# $T$-structures and torsion pairs in a $2-$Calabi-Yau triangulated category[1]

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### Abstract

For a $d-$Calabi-Yau triangulated category $C$ with a $d-$cluster tilting subcategory $\mathcal{T}$, the decomposition of $C$ is determined by the decomposition of $\mathcal{T}$ satisfying "vanishing condition" of negative extension groups, namely, $C = \oplus_{i \in I} C_i$ where $C_i, i \in I$ are triangulated subcategories, if and only if $\mathcal{T} = \oplus_{i \in I} \mathcal{T}_i$ where $\mathcal{T}_i, i \in I$ are subcategories with $\mathrm{Hom}_C(\mathcal{T}_i[t], \mathcal{T}_j) = 0, \forall\, 0 \leq t \leq d - 2$ and $i \neq j$. This induces that for any two cluster tilting objects $T, T'$ in a $2-$Calabi-Yau triangulated category $C$, the Gabriel quiver of the endomorphism algebra $\mathrm{End}_C T$ is connected if and only if so is that of $\mathrm{End}_C T'$. As an application, we prove that indecomposable $2-$Calabi-Yau triangulated categories with cluster tilting objects have no non-trivial t-structures and no non-trivial co-t-structures. This allows us to give a classification of torsion pairs in those triangulated categories, and to determine further the hearts of torsion pairs in the sense of Nakaoka, which are equivalent to the module categories over the endomorphism algebras of the cores of the torsion pairs. We also discuss the relation between mutations of torsion pairs and mutations of cluster tilting objects.

**Key words.** Calabi-Yau triangulated category; $d-$cluster tilting subcategory; (co)torsion pair; t-structure; mutation of cotorsion pair, heart.

**Mathematics Subject Classification.** 16E99; 16D90; 18E30

## 1 Introduction

Torsion pairs (equivalently, cotorsion pairs) give a way to construct the whole categories from certain special subcategories. They are important in the study of triangulated categories and abelian categories. We recall the definition here. Let $\mathscr{X}, \mathscr{Y}$ be (additive) subcategories in a triangulated category $C$ with shift functor $[1]$. The pair $(\mathscr{X}, \mathscr{Y})$ is called a torsion pair in $C$ provided the following conditions are satisfied:

1. $\mathrm{Hom}(X, Y) = 0$ for any $X \in \mathscr{X}, Y \in \mathscr{Y}$; and

2. for any $C \in C$, there is a triangle $X \to C \to Y \to X[1]$ with $X \in \mathscr{X}, Y \in \mathscr{Y}$.

This notion introduced by Iyama-Yoshino [IY], see also [KR], is the triangulated version of the notion with the same name in abelian categories introduced by Dickson [Di] (see the introduction to [ASS] for further details). The notion of torsion pairs unifies the notion of t-structures in the sense of [BBD], co-t-structures in the sense of Pauksztello [P] (see [Bon] for another name "weight structure"), and the notion of cluster tilting subcategories (objects) in the sense of Keller-Reiten [KR], see also [BMRRT].

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Torsion pairs are important in the study of the algebraic structure and geometric structure of triangulated categories. Iyama and Yoshino [IY] use them to study the mutation of cluster tilting subcategories in triangulated categories, see also [KR, BR]. Nakaoka [N] uses them to unify the construction of abelian categories appearing as quotients of triangulated categories by cluster tilting subcategories [BMR, KR, KZ], and the construction of abelian categories as hearts of t-structures [BBD]. There is a relation between t-structures and stability conditions in triangulated categories, see [Bri] for details. As one of important special cases, cluster tilting objects (or subcategories) appeared naturally in the study on the categorification of cluster algebras [BMRRT]. They have many nice algebraic properties and combinatorial properties which have been used in the categorification of cluster algebras and studied in recent years (see the surveys [K2, Re] and the references therein). In this categorification, the cluster tilting objects in the cluster category of an acyclic quiver (or more general a quiver with potential) correspond to the clusters of the corresponding cluster algebra.

Cluster tilting subcategories in triangulated categories are the torsion classes of some special torsion pairs. A triangulated category (even a $2-$Calabi-Yau triangulated category) may not admit any cluster tilting subcategory [KZ, BIKR]. In contrast, it always admits torsion pairs, for example, the trivial torsion pair: (the whole category, the zero category). In a triangulated category $C$ with shift functor [1], when $(\mathscr{X}, \mathscr{Y})$ is a torsion pair, we call the pair $(\mathscr{X}, \mathscr{Y}[-1])$ a cotorsion pair, and call the subcategory $\mathscr{X} \cap \mathscr{Y}[-1]$ the core (denoted by $\mathcal{I}$) of this cotorsion pair (and of the corresponding torsion pair $(\mathscr{X}, \mathscr{Y})$). It follows that $(\mathscr{X}, \mathscr{Y})$ is a cotorsion pair in $C$ if and only if $(\mathscr{X}, \mathscr{Y}[1])$ is a torsion pair.

Recently there are several works on the classification of torsion pairs (or equivalently, cotorsion pairs) in a $2-$Calabi-Yau triangulated category. Ng gives a classification of torsion pairs in the cluster category of type $A_\infty$ [Ng] by Ptolemy diagrams of an $\infty-$gon $P_\infty$. Holm-Jørgensen-Rubery [HJR1] gives a classification of torsion pairs in the cluster category $C_{A_n}$ of type $A_n$ via Ptolemy diagrams of a regular $(n + 3)-$gon $P_{n+3}$. They also do the same thing for cluster tubes [HJR2]. In [ZZ2], we define the mutations of torsion pairs to produce new torsion pairs by generalizing the mutation of cluster tilting subcategories [IY], and show that the mutation of torsion pairs has the geometric meaning when the categories have a geometric model. In [ZZZ], together with Zhang, we give the classification of (co)torsion pairs in the (generalized) cluster categories associated with marked Riemann surfaces without punctures. For classification of torsion pairs in an abelian category, we refer to the recent work of Baur-Buan-Marsh [BBM].

In this paper, we show that an indecomposable $2-$Calabi-Yau triangulated category $C$ with a cluster tilting object has only trivial t-structures, i.e. $(C, 0)$, or $(0, C)$. For this, we prove the fact that the decomposition of $C$ is determined by the decomposition of cluster tilting subcategories. This decomposition result holds for arbitrary $d-$Calabi-Yau triangulated categories, where $d > 1$ is an integer. As an application of the result on t-structures, we give a classification of (co)torsion pairs in $C$ and determine the hearts of (co)torsion pairs in the sense of Nakaoka [N], which are equivalent to the module categories over the endomorphism algebras of their cores. We also discuss the relation between mutation of (co)torsion pairs [ZZ2] and mutation of cluster tilting objects.

This paper is organized as follows: In Section 2, some basic definitions and results on (co)torsion pairs are recalled. In Section 3, the definition of decomposition of triangulated categories is recalled. The decomposition of $d-$cluster tilting categories is defined, which is not only the decomposition of additive categories, but also with some additional vanish condition on negative extension groups; and for $d = 2$, this condition is empty. An example is given to explain in

general the decomposition of triangulated categories is not determined by that of cluster tilting subcategories. It is proved that for any $d-$Calabi-Yau triangulated category, its decomposition is determined by the decomposition of a $d-$cluster tilting subcategory. In Section 4, the first main result is that the indecomposable $2-$Calabi-Yau triangulated categories with cluster tilting objects have no non-trivial t-structures (Theorem 4.1). This allows us to give a classification of (co)torsion pairs in these categories (Theorem 4.4), which is the second main result in this section. In Section 5, we discuss the relation between mutation of cotorsion pairs and mutation of cluster tilting objects. For any cotorsion pair $(\mathscr{X}, \mathscr{Y})$ with core $I$, any basic cluster tilting object $T$ containing $I$ as a direct summand can be written uniquely as $T = T_{\mathscr{X}} \oplus I \oplus T_{\mathscr{Y}}$ such that $T_{\mathscr{X}} \oplus I$ (or $T_{\mathscr{Y}} \oplus I$) is cluster tilting in $\mathscr{X}$ ($\mathscr{Y}$ respectively), which we shall define in this section, and any triple $(M, I, N)$ of objects $M, I, N$ in $C$ with the property above gives a cluster tilting object $M \oplus I \oplus N$ containing $I$ as a direct summand in $C$. The mutation of such $T$ in the indecomposable object $T_0$ can be made inside $T_{\mathscr{X}} \oplus I$ or $T_{\mathscr{Y}} \oplus I$, depending on that $T_0$ is a direct summand of $T_{\mathscr{X}}$ or $T_{\mathscr{Y}}$ respectively, if $T_0$ is not a direct summand of $I$. If $T_0$ is a direct summand of $I$, then the mutation $T'$ of $T$ in $T_0$ is the cluster tilting object which can be written as $T' = T'_{\mathscr{X}'} \oplus I' \oplus T'_{\mathscr{Y}'}$, where $(\mathscr{X}', \mathscr{Y}')$ is the mutation of $(\mathscr{X}, \mathscr{Y})$ and $I'$ is the core of $(\mathscr{X}', \mathscr{Y}')$. In the final section, for any cotorsion pair $(\mathscr{X}, \mathscr{Y})$ with core $I$ in a $2-$Calabi-Yau triangulated category $C$ with cluster tilting objects, we prove that the heart $\underline{\mathcal{H}}$ of $(\mathscr{X}, \mathscr{Y})$, defined by Nakaoka, is equivalent to the module category over the endomorphism algebra End$I$.

## 2   Preliminaries

Throughout this paper, $k$ denotes a field. When we say that $C$ is a triangulated category, we always assume that $C$ is a Hom-finite Krull-Schmidt $k-$linear triangulated category over $k$. Denote by [1] the shift functor of $C$, and by [-1] the inverse of [1]. By a subcategory $\mathcal{D}$, we mean that $\mathcal{D}$ is a full subcategory of $C$ which is closed under isomorphisms, finite direct sums and direct summands. In this sense, $\mathcal{D}$ is determined by the set of indecomposable objects in it. By $X \in C$, we mean that $X$ is an object of $C$. We denote by add$X$ the additive closure generated by the object $X$, which is a subcategory of $C$. Sometimes, we identify an object $I$ with the subcategory add$I$. Moreover, if a subcategory $\mathcal{D}$ is closed under [1], [-1] and extensions, then $\mathcal{D}$ is a triangulated subcategory of $C$ (in fact it is a thick subcategory). We call that a triangulated category $C$ has a Serre functor provided there is an equivalent functor $S$ such that $\mathrm{Hom}_C(X, Y) \cong D\mathrm{Hom}_C(Y, SX)$, which are functorially in both variables, where $D = \mathrm{Hom}_k(-, k)$. If the Serre functor is $[d]$, an integer, $C$ is called a $d-$Calabi-Yau ($d-$CY, for short) triangulated category. We always use $\mathrm{Hom}(X, Y)$ to denote Hom-space of objects $X, Y$ in $C$. We denote by $\mathrm{Ext}^n(X, Y)$ the space $\mathrm{Hom}(X, Y[n])$.
For a subcategory $\mathscr{X}$ of $C$, denoted by $\mathscr{X} \subset C$, let

$$\mathscr{X}^{\perp} = \{Y \in C \mid \mathrm{Hom}(X, Y) = 0 \text{ for any } X \in \mathscr{X}\}$$

and

$$^{\perp}\mathscr{X} = \{Y \in C \mid \mathrm{Hom}(Y, X) = 0 \text{ for any } X \in \mathscr{X}\}.$$

For two subcategories $\mathscr{X}, \mathscr{Y}$, by $\mathrm{Hom}(\mathscr{X}, \mathscr{Y}) = 0$, we mean that $\mathrm{Hom}(X, Y) = 0$ for any $X \in \mathscr{X}$ and any $Y \in \mathscr{Y}$. Similar for the notation $\mathrm{Ext}^n(\mathscr{X}, \mathscr{Y}) = 0$. A subcategory $\mathscr{X}$ of $C$ is said to be a rigid subcategory if $\mathrm{Ext}^1(\mathscr{X}, \mathscr{X}) = 0$. Let

$$\mathscr{X} * \mathscr{Y} = \{Z \in C \mid \exists \text{ a triangle } X \to Z \to Y \to X[1] \text{ in } C \text{ with } X \in \mathscr{X}, Y \in \mathscr{Y}\}.$$

3

It is easy to see that $\mathscr{X} * \mathscr{Y}$ is closed under taking isomorphisms and finite direct sums. A subcategory $\mathscr{X}$ is said to be closed under extensions (or an extension-closed subcategory) if $\mathscr{X} * \mathscr{X} \subset \mathscr{X}$. Note that $\mathscr{X} * \mathscr{Y}$ is closed under taking direct summands if $\text{Hom}(\mathscr{X}, \mathscr{Y}) = 0$ (Proposition 2.1(1) in [IY]). Therefore, $\mathscr{X} * \mathscr{Y}$ can be understood as a subcategory of $C$ in this case.

