6^\perp[-1] &= \{24, 25, 35\} \\
\mathscr{X}_7 &= \{36\} & \mathscr{X}_7^\perp[-1] &= \{13, 35, 36\} \\
\mathscr{Y}_1 &= \{13, 14\} & \mathscr{Y}_1^\perp[-1] &= \{13, 14\} = \mathscr{Y}_1 \\
\mathscr{Y}_2 &= \{24, 25\} & \mathscr{Y}_2^\perp[-1] &= \{24, 25\} = \mathscr{Y}_2 \\
\mathscr{Y}_3 &= \{35, 36\} & \mathscr{Y}_3^\perp[-1] &= \{35, 36\} = \mathscr{Y}_3 \\
\mathscr{Y}_4 &= \{14, 24\} & \mathscr{Y}_4^\perp[-1] &= \{14, 24\} = \mathscr{Y}_4 \\
\mathscr{Y}_5 &= \{25, 35\} & \mathscr{Y}_5^\perp[-1] &= \{25, 35\} = \mathscr{Y}_5 \\
\mathscr{Y}_6 &= \{13, 36\} & \mathscr{Y}_6^\perp[-1] &= \{13, 36\} = \mathscr{Y}_6
\end{aligned}$$

By Example 3.13, we have known the number of torsion pairs in $\mathcal{A}_{2,2}$ is $T_{2+1} = 32$. Next, we should count the number of extension closed subcategory $\mathscr{X}$, which consists of indecomposable rigid objects and contains one and only one of indecomposable rigid objects with maximal length.

The rigid indecomposable objects with maximal length in $\mathcal{A}_{2,1}$ are $[(1, 4)], [(2, 5)], [(3, 6)]$. The subcategories containing the rigid object $[(1, 4)]$ are the followings: $\mathscr{X}_5 = \{14\}$, $\mathscr{X}_5^\perp[-1] = \{13, 14, 24\}$, $\mathscr{Y}_1 = \{13, 14\}$, and $\mathscr{Y}_4 = \{14, 24\}$. Then shifting all the subcategories above, we have all the subcategories containing one of the rigid objects $[(i, i + n + 1)]$ with maximal length. The number of such subcategories is $t_{2,1} = 3 \times 4 = 12 = (2 + 1)s_{2+2}$.

Then the number of torsion pairs in $\mathcal{A}_{2,1}$ is $T_{2+1} - 12 = 20$. It follows that $(\mathscr{X}_i, \mathscr{X}_i^\perp)$ is a torsion pair in $\mathcal{A}_{2,1}$, where $i = 1, 2, 3, \cdots, 7$. Similarly, we can know that $(^\perp\mathscr{X}_i, \mathscr{X}_i)$ is a torsion pair in $\mathcal{A}_{2,1}$, where $i = 1, 2, 3, \cdots, 7$. Moreover, $(\mathscr{Y}_j, \mathscr{Y}_j^\perp[-1]) = (\mathscr{Y}_j, \mathscr{Y}_j)$ is a cotorsion pair in $\mathcal{A}_{2,1}$, where $j = 1, 2, 3, \cdots, 6$. Note that $^\perp\mathscr{Y}_j[1] = \mathscr{Y}_j^\perp[-1]$. It follows that $(\mathscr{Y}_j, \mathscr{Y}_j[1])$ is a torsion pair in $\mathcal{A}_{2,1}$, where $i = 1, 2, 3, \cdots, 6$.
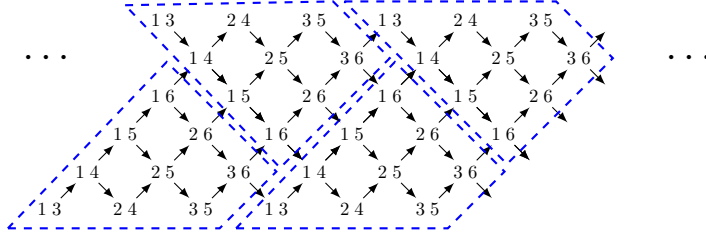
Figure 6: The AR-quiver of $\mathcal{A}_{2,1}$

4 Classification of torsion pairs in $\mathcal{D}_{n,t}$

In this section, we give a classification of torsion pairs in $\mathcal{D}_{n,t}$ and count the number. Let $u = 2t(n+1)$ and $\mathcal{C}_{D_u}$ be the cluster category of type D_u. Then $\mathcal{D}_{n,t} = D^b(\mathbb{K}\vec{D}_u)/\tau^{n+1}\varphi^n$, where φ is induced by an automorphism of D_u of order 2, $n \geq 1, t \geq 1$. It follows from Lemma 2.9 in [BPR] that there exists a covering functor $\pi : \mathcal{C}_{D_u} \to \mathcal{D}_{n,t}$.

Write $F = \tau^{(n+1)}\varphi^n$, then $F : \mathcal{C}_{D_u} \to \mathcal{C}_{D_u}$ can be seen an autoequivalence and $\mathcal{D}_{n,t}$ can be seen as the orbit categories $\mathcal{C}_{D_u}/F$ (compare Lemma 2.9 in [BPR]). Since $\tau^{-u+1} = [1]$ in $D^b(\mathbb{K}D_u)$ by [S], we have $\tau^{-2t(n+1)} = 1 = \tau^{2t(n+1)}$ in $\mathcal{C}_{D_u}$, and π is a $2t$-covering functor and obviously a triangle functor. By Theorem 2.12, we have the following.

