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Frames, Riesz bases, and sampling expansions in Banach spaces via semi-inner products [☆]

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ABSTRACT

Frames in a Banach space $\mathcal{B}$ were defined as a sequence in its dual space $\mathcal{B}^*$ in some recent references. We propose to define them as a collection of elements in $\mathcal{B}$ by making use of semi-inner products. Classical theory on frames and Riesz bases is generalized under this new perspective. We then aim at establishing the Shannon sampling theorem in Banach spaces. The existence of such expansions in translation invariant reproducing kernel Hilbert and Banach spaces is discussed.

1. Introduction

A main purpose of this paper is to provide a language for the study of frames and Riesz bases in Banach spaces, making smoother the passage from Hilbert spaces. The motivation comes from the establishment of a Shannon sampling theorem in Banach spaces of functions. To this end, we first redefine frames in Banach spaces via a compatible semi-inner product, which is a natural substitute for inner products on Hilbert spaces. The classical theory of frames and Riesz bases for Hilbert spaces is then generalized to Banach spaces. Although examples of frames with favorable properties will be implicitly provided in Section 4, we leave out the explicit construction of useful frames for Banach spaces, hoping that our work could set a foundation for such studies in the future.

We start with recalling the definition of frames and Riesz bases for Hilbert spaces. Let $\mathcal{H}$ be a separable Hilbert space and $\mathbb{I}$ a countable index set. A frame for $\mathcal{H}$ is an indexed set of vectors $\{f_j\colon\ j \in \mathbb{I}\} \subseteq \mathcal{H}$ for which there exist positive constants $0 < A \leqslant B < +\infty$ such that

$$A\|f\|_{\mathcal{H}} \leqslant \left\|\left\{(f, f_j)_{\mathcal{H}}\right\}\right\|_{\ell^2(\mathbb{I})} \leqslant B\|f\|_{\mathcal{H}} \quad \text{for all } f \in \mathcal{H}.$$

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Here, $\|\cdot\|_{\mathcal{H}}$ and $(\cdot,\cdot)_{\mathcal{H}}$ denote the norm and inner product on $\mathcal{H}$, respectively. And $\ell^2(\mathbb{I})$ is the Hilbert space of square-summable sequences on $\mathbb{I}$. For simplicity, a set $\{\alpha_j:\ j\in\mathbb{I}\}$ indexed by $\mathbb{I}$ will be abbreviated as $\{\alpha_j\}$ throughout the paper. A Riesz basis $\{f_j\}$ for $\mathcal{H}$ is a frame that is *minimal* in the sense that

$$f_j \notin \overline{\operatorname{span}}\{f_k:\ k\in\mathbb{I},\ k\neq j\}\quad \text{for any } j\in\mathbb{I}.$$

Frames and Riesz bases, as alternatives for orthonormal bases, bring more flexibility in representing elements in a Hilbert space. They were first introduced for the purpose of studying nonharmonic Fourier analysis [10,38,44]. With the advent of the theory of wavelets, they find wide applications in the construction of bases for signal and image processing, time frequency analysis, and sampling theory [8,31]. We are particularly interested in its natural role in the complete reconstruction of a function in a reproducing kernel Hilbert space (RKHS) from its samplings.

An RKHS on a set X is a Hilbert space $\mathcal{H}$ of functions defined on X such that for each $x\in X$ the linear functional of point evaluation at x

$$\delta_x(f):=f(x),\quad f\in\mathcal{H}$$

is continuous [2]. By the Riesz representation theorem, there exists a unique function $K: X\times X\to\mathbb{C}$ such that $\{K(x,\cdot):\ x\in X\}\subseteq\mathcal{H}$ and

$$f(x)=\big(f,K(x,\cdot)\big)_{\mathcal{H}},\quad x\in X,\ f\in\mathcal{H}. \tag{1.1}$$

The function K is called the *reproducing kernel* of $\mathcal{H}$. Many things can be said about RKHS because of the existence of a reproducing kernel (see, for example, [6,39,40,42,43]). As far as sampling is concerned, let us assume that there exist sampling points $x_j\in X$, $j\in\mathbb{I}$ such that $K(x_j,\cdot)$, $j\in\mathbb{I}$ constitute a Riesz basis for $\mathcal{H}$. Then by the standard theory of Riesz bases (see, for example, [8]), the frame operator $S:\mathcal{H}\to\mathcal{H}$ defined by

$$\mathcal{S}f:=\sum_{j\in\mathbb{I}}\big(f,K(x_j,\cdot)\big)_{\mathcal{H}}K(x_j,\cdot),\quad f\in\mathcal{H}$$

is bounded, self-adjoint, positive, and invertible. Applying the inverse S^{-1} to both sides of the above equation and noticing (1.1), we obtain the following sampling expansion on $\mathcal{H}$

$$f(x)=\sum_{j\in\mathbb{I}}f(x_j)\big(S^{-1}K(x_j,\cdot)\big)(x),\quad x\in X,\ f\in\mathcal{H}, \tag{1.2}$$

where the series converges absolutely, and uniformly on any subset of X where $K(x,x)$ is bounded (see [2, p. 344]). When $\mathcal{H}$ is the Paley-Wiener space of square-integrable functions on $\mathbb{R}$ whose Fourier transforms are supported on $[-\pi,\pi]$, and $x_j=j$, $j\in\mathbb{I}=\mathbb{Z}$, the reproducing kernel K is the sinc function and (1.2) becomes the celebrated Shannon sampling series. The general formula (1.2) was first discovered by Nashed and Walter in [34]. Recent developments can be found in Refs. [13,17,18,21,32]. One of the main purposes of this paper is to extend it to Banach spaces of functions. This goal motivates the need of understanding the correspondence of Frames, Riesz bases, and RKHS in Banach spaces.

There have been definitions of frames and Riesz bases for a separable Banach space $\mathcal{B}$ [1,3,4,16]. Two Banach spaces are said to be *isomorphic* to each other if there is a bijective bounded linear operator between them. Since it is no longer true that two Banach spaces of the same dimension must be isomorphic to each other, it is important to choose the appropriate sequence spaces in the definition of frames and Riesz bases for Banach spaces. With this consideration, the notion of *BK*-spaces is needed. A *BK-space* X_d on $\mathbb{I}$ is a Banach space of sequences $c=\{c_j\}\in\mathbb{C}^{\mathbb{I}}$ with the property that the coordinate linear functionals $c\to c_j$, $j\in\mathbb{I}$ are continuous on X_d.