We recall the definition of (co)torsion pairs in a triangulated category $C$ from [IY, N].

**Definition 2.1.** *Let $\mathscr{X}$ and $\mathscr{Y}$ be subcategories of a triangulated category $C$.*

1. *A pair $(\mathscr{X}, \mathscr{Y})$ of subcategories of $C$ is called a torsion pair if $\text{Hom}(\mathscr{X}, \mathscr{Y}) = 0$ and $C = \mathscr{X} * \mathscr{Y}$. The subcategory $\mathcal{I} = \mathscr{X} \bigcap \mathscr{Y}[-1]$ is called the core of the torsion pair.*

2. *The pair $(\mathscr{X}, \mathscr{Y})$ is called a cotorsion pair if*

$$Ext^1(\mathscr{X}, \mathscr{Y}) = 0 \text{ and } C = \mathscr{X} * \mathscr{Y}[1].$$

   *The subcategory $\mathcal{I} = \mathscr{X} \bigcap \mathscr{Y}$ is called the core of the cotorsion pair $(\mathscr{X}, \mathscr{Y})$.*

3. *A t-structure $(\mathscr{X}, \mathscr{Y})$ in $C$ is a cotorsion pair such that $\mathscr{X}$ is closed under $[1]$ (equivalently $\mathscr{Y}$ is closed under $[-1]$). In this case $\mathscr{X} \bigcap \mathscr{Y}[2]$ is an abelian category, which is called the heart of $(\mathscr{X}, \mathscr{Y})$ [BBD, BR]. A t-structure $(\mathscr{X}, \mathscr{Y})$ is called stable if $\mathscr{X}$ is closed under $[-1]$ (equivalently $\mathscr{Y}$ is closed under $[1]$).*

4. *A co-t-structure $(\mathscr{X}, \mathscr{Y})$ in $C$ is a cotorsion pair such that $\mathscr{X}$ is closed under $[-1]$ (equivalently $\mathscr{Y}$ is closed under $[1]$) [Bon, P].*

5. *The subcategory $\mathscr{X}$ is said to be a cluster tilting subcategory if $(\mathscr{X}, \mathscr{X})$ is a cotorsion pair [KR, KZ, IY]. We say that an object $T$ is a cluster tilting object if $\text{add}T$ is a cluster tilting subcategory.*

**Remark 2.2.** *A pair $(\mathscr{X}, \mathscr{Y})$ is a cotorsion pair if and only if $(\mathscr{X}, \mathscr{Y}[1])$ is a torsion pair. In any case, the core $\mathcal{I}$ is a rigid subcategory of $C$.*

**Remark 2.3.** *$(C, 0)$ and $(0, C)$ are t-structures in $C$, which are called trivial t-structures. They are also co-t-structures and are called trivial co-t-structures in $C$.*

**Lemma 2.4.** *[ZZ1] Let $(\mathscr{X}, \mathscr{Y})$ be a cotorsion pair in $C$ with core $\mathcal{I}$. Then*

1. *$(\mathscr{X}, \mathscr{Y})$ is a t-structure if and only if $\mathcal{I} = 0$*

2. *$\mathscr{X}$ is a rigid subcategory if and only if $\mathscr{X} = \mathcal{I}$*

3. *$\mathscr{X}$ is a cluster tilting subcategory if and only if $\mathscr{X} = \mathcal{I} = \mathscr{Y}$.*

Recall that a subcategory $\mathscr{X}$ is contravariantly finite in $C$, if any object $M \in C$ admits a right $\mathscr{X}$−approximation $f : X \to M$, which means that any map from $X' \in \mathscr{X}$ to $M$ factors through $f$. The left $\mathscr{X}$−approximation of $M$ and covariantly finiteness of $\mathscr{X}$ can be defined dually. $\mathscr{X}$ is called functorially finite in $C$ if $\mathscr{X}$ is both covariantly finite and contravariantly finite in $C$. Note that if $(\mathscr{X}, \mathscr{Y})$ is a cotorsion pair, then $\mathscr{X} = {}^\perp(\mathscr{Y}[1])$, $\mathscr{Y} = (\mathscr{X}[-1])^\perp$, and it follows that $\mathscr{X}$ (or $\mathscr{Y}$) is a contravariantly (covariantly, respectively) finite and extension-closed subcategory of $C$. We note that $\text{add}X$ is functorially finite in $C$ for any object $X \in C$.

4

Let $(\mathscr{X}, \mathscr{Y})$ be a cotorsion pair with core $\mathcal{I}$ in a triangulated category $C$. Denote by $\mathcal{H}$ the subcategory $(\mathscr{X}[-1] * \mathcal{I}) \bigcap (\mathcal{I} * \mathscr{Y}[1])$. The image of $\mathcal{H}$ under the natural projection $C \to C/\mathcal{I}$, which denoted by $\underline{\mathcal{H}}$, is called the heart of the cotorsion pair $(\mathscr{X}, \mathscr{Y})$. It is proved by Nakaoka that the heart $\underline{\mathcal{H}}$ is an abelian category, see [N] for more detailed construction.

# 3 Decompositions of Calabi-Yau triangulated categories

In this section, we discuss how the decomposition of a triangulated category is determined by that of a cluster tilting subcategory. We recall the definition of $d-$cluster tilting subcategories from [KR, IY] in the following:

**Definition 3.1.** *Let $C$ be a triangulated category, $d > 1$, an integer.*

1. *A subcategory $\mathcal{T}$ of $C$ is called $d-$rigid provided $Ext^i(\mathcal{T}, \mathcal{T}) = 0$ for all $1 \leq i \leq d-1$.*

2. *A $d-$ rigid subcategory $\mathcal{T}$ is called $d-$cluster tilting provided that $\mathcal{T}$ is functorially finite, and satisfies the property: $T \in \mathcal{T}$ if and only if $Ext^i(\mathcal{T}, T) = 0$ for all $1 \leq i \leq d-1$ if and only if $Ext^i(T, \mathcal{T}) = 0$ for all $1 \leq i \leq d-1$.*

3. *An object $T$ is called a $d-$cluster tilting (respectively $d-$rigid) object if $addT$ is $d-$cluster tilting (respectively $d-$rigid).*

The main examples of $d-$cluster tilting subcategories are $d-$cluster tilting subcategories in $d-$cluster categories (see [IY, T, Zhu]). Other examples can be found in [K1, BIKR]. Note that when $d = 2$, the $d-$cluster tilting subcategories (or $d-$cluster tilting objects) are called cluster tilting subcategories (cluster tilting objects respectively).

**Definition 3.2.** *Let $C$ be a triangulated category, and $C_i, i \in I$ be triangulated subcategories of $C$. We call that $C$ is a direct sum of triangulated subcategories $C_i, i \in I$, provided that*

1. *Any object $M \in C$ is a direct sum of finitely many objects $M_i \in C_i$;*

2. *$Hom(C_i, C_j) = 0, \forall i \neq j$.*

*In this case, we write $C = \oplus_{i \in I} C_i$. We say $C$ is indecomposable if $C$ cannot be written as a direct sum of two nonzero triangulated subcategories.*

**Definition 3.3.** *Let $\mathcal{T}$ be a $d-$cluster tilting subcategory of a triangulated category $C$, and $\mathcal{T}_i, i \in I$, be subcategories of $\mathcal{T}$. We call that $\mathcal{T}$ is a direct sum of subcategories $\mathcal{T}_i, i \in I$, provided that*

1. *Any object $T \in \mathcal{T}$ is a direct sum of finitely many objects $T_i \in \mathcal{T}_i$;*

2. *$Hom(\mathcal{T}_i, \mathcal{T}_j) = 0, \forall i \neq j$;*

3. *$Hom(\mathcal{T}_i[k], \mathcal{T}_j) = 0, \forall i \neq j, 1 \leq k \leq d-2$;*

*In this case, we write $\mathcal{T} = \oplus_{i \in I} \mathcal{T}_i$. We say $\mathcal{T}$ is indecomposable if $\mathcal{T}$ cannot be written as a direct sum of two nonzero subcategories.*

**Remark 3.4.** *When $d = 2$, the third condition is empty.*

The following example shows that there are indecomposable $d-$CY triangulated categories admitting $d-$cluster tilting subcategories, those cluster tilting subcategories can be decomposed as sum of subcategories satisfying the conditions $1, 2$, but not $3$ in Definition 3.3.

**Example 1.** *Let $Q : 3 \to 2 \to 1$ be the quiver of type $A_3$ with linear orientation, and $C$ be the $4-$cluster category of $Q$, i.e. $C = D^b(kQ)/\tau^{-1}[3]$ (compare [K1]). Let $P_1, P_2, P_3$ be the indecomposable projective modules associated to the vertices of $Q$, and $S_1, S_2, S_3$ the corresponding simple modules. Then $T = P_1 \oplus P_2 \oplus P_3$ is a $4-$cluster tilting object, $P_1 \oplus P_3$ is an almost complete $4-$cluster tilting object, it has $4$ complements (compare [Zhu, T]), one is $P_2$, the others are $S_3, S_3[1]$, and $S_3[2]$. Denote by $\mathcal{T} = add(P_1 \oplus P_3 \oplus S_3[1])$, which is a $4-$cluster tilting subcategory of $C$. Set $\mathcal{T}_1 = add(P_1 \oplus P_3)$, $\mathcal{T}_2 = addS_3[1]$. Both are subcategories of $\mathcal{T}$. It is easy to see that $\mathcal{T}, \mathcal{T}_1, \mathcal{T}_2$ satisfy the first two conditions of Definition 3.3, but not the third one: an easy computation shows $Hom(P_3[1], S_3[1]) \neq 0$. We note that this $4-$cluster category $C$ is indecomposable.*

We will discuss the relation between the decomposition of triangulated categories and the decomposition of $d-$cluster tilting subcategories. Firstly we look at two examples:

**Example 2.** *Let $Q$ be a connected quiver without oriented cycles, and $C = D^b(kQ)$ the bounded derived category of $kQ$. It is an indecomposable triangulated category. We know $\mathcal{T} = add\{\tau^n[-n]kQ \mid n \in \mathbf{Z}\}$ is a cluster tilting subcategory containing infinitely many indecomposable objects in $C$. Let $\mathcal{T}_i = add\{\tau^i[-i]kQ\}$ for $i \in \mathbf{Z}$. It is easy to check that $\mathcal{T} = \oplus_{i \in \mathbf{Z}} \mathcal{T}_i$.*

**Example 3.** *Let $Q$ be a connected quiver without oriented cycles, $F = \tau^{-1}[1]$ an automorphism of the derived category $D^b(kQ)$. The repetitive cluster category of $Q$ is defined for any positive integer m, namely, the orbit triangulated category $C = D^b(kQ)/(F^m)$ [K1]. It is an indecomposable triangulated category. Let $m = 2$. Then $kQ \oplus F(kQ)$ is a cluster tilting object in $C$. Let $\mathcal{T} = add(kQ \oplus F(kQ))$, $\mathcal{T}_1 = add(kQ)$, $\mathcal{T}_2 = add(F(kQ))$. Then $\mathcal{T}$ is a cluster tilting subcategory and $\mathcal{T} = \mathcal{T}_1 \oplus \mathcal{T}_2$.*

The two examples above show that in general the indecomposable triangulated category may admit a decomposable $d-$cluster tilting subcategory. In the following, we will prove that the decomposition of $d-$CY triangulated categories is determined by the decomposition of $d-$cluster tilting subcategories. Recall that a $k-$linear triangulated category $C$ is $d-$CY if $[d]$ is the Serre functor.

**Proposition 3.5.** *Let $C$ be a $d-$CY triangulated category with a $d-$cluster tilting subcategory $\mathcal{T}$. Suppose that $\mathcal{T} = \oplus_{i \in I} \mathcal{T}_i$ with $\mathcal{T}_i s$ nonzero subcategories. Then $\mathcal{T}_i * \mathcal{T}_i[1] * \cdots * \mathcal{T}_i[d-1]$, $i \in I$ are triangulated subcategory of $C$ and $C = \oplus_{i \in I}(\mathcal{T}_i * \mathcal{T}_i[1] * \cdots * \mathcal{T}_i[d-1])$.*

Note that by Proposition 2.1 [IY], $\mathcal{T}_i * \mathcal{T}_i[1] * \cdots * \mathcal{T}_i[d-1], i \in I$, are closed under direct summands, so they are subcategories of $C$. We divide our proof into several steps:

**Lemma 3.6.** *Under the same assumption as in Proposition 3.5, every object $X$ in $C$ has a decomposition $X = \oplus_{i \in I} X_i$ with finite many nonzero $X_i \in \mathcal{T}_i * \mathcal{T}_i[1] * \cdots * \mathcal{T}_i[d-1]$, $i \in I$. In particular, every indecomposable object of $C$ lies in some $\mathcal{T}_i * \mathcal{T}_i[1] * \cdots * \mathcal{T}_i[d-1], i \in I$.*

*Proof.* Since $\mathcal{T} = \oplus_{i \in I} \mathcal{T}_i$ is a $d-$cluster tilting subcategory, by Corollary 3.3 in [IY], for each indecomposable object $X$ in $C$, there are $d$ triangles:

$$X^{(n)} \xrightarrow{f^{(n)}} \oplus_{i \in J} B_i^{(n-1)} \to X^{(n-1)} \to X^{(n)}[1], \ n = 1, \cdots, d,$$

6

where $J$ is a finite subset of $I$, $B_i^{(n-1)} \in \mathcal{T}_i$, $X^{(0)} = X$ and $X^{(d)} = 0$. Then $X^{(d-1)} \cong \oplus_{i \in J} B_i^{(d-1)}$. We want to prove that $X \cong \oplus_{i \in J} X_i$ with $X_i \in \mathcal{T}_i * \mathcal{T}_i[1] * \cdots * \mathcal{T}_i[d-1]$.