Corollary 4.1. *There is a bijection between the set of F-periodic torsion pairs in $\mathcal{C}_{D_u}$ and torsion pairs in $\mathcal{D}_{n,t}$.*

Let us recall the definition of Ptolemy diagrams of type D and its relation to torsion pairs in the cluster categories of type D based on [HJR3]. For any $n \geq 1$ we consider a regular $2n$-gon Q_n, we label the vertices of Q_n clockwise by $1, 2, \ldots 2n$ consecutively. In our arguments below vertices will also be numbered by some $r \in \mathbb{N}$ which might not be in the range $1 \leq r \leq 2n$; in this case the numbering of vertices always has to be taken modulo $2n$.

An *arc* is a set $\{i, j\}$ of vertices of Q_n with $j \notin \{i-1, i, i+1\}$, i.e. i and j are different and non-neighboring vertices. The arcs connecting two opposite vertices i and $i+n$ are called *diameters*. We need two different copies of each of these diameters and denote them by $\{i, i+n\}_g$ and $\{i, i+n\}_r$, where $1 \leq i \leq 2n$. The indices should indicate that these diameters are coloured in the colours green and red, which is a convenient way to think about and to visualize the diameters. By a slight abuse of notation, we sometimes omit the indices and just write $\{i, i+n\}$ for diameters, to avoid cumbersome definitions or statements.

Any arc in Q_n which is not a diameter is of the form $\{i, j\}$ where $j \in [i+2, i+n-1]$; here $[i+2, i+n-1]$ stands for the set of vertices of the $2n$-gon Q_n which are met when going clockwise from $i+2$ to $i+n-1$ on the boundary of Q_n. See figure 7 for an example, for better visibility we draw the red diameters in a wavelike form and the green ones as straight lines.

Such an arc has a partner arc $\{i+n, j+n\}$ which is obtained from $\{i, j\}$ by a rotation by 180 degrees. We denote the pair of arcs $\{\{i, j\}, \{i+n, j+n\}\}$ by $\overline{\{i, j\}}$ throughout this section of the paper. The indecomposable objects in $\mathcal{C}_{D_n}$ are in bijection with the union of the set of pairs $\overline{\{i, j\}}$ of non-diameter arcs and the set of diameters $\{i, i+n\}_g$ and $\{i, i+n\}_r$ in two different colours. This bijection extends
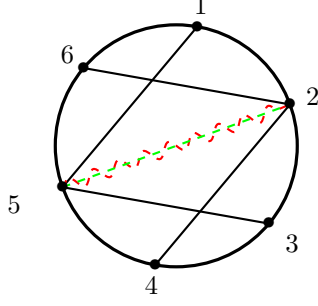
Figure 7: $2n = 6$

to subcategories of $\mathcal{C}_{D_n}$ closed under direct sums and direct summands and collections of arcs of $2n$-gon Q_n.

For the pairs of non-diameter arcs $\overline{\{i, j\}}$ the corresponding indecomposable object has coordinates (i, j); note that the coordinates are only determined modulo n so both arcs $\{i, j\}$ and $\{i + n, j + n\}$ in the pair $\overline{\{i, j\}}$ yield the same coordinate in the Auslander-Reiten quiver of $\mathcal{C}_{D_n}$. The action of τ on non-diameter arc is rotation by one vertex, the action of τ on diameter is rotation by one vertex and changing its colour [HJR3, S]. We note that φ acts on diameters by changing its colour and $\varphi = id$ on non-diameter arcs [BPR].

Back to our consideration, for a coordinate (i, j) corresponding to an indecomposable object of $\mathcal{C}_{D_u}$, we denote $[(i, j)]$ by the image under the covering functor π, then $[(i, j)]$ determines an indecomposable object of $\mathcal{D}_{n,t}$. For any subcategory $\mathscr{X}$ in $\mathcal{D}_{n,t}$, the preimage under the covering functor π corresponds to a F-periodic subcategory $\widetilde{\mathscr{X}} = \pi^{-1}(\mathscr{X})$ in $\mathcal{C}_{D_u}$. Moreover, the subcategory $\widetilde{\mathscr{X}}$ corresponds to set of arcs of $2u$-gon Q_u by the discussion above, we still denote the corresponding set of arcs by $\widetilde{\mathscr{X}}$. Since $F = \tau^{n+1}\varphi^n$, we call that the set of arcs $\widetilde{\mathscr{X}}$ is $F - periodic$, if the following conditions are satisfied: (1) For each non-diameter arc $(i, j) \in \widetilde{\mathscr{X}}$, also all arcs $(i + (n+1)r, j + (n+1)r)$ (modulo u) for $1 \le r \le 2t$ are in $\widetilde{\mathscr{X}}$. (2) For each diameter $(i, i + u) \in \widetilde{\mathscr{X}}$, all diameters $(i + (n+1)r, j + u + (n+1)r)$ (modulo u) with the same colour as $(i, i + u)$, where $1 \le r \le 2t$ and r is even, are in $\widetilde{\mathscr{X}}$, and all diameters $(i + (n+1)r, j + u + (n+1)r)$ (modulo u) with the opposite colour as $(i, i + u)$, where $1 \le r \le 2t$ and r is odd, are in $\widetilde{\mathscr{X}}$.

Lemma 4.2. *There is a bijection between the following sets:*

(1) *Subcategory $\mathscr{X}$ of $\mathcal{D}_{n,t}$;*

(2) *Collection of arcs $\widetilde{\mathscr{X}}$ of the $2u$-gon Q_u which are F-periodic.*

We recall the definition of Ptolemy diagrams of type D for a $2n$-gon Q_n from [HJR3].

Definition 4.3. (a) We say that two non-diameter arcs $\{i, j\}$ and $\{k, \ell\}$ *cross* precisely if the elements i, j, k, ℓ are all distinct and come in the order i, k, j, ℓ when moving around the $2n$-gon Q_n in one direction or the other (i.e. counterclockwise or clockwise). In particular, the two arcs in $\overline{\{i, j\}}$ do not cross.