Definition 1.1. (See [1,3,4].) An indexed set $\{f_j\}\subseteq\mathcal{B}$ is an X_d-Riesz basis for $\mathcal{B}$ if $\overline{\operatorname{span}}\{f_j\}=\mathcal{B}$, $\sum_{j\in\mathbb{I}}c_jf_j$ converges in $\mathcal{B}$ for all $c\in X_d$ and there exists $0<A\leqslant B<+\infty$ such that

$$A\|c\|_{X_d}\leqslant\Big\|\sum_{i\in\mathbb{I}}c_jf_j\Big\|_{\mathcal{B}}\leqslant B\|c\|_{X_d}\quad\text{for all } c\in X_d. \tag{1.3}$$

Due to the lack of an inner product in a general Banach space $\mathcal{B}$, a frame for $\mathcal{B}$ was defined as an indexed set of linear functionals from the dual space $\mathcal{B}^*$ in [1,3,4,16]. Specifically, $\{\mu_j\}\subseteq\mathcal{B}^*$ was called an X_d-frame for $\mathcal{B}$ if $\{\mu_j(f)\}\in X_d$ for every $f\in\mathcal{B}$ and there exist constants $0<A\leqslant B<+\infty$ such that

$$A\|f\|_{\mathcal{B}}\leqslant\big\|\{\mu_j(f)\}\big\|_{X_d}\leqslant B\|f\|_{\mathcal{B}}\quad\text{for all } f\in\mathcal{B}. \tag{1.4}$$

Thus, according to the above definition, a frame for $\mathcal{B}$ consists of elements from the dual space $\mathcal{B}^*$, not of elements in the original space $\mathcal{B}$ as one might have expected. However, this inconvenience can be avoided using the tool of semi-inner products [27] for Banach spaces.

A semi-inner product on $\mathcal{B}$ is a function on $\mathcal{B}\times\mathcal{B}$, usually denoted by $[\cdot,\cdot]$, such that for all $f,g,h\in\mathcal{B}$ and $\alpha\in\mathbb{C}$

1. $[f+g,h]=[f,h]+[g,h]$,
2. $[\alpha f,g]=\alpha[f,g]$, $[f,\alpha g]=\bar{\alpha}[f,g]$,
3. $[f,f]>0$ for $f\neq 0$,
4. $|[f,g]|^2\leqslant [f,f][g,g]$.

Every Banach space $\mathcal{B}$ has a semi-inner product $[\cdot,\cdot]$ that is *compatible* in the sense that (see [15,27])

$$[f,f]^{1/2}=\|f\|_{\mathcal{B}}\quad\text{for all } f\in\mathcal{B}.$$

The striking difference between a semi-inner product $[\cdot,\cdot]$ and an inner product is that $[\cdot,\cdot]$ is nonadditive with respect to its second variable unless it becomes an inner product [36]. Semi-inner products make possible the development of Hilbert space type arguments in Banach spaces (see, for example, [26–28,37]). They have recently been applied to machine learning. With the aid of semi-inner products, Der and Lee [9] studied hard margin classification in Banach spaces, and we established the theory of reproducing kernel Banach spaces (RKBS) in a recent work [45]. The detailed definition of RKBS will be introduced in Section 4. We present our definition of frames for Banach spaces via semi-inner products below.

Definition 1.2. Let $[\cdot,\cdot]$ be a compatible semi-inner product on $\mathcal{B}$. We call $\{f_j\}\subseteq\mathcal{B}$ an X_d*-frame* for $\mathcal{B}$ if $\{[f,f_j]\}\in X_d$ for all $f\in\mathcal{B}$ and there exist two positive constants A,B such that

$$A\|f\|_{\mathcal{B}}\leqslant\big\|\{[f,f_j]\}\big\|_{X_d}\leqslant B\|f\|_{\mathcal{B}}\quad\text{for all } f\in\mathcal{B}. \tag{1.5}$$

We shall discuss the connection of the above definition of frames with that in [1,3,4,16] in Section 2, where we shall generalize the classical theory of frames and Riesz bases for Hilbert spaces to Banach spaces. Many of the results and arguments for their proofs in Section 2 are merely translations of those in [3,4] in the language of semi-inner products and duality mappings. Besides making the exposition of the paper self-contained, another reason for including the results and proofs is that under the new language they seem to be natural extensions of the counterparts in Hilbert spaces. We shall illustrate two such results here. Let $\mathcal{B}$ and X_d have properties that will be described at the beginning of the next section. Also denote by $[\cdot,\cdot]$ a compatible semi-inner product on $\mathcal{B}$. By properties 3 and 4 of semi-inner products, for each $f\in\mathcal{B}$ the function that sends $g\in\mathcal{B}$ to $[g,f]$ is a bounded linear functional on $\mathcal{B}$. We shall denote this functional associated with f by f^* and call it the *dual element* of f. The mapping $f\to f^*$ will be called the *duality mapping* from $\mathcal{B}$ to $\mathcal{B}^*$. The following two results will be proved in Proposition 2.13 and Theorem 2.15 respectively:

1. An indexed set $\{f_j\}\subseteq\mathcal{B}$ is an X_d-Riesz basis for $\mathcal{B}$ if and only if it is minimal in $\mathcal{B}$ and $\{f_j^*\}$ forms an X_d^*-frame for $\mathcal{B}^*$.
2. If $\{f_j^*\}$ is an X_d^*-Riesz basis for $\mathcal{B}^*$ then there exists a unique X_d-Riesz basis $\{g_j\}$ for $\mathcal{B}$ such that

$$[g_j,f_k]=\delta_{j,k},\quad j,k\in\mathbb{I},$$

where $\delta_{j,k}$ is the Kronecker delta, and

$$f=\sum_{j\in\mathbb{I}}[f,f_j]g_j,\qquad f^*=\sum_{j\in\mathbb{I}}[g_j,f]f_j^*\quad\text{for all } f\in\mathcal{B}.$$

In Section 3, we shall investigate the conditions for the frame operator on Banach spaces to be invertible. A frame $\{f_j\}$ for a Hilbert space $\mathcal{H}$ has the remarkable property that $(f,f_j)_{\mathcal{H}}$ are the most economical coefficients for a decomposition of $f\in\mathcal{H}$ into $S^{-1}f_j$, where S is the frame operator associated with $\{f_j\}$. We shall also establish this result for Banach spaces in Section 3. Our main focus is on Section 4, where we discuss sampling expansions of the form (1.2) in RKBS. Examples based on existing research on weighted Paley–Wiener spaces [29,30,35] and generalized interpolating refinable function vectors [19,23,24] will be presented. The main finding of the paper is the negative result that such expansions do not exist for some common translation invariant RKBS. In particular, as a corollary of this fact, the RKHS of the Gaussian kernels on $\mathbb{R}^d$ do not possess a complete sampling expansion (1.2). The last section is devoted to the discussion of finite-dimensional Banach spaces. Especially, we shall present a nonlinear Gram–Schmidt process to generate a Riesz basis for $\mathcal{B}$ whose dual elements automatically form a Riesz basis for $\mathcal{B}^*$.