Assume that $X^{(n)} \cong \oplus_{i \in J} X_i^{(n)}$ with $X_i^{(n)} \in \mathcal{T}_i * \mathcal{T}_i[1] * \cdots * \mathcal{T}_i[d-1-n]$ for some $1 \le n \le d-1$. By Definition 3.3, $\text{Hom}(\mathcal{T}_i[k], \mathcal{T}_j) = 0$ for $i \ne j$, $0 \le k \le d-1-n \le d-2$, then $\text{Hom}(X_i^{(n)}, \mathcal{T}_j) = 0$ for $j \ne i$. So $f^{(n)}$ is a diagonal map, say $\begin{pmatrix} f_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & f_{|J|} \end{pmatrix}$, where $f_i : X_i^{(n)} \to B_i^{(n-1)}$. Extend each $f_i$ to triangle:

$$X_i^{(n)} \xrightarrow{f_i} B_i^{(n-1)} \to X_i^{(n-1)} \to X_i^{(n)}[1].$$

Then we have that $X^{(n-1)} \cong \oplus_{i \in J} X_i^{(n-1)}$ and $X_i^{(n-1)} \in \mathcal{T}_i * \mathcal{T}_i[1] * \cdots * \mathcal{T}_i[d-n]$, $i \in J$. By induction on $n$ (from $d-1$ to 0), $X = X^{(0)} \cong \oplus_{i \in J} X_i^{(0)}$ with $X_i^{(0)} \in \mathcal{T}_i * \mathcal{T}_i[1] * \cdots * \mathcal{T}_i[d-1]$. $\square$

**Lemma 3.7.** *Under the same assumption as in Proposition 3.5, $\mathcal{T}_i * \mathcal{T}_i[1] * \cdots * \mathcal{T}_i[d-1] = \bigcap_{j \ne i} \bigcap_{k=1}^{2d-2} {}^\perp \mathcal{T}_j[k]$ holds for any $i \in I$.*

*Proof.* By Definition 3.3, $\text{Hom}(\mathcal{T}_i, \mathcal{T}_j[l]) = 0$ and $\text{Hom}(\mathcal{T}_j, \mathcal{T}_i[l]) = 0$, for $-(d-2) \le l \le d-1$, $i \ne j$. Then for $0 \le m \le d-1$, $1 \le k \le d-1$, we have that $\text{Hom}(\mathcal{T}_i[m], \mathcal{T}_j[k]) \cong \text{Hom}(\mathcal{T}_i, \mathcal{T}_j[k-m]) = 0$ due to $-(d-2) \le k-m \le d-1$, and $\text{Hom}(\mathcal{T}_i[m], \mathcal{T}_j[d+k-1]) \cong D\text{Hom}(\mathcal{T}_j[k-1], \mathcal{T}_i[m]) = 0$ as $-(d-2) \le m-k+1 \le d-1$. So $\text{Hom}(\mathcal{T}_i * \mathcal{T}_i[1] \cdots * \mathcal{T}_i[d-1], \mathcal{T}_j[k]) = 0$ for $1 \le k \le 2d-2$, $i \ne j$. That is, $\mathcal{T}_i * \mathcal{T}_i[1] * \cdots * \mathcal{T}_i[d-1] \subset \bigcap_{j \ne i} \bigcap_{k=1}^{2d-2} {}^\perp \mathcal{T}_j[k]$.

Fix an element $i \in I$. Let $X$ be an object satisfying $\text{Hom}(X, \mathcal{T}_j[k]) = 0$ for $1 \le k \le 2d-2$, $j \ne i$. By Lemma 3.6, $X$ has a decomposition $X = \oplus_{j \in J} X_j$, $X_j \in \mathcal{T}_j * \mathcal{T}_j[1] * \cdots * \mathcal{T}_j[d-1]$, for some finite subset $J$ of $I$. Then there is a triangle

$$M \to X_j \to N \to M[1]$$

with $M \in \mathcal{T}_j$, $N \in \mathcal{T}_j[1] * \cdots * \mathcal{T}_j[d-1]$. If $j \ne i$, we have that $\text{Hom}(M, X_j) = 0$ by $\text{Hom}(\mathcal{T}_j, X) \cong D\text{Hom}(X, \mathcal{T}_j[d]) = 0$ and that $\text{Hom}(X_j, N) = 0$ by $\text{Hom}(X, \mathcal{T}_j[1] * \cdots * \mathcal{T}_j[d-1]) = 0$. Therefore $X_j \cong 0$. Hence $X \cong X_i \in \mathcal{T}_i * \mathcal{T}_i[1] * \cdots * \mathcal{T}_i[d-1]$. $\square$

**Lemma 3.8.** *Under the same assumption as in Proposition 3.5, all $\mathcal{T}_i * \mathcal{T}_i[1] * \cdots * \mathcal{T}_i[d-1]$, $i \in I$, are triangulated subcategories of $C$.*

*Proof.* Let $X \to Z \to Y \to X[1]$ be a triangle with $X, Y \in \mathcal{T}_i * \mathcal{T}_i[1] * \cdots * \mathcal{T}_i[d-1]$. By Lemma 3.7, we have that for $1 \le k \le 2d-2$, $j \ne i$, $\text{Hom}(X, \mathcal{T}_j[k]) = 0$ and $\text{Hom}(Y, \mathcal{T}_j[k]) = 0$, and then $\text{Hom}(Z, \mathcal{T}_j[k]) = 0$. By Lemma 3.7 again, we have that $Z \in \mathcal{T}_i * \mathcal{T}_i[1] * \cdots * \mathcal{T}_i[d-1]$. Therefore, $\mathcal{T}_i * \mathcal{T}_i[1] * \cdots * \mathcal{T}_i[d-1]$ is closed under extensions.

For any $i$, $\mathcal{T}_i, \mathcal{T}_i[1], \cdots, \mathcal{T}_i[d-1]$ are included in $\mathcal{T}_i * \mathcal{T}_i[1] * \cdots * \mathcal{T}_i[d-1]$. We claim that $\mathcal{T}_i[d]$ is a subcategory of $\mathcal{T}_i * \mathcal{T}_i[1] * \cdots * \mathcal{T}_i[d-1]$. Otherwise, there is an indecomposable object of $\mathcal{T}_i$, say $X$, such that $X[d]$ is not an object of $\mathcal{T}_i * \mathcal{T}_i[1] * \cdots * \mathcal{T}_i[d-1]$. Then by Lemma 3.6, $X[d]$ is in $\mathcal{T}_j * \mathcal{T}_j[1] * \cdots * \mathcal{T}_j[d-1]$ for some $j \ne i$. Note that $\text{Hom}(X, X[d]) \cong D\text{Hom}(X, X) \ne 0$ which contradicts with $\text{Hom}(\mathcal{T}_i, \mathcal{T}_j * \mathcal{T}_j[1] * \cdots * \mathcal{T}_j[d-1]) = 0$ by Lemma 3.7 and $d-$CY property. Thus, we have proved that $\mathcal{T}_i[d]$ is included in $\mathcal{T}_i * \mathcal{T}_i[1] * \cdots * \mathcal{T}_i[d-1]$. Hence

$(\mathcal{T}_i * \mathcal{T}_i[1] * \cdots * \mathcal{T}_i[d-1])[1] \subset \mathcal{T}_i * \mathcal{T}_i[1] * \cdots * \mathcal{T}_i[d-1]$, that is, $\mathcal{T}_i * \mathcal{T}_i[1] * \cdots * \mathcal{T}_i[d-1]$ is closed under [1]. Dually, one can prove that $\mathcal{T}_i * \mathcal{T}_i[1] * \cdots * \mathcal{T}_i[d-1]$ is closed under [-1]. Therefore, $\mathcal{T}_i * \mathcal{T}_i[1] * \cdots * \mathcal{T}_i[d-1]$ is a triangulated subcategory of $C$. $\square$

**Proof of Proposition 3.5**. It is sufficient to verify that $\text{Hom}(\mathcal{T}_i * \mathcal{T}_i[1] * \cdots * \mathcal{T}_i[d-1], \mathcal{T}_j * \mathcal{T}_j[1] * \cdots * \mathcal{T}_j[d-1]) = 0$, for $i \neq j$. By Lemma 3.8, $(\mathcal{T}_i * \mathcal{T}_i[1] * \cdots * \mathcal{T}_i[d-1])[-1] = \mathcal{T}_i * \mathcal{T}_i[1] * \cdots * \mathcal{T}_i[d-1]$, then $\text{Hom}(\mathcal{T}_i * \mathcal{T}_i[1] * \cdots * \mathcal{T}_i[d-1], \mathcal{T}_j) = \text{Hom}((\mathcal{T}_i * \mathcal{T}_i[1] * \cdots * \mathcal{T}_i[d-1])[-1], \mathcal{T}_j) = \text{Hom}(\mathcal{T}_i * \mathcal{T}_i[1] * \cdots * \mathcal{T}_i[d-1], \mathcal{T}_j[1]) = 0$ for $i \neq j$, where the last equality is due to Lemma 3.7. Then $\text{Hom}(\mathcal{T}_i * \mathcal{T}_i[1] * \cdots * \mathcal{T}_i[d-1], \mathcal{T}_j * \mathcal{T}_j[1] * \cdots * \mathcal{T}_j[d-1]) = 0$.

The following lemma is a generalization of Remark 2.3 in [ZZ1].

**Lemma 3.9.** *Let $C$ be a triangulated category and $\mathcal{T}$ be a $d-$rigid subcategory of $C$ satisfying $C = \mathcal{T} * \mathcal{T}[1] * \cdots * \mathcal{T}[d-1]$. Then $\mathcal{T}$ is a $d-$cluster tilting subcategory of $C$.*

*Proof.* Note that $(\mathcal{T}, \mathcal{T}[1] * \cdots * \mathcal{T}[d-1])$ and $(\mathcal{T} * \cdots * \mathcal{T}[d-2], \mathcal{T}[d-1])$ form two torsion pairs. So $\mathcal{T}$ is contravariantly finite in $C$ and $\mathcal{T}[d]$ is covariantly finite in $C$. Therefore $\mathcal{T}$ is functorially finite in $C$. Take an object $X$ in $C$ with $\text{Hom}(X, \mathcal{T}[t]) = 0$ for $1 \leq t \leq d-1$. Then $\text{Hom}(X, \mathcal{T}[1] * \cdots * \mathcal{T}[d-1]) = 0$. Hence $X \in \mathcal{T}$. Similar proof for $X \in \mathcal{T}$ if $\text{Hom}(\mathcal{T}, X[t]) = 0$ for $1 \leq t \leq d-1$. Hence $\mathcal{T}$ is $d-$cluster tilting in $C$. $\square$
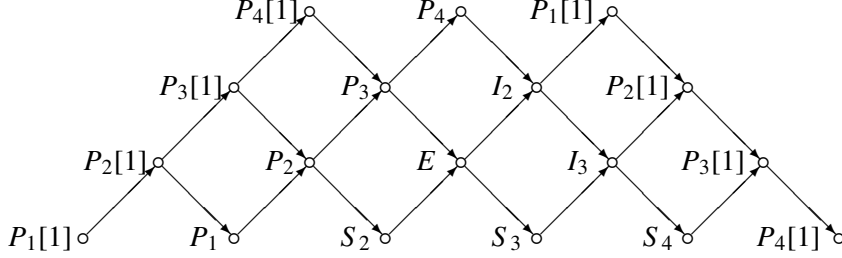
Now we prove our main result in this section.