Similarly, in the case $j = i + n$, the above condition defines when a diameter $\{i, i + n\}_g$ (or $\{i, i + n\}_r$) crosses the non-diameter arc $\{k, \ell\}$.

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(b) We say that two pairs $\overline{\{i,j\}}$ and $\overline{\{k,\ell\}}$ of non-diameter arcs *cross* if there exist two arcs in these two pairs which cross in the sense of part (a). (Note that then necessarily the other two rotated arcs also cross.)

Similarly, the diameter $\{i,i+n\}_g$ (or $\{i,i+n\}_r$) crosses the pair $\overline{\{k,\ell\}}$ of non-diameter arcs if it crosses one of the arcs in $\overline{\{k,\ell\}}$. (Note that it then necessarily crosses both arcs in $\overline{\{k,\ell\}}$.)

(c) Two diameters $\{i,i+n\}_g$ and $\{j,j+n\}_r$ of different colour *cross* if $j \notin \{i,i+n\}$, i.e. if they have different endpoints. But $\{i,i+n\}_g$ and $\{i,i+n\}_r$ do not cross. Moreover, any diameters of the same colour do not cross.

Definition 4.4. (a) Let $\mathscr{X}$ be a collection of arcs of the $2n$-gon Q_n, $n > 1$, which is invariant under rotation of 180 degrees. Then $\mathscr{X}$ is called a *Ptolemy diagram of type D_n* if it satisfies the following conditions. Let $\alpha = \{i,j\}$ and $\beta = \{k,\ell\}$ be crossing arcs in $\mathscr{X}$ (in the sense of Definition 4.3).

(Pt1) If α and β are not diameters, then those of $\{i,k\}$, $\{i,\ell\}$, $\{j,k\}$, $\{j,\ell\}$ which are arcs in P are also in $\mathscr{X}$. In particular, if two of the vertices i,j,k,ℓ are opposite vertices (i.e. one of k and ℓ is equal to $i+n$ or $j+n$), then both the green and the red diameter connecting them are also in $\mathscr{X}$.

(Pt2) If both α and β are diameters (necessarily of different colour by Definition 4.3 (c)) then those of $\{i,k\}$, $\{i,k+n\}$, $\{i+n,k\}$, $\{i+n,k+n\}$ which are arcs of P are also in $\mathscr{X}$.

(Pt3) If α is a diameter while β is not a diameter, then those of $\{i,k\}$, $\{i,\ell\}$, $\{j,k\}$, $\{j,\ell\}$ which are arcs and do not cross the arc $\{k+n,\ell+n\}$ are also in $\mathscr{X}$. Additionally, the diameters $\{k,k+n\}$ and $\{\ell,\ell+n\}$ of the same colour as α are also in $\mathscr{X}$.

A collection $\mathscr{X}$ of arcs is called a *Ptolemy diagram of type D* if it is a Ptolemy diagram of type D_n for some $n \geq 1$.

These conditions are illustrated in figure 8.

Now we define the *F-periodic Ptolemy diagram* of type D for $2u$-gon Q_u which we will use to give a classification of torsion pairs in $\mathcal{D}_{n,t}$.

Definition 4.5. Let $\widetilde{\mathscr{X}}$ be a collection of arcs in $2u$-gon Q_u, and $F = \tau^{n+1}\varphi^n$. $\widetilde{\mathscr{X}}$ is called *F-periodic Ptolemy diagram* of type D if $\widetilde{\mathscr{X}}$ is a Ptolemy diagram of type D and is F-periodic.

Definition 4.6. In the coordinate system of the AR-quiver of $\mathcal{D}_{n,t}$, if a vertex of the AR-quiver has coordinate $[(i,j)]$ with $i < j$, we call the number $j - i - 1$ the *level* of the vertex. In particular, if $(i,i+u)$ is a diameter, we call $u-1$ the *level* of the diameter, note that $u - 1 > n$. The *length* of $[(i,j)]$ is defined by $j - i$.
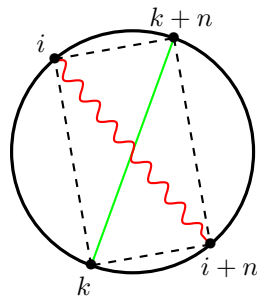
Lemma 4.7. *Let $(\mathscr{X}, \mathscr{X}^\perp)$ be a torsion pair in $\mathcal{D}_{n,t}$ and $\widetilde{\mathscr{X}}$ be the corresponding F-periodic collection of arcs of the $2u$-gon Q_u, where $u = 2t(n+1)$. If $t > 1$, then precisely one of the following situation occurs:*

(i) *The level of all indecomposable objects of $\mathscr{X} \leq n$;*

(ii) *The level of all indecomposable objects of $\mathscr{X}^\perp \leq n$.*

(Pt1) The first Ptolemy condition in type D:
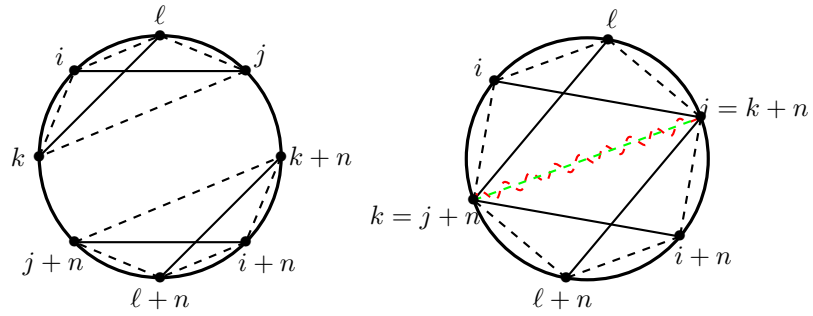
(Pt2) The second Ptolemy condition in type D:

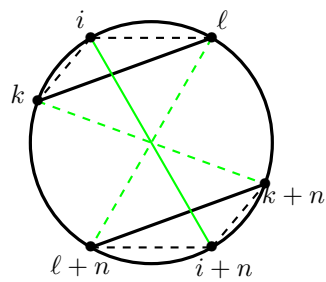(Pt3) The third Ptolemy condition in type D:
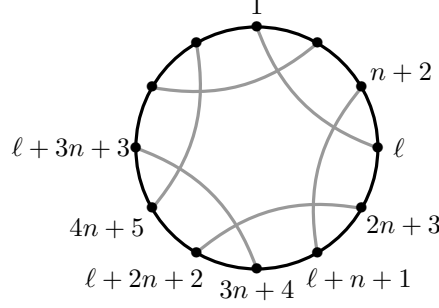

Figure 8: The Ptolemy conditions in type D.