2. Frames and Riesz bases via semi-inner products

We start with introducing necessary preliminaries on semi-inner products, and desired properties of the Banach space $\mathcal{B}$ and *BK*-space X_d under consideration.

Let $\mathcal{B}$ be a separable Banach space and $[\cdot,\cdot]$ a compatible semi-inner product on $\mathcal{B}$. We require that $\mathcal{B}$ be reflexive and strictly convex. In other words, $(\mathcal{B}^*)^*=\mathcal{B}$, and whenever $\|f+g\|_{\mathcal{B}}=\|f\|_{\mathcal{B}}+\|g\|_{\mathcal{B}}$ where $f,g\neq 0$ then $f=\alpha g$ for some

$\alpha > 0$. An important consequence [12] is that the duality mapping from $\mathcal{B}$ and $\mathcal{B}^*$ is bijective. In other words, for every linear functional $\mu \in \mathcal{B}^*$ there exists a unique $f \in \mathcal{B}$ such that

$$\mu(g) = f^*(g) = [g, f] \quad \text{for all } g \in \mathcal{B}. \tag{2.1}$$

We also note that the duality mapping is isometric, namely,

$$\|f^*\|_{\mathcal{B}^*} = \|f\|_{\mathcal{B}} \quad \text{for all } f \in \mathcal{B}. \tag{2.2}$$

Furthermore, it was observed in [15] that the function $[\cdot,\cdot]_* : \mathcal{B}^* \times \mathcal{B}^* \to \mathbb{C}$ defined by

$$[f^*, g^*]_* := [g, f], \quad f, g \in \mathcal{B} \tag{2.3}$$

is a compatible semi-inner product on $\mathcal{B}^*$.

Let $\mathbb{I}$ be a countable index set that has been well-ordered. We shall denote by $\mathbb{I}_n$, $n \in \mathbb{N}$, the subset of the first n indices in $\mathbb{I}$. If $\#\mathbb{I} < +\infty$ then $\mathbb{I}_n = \mathbb{I}$ for $n \geqslant \#\mathbb{I}$. The notation X_d will always be reserved to denote a *BK*-space on $\mathbb{I}$. We shall also require that the canonical unit vectors e_j, $j \in \mathbb{I}$ form a Schauder basis for X_d. In other words, every $c \in X_d$ equals $\sum_{j\in\mathbb{I}} c_j e_j$ in the sense that

$$\lim_{n\to\infty} \Big\| c - \sum_{j\in\mathbb{I}_n} c_j e_j \Big\|_{X_d} = 0,$$

and the coefficients in a decomposition of c into e_j are unique. By a result in [25], the dual space X_d^* of X_d is also a *BK*-space of sequences $d = \{d_j\} \subseteq \mathbb{C}$ such that

$$d(c) = \sum_{j\in\mathbb{I}} c_j d_j, \quad c \in X_d,\ d \in X_d^*.$$

For instance, if $X_d = \ell^p(\mathbb{I})$, $1 < p < +\infty$ then $X_d^* = \ell^q(\mathbb{I})$, where $1/p + 1/q = 1$. We impose several more crucial assumptions on X_d, which are satisfied by $\ell^p(\mathbb{I})$, $p \in (1, +\infty)$. Specifically, we require that X_d be reflexive, the canonical unit vectors e_j, $j \in \mathbb{I}$ form a Schauder basis for X_d^* as well, if $d = \{d_j\} \in \mathbb{C}^{\mathbb{I}}$ satisfies

$$\sum_{j\in\mathbb{I}} c_j d_j \tag{2.4}$$

converges for every $c \in X_d$ then $d \in X_d^*$, and if the above series converges for all $d \in X_d^*$ then $c \in X_d$.

The above notations and requirements about the spaces $\mathcal{B}$ and X_d are assumed throughout the rest of the paper.

2.1. Frames

We shall see that the lower bound inequality in Definition 1.1 and the upper bound inequality in Definition 1.2 each lead to a new object in Banach spaces, whose precise definitions are given below.

Definition 2.1. An indexed set $\{f_j\}$ is called an X_d-*Bessel sequence* for $\mathcal{B}$ if $\{[f, f_j]\} \in X_d$ for all $f \in \mathcal{B}$. It is said to be an X_d-*Riesz–Fischer sequence* for $\mathcal{B}$ if for all $c \in X_d$ there exists some $f \in \mathcal{B}$ such that

$$[f, f_j] = c_j, \quad j \in \mathbb{I}. \tag{2.5}$$

One might replace X_d by X_d^*, $\mathcal{B}$ by $\mathcal{B}^*$, or both in Definitions 1.1, 1.2, or 2.1 to obtain other definitions. For instance, we get by (2.2) and (2.3) the following useful fact.

Lemma 2.2. *Let $\{f_j\} \subseteq \mathcal{B}$. Then $\{f_j^*\}$ is an X_d^*-frame for $\mathcal{B}^*$ if and only if $\{[f_j, f]\} \in X_d^*$ for all $f \in \mathcal{B}$ and there exist two positive constants A, B such that*

$$A\|f\|_{\mathcal{B}} \leqslant \big\|\{[f_j, f]\}\big\|_{X_d^*} \leqslant B\|f\|_{\mathcal{B}} \quad \text{for all } f \in \mathcal{B}. \tag{2.6}$$

The purpose of this subsection is to explore the properties of frames, Bessel sequences and Riesz–Fischer sequences, and relationships among them. Before moving on, let us make connections with the existing definitions [1,3,4,16] of frames and Bessel sequences in Banach spaces, and the classical ones [8,10,31,44] for Hilbert spaces. We shall discuss frames only. Recall that in Refs. [1,3,4,16], an X_d-frame for $\mathcal{B}$ consists of elements $\mu_j \in \mathcal{B}^*$, $j \in \mathbb{I}$ that satisfies $\{\mu_j(f)\} \in X_d$ for all $f \in \mathcal{B}$ and Eq. (1.4). One sees from (2.1) that $\{f_j\}$ is an X_d-frame for $\mathcal{B}$ satisfying Definition 1.2 if and only if $\{f_j^*\}$ is an X_d-frame for $\mathcal{B}$ in the sense of [1,3,4,16]. When $[\cdot,\cdot]$ is an inner product on $\mathcal{B}$ and $X_d = X_d^* = \ell^2(\mathbb{I})$, by

$$[f,g]=\overline{[g,f]},\quad f,g\in\mathcal{B},$$

$\{f_j\}$ is an X_d-frame for $\mathcal{B}$ if and only if $\{f_j^*\}$ is an X_d^*-frame for $\mathcal{B}^*$. Example 5.3 in Section 5 illustrates that this in general does not hold if $\mathcal{B}$ is a Banach space.

We work toward our goal by starting with Bessel and Riesz–Fischer sequences. The results below generalize the classical one for Hilbert spaces (see, for example, [44]).