**Theorem 3.10.** *Let $C$ be a $d-$CY triangulated category with a $d-$cluster tilting subcategory $\mathcal{T}$. Then $C$ is a direct sum of indecomposable triangulated subcategories $C_i$, $i \in I$ if and only if the $d-$cluster tilting subcategory $\mathcal{T}$ is a direct sum of indecomposable subcategories $\mathcal{T}_i, i \in I$. Moreover $C_i = \mathcal{T}_i * \mathcal{T}_i[1] * \cdots * \mathcal{T}_i[d-1]$ and $\mathcal{T}_i$ is a $d-$cluster tilting subcategory in $C_i$, $i \in I$.*

*Proof.* We first show the "only if" part. By the definition of direct sums of triangulated subcategories, any object $T$ in $\mathcal{T}$ has a decomposition $T = \oplus_{i \in J} T_i$ with $J$ a finite subset of $I$, $T_i \in C_i$ and $\text{Hom}(T_i[k], T_j) = 0$ for $0 \leq k \leq d-2$, $i \neq j$. Then $\mathcal{T} = \oplus_{i \in I} \mathcal{T}_i$ where $\mathcal{T}_i = \mathcal{T} \bigcap C_i$. By Definition 3.2, for any object $X \in C_i$, $\text{Hom}(X, \mathcal{T}_j[k]) = 0$ for $j \neq i$ and any $k$. Then by Lemma 3.7, $X \in \mathcal{T}_i * \mathcal{T}_i[1] * \cdots * \mathcal{T}[d-1]$. Hence $C_i = \mathcal{T}_i * \mathcal{T}_i[1] * \cdots * \mathcal{T}[d-1]$. By Lemma 3.9, $\mathcal{T}_i$ is a $d-$cluster tilting subcategory of $C_i$. It follows from the indecomposability of $C_i$ and Proposition 3.5 that $\mathcal{T}_i$ is indecomposable.

To prove the "if" part. Let $C_i = \mathcal{T}_i * \mathcal{T}_i[1] * \cdots * \mathcal{T}_i[d-1]$. It follows from Proposition 3.5 that $C_i$s are triangulated categories and there is a decomposition $C = \oplus_{i \in I} C_i$. By Lemma 3.9, $\mathcal{T}_i$ is a $d-$cluster tilting subcategory in $C_i$. If $C_i$ is not indecomposable, say $C_i = C'_i \oplus C''_i$ with nonzero triangulated subcategories $C'_i, C''_i$, then by the proof of the "only if" part, we have $\mathcal{T}_i = \mathcal{T}'_i \oplus T''_i$, and $C'_i = \mathcal{T}'_i * \mathcal{T}'_i[1] * \cdots * \mathcal{T}'_i[d-1], C''_i = \mathcal{T}''_i * \mathcal{T}''_i[1] * \cdots * \mathcal{T}''_i[d-1]$. It follows that $\mathcal{T}'_i, \mathcal{T}''_i$ are nonzero subcategories, a contradiction to the indecomposability of $\mathcal{T}_i$. The other assertion follows from Lemma 3.9. $\square$

We give a simple example for $d = 2$.

**Example 4.** *Let $Q$ be the quiver:$4 \to 3 \to 2 \to 1$, and $C = C_Q$, the cluster category of $Q$ whose Auslander-Reiten quiver is the following:*

$$P_4[1] \quad P_4 \quad P_1[1]$$
$$P_3[1] \quad P_3 \quad I_2 \quad P_2[1]$$
$$P_2[1] \quad P_2 \quad E \quad I_3 \quad P_3[1]$$
$$P_1[1] \quad P_1 \quad S_2 \quad S_3 \quad S_4 \quad P_4[1]$$

*We take $X = add(E)$, $^{\perp}(X[1]) = add(\{E, P_3, P_4[1], P_4, I_2, P_1[1], S_2, S_3\})$ By [IY], the subquotient category $^{\perp}(X[1])/X = add(\{P_3, P_4[1], P_4, I_2, P_1[1], S_2, S_3\})$ is triangulated, and $2-CY$. This subquotient category admits cluster tilting objects, for example, the object $T = P_4[1] \oplus P_3 \oplus S_3$. We have that in this subquotient category, $addT = add(S_3) \oplus add(P_3 \oplus P_4[1])$. Then by Theorem 3.10, this subquotient category $^{\perp}(X[1])/X = add(\{S_2, S_3\}) \oplus add(\{P_3, P_4[1], P_4, I_2, P_1[1]\})$, in which, the first direct summand is equivalent to the cluster category of type $A_1$, the second one is equivalent to the cluster category of type $A_2$.*

**Corollary 3.11.** *Let $C$ be a $d-CY$ triangulated category admitting a $d-$cluster tilting subcategory $\mathcal{T}$. Then $C$ is indecomposable if and only if $\mathcal{T}$ is indecomposable.*

**Corollary 3.12.** *Let $C$ be a $d-CY$ triangulated category, $\mathcal{T}$ and $\mathcal{T}'$ be two $d-$cluster tilting subcategories. Then $\mathcal{T}$ is indecomposable if and only if $\mathcal{T}'$ is indecomposable.*

**Corollary 3.13.** *Let $C$ be a $d-CY$ triangulated category with a $d-$cluster tilting object $T$. Then $C$ is a direct sum of finitely many indecomposable triangulated subcategories $C_i$, $i = 1, \cdots m$. Moreover the cluster tilting object $T$ is a direct sum of $T_i, i = 1, \cdots, m$ such that $C_i = addT_i * addT_i[1] * \cdots * addT_i[d-1]$ and $T_i$ is $d-$cluster tilting in $C_i$.*

*Proof.* Any triangulated category can be decomposed as a direct sum of triangulated subcategories. For the $d-CY$ triangulated category $C$ with a $d-$cluster tilting object $T$, the number of direct summands of the decomposition of $C$ is finite since that the number of indecomposable direct summands of $T$ is finite. Then we have the decomposition of $C = \oplus_{i=1}^{m} C_i$. The other assertion follows directly from Theorem 3.10.

$\square$

For the special case of $d = 2$, i.e., $C$ is $2-CY$ triangulated category with a cluster tilting object $T$, the decomposition of $C$ corresponds to the partition of connected components of the Gabriel quiver of $End(T)$.

**Definition 3.14.** *A basic rigid object $T$ in $C$ is called connected provided $T$ cannot written as $T = T_1 \oplus T_2$ with property that $T_i \neq 0$, and $Hom(T_i, T_j) = 0$, for $i \neq j \in \{1, 2\}$. Any cluster tilting object in $C$ can be decomposed as a direct sum of connected summands: $T = \oplus_{i=1}^{m} T_i$ with $T_i$ being connected. We call such decomposition a complete decomposition of $T$.*

Every $C$ can be decomposed uniquely to a direct sum of nonzero indecomposable triangulated subcategories. We call this decomposition is the complete decomposition of $C$ and denote by $ns(C)$ the number of indecomposable direct summands of such decomposition of $C$. For a cluster

tilting object $T$ in $C$, the Gabriel quiver of $\mathrm{End}(T)$ is denoted by $\Gamma_T$ and the number of connected components of $\Gamma_T$ is denoted by $nc(\Gamma_T)$. Note that the complete decomposition of $T$ corresponds to the connected components of Gabriel quiver of the $2-$CY tilted algebra $\mathrm{End}(T)$. So by applying the theorem above, we have the following result immediately.

**Corollary 3.15.** *Let $C$ be a $2-$CY triangulated category admitting a cluster tilting object $T$. Then the number $nc(\Gamma_T)$ of connected components of the quiver $\Gamma_T$ is equal to $ns(C)$. In particular, $C$ is indecomposable if and only if $\Gamma_T$ is connected.*

**Corollary 3.16.** *Let $C$ be a $2-$CY triangulated category and let $T, T'$ be cluster tilting objects in $C$. Then $\Gamma_T$ is connected if and only if $\Gamma_{T'}$ is connected.*

*Proof.* $\Gamma_T$ is connected $\Leftrightarrow C$ is indecomposable $\Leftrightarrow \Gamma_{T'}$ is connected. $\qquad\qquad\square$

**Remark 3.17.** *Let $(S, M)$ be a marked surface and $nc(S)$ denote the number of connected components of $S$. Then $nc(S) = ns(C(S, M))$ (compare [ZZ2]).*

# 4  Classification of Cotorsion pairs in 2-Calabi-Yau categories

From now on, except Proposition 4.6, we always suppose that the triangulated category $C$ is $2-$Calabi-Yau ($2-$CY for short), i.e. [2] is the Serre functor of $C$. The main examples of $2-$CY triangulated categories are the followings:

1. Cluster categories of hereditary abelian $k-$categories in the sense of [BMRRT] (also [CCS] for type $A$); and generalized cluster categories of algebras with global dimension at most 2 (including the case of quivers with potentials) in the sense of Amoit [Am]. All these $2-$CY triangulated categories have cluster tilting objects.

2. The stable categories of preprojective algebras of Dynkin quivers. They also have cluster tilting objects [GLS, BIRS].

3. The cluster category of type $A_\infty$. It has cluster tilting subcategories, which contains infinitely many indecomposable objects [KR, HJ, Ng].

4. The bounded derived categories $D^b(mod_{f.l.}\Lambda)$ of modules with finite length over preprojective algebras $\Lambda$ of non-Dynkin quivers. They have no cluster tilting subcategories. There are many stable subcategories of $mod_{f.l.}\Lambda$ associated to elements in the Coxeter groups of the quivers. Their stable categories are $2-$CY, and have cluster tilting objects. See [GLS, BIRS] for details.

5. Stable categories of Cohen-Macaulay modules over three-dimensional complete local commutative noetherian Gorenstein isolated singularity containing the residue field [BIKR].

We shall first decide a special kind of cotorsion pairs: t-structures. Recall that $(\mathscr{X}, \mathscr{Y})$ is a t-structure in $C$, if $\mathrm{Ext}^1(\mathscr{X}, \mathscr{Y}) = 0$, $C = \mathscr{X} * \mathscr{Y}[1]$, $\mathscr{X}[1] \subset \mathscr{X}$ and $\mathscr{Y}[-1] \subset \mathscr{Y}$. The first main result in this section is the following result.

**Theorem 4.1.** *Let $C$ be an indecomposable 2-CY triangulated category with a cluster tilting object $T$. Then $C$ has no non-trivial t-structures, i.e. the t-structures in $C$ are $(C, 0)$ and $(0, C)$.*

*Proof.* Let $(\mathscr{X}, \mathscr{Y})$ be a t-structure in $C$. Put $\mathcal{T} = \text{add}T$. Then for each indecomposable object $T_i \in \mathcal{T}$, $i \in I$, there is a triangle

$$X_i \xrightarrow{f_i} T_i \xrightarrow{g_i} Y_i[1] \xrightarrow{h_i} X_i[1]$$

with $X_i \in \mathscr{X}$, $Y_i \in \mathscr{Y}$. Let $\mathcal{R}$ be the subcategory of $C$ generated additively by $X_i$, $Y_i$, $i \in I$. Then $\mathcal{T} \subset \mathcal{R} * \mathcal{R}[1]$. We shall prove that $\mathcal{R}$ is a cluster tilting subcategory of $C$.

For any map $\alpha \in \text{Hom}(Y_i[1], Y_j[2])$, consider the following diagram:



The composition $-h_j[1] \circ \alpha \in \text{Hom}(Y_i[1], X_j[2]) \cong D\text{Hom}(X_j, Y_i[1]) = 0$, then $\alpha$ factors through $-g_j[1]$. So $\alpha \circ g_i = 0$ due to $\text{Hom}(T_i, T_j[1]) = 0$. Therefore $\alpha$ factors through $h_i$, i.e. there is a morphism $\beta \in \text{Hom}(X_i[1], Y_j[2])$ such that $\alpha = \beta \circ h_i$. But $\text{Hom}(X_i[1], Y_j[2]) = 0$, so $\alpha = 0$. Then $\text{Ext}^1(Y_i, Y_j) = 0$. Dually, we have that $\text{Ext}^1(X_i, X_j) = 0$. By the definition of t-structure and 2-CY property, we also have $\text{Ext}^1(X_i, Y_j) = 0$ and $\text{Ext}^1(Y_i, X_j) = 0$. Hence $\mathcal{R}$ is a rigid subcategory.

Given an object $M$ with $\text{Ext}^1(M, X_i) = 0$, $\text{Ext}^1(M, Y_i) = 0$ for $i \in I$. Since $\mathcal{T}$ is cluster tilting, there is a triangle $M \xrightarrow{w} A \xrightarrow{u} B \xrightarrow{v} M[1]$ with $A, B \in \mathcal{T}$. Since $\mathcal{T} \subset \mathcal{R} * \mathcal{R}[1]$,

$$X_A \xrightarrow{f_A} A \xrightarrow{g_A} Y_A[1] \xrightarrow{h_A} X_A[1],$$

$$X_B \xrightarrow{f_B} B \xrightarrow{g_B} Y_B[1] \xrightarrow{h_B} X_B[1],$$

where $f_A$ (resp. $f_B$) is the minimal right $\mathscr{X}$−approximation of $A$ (resp. $B$) and $g_A$ (resp. $g_B$) is the minimal left $\mathscr{Y}[1]$−approximation of $A$ (resp. $B$). Then the composition $u \circ f_A$ factors through $f_B$, that is, there exists $s$ such that $f_B \circ s = u \circ f_A$.