Figure 9: A case when $t = 3$

Proof. Note that the elements with level $\leq n$ in the AR-quiver of $\mathcal{D}_{n,t}$ are exactly all the rigid indecomposable objects [BPR].

(I) If the level of all the indecomposable objects of $\mathscr{X} \leq n$, we claim that $\mathscr{X}^{\perp}$ must contain an element (not diameters) with level$> n$. Indeed, since $\mathscr{X}$ contains only (finitely many) indecomposable rigid objects, we pick an arc from $\widetilde{\mathscr{X}}$ with maximal length. We can suppose that its coordinate is $(1, \ell)$ with $3 \leq \ell \leq n+2$, since that $(1, \ell)$ corresponds to a rigid indecomposable object. If we can show $(1, \ell + n + 1) \in \mathscr{X}^{\perp}[-1]$, then $\mathscr{X}^{\perp}$ contains an element with level$> n$. We first note that $(1, \ell + n + 1)$ represents a non-diameter arc in the AR-quiver of $\mathcal{D}_{n,t}$, and the level of $(1, \ell + n + 1)$ is $\ell + n - 1 \geq 2 + n - 1 = n + 1$, since $t > 1$, $\ell + n + 1 \leq n + 2 + n + 1 = 2n + 3 < u + 1$. Secondly, since $(1, \ell)$ is in $\widetilde{\mathscr{X}}$ with maximal length, there is no arc $(i, j) \in \widetilde{\mathscr{X}}$ with $i < \ell < j$. Otherwise $(1, \ell)$ has to cross (i, j), but the Ptolemy condition yields an arc longer than $(1, \ell)$, a contradiction. This means $(1, \ell + n + 1)$ cannot cross any arc from $\widetilde{\mathscr{X}}$, i.e., $(1, \ell + n + 1) \in \mathscr{X}^{\perp}[-1]$.

(II) Suppose $\mathcal{X}$ contains an element with level$> n$. Note that all indecomposable objects of $\mathcal{D}_{n,t}$ with level$> n$ may diameters or non-diameters arcs, we claim that $\mathscr{X}^{\perp}$ only contains elements with level$\leq n$.

1) If $\mathscr{X}$ contains a non-diameter arc with level$> n$, without losing generality, we suppose $[(1, \ell)]$ is such an element. By the AR-quiver of $\mathcal{D}_{n,t}$, we can choose $n + 3 \leq \ell \leq u$. We choose ℓ for different intervals $[n + 3, 2n + 3]$, $[2n + 3, 3n + 4]$, $[3n + 4, 4n + 5]$, ..., $[(2t - 1)(n + 1) + 1, 2t(n + 1) + 1]$.

If $n + 3 \leq \ell \leq 2n + 3$, then the corresponding arcs in $\widetilde{\mathscr{X}}$ are shown in figure 9.

If $2n + 3 \leq \ell \leq 3n + 4$, then the corresponding arcs in $\widetilde{\mathscr{X}}$ are shown in figure 10.

The other cases are similar. This shows the level of indecomposable objects in $[(1, \ell)]^{\perp}[-1] \leq n$. Therefore the level of indecomposable objects in $\mathscr{X}^{\perp} \leq n$, since $\mathscr{X}^{\perp}[-1] \subseteq [(1, \ell)]^{\perp}[-1]$.

2) If $\mathscr{X}$ contains a diameter $[(1, u + 1)]$ (green or red) in $\mathscr{X}$. without losing generality, we suppose $[(1, u + 1)]_g \in \mathscr{X}$. Then $(1, u + 1)_g, (n + 2, u + n + 2)_r, \cdots, \in \widetilde{\mathscr{X}}$ and the first two objects cross. Then (Pt2) implies that $(n + 2, u + 1) \in \widetilde{\mathscr{X}}$ which is not a diameter and then the non-diameter arc $[(n + 2, u + 1)] \in \mathscr{X}$, its level is $u + 1 - (n + 2) - 1 = u - (n + 2) = (2t - 1)(n + 1) - 1 \geq 3n + 2 > n$, since $t > 1$. Thus $\mathscr{X}^{\perp}$ contains only indecomposable rigid objects by case 1).
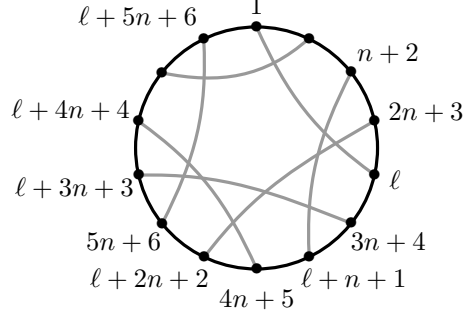
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Figure 10: A case when $t = 3$

As a consequence, for a torsion pair $(\mathscr{X}, \mathscr{X}^{\perp})$ in $\mathcal{D}_{n,t}$, precisely one of (1) and (2) occurs. If neither (1) nor (2) occurs, this is a contradiction with II. Contrarily, if both (1) and (2) occur, this will be a contradiction with I. Thus we complete the proof. $\qquad\square$

Definition 4.8. Let (i, j) be a non-diameter arc of $2u$-gon Q_u. The *wing* $W(i, j)$ of (i, j) consists of all arcs (r, s) of the $2u$-gon such that $i \leq r \leq s \leq j$, that is all arcs which are overarched by (i, j). $[(i, j)]$ represents a vertex in the AR-quiver of $\mathcal{D}_{n,t}$, the corresponding wing is denoted by $W[(i, j)]$.