For an indexed set $\{f_j\}\subseteq\mathcal{B}$, we introduce two linear operators $U:\mathcal{B}\to\mathbb{C}^{\mathbb{I}}$ and $V:\mathcal{B}^*\to\mathbb{C}^{\mathbb{I}}$ by setting

$$Uf:=\big\{[f,f_j]\big\},\quad f\in\mathcal{B}\tag{2.7}$$

and

$$V(\mu):=\big\{\mu(f_j)\big\},\quad \mu\in\mathcal{B}^*.$$

One observes that

$$V\big(f^*\big)=\big\{[f_j,f]\big\},\quad f\in\mathcal{B}.$$

Using the operator U, Definition 1.2 might be abbreviated as $\{f_j\}\subseteq\mathcal{B}$ is an X_d-frame for $\mathcal{B}$ if and only if $Uf\in X_d$ for all $f\in\mathcal{B}$ and

$$A\|f\|_{\mathcal{B}}\leqslant\|Uf\|_{X_d}\leqslant B\|f\|_{\mathcal{B}}.\tag{2.8}$$

Similar formulations hold for X_d-Bessel sequences for $\mathcal{B}$, X_d^*-frames for $\mathcal{B}^*$, and X_d^*-Bessel sequences for $\mathcal{B}^*$.

Proposition 2.3. *If $\{f_j\}$ is an X_d-Bessel sequence for $\mathcal{B}$ then there exists some $B>0$ such that*

$$\big\|\big\{[f,f_j]\big\}\big\|_{X_d}\leqslant B\|f\|_{\mathcal{B}}\quad\text{for all } f\in\mathcal{B}.\tag{2.9}$$

If $\{f_j\}$ is an X_d-Riesz–Fischer sequence for $\mathcal{B}$ then there is some $A>0$ such that for every $c\in X_d$ there exists $f\in\mathcal{B}$ that satisfies (2.5) *and*

$$A\|f\|_{\mathcal{B}}\leqslant\|c\|_{X_d}.\tag{2.10}$$

Proof. Suppose that $\{f_j\}$ is an X_d-Bessel sequence for $\mathcal{B}$, that is, $Uf\in X_d$. Then it is obvious that U has a closed graph. By the closed graph theorem, U is bounded. Thus, (2.9) holds true for some $B>0$.

Let us deal with the second claim. Let $\{f_j\}$ be an X_d-Riesz–Fischer sequence for $\mathcal{B}$, and $\mathcal{C}$ the set of all the elements $f\in\mathcal{B}$ such that $f_j^*(f)=0$, $j\in\mathbb{I}$. Clearly, $\mathcal{C}$ is a closed subspace of $\mathcal{B}$. We denote for each $f\in\mathcal{B}$ by $\tilde f$ the element $f+\mathcal{C}$ in the quotient space $\mathcal{B}/\mathcal{C}$. Introduce a mapping $T:X_d\to\mathcal{B}/\mathcal{C}$ which sends $c\in X_d$ to $\tilde f$ where f is some element in $\mathcal{B}$ satisfying (2.5). Clearly, T is well-defined and has a closed graph. It is hence bounded. In other words, for each $c\in X_d$ there exists $g\in\mathcal{B}$ satisfying $\{[g,f_j]\}=c$ and

$$A\|\tilde g\|_{\mathcal{B}/\mathcal{C}}\leqslant\|c\|_{X_d}.$$

Since $\mathcal{B}$ is reflexive, there exists $h\in\mathcal{B}$ such that (see [5, p. 133])

$$\|g-h\|_{\mathcal{B}}=\inf\big\{\big\|g-h'\big\|_{\mathcal{B}}:\ h'\in\mathcal{C}\big\}=\|\tilde g\|_{\mathcal{B}/\mathcal{C}}.$$

By the above two equations, $f:=g-h$ satisfies our requirements. □

There is a characterization of Riesz–Fischer sequences that is easy to apply.

Proposition 2.4. *An indexed set $\{f_j\}$ is an X_d-Riesz–Fischer sequence for $\mathcal{B}$ with* (2.10) *if and only if*

$$A\|d\|_{X_d^*}\leqslant\bigg\|\sum_{j\in\mathbb{I}}d_jf_j^*\bigg\|_{\mathcal{B}^*}\tag{2.11}$$

for all $d\in X_d^*$ with at most finitely many nonzero components.

Proof. Suppose that $\{f_j\}$ is an X_d-Riesz–Fischer sequence for $\mathcal{B}$ with (2.10). Let $d\in X_d$ be of finitely many nonzero components. We find $c\in X_d$ with

$$\|c\|_{X_d}=1\quad\text{and}\quad\bigg|\sum_{j\in\mathbb{I}}c_jd_j\bigg|=\|d\|_{X_d^*}.$$

By the assumption, there exists some $f \in \mathcal{B}$ with (2.5) and $\|f\|_{\mathcal{B}} \leqslant 1/A$. We get that

$$\left|\sum_{j\in\mathbb{I}} c_j d_j\right| = \left|\sum_{j\in\mathbb{I}} [f, f_j] d_j\right| = \left|\left(\sum_{j\in\mathbb{I}} d_j f_j^*\right)(f)\right| \leqslant \|f\|_{\mathcal{B}} \left\|\sum_{j\in\mathbb{I}} d_j f_j^*\right\|_{\mathcal{B}^*} \leqslant \frac{1}{A}\left\|\sum_{j\in\mathbb{I}} d_j f_j^*\right\|_{\mathcal{B}^*}.$$

Combining the above two equations yields (2.11).

On the other hand, suppose that (2.11) holds true for all $d \in X_d$ with finitely many nonzero components. Let $\mathcal{C} = \operatorname{span}\{f_j^*\}$ and define a linear functional ν on $\mathcal{C}$ by setting for each $d \in X_d^*$ with finitely many nonzero components

$$\nu\left(\sum_{j\in\mathbb{I}} d_j f_j^*\right) = \sum_{j\in\mathbb{I}} d_j c_j.$$

By (2.11),

$$\left|\sum_{j\in\mathbb{I}} d_j c_j\right| \leqslant \|d\|_{X_d^*}\|c\|_{X_d} \leqslant \frac{1}{A}\|c\|_{X_d}\left\|\sum_{j\in\mathbb{I}} d_j f_j^*\right\|_{\mathcal{B}^*}, \tag{2.12}$$

which implies that ν is bounded. We extend ν by the Hahn–Banach theorem to be on the whole space $\mathcal{B}^*$. The resulting linear functional on $\mathcal{B}^*$ is still denoted by ν. By (2.12), its norm $\|\nu\|_{\mathcal{B}^{**}}$ is bounded by $\|c\|_{X_d}/A$. Since $\mathcal{B}$ is reflexive, there exists $f \in \mathcal{B}$ such that

$$\|f\|_{\mathcal{B}} = \|\nu\|_{\mathcal{B}^{**}} \leqslant \frac{1}{A}\|c\|_{X_d} \tag{2.13}$$

and

$$\left(\sum_{j\in\mathbb{I}} d_j f_j^*\right)(f) = \nu\left(\sum_{j\in\mathbb{I}} d_j f_j^*\right) = \sum_{j\in\mathbb{I}} d_j c_j.$$

In particular, the above equation implies that

$$f_j^*(f) = [f, f_j] = c_j, \quad j \in \mathbb{I}. \tag{2.14}$$

We conclude by (2.13) and (2.14) that $\{f_j\}$ is an X_d-Riesz–Fischer sequence for $\mathcal{B}$ with (2.10). □

We then study the two inequalities in the definition of frames. The following proposition shows that the lower bound inequality leads to a completeness condition in the dual space.