Due to $\text{Hom}(X_B, M[1]) = 0$, we have $v \circ f_B = 0$, then $f_B$ factors through $u$. Since any morphism from $X_B$ to $A$ factors through $f_A$, then there is a morphism $r \in \text{Hom}(X_B, X_A)$ such that $f_B = u \circ f_A \circ r$. Replace $u \circ f_A$ by $f_B \circ s$, we have $f_B = f_B \circ s \circ r$. Then $s \circ r$ is an isomorphism by the right minimality of $f_B$. Thus $s$ is a retraction and we have the triangle $X_A \xrightarrow{s} X_B \xrightarrow{0} X_C[1] \longrightarrow X_A[1]$, where $X_C$ is a direct summand of $X_A$. From $f_B \circ s$ and $u \circ f_A$ respectively, by the octahedral axiom,

we have the following two commutative diagrams of triangles:

$$
\begin{array}{ccccccc}
X_A & = & X_A & & & & \\
s \downarrow & & f_B \circ s \downarrow & & & & \\
Y_B \longrightarrow & X_B & \xrightarrow{f_B} & B & \xrightarrow{g_B} & Y_B[1] & \\
\| & 0 \downarrow & & \downarrow & & \| & (*) \\
Y_B \longrightarrow & X_C[1] & \longrightarrow & N & \longrightarrow & Y_B[1] & \\
& \downarrow & & \downarrow & & & \\
& X_A[1] & = & X_A[1] & & &
\end{array}
$$

and

$$
\begin{array}{ccccccc}
& M & = & M & & & \\
& w \downarrow & & \downarrow & & & \\
X_A & \xrightarrow{f_A} & A & \xrightarrow{g_A} & Y_A[1] & \xrightarrow{h_A} & X_A[1] \\
\| & u \downarrow & & \downarrow & & \| & (**) \\
X_A & \xrightarrow{u \circ f_A} & B & \longrightarrow & N & \longrightarrow & X_A[1] \\
& v \downarrow & & \downarrow & & & \\
& M[1] & = & M[1]. & & &
\end{array}
$$

Since the morphism from $Y_B$ to $X_C[1]$ in the third row of the diagram $(*)$ is zero, then $N \cong X_C[1] \oplus Y_B[1] \in \mathcal{R}[1]$. On the other hand, the morphism from $M$ to $Y_A[1]$ in the third column of the diagram $(**)$ is zero due to $\mathrm{Hom}(M, Y_A[1]) = 0$, then $M[1]$ is isomorphic to a direct summand of $N$, and then it is in $\mathcal{R}[1]$. Hence $M \in \mathcal{R}$. The functorially finiteness of $\mathcal{R}$ follows from that the number of indecomposable objects (up to isomorphism) in it is finite and $C$ is Hom-finite. Therefore $\mathcal{R}$ is cluster tilting in $C$. By Corollary 3.11, $\mathcal{R}$ is indecomposable.

Now we replace $\mathcal{T}$ by $\mathcal{R}$, repeat the proof above. Namely, we consider the following split triangles:

$$X_i \longrightarrow X_i \longrightarrow 0 \xrightarrow{0} X_i[1],$$

$$0 \rightarrow Y_i \rightarrow Y_i[-1][1] \xrightarrow{0} 0.$$

In these triangles, $X_i \in \mathcal{X}$, $Y_i[-1] \in \mathcal{Y}$. We have that the subcategory $\mathcal{R}'$ generated by $X_i$, $Y_i[-1]$, $i \in I$ is a cluster tilting subcategory. It is an indecomposable by Corollary 3.11. For any $i, j \in I$, $Y_j[-2] \in \mathcal{Y}$, then $\mathrm{Hom}(X_i, Y_j[-1]) = \mathrm{Ext}^1(X_i, Y_j[-2]) = 0$. Note that $\mathrm{Hom}(Y_j[-1], X_i) = \mathrm{Ext}^1(Y_j, X_i) \cong D\mathrm{Ext}^1(X_i, Y_j) = 0$. Therefore $X_i \cong 0$ for all $i$ or $Y_i \cong 0$ for all $i$ as $\mathcal{R}'$ is indecomposable. Then $\mathcal{R}' \subseteq \mathcal{Y}$ or $\mathcal{R}' \subseteq \mathcal{X}$. Hence $C = \mathcal{Y}$ or $C = \mathcal{X}$.

$\square$

**Remark 4.2.** *The above result is not true for $2-CY$ triangulated categories without cluster tilting objects. For example: the derived category of coherent sheaves on an algebraic K3 surface is 2-CY and admits no cluster tilting objects. It admits a non trivial t-structure (the canonical t-structure whose heart is the category of coherent sheaves). There are also examples (e. g. the cluster category $C_{A_\infty}$ of type $A_\infty$, see [HJ, KR, Ng]) that there are nontrivial t-structures in a 2-CY triangulated category admitting cluster tilting subcategories which contains infinitely many indecomposables (up to isomorphism).*

**Corollary 4.3.** *Let $C$ be a 2-CY triangulated category with a cluster tilting object $T$ and let $C = \oplus_{j \in J} C_j$ be the complete decomposition of $C$. Then the t-structures in $C$ are of the form $(\oplus_{j \in L} C_j, \oplus_{j \in J-L} C)$ where $L$ is a subset of $J$. In particular, each t-structure is stable and has a trivial heart.*

The following theorem is the second main result in this section, which gives a classification of cotorsion pairs (equivalently torsion pairs) in $2-$CY triangulated categories $C$ with cluster tilting objects. We note that in any $2-$CY triangulated category $C$ with a cluster tilting object, any rigid subcategory $\mathcal{I}$ contains only finitely many indecomposables (up to isomorphism) [DK]. So we identify $\mathcal{I}$ with the object $I$ obtained as the direct sum of representatives of isoclasses of indecomposables in it. We also note that for any rigid subcategory $I$ in $C$, the subquotient category $^\perp(I[1])/I$ is again a $2-$CY triangulated category with cluster tilting objects (see Theorem 4.7 and Theorem 4.9 in [IY]).

**Theorem 4.4.** *Let $C$ be a 2-CY triangulated category admitting cluster tilting objects and $I$ a rigid subcategory of $C$. Let $^\perp(I[1])/I = \oplus_{j \in J} I_j$ be the complete decomposition of $^\perp(I[1])/I$. Then all cotorsion pairs with core $I$ are obtained as preimages under $\pi : {}^\perp(I[1]) \to {}^\perp(I[1])/I$ of the pairs $(\oplus_{j \in L} I_j, \oplus_{j \in J-L} I_j)$ where $L$ is a subset of $J$. There are $2^{ns(^\perp(I[1])/I)}$ cotorsion pairs with core $I$.*

*Proof.* By Theorem 3.5 and Corollary 3.6 in [ZZ2], a pair $(\mathcal{X}_1, \mathcal{X}_2)$ of subcategories of $C$ is a cotorsion pair with core $I$ if and only if $I \subset \mathcal{X}_i \subset {}^\perp(I[1])$, $i = 1, 2$, and $(\pi(\mathcal{X}_1), \pi(\mathcal{X}_2))$ is a t-structure in $^\perp(I[1])/I$. Then by Corollary 4.3, the t-structures in $^\perp(I[1])/I$ are of the form $(\oplus_{j \in L} I_j, \oplus_{j \in J-L} I_j)$. Therefore the cotorsion pairs with core $I$ are the preimages under $\pi : {}^\perp(I[1]) \to {}^\perp(I[1])/I$ of the $t-$structure $(\oplus_{j \in L} I_j, \oplus_{j \in J-L} I_j)$ in $^\perp(I[1])/I$. $\qquad\square$

Indeed, this correspondence is the same as that in Theorem II.2.5 in [BIRS] under the following result: every cotorsion pair is symmetric, i.e.

**Corollary 4.5.** *Let $C$ be a 2-CY triangulated category admitting a cluster tilting object and let $(\mathcal{X}, \mathcal{Y})$ be a cotorsion pair with core $I$. Then $(\mathcal{Y}, \mathcal{X})$ is also a cotorsion pair with the same core.*

*Proof.* By Theorem 4.4, $(\mathcal{X}, \mathcal{Y}) = (\pi^{-1}(\oplus_{j \in L} I_j), \pi^{-1}(\oplus_{j \in J-L} I_j))$ for some subset $J$, then $(\mathcal{Y}, \mathcal{X}) = (\pi^{-1}(\oplus_{j \in J-L} I_j), \pi^{-1}(\oplus_{j \in L} I_j))$ is also a cotorsion pair with core $I$. $\qquad\square$

Recall that $(\mathcal{X}, \mathcal{Y})$ is a co-t-structure in $C$, if $\text{Ext}^1(\mathcal{X}, \mathcal{Y}) = 0$, $C = \mathcal{X} * \mathcal{Y}[1]$ and $\mathcal{X}[-1] \subset \mathcal{X}$, $\mathcal{Y}[1] \subset \mathcal{Y}$. Using Theorem 4.4 and Corollary 4.3, one can prove that there are no non-trivial co-t-structures in an indecomposable 2-CY triangulated category with a cluster tilting object in the similar way as [ZZZ]. Indeed, if $(\mathcal{X}, \mathcal{Y})$ is a co-t-structure in $C$, then $(\mathcal{X}, \mathcal{Y})$ is a cotorsion pair by the definition of co-t-structure. By Corollary 4.5, $(\mathcal{Y}, \mathcal{X})$ is also a cotorsion pair. Since $\mathcal{Y}$ is closed under [1] and $\mathcal{X}$ is closed under [-1], $(\mathcal{Y}, \mathcal{X})$ is a t-structure. Then by Theorem 4.1, $\mathcal{X} = 0$ or $\mathcal{Y} = 0$.

In fact, we have the following more general result on t-structures or co-t-structures in a $d-$CY triangulated category, generalizing a recent result in [HJY].

**Proposition 4.6.** *Let $C$ be an indecomposable $d-$CY triangulated category. If $d \geq 1$, then the co-t-structures in $C$ are $(C, 0)$ and $(0, C)$. Dually, if $d \leq -1$, Then the t-structures in $C$ are $(C, 0)$ and $(0, C)$.*

*Proof.* We only prove the case of $d \geq 1$. Let $(\mathscr{X}, \mathscr{Y})$ be a co-t-structure in $C$. For any object $M \in \mathscr{X} \cap \mathscr{Y}$, we have $\mathrm{Hom}(M, M) \cong D\mathrm{Ext}^1(M, M[d-1]) = 0$ by $M \in \mathscr{X}$ and $M[d-1] \in \mathscr{Y}$. This implies the core of $(\mathscr{X}, \mathscr{Y})$ is zero. Then by Lemma 2.3, $(\mathscr{X}, \mathscr{Y})$ is a t-structure. Thus $\mathscr{X}, \mathscr{Y}$ are triangulated subcategories of $C$. For any $X \in \mathscr{X}, Y \in \mathscr{Y}$, we have that $\mathrm{Hom}(X, Y) = \mathrm{Ext}^1(X, Y[-1]) = 0$ and $\mathrm{Hom}(Y, X) \cong D\mathrm{Ext}^1(X, Y[d-1]) = 0$ by $Y[-1], Y[d-1] \in \mathscr{Y}$. Due to $C = \mathscr{X} * \mathscr{Y}[1]$, we have $C = \mathscr{X} \oplus \mathscr{Y}$. Therefore $\mathscr{X} = 0$ or $\mathscr{Y} = 0$. $\qquad\square$

# 5 Mutations

In this section, all cluster tilting objects we considered are basic. We shall discuss the relation between mutation of cotorsion pairs and that of cluster tilting objects contained in those cotorsion pairs in a $2-$CY triangulated category with a cluster tilting object. First we introduce a notion of cluster tilting subcategories in a subcategory.

**Definition 5.1.** *Let $\mathscr{X}$ be a contravariantly finite (or covariantly finite) extension-closed sub-category of a triangulated category $C$ and let $\mathcal{D}$ be a subcategory of $\mathscr{X}$. We call that $\mathcal{D}$ is a $\mathscr{X}-$cluster tilting subcategory provided that $\mathcal{D}$ is functorially finite in $\mathscr{X}$, and satisfies that for any object $M \in \mathscr{X}$, $M \in \mathcal{D}$ if and only if $\mathrm{Ext}^1(\mathcal{D}, M) = 0$ if and only if $\mathrm{Ext}^1(M, \mathcal{D}) = 0$. An object $D$ in $\mathscr{X}$ is called a $\mathscr{X}-$cluster tilting object if $\mathrm{add}D$ is a $\mathscr{X}-$cluster tilting subcategory.*

When $\mathscr{X} = C$, then $C-$cluster tilting subcategories are exactly cluster tilting in $C$. When $\mathscr{X}$ is a contravariantly finite (or covariantly finite) rigid subcategory, then $\mathscr{X}$ is the only $\mathscr{X}-$cluster tilting subcategory.