Combining Lemma 4.7, we have the classification of torsion pairs in $\mathcal{D}_{n,t}, t > 1$, whose proof is same as Theorem 3.10 (compare [HJR2]).

Theorem 4.9. *There are bijections between the following sets for $t > 1$:*

(1) *Torsion pairs $(\mathscr{X}, \mathscr{X}^{\perp})$ in $\mathcal{D}_{n,t}$ such that the level of all the indecomposable objects in $\mathscr{X} \leq n$;*

(2) *F-periodic Ptolemy diagrams $\widetilde{\mathscr{X}}$ of type D of $2u$-gon Q_u such that all arcs in $\widetilde{\mathscr{X}}$ have length at most $n + 1$;*

(3) *Collections $\{([(i_1, j_1)], [W_1]), \ldots, ([(i_r, j_r)], [W_r])\}$ of pairs consisting of vertices $[(i_\ell, j_\ell)]$ of level$\leq n$ in the AR-quiver of $D_{n,t}$ and subset $[W_\ell] \subset W[(i_\ell, j_\ell)]$ of their wings such that for any different $k, \ell \in \{1, 2, \ldots, r\}$, we have*
$$W[(i_k, j_k)][1] \cap W[(i_\ell, j_\ell)] = \emptyset,$$
and the F-periodic collection W_ℓ corresponding to $[W_\ell]$ is a Ptolemy diagram of type D.

Next, we describe torsion pairs in $\mathcal{D}_{n,1}$. Recall that two diameters are called *paired* if they connect the same two vertices (and thus of different colour).

Theorem 4.10. *For a torsion pair $(\mathscr{X}, \mathscr{Y})$ in $\mathcal{D}_{n,1}$, precisely one of the following situations occurs:*

1. *$\mathscr{X}$ (resp. $\mathscr{Y}$) contains one paired diameters and $\mathscr{Y}$ (resp. $\mathscr{X}$) contains only indecomposable rigid objects.*

2. *Both $\mathscr{X}$ and $\mathscr{Y}$ contain only one non-paired diameter and some indecomposable rigid objects.*

Proof. Let $(\mathscr{X}, \mathscr{Y})$ be a torsion pair in $\mathcal{D}_{n,1}$ and $\widetilde{\mathscr{X}}, \widetilde{\mathscr{Y}}$ be the corresponding F-periodic collection of arcs of the $2u$-gon Q_u, where $u = 2(n + 1)$. We note that $(\mathscr{X}, \mathscr{Y})$ is a torsion pair if and only if so is $(\mathscr{Y}, \mathscr{X}[2])$, since $\mathcal{C}$ is 2-CY.
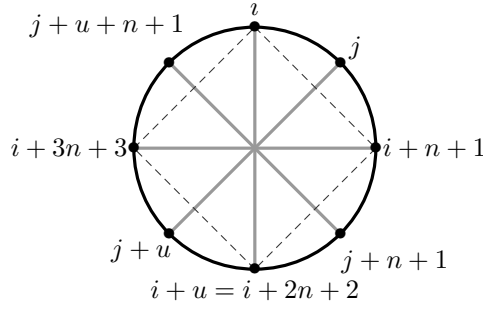
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Figure 11: The case when $t = 1$

(1) If $\widetilde{\mathscr{X}}$ contains a non-diameter arc with length longer than $n+1$, then $\mathscr{Y}$ contains only indecomposable rigid objects. The proof is the same as (II) in the proof of Lemma 4.7.

(2) If $\widetilde{\mathscr{X}}$ contains a paired diameters, then the indecomposable objects in $\mathscr{Y}$ are all rigid. Suppose $[(1, u+1)] \in \widetilde{\mathscr{X}}$ (green and red) is one paired diameters. Because $\widetilde{\mathscr{X}}$ is F-periodic, $(1, u+1)_{r,g} \in \widetilde{\mathscr{X}}$, $(n+2, 3n+4)_{r,g} \in \widetilde{\mathscr{X}}$ and they cross. So $(1, n+2)$ and $(n+2, 2n+3)$ are in $\widetilde{\mathscr{X}}$ by (Pt2). Thus the maximal length of arcs that do not cross any arc in $\widetilde{\mathscr{X}}$ is $n+1$ and any diameter crosses $(1, u+1)$ in $\widetilde{\mathscr{X}}$, that is $\mathscr{X}^{\perp}[-1]$ contains only indecomposable rigid objects, so does $\mathscr{Y}$.

(3) If $\mathscr{X}$ contains only indecomposable rigid objects, then $\widetilde{\mathscr{Y}}$ will contain a paired diameters. Suppose the arc in $\widetilde{\mathscr{X}}$ with maximal length is $(1, \ell)$ (up to shifting), then $3 \leq \ell \leq n+2$ since $(1, \ell)$ corresponds to a rigid object. We claim the paired diameters $(2, u+2)$ is in $\widetilde{\mathscr{Y}}$. Otherwise, there is an arc in $\widetilde{\mathscr{X}}$ crosses the paired diameters $(1, u+1)$, then there will exist an arc with length longer than $(1, \ell)$ similarly as the proof in Lemma 4.7. So the paired diameters $(1, u+1)$ is in $\mathscr{Y}[-1]$, that is, $(2, u+2)$ (red and green) is in $\mathscr{Y}$.