Proposition 2.5. *Let $\{f_j\} \subseteq \mathcal{B}$. If there exists a positive constant A such that*

$$A\|f\|_{\mathcal{B}} \leqslant \left\|\left\{[f, f_j]\right\}\right\|_{X_d} \quad \text{for all } f \in \mathcal{B} \tag{2.15}$$

then

$$\overline{\operatorname{span}}\left\{f_j^*\right\} = \mathcal{B}^*. \tag{2.16}$$

Similarly, if for some $A > 0$

$$A\|f\|_{\mathcal{B}} \leqslant \left\|\left\{[f_j, f]\right\}\right\|_{X_d} \quad \text{for all } f \in \mathcal{B}$$

then there holds

$$\overline{\operatorname{span}}\{f_j\} = \mathcal{B}. \tag{2.17}$$

Proof. We shall rely on the well-known fact that a subset Y' of a Banach space Y satisfies $\overline{\operatorname{span}} Y' = Y$ if and only if there does not exist a nonzero $\mu \in Y^*$ that vanishes everywhere on Y'. Suppose that (2.15) holds true. We assume to the contrary that (2.16) is not true. By the fact and the reflexivity of $\mathcal{B}$, there exists some nonzero $f \in \mathcal{B}$ such that

$$[f, f_j] = f\left(f_j^*\right) = 0 \quad \text{for all } j \in \mathbb{I},$$

which leads by (2.15) to the contradiction that $\|f\|_{\mathcal{B}} = 0$. The second claim can be proved similarly using the additional fact that $\mathcal{B}^* = \{f^*\colon f \in \mathcal{B}\}$. □

Let us turn to the upper bound inequality.

Proposition 2.6. *A subset $\{f_j\} \subseteq \mathcal{B}$ forms an X_d-Bessel sequence for $\mathcal{B}$ satisfying* (2.9) *if and only if $\sum_{j\in\mathbb{I}} d_j f_j^*$ converges in $\mathcal{B}^*$ for all $d \in X_d^*$ and*

$$\left\|\sum_{j\in\mathbb{I}} d_j f_j^*\right\|_{\mathcal{B}^*} \leqslant B\|d\|_{X_d^*}. \tag{2.18}$$

Proof. (See, also, Proposition 3.2 of [3] and Proposition 2.2 of [4].[2]) Suppose that $\{f_j\}$ is an X_d-Bessel sequence for $\mathcal{B}$ satisfying (2.9). Let $d \in X_d^*$. We estimate for positive integers $m > n$ that

$$\begin{aligned}\left\|\sum_{j\in I_m\setminus I_n} d_j f_j^*\right\|_{\mathcal{B}^*} &= \sup_{f\in\mathcal{B},\,\|f\|_{\mathcal{B}}\leqslant 1}\left|\sum_{j\in I_m\setminus I_n} d_j f_j^*(f)\right| = \sup_{f\in\mathcal{B},\,\|f\|_{\mathcal{B}}\leqslant 1}\left|\sum_{j\in I_m\setminus I_n} d_j [f, f_j]\right| \\ &\leqslant \left\|\sum_{j\in I_m\setminus I_n} d_j e_j\right\|_{X_d^*} \sup_{f\in\mathcal{B},\,\|f\|_{\mathcal{B}}\leqslant 1} \left\|\{[f, f_j]\}\right\|_{X_d} \\ &\leqslant \left\|\sum_{j\in I_m\setminus I_n} d_j e_j\right\|_{X_d^*} B. \end{aligned} \tag{2.19}$$

As e_j form a Schauder basis for X_d^*, $\|\sum_{j\in I_m\setminus I_n} d_j e_j\|_{X_d^*}$ goes to zero as m, n tend to infinity. As a result, $\sum_{j\in\mathbb{I}} d_j f_j^*$ converges in $\mathcal{B}^*$.

Let $\varepsilon > 0$. Then for large enough n,

$$\left\|\sum_{j\in I_n} d_j e_j\right\|_{X_d^*} \leqslant \|d\|_{X_d^*} + \varepsilon.$$

Using the same technique as that engaged in (2.19), we obtain for such n that

$$\left\|\sum_{j\in I_n} d_j f_j^*\right\|_{\mathcal{B}^*} \leqslant B\left\|\sum_{j\in I_n} d_j e_j\right\|_{X_d^*} \leqslant B\big(\|d\|_{X_d^*} + \varepsilon\big).$$

Eq. (2.18) follows immediately from the above equation.

Conversely, assume that $\sum_{j\in\mathbb{I}} d_j f_j^*$ converges in $\mathcal{B}^*$ for all $d \in X_d^*$ and (2.18) holds true. Then

$$\lim_{n\to\infty}\sum_{j\in\mathbb{I}_n} d_j[f, f_j] = \lim_{n\to\infty}\left(\sum_{j\in\mathbb{I}_n} d_j f_j^*\right)(f) = \left(\lim_{n\to\infty}\sum_{j\in\mathbb{I}_n} d_j f_j^*\right)(f) = \left(\sum_{j\in\mathbb{I}} d_j f_j^*\right)(f), \quad f \in \mathcal{B}.$$

By our requirements on X_d and X_d^*, $\{[f, f_j]\} \in X_d$ for all $f \in \mathcal{B}$. We also estimate by (2.18) for every $d \in X_d^*$ that

$$\left|\sum_{j\in\mathbb{I}} d_j[f, f_j]\right| = \left|\left(\sum_{j\in\mathbb{I}} d_j f_j^*\right)(f)\right| \leqslant \left\|\sum_{j\in\mathbb{I}} d_j f_j^*\right\|_{\mathcal{B}^*} \|f\|_{\mathcal{B}} \leqslant B\|d\|_{X_d^*}\|f\|_{\mathcal{B}},$$

from which (2.9) follows. The proof is complete. □

Let $\{f_j\}$ be an X_d-Bessel sequence for $\mathcal{B}$. One can see from the above proof that if X_d and X_d^* possess the additional property that for all $c \in X_d$ and $d \in X_d^*$, series (2.4) converges absolutely then $\sum_{j\in\mathbb{I}} d_j f_j^*$ converges unconditionally in $\mathcal{B}^*$. In other words, $\sum_{j\in\mathbb{I}} d_j f_j^*$ converges to the same element in $\mathcal{B}^*$ independent of the arrange of the summation order. The observation applies to most of the convergence in the paper and we shall not point it out explicitly any more.