From now on to the rest of the section, $C$ denotes a $2-$CY triangulated category with a cluster tilting object, $(\mathscr{X}, \mathscr{Y})$ denotes a cotorsion pair with core $I$ in $C$. We shall show that any cluster tilting object containing $I$ as a direct summand in $C$ gives a $\mathscr{X}-$cluster tilting object and a $\mathscr{Y}-$cluster tilting object respectively. First we prove some lemmas.

**Lemma 5.2.** *Let $C$, $(\mathscr{X}, \mathscr{Y})$ and $I$ be above. Then $^{\perp}(I[1])/I = \mathscr{X}/I \oplus \mathscr{Y}/I$ as triangulated categories.*

*Proof.* The proof of this lemma can be deduced from the proof of Theorem 4.4. $\qquad\square$

**Lemma 5.3.** *Let $(\mathscr{X}, \mathscr{Y})$ be a cotorsion pair with core $I$ in a $2-$CY triangulated category $C$ with a cluster tilting object. Then any map $f : X \to Y$ and any map $g : Y \to X$ with $X \in \mathscr{X}, Y \in \mathscr{Y}$ factor through $I$.*

*Proof.* This dues to the fact that the images of $f$ and $g$ under the projection $\pi : {}^{\perp}(I[1]) \to {}^{\perp}(I[1])/I$ are zero (since $^{\perp}(I[1])/I = \mathscr{X}/I \oplus \mathscr{Y}/I$ by Lemma 5.2). $\qquad\square$

**Lemma 5.4.** *Let $C$, $(\mathscr{X}, \mathscr{Y})$ and $I$ be above. Let $T$ be a cluster tilting object in $C$. Suppose $T$ can be written as $T = T_{\mathscr{X}} \oplus I \oplus T_{\mathscr{Y}}$ with $T_{\mathscr{X}} \in \mathscr{X}$ and $T_{\mathscr{Y}} \in \mathscr{Y}$. Then $T_{\mathscr{X}} \oplus I$ is $\mathscr{X}-$cluster tilting and $T_{\mathscr{Y}} \oplus I$ is $\mathscr{Y}-$cluster tilting.*

*Proof.* We prove the assertion for $\mathscr{X}-$cluster tilting, the proof for $\mathscr{Y}-$cluster tilting is similar. Suppose that $\mathrm{Ext}^1(T_{\mathscr{X}} \oplus I, X) = 0$ for $X \in \mathscr{X}$, then $\mathrm{Ext}^1(T, X) = 0$ since $\mathrm{Ext}^1(T_{\mathscr{Y}}, X) \cong D\mathrm{Ext}^1(X, T_{\mathscr{Y}}) = 0$, where the first isomorphism dues to $2-$CY property and the second one dues to that $(\mathscr{X}, \mathscr{Y})$ is a cotorsion pair. Hence $X \in \mathrm{add}T$. It follows that $X \in \mathrm{add}(T_{\mathscr{X}} \oplus I)$. $\mathrm{add}(T_{\mathscr{X}} \oplus I)$ is automatically functorially finite in $\mathscr{X}$. Therefore $T_{\mathscr{X}} \oplus I$ is a $\mathscr{X}-$cluster tilting object. $\qquad\square$

14

The following result gives the precise relation between the cluster tilting objects containing $I$ as a direct summand and the $\mathscr{X}$−cluster tilting objects, $\mathscr{Y}$−cluster tilting objects.

**Proposition 5.5.** *Let $C$ be a $2−CY$ triangulated category with a cluster tilting object, and $(\mathscr{X},\mathscr{Y})$ be a cotorsion pair in $C$ with core $I$. Then*
*1. Any cluster tilting object $T$ containing $I$ as a direct summand can be written uniquely as: $T = T_{\mathscr{X}} \oplus I \oplus T_{\mathscr{Y}}$, such that $T_{\mathscr{X}} \oplus I$ is $\mathscr{X}$−cluster tilting, and $T_{\mathscr{Y}} \oplus I$ is $\mathscr{Y}$−cluster tilting.*
*2. Any $\mathscr{X}$−cluster tilting object $M$ (or $\mathscr{Y}$−cluster tilting object $N$) contains $I$ as a direct summand, and can be written as $M = M_{\mathscr{X}} \oplus I$ ( $N = N_{\mathscr{Y}} \oplus I$ respectively). Furthermore $M_{\mathscr{X}} \oplus I \oplus N_{\mathscr{Y}}$ is a cluster tilting object in $C$.*
*3. There is a bijection between the set of cluster tilting objects containing $I$ as a direct summand in $C$ and the product of the set of $\mathscr{X}$−cluster tilting objects with the set of $\mathscr{Y}$−cluster tilting objects. The bijection is given by $T \mapsto T_{\mathscr{X}} \oplus I \oplus T_{\mathscr{Y}}$.*

*Proof.* 1. Let $T$ be any cluster tilting object containing $I$ as a direct summand, we write $T$ as $T = I \oplus T_0$. Then $T_0 \in {}^{\perp}(I[1])$, and by passing from ${}^{\perp}(I[1])$ to the quotient triangulated category ${}^{\perp}(I[1])/I = \mathscr{X}/I \oplus \mathscr{Y}/I$ (Lemma 5.2), we have that $T_0 = T_{\mathscr{X}} \oplus T_{\mathscr{Y}}$, where $T_{\mathscr{X}} \in \mathscr{X}, T_{\mathscr{Y}} \in \mathscr{Y}$. Therefore $T = T_{\mathscr{X}} \oplus I \oplus T_{\mathscr{Y}}$. By Lemma 5.4, $T_{\mathscr{X}} \oplus I, T_{\mathscr{Y}} \oplus I$ are $\mathscr{X}$−cluster tilting, $\mathscr{Y}$−cluster tilting respectively.

2. Let $M$ be a $\mathscr{X}$−cluster tilting object. Then by $\text{Ext}^1(M, I) = 0$, we have that $I \in \text{add}M$, i.e. $I$ is a direct summand of $M$. Then $M = M_{\mathscr{X}} \oplus I$. Similarly, any $\mathscr{Y}$−cluster tilting object $N$ can be written as $N = N_{\mathscr{Y}} \oplus I$. Now $M_{\mathscr{X}}, N_{\mathscr{Y}}$ are cluster tiltings in $\mathscr{X}/I, \mathscr{Y}/I$ respectively, and then $M_{\mathscr{X}} \oplus N_{\mathscr{Y}}$ is a cluster tilting object in ${}^{\perp}(I[1])/I$ since ${}^{\perp}(I[1])/I = \mathscr{X}/I \oplus \mathscr{Y}/I$. It follows that $M_{\mathscr{X}} \oplus I \oplus N_{\mathscr{Y}}$ is a cluster tilting object in $C$.

3. It follows from 1 and 2. $\square$

We know that one can mutate cluster tilting objects to get new ones. In the following we shall see that the mutation of cluster tilting objects containing $I$ as a direct summand is related to the mutation of cotorsion pairs introduced in [ZZ2]. We recall the notion of mutation of cotorsion pairs in $2−CY$ triangulated categories. This notion is defined in a general triangulated category in [ZZ2].
Let $C$ be a $2−CY$ triangulated category with a cluster tilting object $T$. We denote by $\delta(M)$ the number of indecomposable direct summands (up to isomorphism) of an object $M$. We assume that $\delta(T) = n$.
Let $(\mathscr{X},\mathscr{Y})$ be a cotorsion pair with core $I$. Then $0 \leq \delta(I) \leq n$ [DK]. It follows from Lemma 2.4 that $\delta(I) = 0$ if and only if $(\mathscr{X},\mathscr{Y})$ is a t-structure in $C$, while $\delta(I) = n$ if and only if $\mathscr{X} = \mathscr{Y} = \text{add}(I)$ is cluster tilting in $C$. In the later case, $I$ is a cluster tilting object in $C$.

**Definition 5.6.** *Let $C$ be an indecomposable $2−CY$ triangulated category with a cluster tilting object $T$, and $\delta(T) = n$. Assume that $0 \leq d \leq n$ is an integer. A cotorsion pair $(\mathscr{X},\mathscr{Y})$ with core $I$ is called a $d$−cotorsion pair if $\delta(I) = d$.*
*Denote by $CTN_d(C)$ the set of all $d$−cotorsion pairs.*

From the definition above and Theorem 4.1, $CTN_0(C) = \{(C, 0), (0, C)\}$. $CTN_n(C)$ consists of cluster tilting objects in $C$.

Throughout the remainder of this section, let $D$ be a direct summand of $I$ (maybe zero summand). Denote by $\mathcal{D} = \text{add}D$. Put:

$$\mu^{-1}(\mathscr{X};\mathcal{D}) := (\mathcal{D} * \mathscr{X}[1]) \cap {}^{\perp}(\mathcal{D}[1]);$$
$$\mu^{-1}(\mathscr{Y};\mathcal{D}) := (\mathcal{D} * \mathscr{Y}[1]) \cap {}^{\perp}(\mathcal{D}[1]);$$
$$\mu^{-1}(I;\mathcal{D}) := (\mathcal{D} * I[1]) \cap {}^{\perp}(\mathcal{D}[1]).$$

The following proposition is proved in [ZZ2].

**Proposition 5.7.** *With the assumption above, we have that $(\mu^{-1}(\mathscr{X};\mathcal{D}), \mu^{-1}(\mathscr{Y};\mathcal{D}))$ is also a cotorsion pair with the core $\mu^{-1}(I;\mathcal{D})$ in C. Moreover $(\mu^{-1}(\mathscr{X};\mathcal{D}), \mu^{-1}(\mathscr{Y};\mathcal{D})) \in CTN_d(C)$ if and only if $(\mathscr{X},\mathscr{Y}) \in CTN_d(C)$.*

**Definition 5.8.** *We call the cotorsion pair $(\mu^{-1}(\mathscr{X};\mathcal{D}), \mu^{-1}(\mathscr{Y};\mathcal{D}))$ is a $\mathcal{D}-$mutation of cotorsion pair $(\mathscr{X},\mathscr{Y})$. Sometimes denote this cotorsion pair by $(\mathscr{X}',\mathscr{Y}')$, denote its core by $I'$.*

**Corollary 5.9.** *Let $(\mathscr{X},\mathscr{Y})$ be a cotorsion pair with core $I$, and $(\mathscr{X}',\mathscr{Y}')$ with core $I'$ be the $\mathcal{D}-$mutation of $(\mathscr{X},\mathscr{Y})$. Then $(\mathscr{X}',\mathscr{Y}') = (\mathscr{X},\mathscr{Y})$ if and only if $I' = I$.*

*Proof.* The "only if" part is obviously. We prove the "if" part. Suppose $I' = I$. Then by Theorem 3.11(2) in [ZZ1], $\mathcal{D} = I$. It follows that passing to the quotient category ${}^{\perp}(I[1])/I$, $(\underline{\mathscr{X}'},\underline{\mathscr{Y}'})$ is $0-$mutation of the t-structure $(\underline{\mathscr{X}},\underline{\mathscr{Y}})$ in the quotient triangulated category ${}^{\perp}(I[1])/I$. By Corollary 4.3, both $(\underline{\mathscr{X}'},\underline{\mathscr{Y}'})$ and $(\underline{\mathscr{X}},\underline{\mathscr{Y}})$ are stable, then $(\underline{\mathscr{X}'},\underline{\mathscr{Y}'}) = (\underline{\mathscr{X}},\underline{\mathscr{Y}})$ in this quotient category. Hence $(\mathscr{X}',\mathscr{Y}') = (\mathscr{X},\mathscr{Y})$ in C. $\square$
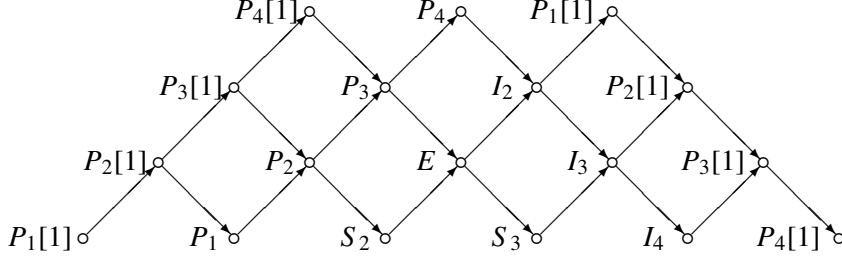
This corollary was proved for finite triangulated categories in [ZZ2].

Note that there are many choices for $\mathcal{D}$. Two extreme cases are: when $\mathcal{D} = \{0\}$, then the $\mathcal{D}-$mutation of $(\mathscr{X},\mathscr{Y})$ is $(\mathscr{X}[1],\mathscr{Y}[1])$; when $\mathcal{D} = \mathrm{add}I$, then the $\mathcal{D}-$mutation of $(\mathscr{X},\mathscr{Y})$ is $(\mathscr{X},\mathscr{Y})$ itself.