If $\mathscr{X}$ contains diameters, but no paired diameter, then $\widetilde{\mathscr{X}}$ contains no paired diameters either. We claim that $\mathscr{X}$ contains only one diameter (red or green). In fact, if $\mathscr{X}$ contains 2 non-paired diameters $[(i, i+u)], [(j, j+u)]$ (red or green) $(j \neq i, j \neq i+n+1)$. Then $\widetilde{\mathscr{X}}$ will contain 4 non-paired diameter with different colours, see figure 11.

Note that $(i, i+u)$ and $(i+n+1, i+3n+3)$ are two different coloured diameters and they cross, then $(i, i+n+1) \in \widetilde{\mathscr{X}}$ by (Pt2). Moreover, the arc $(i, i+n+1)$ crosses $(j, j+u)$, then $(i, i+u)$ and $(i+n+1, i+3n+3)$ with the same colour as $(j, j+u)$ are in $\widetilde{\mathscr{X}}$, but $(i, i+u)$ and $(i+n+1, i+3n+3)$ are different colours, this implies that $[(i, i+u)]$ are paired diameters in $\mathscr{X}$, a contradiction.

Moreover, if $\mathscr{X}$ contains only one diameter, without losing generality, suppose its coordinate is $[(1, 1+u)]$, then $(1, 1+u)$ and $(n+2, u+n+1)$ are in $\widetilde{\mathscr{X}}$ and they are different colours, so they cross. The Ptolemy condition implies that $(1, n+2)$ and $(n+2, 2n+3)$ are in $\widetilde{\mathscr{X}}$, so the maximal length of arcs that do not cross any arc in $\widetilde{\mathscr{X}}$ is $n+1$, and the diameter $[(1, 1+u)]$ with different colour as it in $\mathscr{X}$ does not cross any arc in $\widetilde{\mathscr{X}}$ either. That means that $\mathscr{X}^{\perp}[-1]$ contains only indecomposable rigid objects and one diameter, so does $\mathscr{Y} = \mathscr{X}^{\perp}$. $\qquad\square$

Theorem 4.11. *Let $D_{n,t}$ be the number of torsion pairs in $\mathcal{D}_{n,t}$.*

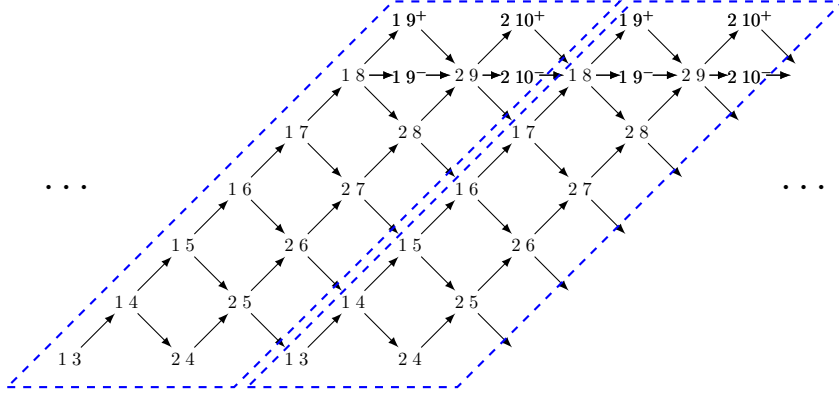   *1 If $t > 1$, then $D_{n,t} = T_{n+1}$, the number of torsion pairs in $\mathcal{A}_{n,t}$.*

Figure 12: The AR-quiver of $\mathcal{D}_{1,2}$

2 If $t = 1$, then $D_{n,t} = T_{n+1} + 2t_{n,1} = \sum_{\ell \geq 0} 2^{\ell+1} \binom{n+\ell}{\ell} \left[\binom{2n+1}{n-2\ell} + \binom{2n}{n-2\ell} \right]$

Proof. For $t > 1$, we only consider the subcategories $\mathcal{X}$ of $\mathcal{D}_{n,t}$ with the level of indecomposable objects in $\mathcal{X} \leq n$. Because such subcategories $\mathcal{X}$ cannot contain any diameters and (Pt1) in Definition 4.4 coincides with the Ptolemy condition in type A, the Ptolemy diagram of type D is the same as Ptolemy diagram of type A. Moreover, because $\mathcal{X}$ contains only indecomposable rigid objects, the F-periodic of the corresponding collection of non-diameter arcs are $(n+1)$-periodic. By Theorem 4.9 and Theorem 3.10, we know that the number of torsion pairs in $\mathcal{D}_{n,t}$ equals to the number of torsion pairs in $\mathcal{A}_{n,t}$ and equals to the number of torsion pairs in the cluster tube of rank $n + 1$.

For $t = 1$, The torsion pairs in $\mathcal{D}_{n,1}$ divide into two subclasses: one is the torsion pairs $(\mathcal{X}, \mathcal{Y})$ in $\mathcal{D}_{n,1}$ with $\mathcal{X}$ or $\mathcal{Y}$ (not both) containing a paired diameters, counting the number of this case reduces to counting of the possible halves $\mathcal{X}$ or $\mathcal{Y}$ of a torsion pair, whose all indecomposable objects are rigid by Theorem 4.10. The number of such torsion pairs in $\mathcal{D}_{n,1}$ equals to the number of torsion pairs in $\mathcal{A}_{n,t}$ with $t > 1$. Another one is the torsion pairs $(\mathcal{X}, \mathcal{Y})$ in $\mathcal{D}_{n,1}$ with both $\mathcal{X}$ and $\mathcal{Y}$ containing one non-paired diameter and some indecomposable rigid objects. The number of such torsion pairs is $2t_{n,1}$. $\qquad\square$