We have a parallel result for X_d^*-Bessel sequence for $\mathcal{B}^*$.

Proposition 2.7. *Let $\{f_j\} \subseteq \mathcal{B}$. Then $\{f_j^*\} \subseteq \mathcal{B}$ is an X_d^*-Bessel sequence for $\mathcal{B}^*$, that is, $\{[f_j, f]\} \in X_d^*$ for all $f \in \mathcal{B}$ and there exists $B > 0$ such that*

$$\left\|\{[f_j, f]\}\right\|_{X_d^*} \leqslant B\|f\|_{\mathcal{B}} \quad \text{for all } f \in \mathcal{B}$$

if and only if $\sum_{j\in\mathbb{I}} c_j f_j$ converges in $\mathcal{B}$ for all $c \in X_d$ and

[2] The proof provided here can be considered as a translation of those of Proposition 3.2 of [3] and Proposition 2.2 of [4] in the language of semi-inner products and duality mappings. We shall not repeat this footnote.

$$\left\|\sum_{j\in\mathbb{I}} c_j f_j\right\|_{\mathcal{B}} \leqslant B\|c\|_{X_d}.$$

By Proposition 2.6, if $\{f_j\}$ is an X_d-Bessel sequence for $\mathcal{B}$ then U is a bounded linear operator from $\mathcal{B}$ to X_d and its adjoint $U^*: X_d^* \to \mathcal{B}^*$ has the form

$$U^*d = \sum_{j\in\mathbb{I}} d_j f_j^*, \quad d \in X_d^*. \tag{2.20}$$

Proposition 2.7 implies that if $\{f_j^*\}$ is an X_d^*-Bessel sequence for $\mathcal{B}^*$ then V is bounded from $\mathcal{B}^*$ to X_d^* and $V^*: X_d \to \mathcal{B}$ is of the form

$$V^*c = \sum_{j\in\mathbb{I}} c_j f_j, \quad c \in X_d. \tag{2.21}$$

There is a characterization of X_d-frames for $\mathcal{B}$ and X_d^*-frame for $\mathcal{B}^*$ in terms of U^* and V^*, respectively.

Lemma 2.8. *A sequence $\{f_j\}$ is an X_d-frame for $\mathcal{B}$ if and only if the operator U is bounded from $\mathcal{B}$ to X_d and has a bounded inverse on $\mathcal{R}(U)$. Likewise, $\{f_j^*\}$ is an X_d^*-frame for $\mathcal{B}^*$ if and only if V is bounded and possess a bounded inverse on $\mathcal{R}(V)$.*

Proof. The results are straightforward reformulations of the definitions. $\square$

We remark that U is bounded and has a bounded inverse on $\mathcal{R}(U)$ implies that $\mathcal{R}(U)$ is closed in X_d. Thus, if $\{f_j\}$ is an X_d-frame for $\mathcal{B}$ then $\mathcal{R}(U)$ is a closed subspace of X_d.

Proposition 2.9. *A sequence $\{f_j\}$ is an X_d-frame for $\mathcal{B}$ if and only if U^* defined by* (2.20) *is bounded and surjective from X_d^* to $\mathcal{B}^*$. Similarly, $\{f_j^*\}$ is an X_d^*-frame for $\mathcal{B}^*$ if and only if V^* given by* (2.21) *is bounded and surjective from X_d to $\mathcal{B}$.*

Proof. (See, also, Theorem 2.4 of [4].) The results follow from Lemma 2.8 and the fact that a bounded linear operator between two Banach spaces has a bounded inverse on its range if and only if its adjoint is surjective. $\square$

2.2. Riesz bases

Recall Definition 1.1 of X_d-Riesz bases for $\mathcal{B}$. By the definition, $\{f_j^*\}$ is an X_d^*-Riesz basis for $\mathcal{B}^*$ if and only if $\overline{\operatorname{span}}\{f_j^*\} = \mathcal{B}^*$, $\sum_{j\in\mathbb{I}} d_j f_j^*$ converges in $\mathcal{B}^*$ for all $d \in X_d^*$ and there exists $0 < A \leqslant B < +\infty$ such that

$$A\|d\|_{X_d^*} \leqslant \left\|\sum_{j\in\mathbb{I}} d_j f_j^*\right\|_{\mathcal{B}^*} \leqslant B\|d\|_{X_d^*} \quad \text{for all } d \in X_d^*. \tag{2.22}$$

It is straightforward that an X_d-Riesz basis for $\mathcal{B}$ must be a Schauder basis. We next show that a Riesz basis automatically generates a frame in the dual space.

Proposition 2.10. *If $\{f_j^*\}$ is an X_d^*-Riesz basis for $\mathcal{B}^*$ satisfying* (2.22) *then $\{f_j\}$ is an X_d-frame for $\mathcal{B}$ satisfying* (1.5)*. On the other hand, if $\{f_j\}$ is an X_d-Riesz basis for $\mathcal{B}$ satisfying* (1.3) *then $\{f_j^*\}$ is an X_d^*-frame for $\mathcal{B}^*$ satisfying* (2.6)*.*

Proof. (See, also, Corollary 2.5 of [4].) We shall only provide the proof for the first result. Suppose that $\{f_j^*\}$ is an X_d^*-Riesz basis for $\mathcal{B}^*$ satisfying (2.22). Eq. (2.22) implies that U^* given by (2.20) is bounded from X_d^* to $\mathcal{B}^*$ and has a bounded inverse on $\mathcal{R}(U^*)$. Thus, $\mathcal{R}(U^*)$ is closed in $\mathcal{B}^*$. This together with $\overline{\operatorname{span}}\{f_j^*\} = \mathcal{B}^*$ implies that U^* is surjective. By Proposition 2.9, $\{f_j\}$ is an X_d-frame for $\mathcal{B}$. By (2.22), U^* is also injective. Therefore, U^* is bijective from X_d^* to $\mathcal{B}^*$. As a result, U is bijective from $\mathcal{B}$ to X_d. That the X_d^*-Riesz basis $\{f_j^*\}$ for $\mathcal{B}^*$ and the X_d-frame $\{f_j\}$ for $\mathcal{B}$ share the same bounds A, B follows from the fact that

$$\|U\| = \|U^*\| \quad \text{and} \quad \|U^{-1}\| = \|(U^{-1})^*\| = \|(U^*)^{-1}\|. \tag{2.23}$$