When $D$ is a direct summand of $I$ with $\delta(D) = \delta(I) - 1$, the $\mathcal{D}-$mutation is the usually one, which was defined and studied for cluster tilting objects (subcategories) in [BMRRT, KR, IY], for rigid objects(subcategories) in [MP], for maximal rigid objects(subcategories) in [ZZ1]. We call the $\mathcal{D}-$mutation with $\delta(D) = \delta(I) - 1$ just mutation, for simplicity. Denote this mutation by $\mu_{I_0}$, where $I_0$ is the missing indecomposable object of $D$ in $I$.

**Remark 5.10.** *For a cluster tilting object $T$, the mutation $\mu$ is an involution. But the mutation of cotorsion pairs is not an involution in general (compare [MP]), see the following example.*

**Example 5.** *Let $Q$ be the quiver $4 \to 3 \to 2 \to 1$, and $C = D^b(kQ)/\tau^{-1}[1]$, the cluster category of $Q$, see the AR-quiver below. Set $\mathscr{X} = \mathrm{add}(P_1 \oplus P_2 \oplus P_3 \oplus S_2)$, $\mathscr{Y} = \mathrm{add}(P_2 \oplus P_3 \oplus P_4 \oplus P_4[1])$, $I = P_2 \oplus P_3$. Then $(\mathscr{X},\mathscr{Y})$ is a cotorsion pair with core $I$. We mutate the cotorsion pair $(\mathscr{X},\mathscr{Y})$ at $P_2$ to get a new cotorsion pair $(\mathscr{X}_1,\mathscr{Y}_1)$ with core $I_1$, where $\mathscr{X}_1 = \mathrm{add}(E \oplus S_3 \oplus P_3 \oplus P_1)$, $\mathscr{Y}_1 = \mathrm{add}(S_3 \oplus P_3 \oplus P_4[1] \oplus P_4)$, $I_1 = S_3 \oplus P_3$. Now we continues to mutate $(\mathscr{X}_1,\mathscr{Y}_1)$ at $S_3$. We get another new cotorsion pair $(\mathscr{X}_2,\mathscr{Y}_2)$ with core $I_2$, where $\mathscr{X}_2 = \mathrm{add}(P_2 \oplus S_2 \oplus P_3 \oplus E)$, $\mathscr{Y}_2 = \mathrm{add}(S_2 \oplus P_3 \oplus P_4 \oplus P_4[1])$, $I_2 = S_2 \oplus P_3$. We conclude that $(\mathscr{X}_2,\mathscr{Y}_2) \neq (\mathscr{X},\mathscr{Y})$.*

$P_4[1]$    $P_4$    $P_1[1]$

$P_3[1]$    $P_3$    $I_2$    $P_2[1]$

$P_2[1]$    $P_2$    $E$    $I_3$    $P_3[1]$

$P_1[1]$    $P_1$    $S_2$    $S_3$    $I_4$    $P_4[1]$

We define mutation quiver of cotorsion pairs in $C$. It is a quiver whose vertices are cotorsion pairs, there is an arrow from the vertex to another vertex if the target cotorsion pair is a mutation of the initial one. This quiver is denoted by $\mathscr{M}(C)$. It is not connected from Proposition 5.5. Denoted by $\mathscr{M}_d(C)$ the subquiver of $\mathscr{M}(C)$ consisting of vertices belong to $CTN_d(C)$. $\mathscr{M}(C) = \bigsqcup_{d=0}^{n} \mathscr{M}_d(C)$. Note that if we replace the each double anti-arrows by an edge, then $\mathscr{M}_n(C)$ is the exchange graph of cluster tilting objects in $C$.

Now we give the relation of mutation of cluster tilting objects containing $I$ as a direct summand with mutation of cotorsion pairs.

**Proposition 5.11.** *Let $(\mathscr{X}, \mathscr{Y})$ be a cotorsion pair with core $I$ in $C$, $T = T_{\mathscr{X}} \oplus I \oplus T_{\mathscr{Y}}$ a cluster tilting object. Suppose $(\mathscr{X}', \mathscr{Y}')$ is a $\mathcal{D}-$mutation of $(\mathscr{X}, \mathscr{Y})$, $I'$ is the core of $(\mathscr{X}', \mathscr{Y}')$. Then the $\mathcal{D}-$mutation $T'$ of $T$ is $T'_{\mathscr{X}'} \oplus I' \oplus T'_{\mathscr{Y}'}$.*

*Proof.* For $\mathcal{D} = \mathrm{add}D$, where $D$ is a direct summand of $I$, we consider the subquotient category $^{\perp}(\mathcal{D}[1])/\mathcal{D}$. It is a triangulated category by [IY] with shift functor $< 1 >$. In this subquotient category, $(\underline{\mathscr{X}}, \underline{\mathscr{Y}})$ is a cotorsion pair with core $\underline{I}$ in [ZZ2] and $\underline{T} = \underline{T_{\mathscr{X}} \oplus I \oplus T_{\mathscr{Y}}}$ is a cluster tilting object by [IY]. The images of their $\mathcal{D}-$mutations are $(\underline{\mathscr{X}'}, \underline{\mathscr{Y}'}) = (\underline{\mathscr{X}} < 1 >, \underline{\mathscr{Y}} < 1 >)$, $\underline{T'} = \underline{T} < 1 >= \underline{T_{\mathscr{X}}} < 1 > \oplus \underline{I} < 1 > \oplus \underline{T_{\mathscr{Y}}} < 1 >$ respectively. It follows that $\underline{T_{\mathscr{X}}} < 1 > \in \underline{\mathscr{X}} < 1 >$, $\underline{T_{\mathscr{Y}}} < 1 > \in \underline{\mathscr{Y}} < 1 >$. Therefore $\underline{T'} = \underline{T'_{\mathscr{X}'} \oplus I' \oplus T'_{\mathscr{Y}'}}$, where $T'_{\mathscr{X}'} \oplus I'$, $T'_{\mathscr{Y}'} \oplus I'$ are $\mathscr{X}'-$cluster tilting object in $\mathscr{X}'$, $\mathscr{Y}'-$cluster tilting object in $\mathscr{Y}'$ respectively.

$\square$

We state and prove the main result in this section.

**Theorem 5.12.** *Let $(\mathscr{X}, \mathscr{Y})$ be a cotorsion pair with core $I$ in a $2-CY$ triangulated category $C$ with a cluster tilting object. Let $T = T_{\mathscr{X}} \oplus I \oplus T_{\mathscr{Y}}$ be a cluster tilting object containing $I$ as a direct summand. Suppose that $T_0$ is an indecomposable direct summand of $T$. We consider the mutation $\mu_{T_0}(T)$ of $T$ in $T_0$.*
*1. If $T_0$ is a direct summand of $I$, denoted $\mathrm{add}(I/T_0)$ by $\mathcal{D}$, then the $\mathcal{D}-$mutation of $T$ is $T'_{\mathscr{X}'} \oplus I' \oplus T'_{\mathscr{Y}'}$, where $(\mathscr{X}', \mathscr{Y}') = \mu_{T_0}(\mathscr{X}, \mathscr{Y})$ is the mutation of $(\mathscr{X}, \mathscr{Y})$, $I'$ is the core of cotorsion pair $(\mathscr{X}', \mathscr{Y}')$.*
*2. If $T_0$ is not the direct summand of $I$, then $\mu_{T_0}(T) = \mu_{T_0}(T_{\mathscr{X}} \oplus I) \oplus T_{\mathscr{Y}}$ when $T_0$ is a direct summand of $T_{\mathscr{X}}$, and $\mu_{T_0}(T) = T_{\mathscr{X}} \oplus \mu_{T_0}(I \oplus T_{\mathscr{Y}})$ when $T_0$ is a direct summand of $T_{\mathscr{Y}}$.*

*Proof.* 1. The assertion follows from Proposition 5.10.
2. We will prove the case of that $T_0$ is a direct summand of $T_{\mathscr{X}}$, the proof for the other case is similar. For the minimal left $T/T_0-$approximation of $T_0$, say $g : T_0 \to B$, we have $B \in \mathrm{add}(T_{\mathscr{X}} \oplus I)$ by Lemma 5.3. Then $g : T_0 \to B$ is a minimal left $(T_{\mathscr{X}} \oplus I)-$approximation. Extend

$g$ to a triangle $T_0 \xrightarrow{g} B \to T_0' \to T_0[1]$. It induces a triangle in the subfactor triangulated category $^{\perp}(I[1])/I : T_0 \xrightarrow{g} B \to T_0' \to T_0 < 1 > $ [IY]. It follows that $T_0' \in \mathscr{X}/I$ and $T_0' \in \mathscr{X}$. Then $\mu_{T_0}(T) = (T/T_0) \oplus T_0' = T_0' \oplus (T_{\mathscr{X}}/T_0) \oplus I \oplus T_{\mathscr{Y}} = \mu_{T_0}(T_{\mathscr{X}} \oplus I) \oplus T_{\mathscr{Y}}$.

$\square$

**Remark 5.13.** *For any cotorsion pair $(\mathscr{X}, \mathscr{Y})$ with core $I$ in $C$. From the theorem above, $\mathscr{X}$ (or $\mathscr{Y}$) has weak cluster structure in the sense of [BIRS], i.e. the $\mathscr{X}-$cluster tiltings $T_{\mathscr{X}} \oplus I$ are the candidates of extended clusters, where $I$ is the set of coefficients; one can mutate the $\mathscr{X}-$cluster tiltings at $T_0$ to get a new $\mathscr{X}-$cluster tilting object by the above theorem; and one also have exchange triangles. There is a substructure of $C$ induced by a $\mathscr{X}-$cluster tilting object and a $\mathscr{Y}-$cluster tilting object: Let $T_{\mathscr{X}} \oplus I$ be a $\mathscr{X}-$cluster tilting object, $T_{\mathscr{Y}} \oplus I$ a $\mathscr{Y}-$cluster tilting object. Then $T_{\mathscr{X}} \oplus I \oplus T_{\mathscr{Y}}$ is the cluster tilting object in $C$ by Proposition 5.4. We call that $T_{\mathscr{X}} \oplus I$ and $T_{\mathscr{Y}} \oplus I$ give a substructure of $C$ (compare [BIRS]) if for any $\mathscr{X}-$cluster tilting object $T'_{\mathscr{X}} \oplus I$, $\mathscr{Y}-$cluster tilting object $T'_{\mathscr{Y}} \oplus I$, both of which are obtained from $T_{\mathscr{X}} \oplus I$ and $T_{\mathscr{Y}} \oplus I$ respectively via a finite number of mutations, then the cluster tilting object $T'_{\mathscr{X}} \oplus I \oplus T'_{\mathscr{Y}}$ in $C$ can be obtained from $T_{\mathscr{X}} \oplus I \oplus T_{\mathscr{Y}}$ via a finite number of mutations in $C$.*

# 6 Hearts of cotorsion pairs

As an application of the classification theorem of cotorsion pairs, we determine the hearts of cotorsion pairs in $2-$CY triangulated categories with cluster tilting objects in this section. Hearts of cotorsion pairs in any triangulated category were introduced by Nakaoka [N], which unify the construction of hearts of t-structures [BBD] and construction of the abelian quotient categories by cluster tilting subcategories [BMRRT, KR, KZ].

We recall the construction of hearts of cotorsion pairs from [N]. Recall that this construction works for cotorsion pairs in any triangulated category $C$. In the second half of this section we restrict our attention to $2-$CY triangulated categories. Let $(\mathscr{X}, \mathscr{Y})$ a cotorsion pair with core $\mathcal{I}$ in $C$. Denote by $\mathcal{H}$ the subcategory $(\mathscr{X}[-1] * \mathcal{I}) \cap (\mathcal{I} * \mathscr{Y}[1])$. The heart of the cotorsion pair $(\mathscr{X}, \mathscr{Y})$ is defined as the quotient category $\mathcal{H}/\mathcal{I}$, denoted by $\underline{\mathcal{H}}$.

It was proved that $\underline{\mathcal{H}}$ is an abelian category [N]. There is a cohomology functor $H = h\pi$ from $C$ to $\underline{\mathcal{H}}$, where $\pi$ is the quotient functor from $C$ to $\underline{C} = C/\mathcal{I}$ and $h$ is a functor from $\underline{C}$ to $\underline{\mathcal{H}}$. Those constructions were given in Proposition 3.4 and Proposition 4.2 in [AN] combined with Construction 4.2, Proposition 4.3 and Remark 4.5 in [N]. For the convenience of reader, we recall the definition of the functor $h$ from [AN] as follows.