Example 4.12. When $n = 1, t = 2$, $\mathcal{D}_{1,2} = D^b(\mathbb{K}D_8)/\tau^2\varphi$ is 2-Calabi-Yau with maximal rigid objects, whose Auslander-Reiten quiver is shown in figure 12. By Theorem 4.11, the number of torsion pairs in $\mathcal{D}_{1,2}$ is $T_{1+1} = 6$. We list the torsion pairs according to Theorem 4.9:

$$\mathcal{X}_1 = \{0\} \qquad\qquad \mathcal{X}_1^\perp[-1] = \mathcal{D}_{1,2}$$
$$\mathcal{X}_2 = \{13\} \qquad\qquad \mathcal{X}_2^\perp[-1] = \{13, 15, 17, 19^+, 19^-\}$$
$$\mathcal{X}_3 = \{24\} \qquad\qquad \mathcal{X}_3^\perp[-1] = \{24, 26, 28, 210^+, 210^-\}$$

We only need to consider the subcategories containing only indecomposable rigid objects in $\mathcal{D}_{1,2}$. By Theorem 4.9, all the torsion pairs $(\mathcal{X}_i, \mathcal{X}_i^\perp)$ and $(^\perp\mathcal{X}_i, \mathcal{X}_i)$ are listed above for $i = 1, 2, 3$.

Example 4.13. When $n = 1, t = 1$, $\mathcal{D}_{1,1} = D^b(\mathbb{K}D_4)/\tau^2\varphi$ is 2-Calabi-Yau with maximal rigid objects, whose Auslander-Reiten quiver is shown in figure 13. By Theorem 4.11, the number of torsion pairs in $\mathcal{D}_{1,1}$ is $T_{1+1} + 2t_{1,1} = 10$. We list the torsion pairs according to Theorem 4.10:
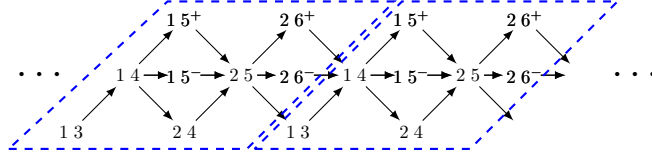
Figure 13: The AR-quiver of $\mathcal{D}_{1,1}$

$$\mathscr{X}_1 = \{0\} \qquad \mathscr{X}_1^\perp[-1] = \{\mathcal{D}_{1,1}\}$$
$$\mathscr{X}_2 = \{13\} \qquad \mathscr{X}_2^\perp[-1] = \{13, 15^+, 15^-\}$$
$$\mathscr{X}_3 = \{24\} \qquad \mathscr{X}_3^\perp[-1] = \{24, 26^+, 26^-\}$$
$$\mathscr{X}_4 = \{13, 15^+\} \qquad \mathscr{X}_4^\perp[-1] = \{13, 15^-\}$$
$$\mathscr{X}_5 = \{24, 26^+\} \qquad \mathscr{X}_5^\perp[-1] = \{24, 26^-\}$$

Note that $\mathscr{X}_4 = \{13, 15^+\}$, $\mathscr{X}_4^\perp[-1] = \{13, 15^-\}$, $\mathscr{X}_5 = \{24, 26^+\}$, and $\mathscr{X}_5^\perp[-1] = \{24, 26^-\}$ are all the subcategories containing one diameter and some indecomposable rigid objects, its number is $4 = 2t_{1,1}$. All the torsion pairs $(\mathscr{X}_i, \mathscr{X}_i^\perp)$ and $(^\perp\mathscr{X}_i, \mathscr{X}_i)$ are listed above for $i = 1, 2, 3, 4, 5$.

5 Hearts of torsion pairs

In this section, we determine the hearts of torsion pairs in finite 2-CY triangulated categories with maximal rigid objects. Hearts of cotorsion pairs in any triangulated category were introduced by Nakaoka [N1], which unify the construction of hearts of t-structures [BBD] and construction of the abelian quotient categories by cluster tilting subcategories [BMRRT, KR, KZ]. For two subcategories $\mathscr{X}, \mathscr{Y}$ in a triangulated category $\mathcal{C}$, the pair $(\mathscr{X}, \mathscr{Y})$ is a torsion pair in $\mathcal{C}$ if and only if $(\mathscr{X}, \mathscr{Y}[-1])$ is a cotorsion pair in $\mathcal{C}$. The heart of torsion pair $(\mathscr{X}, \mathscr{Y})$ is by definition the heart of cotorsion pair $(\mathscr{X}, \mathscr{Y}[-1])$. We will use the notation of cotorsion pairs in this section.

We recall the construction of hearts of cotorsion pairs from [N1]: Let $\mathcal{C}$ be a triangulated category and $(\mathscr{X}, \mathscr{Y})$ a cotorsion pair with core $\mathcal{I}$ in $\mathcal{C}$. Denote by $\mathcal{H}$ the subcategory $(\mathscr{X}[-1] * \mathcal{I}) \cap (\mathcal{I} * \mathscr{Y}[1])$. The heart of the cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is defined as the quotient category $\mathcal{H}/\mathcal{I}$, denoted by $\underline{\mathcal{H}}$.

It was proved that $\underline{\mathcal{H}}$ is an abelian category [N1]. There is a cohomology functor $H = h\pi$ from $\mathcal{C}$ to $\underline{\mathcal{H}}$, where π is the quotient functor from $\mathcal{C}$ to $\underline{\mathcal{C}} = \mathcal{C}/\mathcal{I}$ and h is a functor from $\underline{\mathcal{C}}$ to $\underline{\mathcal{H}}$. See [AN, N1] for the details of the constructions. We give the main result in this section.