The proof is complete. $\square$

We next give a characterization of Riesz bases. To this end, we note by the Hahn–Banach theorem that $\{v_j\} \subseteq V$ is minimal in a Banach space Y if and only if there exists $\mu_j \in Y^*$, $j \in \mathbb{I}$ such that

$$\mu_j(v_k) = \delta_{j,k}, \quad j, k \in \mathbb{I}.$$

Proposition 2.11. *Let $\{f_j\} \subseteq \mathcal{B}$. Then $\{f_j^*\} \subseteq \mathcal{B}^*$ is an X_d^*-Riesz basis for $\mathcal{B}^*$ satisfying* (2.22) *if and only if $\{f_j^*\}$ is minimal in $\mathcal{B}^*$ and $\{f_j\}$ is an X_d-frame for $\mathcal{B}$ satisfying* (1.5)*.*

Proof. (See, also, Proposition 2.7 of [4].) Suppose first that $\{f_j^*\} \subseteq \mathcal{B}^*$ is an X_d^*-Riesz basis for $\mathcal{B}^*$ satisfying (2.22). By Proposition 2.10, (1.5) holds true. Assume to the contrary that $\{f_j^*\}$ is not minimal, that is, there exists some $j \in \mathbb{I}$ such that

$$f_j^* \subseteq \overline{\operatorname{span}}\{f_k^*\colon k \in \mathbb{I},\ k \neq j\}.$$

Then there exists $d^n \in X_d^*$, $n \in \mathbb{N}$, with $d_j^n = 0$ and $d_k^n \neq 0$ for at most finitely many indices $k \in \mathbb{I}$ such that

$$\lim_{n\to\infty} \sum_{k\in\mathbb{I}} d_k^n f_k^* = f_j^*.$$

We get by (2.22) that

$$A\|d^n - d^m\|_{X_d^*} \leqslant \left\| \sum_{k\in\mathbb{I}} d_k^n f_k^* - \sum_{k\in\mathbb{I}} d_k^m f_k^* \right\|_{\mathcal{B}^*}, \quad m, n \in \mathbb{N},$$

which implies that d^n converges to some $d \in X_d^*$ as $n \to \infty$. Since $d_j^n = 0$ for every $n \in \mathbb{N}$ and coordinate functionals are continuous on X_d^*, $d_j = 0$. We reach that

$$f_j^* - \sum_{k\in\mathbb{I}, k\neq j} d_k f_k^* = 0,$$

which contradicts (2.22).

Conversely, suppose that $\{f_j^*\}$ is minimal and (1.5) holds true. By Proposition 2.5, $\overline{\operatorname{span}}\{f_j^*\} = \mathcal{B}^*$. Note that Eq. (1.5) implies that f_j form an X_d-frame for $\mathcal{B}$. By Propositions 2.6 and 2.9, $\sum_{j\in\mathbb{I}} d_j f_j^*$ converges in $\mathcal{B}^*$ for all $d \in X_d^*$, $\|U^*\| \leqslant B$ and U^* is surjective. That $\{f_j^*\}$ being minimal implies that U^* is also injective and is hence bijective. Thus, U is bijective as well. The first inequality in (2.22) then follows from Eq. (2.23). $\square$

Another characterization of X_d^*-Riesz bases for $\mathcal{B}^*$ is given below.

Proposition 2.12. *An indexed set $\{f_j^*\}$ is an X_d^*-Riesz basis for $\mathcal{B}^*$ if and only if $\overline{\operatorname{span}}\{f_j^*\} = \mathcal{B}^*$ and U is surjective onto X_d.*

Proof. If $\{f_j^*\}$ is an X_d^*-Riesz basis for $\mathcal{B}^*$ then by Proposition 2.10, U is bounded from $\mathcal{B}$ to X_d. Moreover, U^* is bijective. So is U. In particular, U is surjective. On the other hand, suppose that $\overline{\operatorname{span}}\{f_j^*\} = \mathcal{B}^*$ and U is surjective onto X_d. Then U is injective and by the closed graph theorem, U is bounded. Thus, U is bounded and bijective. It follows that U^* is bounded and bijective as well. Therefore, $\{f_j^*\}$ is an X_d^*-Riesz basis for $\mathcal{B}^*$. $\square$

The following result about X_d-Riesz bases for $\mathcal{B}$ can be proved in a similar way.

Proposition 2.13. *A sequence $\{f_j\}$ is an X_d-Riesz basis for $\mathcal{B}$ satisfying* (1.3) *if and only if f_j are minimal in $\mathcal{B}$ and $\{f_j^*\}$ is an X_d^*-frame for $\mathcal{B}^*$ satisfying* (2.6)*.*

2.3. Reconstruction

Let $\{f_j\}$ be an X_d-frame for $\mathcal{B}$. By Lemma 2.8 and the remark following it, $U : \mathcal{B} \to X_d$ given by (2.7) is bounded linear, injective, has a closed range $\mathcal{R}(U)$ and a bounded inverse on $\mathcal{R}(U)$. We are concerned with the reconstruction of an element $f \in \mathcal{B}$ from its data $Uf \in X_d$. Following [16], we call $\{f_j\}$ a *Banach frame* for $\mathcal{B}$ if there exists a bounded linear operator $T : X_d \to \mathcal{B}$ such that

$$TUf = f \quad \text{for all } f \in \mathcal{B}. \tag{2.24}$$

We say that $\mathcal{R}(U)$ has an *algebraic complement* in X_d if there exists another closed linear subspace $\mathcal{C}$ of X_d such that $X_d = \mathcal{R}(U) \oplus \mathcal{C}$ in the sense that $\mathcal{R}(U) \cap \mathcal{C} = \{0\}$ and for every $c \in X_d$ there exists $c_1 \in \mathcal{R}(U)$ and $c_2 \in \mathcal{C}$ such that $c = c_1 + c_2$. By a result in [16] (see, also, [3,4]), an X_d-frame $\{f_j\}$ for $\mathcal{B}$ is a Banach frame if and only if $\mathcal{R}(U)$ has an algebraic complement in X_d. If $X_d = \ell^2(\mathbb{I})$ then every closed linear subspace $\mathcal{C}$ of it has an orthogonal complement, which is an algebraic complement of $\mathcal{C}$. As a consequence, every $\ell^2(\mathbb{I})$-frame for $\mathcal{B}$ is a Banach frame. Conversely, if every closed linear subspace of X_d has an algebraic complement then X_d must be isomorphic to an Hilbert space [5].