For any $M \in C$, there is a triangle $Y_M \to X_M \to M \to Y_M[1]$ with $X_M \in \mathscr{X}, Y_M \in \mathscr{Y}$, since $(\mathscr{X}, \mathscr{Y})$ is a cotorsion pair. Then there is a triangle $X_M'[-1] \to X_M \to Y_M' \to X_M'$ with $X_M' \in \mathscr{X}, Y_M' \in \mathscr{Y}$, since $(\mathscr{X}[-1], \mathscr{Y}[-1])$ is a cotorsion pair. Composing the morphism from $X_M'[-1]$ to $X_M$ and the morphism from $X_M$ to $M$, we have the following commutative diagram of

triangles in $C$ by the octahedral axiom, in which we get $\widetilde{M}$ and $s_M : M \to \widetilde{M}$:

$$
\begin{array}{ccc}
X'_M[-1] & = & X'_M[-1] \\
\downarrow & & \downarrow \\
\end{array}
$$

$$
\begin{array}{ccccccc}
Y_M & \longrightarrow & X_M & \longrightarrow & M & \longrightarrow & Y_M[1] \\
\| & & \downarrow & & s_M \downarrow & & \| \quad (\star). \\
Y_M & \longrightarrow & Y'_M & \longrightarrow & \widetilde{M} & \longrightarrow & Y_M[1] \\
& & \downarrow & & \downarrow & & \\
& & X'_M & = & X'_M & &
\end{array}
$$

Using the definition of cotorsion pair $(\mathscr{X}[-1], \mathscr{Y}[-1])$ again, we have a triangle $X''_M[-1] \to \widetilde{M} \to Y''_M \to X''_M$ and then we have another triangle $Y''_M \to X'''_M \to Y'''_M \to Y''_M[1]$ with $X''_M, X'''_M \in \mathscr{X}$ and $Y''_M, Y'''_M \in \mathscr{Y}$. Compose the morphism from $X'''_M$ to $Y'''_M$ and the morphism from $Y''_M$ to $X''_M$, by the octahedral axiom, we have the following commutative diagram of triangles in $C$, in which we have $\overline{M}$ and $t_M : \overline{M} \to \widetilde{M}$:

$$
\begin{array}{ccc}
Y'''_M & = & Y'''_M \\
\downarrow & & \downarrow \\
\end{array}
$$

$$
\begin{array}{ccccccc}
X''_M[-1] & \to & \overline{M} & \to & X'''_M & \to & X''_M \\
\| & & t_M \downarrow & & \downarrow & & \| \quad (\star\star). \\
X''_M[-1] & \to & \widetilde{M} & \to & Y''_M & \to & X''_M \\
& & \downarrow & & \downarrow & & \\
& & Y'''_M[1] & = & Y'''_M[1] & &
\end{array}
$$

The image of $M$ under $h$ is defined as $\overline{M}$. Abe and Nakaoka proved that $\overline{M} \in \underline{\mathcal{H}}$. It is easy to see that up to isomorphisms in $\underline{\mathcal{H}}$, $\overline{M}$ does not depend on the choice of $X_M, X'_M, X''_M, X'''_M$ and $Y_M, Y'_M, Y''_M, Y'''_M$ (See Section 4 in [AN] for details).

For any morphism $f : M \to N$ in $\underline{C}$, there is a unique morphism $\widetilde{f}$ in $\underline{C}$ such that the left square of the following diagram commutate (Proposition 4.3 in [N]) and then there is a unique morphism $\overline{f}$ in $\underline{C}$ such that the right square in the following diagram commutate (Remark 4.5 in [N]):

$$
\begin{array}{ccccc}
M & \overset{s_M}{\to} & \widetilde{M} & \overset{t_M}{\leftarrow} & \overline{M} \\
f \downarrow & & \widetilde{f} \downarrow & & \overline{f} \downarrow \quad (\star\star\star). \\
N & \underset{s_N}{\to} & \widetilde{N} & \underset{t_N}{\leftarrow} & \overline{N}
\end{array}
$$

The image of $f$ under $h$ is defined as $\overline{f}$.

We state two simple facts followed from the constructions above.

**Lemma 6.1.** $H(\mathscr{X}) = 0$ and $H(\mathscr{Y}) = 0$ hold.

*Proof.* We give a proof for $H(\mathscr{X}) = 0$, $H(\mathscr{Y}) = 0$ can be proved dually. Let $M$ be an object in $\mathscr{X}$. One can choose $Y_M = 0$. Then $\widetilde{M} \cong Y'_M$. So one can choose $X''_M = 0$. Then $h(M) = \overline{M} \cong X'''_M$. Note that $X'''_M \in \mathscr{Y} * \mathscr{Y} \subset \mathscr{Y}$ by the triangle in the third column in the diagram $(\star\star)$. We have that $h(M) \in \mathcal{I}$ and hence $h(M) \cong 0$ in $\underline{\mathcal{H}}$. $\quad\square$

**Lemma 6.2.** $h|_{\underline{\mathcal{H}}} = id_{\underline{\mathcal{H}}}$.

*Proof.* By the definition of $h$, one only need to check that $h(M) \cong M$ for any $M \in \mathcal{H}$. In this case, we have that $X_M \in \mathcal{I}$ by Corollary 3.3 in [N]. One can choose $Y'_M = X_M$ and then $\widetilde{M} \cong M$. By the dual, one can have that $\overline{M} \cong \widetilde{M}$. Thus this lemma holds. □

Let $(\mathcal{X}_1, \mathcal{Y}_1)$ and $(\mathcal{X}_2, \mathcal{Y}_2)$ be two cotorsion pairs with the same core $\mathcal{I}$ in a triangulated category $C$. Denote by $\mathcal{H}_i$ the heart of $(\mathcal{X}_i, \mathcal{Y}_i)$, $i = 1, 2$. Let $H_i = h_i \pi$ be the cohomology functor from $C$ to $\mathcal{H}_i$ given in [AN], and $\iota_i$ be the inclusion functor from $\mathcal{H}_i$ to $C$, $i = 1, 2$. The composition functors $h_1 \iota_2$ and $h_2 \iota_1$ are denoted by $E$ and $F$ respectively.
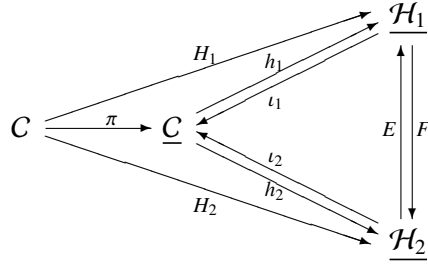


**Lemma 6.3.** *If $H_1({}^{\perp}(\mathcal{I}[1])) = 0$ and $H_1((\mathcal{I}[-1])^{\perp}) = 0$, then $EF \simeq id_{\mathcal{H}_1}$.*

*Proof.* For any $M \in \mathcal{H}_1$, we have the above commutative diagrams $(\star)$ and $(\star\star)$ with $X_M, X'_M, X''_M, X'''_M \in \mathcal{X}_2$ and $Y_M, Y'_M, Y''_M, Y'''_M \in \mathcal{Y}_2$. Then $h_2(M) = \overline{M}$ by the definition. The first and the last morphisms in the third column of the diagram $(\star)$ and in the second column of the diagram $(\star\star)$ factor through ${}^{\perp}(\mathcal{I}[1])$ or $(\mathcal{I}[-1])^{\perp}$ respectively, by $\mathcal{X}_2 \subset {}^{\perp}(I[1])$ and $\mathcal{Y}_2 \subset (I[-1])^{\perp}$. Then the image of these morphisms under $H_1$ are zero. Applying the cohomology functor $H_1$ to these two triangles (in the third column of the diagram $(\star)$ and in the second column of the diagram $(\star\star)$), one has two isomorphisms in $\mathcal{H}_1$:

$$H_1 M \overset{H_1(s_M)}{\longrightarrow} E\widetilde{M}$$

and

$$EFM \overset{H_1(t_M)}{\longrightarrow} E\widetilde{M}.$$

Since $M \in \mathcal{H}_1$, so $H_1 M = M$ by Lemma 6.2.

For any morphism $f : M \to N$ in $\mathcal{H}_1$, applying the functor $h_1$ to the diagram $(\star \star \star)$, we have the following commutative diagram in $\mathcal{H}_1$:

$$
\begin{array}{ccc}
h_1 M & \overset{H_1(t_M)^{-1} H_1(s_M)}{\longrightarrow} & EFM \\
h_1 f \downarrow & & EFf \downarrow \\
h_1 N & \overset{H_1(t_N)^{-1} H_1(s_N)}{\longrightarrow} & EFN
\end{array}
.
$$

Since $M, N \in H_1$, then $h_1 M = M$, $h_1 N = N$ and $h_1 f = f$ by Lemma 6.2. Therefore, $id_{\mathcal{H}_1} \simeq EF$. □

From now on to the end of the section, we assume that $C$ is a $2-CY$ triangulated category with cluster tilting objects. The main result of this section is to determine the hearts of any cotorsion pairs in $C$.

**Theorem 6.4.** *Let C be a 2-CY triangulated category with cluster tilting objects and $(\mathscr{X}, \mathscr{Y})$ be a cotorsion pair in C with core I. Then we have an equivalence of abelian categories*

$$\underline{\mathcal{H}} \simeq mod\ EndI,$$

*and in particular, the hearts of any two cotorsion pairs with the same core are equivalent.*

*Proof.* Denote by $\mathcal{I}$ the subcategory add$I$. Let $(\mathscr{X}_1, \mathscr{Y}_1)$ be the cotorsion pair $(\mathcal{I}, {}^{\perp}(\mathcal{I}[1]))$ and $(\mathscr{X}_2, \mathscr{Y}_2) = (\mathscr{X}, \mathscr{Y})$. By Lemma 5.2, ${}^{\perp}(\mathcal{I}[1])/\mathcal{I} = \mathscr{X}/\mathcal{I} \oplus \mathscr{Y}/\mathcal{I}$. Then by Lemma 6.1, we have that $H_1({}^{\perp}(\mathcal{I}[1])) = 0$ and $H_2({}^{\perp}(\mathcal{I}[1])) = 0$. By the 2$-$CY property of $C$ and Lemma 6.3, we have $EF \simeq \mathrm{id}_{\mathcal{H}_1}$ and $FE \simeq \mathrm{id}_{\mathcal{H}_2}$. Thanks to that the heart of $(\mathcal{I}, {}^{\perp}(\mathcal{I}[1]))$ is equivalent to the module category $\underline{\mathrm{mod}}\ EndI$ [IY], we have the equivalence $\underline{\mathcal{H}} \simeq \underline{\mathrm{mod}}\ \mathcal{I}$. Note that both categories $\underline{\mathcal{H}}$ and mod $\mathcal{I}$ are abelian and $E, F$ are additive functors. So $\underline{\mathcal{H}}$ and mod $EndI$ are equivalent as abelian categories. $\square$

**Example 6.** *Let $Q$ be the quiver $4 \rightarrow 3 \rightarrow 2 \rightarrow 1$, and $C$ the cluster category of $Q$. Set $I = add(P_2[1] \oplus P_3[1])$. Then the subcategory ${}^{\perp}(I[1]) = add(P_1[1] \oplus P_2[1] \oplus P_3[1] \oplus P_4[1] \oplus I_4 \oplus P_1)$. We mark the indecomposable objects in ${}^{\perp}(I[1])$ by $\square$ in the following AR-quiver of C.*
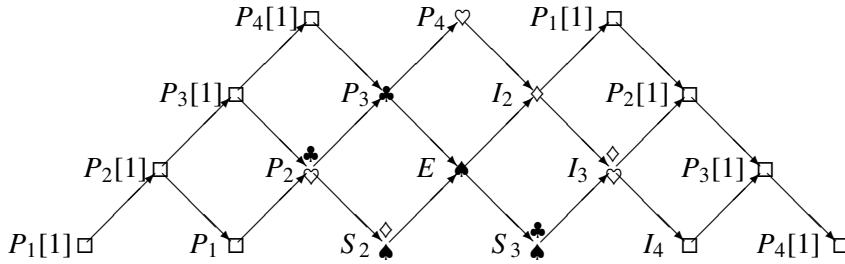


*There are four cotorsion pairs with core I in this category, we list them together with their hearts in the following and mark the indecomposable objects in each heart by ♣, ◇, ♡ and ♠ respectively in order in the AR-quiver above.*

| Cotorsion pairs | Hearts |
|---|---|
| $(I, {}^{\perp}I[1])$ | $add(P_2 \oplus P_3 \oplus S_3)$ |
| $({}^{\perp}I[1], I)$ | $add(S_2 \oplus I_2 \oplus I_3)$ |
| $(add(P_2[1] \oplus P_3[1] \oplus P_4[1] \oplus I_4), add(P_2[1] \oplus P_3[1] \oplus P_1[1] \oplus P_1))$ | $add(P_2 \oplus P_4 \oplus I_3)$ |
| $(add(P_2[1] \oplus P_3[1] \oplus P_1[1] \oplus P_1), add(P_2[1] \oplus P_3[1] \oplus P_4[1] \oplus I_4))$ | $add(S_2 \oplus E \oplus S_3)$ |

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