Proposition 5.1. *Let $\mathcal{C}$ be a finite 2-Calabi-Yau triangulated category, and $(\mathscr{X}, \mathscr{Y})$ be a cotorsion pair in $\mathcal{C}$ with core $\mathcal{I} = addI$, where I is a rigid object, Then we have an equivalence of abelian categories*

$$\underline{\mathcal{H}} \simeq mod\ EndI$$

Proof. : For any cotorsion pair $(\mathscr{X}, \mathscr{Y})$ in $\mathcal{C}$ with core $\mathcal{I}$, $(^\perp\mathcal{I}[1])/\mathcal{I}$ is also a finite 2-CY triangulated category with shift functor $< 1 >$ [IY], and $(\mathscr{X}/\mathcal{I}, \mathscr{Y}/\mathcal{I})$ is a t-structure by [ZZ2]. It follows from

Proposition2.13 that $^{\perp}\mathcal{I}[1])/\mathcal{I} = \mathscr{X}/\mathcal{I} \bigoplus \mathscr{Y}/\mathcal{I}$.

On the other hand, $(\mathcal{I}, {}^{\perp}\mathcal{I}[1])$ is a cotorsion pair with the same core $\mathcal{I}$, and the heart $\underline{H}_1$ of $(\mathcal{I}, {}^{\perp}\mathcal{I}[1])$ is equivalent to the module category mod End I by [IY], i.e, $\underline{H}_1 \simeq$ mod $EndI$. By the same proof as Theorem 6.4 in [ZZ2], the heart $\underline{H}$ of $(\mathscr{X}, \mathscr{Y})$ is equivalent to $\underline{H}_1$. Thus $\underline{H} \simeq$ mod $EndI$. $\qquad\square$

Now we have the following conclusions about the hearts of cotorsion pairs in finite 2-CY triangulated categories $\mathcal{C}$: 1. If $\mathcal{C}$ contains cluster tilting objects, then hearts had been determined in [ZZ2]. 2. If $\mathcal{C}$ has only zero maximal rigid objects, then any cotorsion pair $(\mathscr{X}, \mathscr{Y})$ is a t-structure, and $\mathscr{X}[1] = \mathscr{X}, \mathscr{Y}[1] = \mathscr{Y}$. Then the heart $\mathcal{H} = \mathscr{X}[-1] \bigcap \mathscr{Y}[1] = 0$. 3. If $\mathcal{C}$ has non-zero maximal rigid objects which are not cluster tilting, then the hearts are determined in the following result combining Proposition 5.1:

Corollary 5.2. *1. The heart of any cotorsion pair in $\mathcal{A}_{n,t}$ is module category over the algebras given by one of the following quivers with relations:*

(1) $1 \longrightarrow 2 \longrightarrow \cdots\cdots k-1 \longrightarrow k$ *with* $1 \leq k \leq n$.

(2) $1 \longrightarrow 2 \longrightarrow \cdots\cdots k-1 \longrightarrow k \circlearrowright \alpha$ *with relation* α^2, $1 \leq k \leq n$.

(3) Mutations of the quiver occurred in the above (1) or (2).

2. The heart of any cotorsion pair in $\mathcal{D}_{n,t}$ is module category over the algebras given by one of the following quivers with relations:

(1) $1 \longrightarrow 2 \longrightarrow \cdots\cdots k-1 \longrightarrow k$ *with* $1 \leq k \leq n$.

(2) $1 \longrightarrow 2 \longrightarrow \cdots\cdots k-1 \longrightarrow k \circlearrowright \alpha$ *with relation* α^2, $1 \leq k \leq n$.

(3) Mutations of the quiver occurred in the above (1) or (2).

3. The heart of any cotorsion pair in $\mathcal{D}^b(\mathbb{K}E_7)/\tau^2$ is module category over the algebras given by the following quiver with relation:

$$\cdot \circlearrowright \alpha$$

with relation α^3.

4. The heart of any cotorsion pair in $\mathcal{D}^b(\mathbb{K}E_7)/\tau^5$ is module category over the algebras given by one of the following quivers with relations::

(1) $\cdot \circlearrowright \alpha$ *with relation* α^2.

(2) $\alpha \circlearrowleft \cdot \xrightarrow{\beta} \cdot \circlearrowright \gamma$ *with relations* $\beta\alpha - \gamma\beta$, α^2, γ^2.

Proof. By Proposition 5.1, for any cotorsion pair $(\mathscr{X}, \mathscr{Y})$ with core $\mathcal{I}$ in a finite 2-CY triangulated category, the heart $\underline{H} \simeq mod\ EndI$, I is a rigid object in $\mathcal{C}$. So the heart is determined by the endomorphism algebra of some rigid objects, a subalgebra of endomorphism algebra of a maximal rigid object. Since $[(1,3)] \bigoplus [(1,4)] \bigoplus \ldots \bigoplus [(1, n+2)]$ is a maximal rigid object in $\mathcal{A}_{n,t}$, and its endomorphism algebra is given by $1 \longrightarrow 2 \longrightarrow 3 \cdots\cdots n-1 \longrightarrow n \circlearrowright \alpha$ with relation α^2 by [BPR], we can get any endomorphism algebra of a maximal rigid object through some mutations, as a result, the endomorphism algebra of any rigid objects is obtained. We prove the assertion in 1.

Statement in 2. can be proved similarly. $\mathcal{D}^b(\mathbb{K}E_7)/\tau^2$ has only two indecomposable rigid objects and each is maximal, the endomorphism algebra is given in Proposition 2.14 in [BPR], so the result in

3. is clear. For 4., $\mathcal{D}^b(\mathbb{K}E_7)/\tau^5$ has five indecomposable rigid objects, the endomorphism algebra of any maximal rigid object is given in Proposition 2.12 in [BPR]. $\qquad\square$

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