Assume that $\mathcal{R}(U)$ has an algebraic complement in X_d and T is a bounded linear operator from X_d to $\mathcal{B}$ satisfying (2.24). Setting $g_j := Te_j$, $j \in \mathbb{I}$, we obtain for each $f \in \mathcal{B}$ that

$$f = TUf = T\left(\sum_{j\in\mathbb{I}}[f, f_j]e_j\right) = \sum_{j\in\mathbb{I}}[f, f_j](Te_j) = \sum_{j\in\mathbb{I}}[f, f_j]g_j. \tag{2.25}$$

It is observed for all $c \in X_d$ that

$$\left\|\sum_{j\in\mathbb{I}} c_j g_j\right\|_{\mathcal{B}} = \left\|T\left(\sum_{j\in\mathbb{I}} c_j e_j\right)\right\|_{\mathcal{B}} \leqslant \|T\|\|c\|_{X_d}.$$

Moreover, (2.24) implies that T is surjective onto $\mathcal{B}$. Thus, every element in $\mathcal{B}$ must be of the form $\sum_{j\in\mathbb{I}} c_j g_j$ for some $c \in X_d$. By Proposition 2.9, $\{g_j^*\}$ is an X_d^*-frame for $\mathcal{B}$. We hence reach the following result.

Theorem 2.14. *If $\{f_j\}$ is an X_d-frame for $\mathcal{B}$ and $\mathcal{R}(U)$ has an algebraic complement in X_d then there exists an X_d^*-frame $\{g_j^*\}$ for $\mathcal{B}^*$ such that*

$$f = \sum_{j\in\mathbb{I}} [f, f_j]g_j \quad \text{for all } f \in \mathcal{B} \tag{2.26}$$

and

$$f^* = \sum_{j\in\mathbb{I}} [g_j, f]f_j^* \quad \text{for all } f \in \mathcal{B}. \tag{2.27}$$

Proof. It remains to prove (2.27). By (2.26),

$$g^*(f) = [f, g] = \sum_{j\in\mathbb{I}}[f, f_j][g_j, g] \quad \text{for all } f, g \in \mathcal{B}. \tag{2.28}$$

We estimate that there exists some $B > 0$ such that for all $m, n \in \mathbb{N}$

$$\begin{aligned}\left|\left(\sum_{j\in\mathbb{I}_m\setminus\mathbb{I}_n} [g_j, g]f_j^*\right)(f)\right| &= \left|\sum_{j\in\mathbb{I}_m\setminus\mathbb{I}_n} [g_j, g][f, f_j]\right| \leqslant \left\|\{[f, f_j]\}\right\|_{X_d}\left\|\sum_{j\in\mathbb{I}_m\setminus\mathbb{I}_n} [g_j, g]e_j\right\|_{X_d^*} \\ &\leqslant B\|f\|_{\mathcal{B}}\left\|\sum_{j\in\mathbb{I}_m\setminus\mathbb{I}_n} [g_j, g]e_j\right\|_{X_d^*},\end{aligned}$$

which implies that $\sum_{j\in\mathbb{I}_n}[g_j, g]f_j^*$ converges in $\mathcal{B}^*$ as $n \to \infty$. By (2.28), it converges to g^*. Eq. (2.27) is hence proved. □

If additionally, f_j^* are minimal then by Proposition 2.11, $\{f_j^*\}$ is an X_d^*-Riesz basis for $\mathcal{B}^*$. Moreover, we have in this case that U is bijective. Thus, the only $T : X_d \to \mathcal{B}$ satisfying (2.24) is U^{-1}. We then get that $g_j = U^{-1}e_j$ are minimal. By Proposition 2.13 and Theorem 2.14, $\{g_j\}$ is an X_d-Riesz basis for $\mathcal{B}$. We draw the conclusion below.

Theorem 2.15. *If $\{f_j\}$ is an X_d-frame for $\mathcal{B}$ for which $\{f_j^*\}$ is minimal in $\mathcal{B}^*$ then there exists a unique X_d-Riesz basis $\{g_j\}$ for $\mathcal{B}$ such that there hold* (2.26), (2.27), *and*

$$[g_j, f_k] = \delta_{j,k}, \quad j, k \in \mathbb{I}. \tag{2.29}$$

Proof. It remains to prove (2.29). By (2.26),

$$g_j = \sum_{k\in\mathbb{I}} [g_j, f_k]g_k.$$

Since $\{g_j\}$ is minimal, we obtain (2.29). □

3. The standard reconstruction operator

It is well-known that if $\mathcal{B}$ is a Hilbert space and $\{f_j\} \subseteq \mathcal{B}$ is an $\ell^2(\mathbb{I})$-frame for $\mathcal{B}$ then the operator $S : \mathcal{B} \to \mathcal{B}$ defined by

$$Sf := \sum_{j\in\mathbb{I}}[f, f_j]f_j, \quad f \in \mathcal{B} \tag{3.1}$$

is bijective and bounded. As a consequence, there holds

$$f=\sum_{j\in\mathbb{I}}[f,f_j]S^{-1}f_j,\quad f\in\mathcal{B}. \tag{3.2}$$

The purpose of this section is to examine conditions for the above reconstruction strategy to hold in separable Banach spaces.

Let us return to the Banach space setting. For (3.1) to be a well-defined bounded linear operator from $\mathcal{B}$ to itself, we shall assume that the operator $U:\mathcal{B}\to X_d$ defined by (2.7) and the operator $V^*:X_d\to\mathcal{B}$ defined by (2.21) are both bounded. By Propositions 2.6 and 2.7, the operator U^* given by (2.20) is bounded from X_d^* to $\mathcal{B}^*$ and f_j^* form an X_d^*-Bessel sequence for $\mathcal{B}^*$.

We first present necessary and sufficient conditions for the operator S given as (3.1) to be bijective. Before that, it is worthwhile to point out that when $\mathcal{B}$ is a Banach space, that $\{f_j\}$ is an X_d-frame for $\mathcal{B}$ alone in general is insufficient to guarantee the bijectivity of S. We shall construct in Example 5.3 an X_d-frame $\{f_j\}$ for a finite-dimensional $\mathcal{B}$ for which $\operatorname{span}\{f_j\}\neq\mathcal{B}$. As a consequence, the operator $\mathcal{S}$ is not surjective in this example.

Theorem 3.1. *Suppose that $U:\mathcal{B}\to X_d$ defined by* (2.7) *and $V^*:X_d\to\mathcal{B}$ by* (2.21) *are bounded. Then the operator S given by* (3.1) *is bijective and bounded if and only if $\{f_j^*\}$ is an X_d^*-frame for $\mathcal{B}^*$, $\{f_j\}$ is an X_d-frame for $\mathcal{B}$, and $\{f_j\}$ is an $\mathcal{R}(U)$-Riesz basis for $\mathcal{B}$.*

Proof. Note that $S=V^*U$. Thus, S is bounded. Suppose that S is bijective. Then V^* is surjective, which implies by Proposition 2.9 that f_j^* form an X_d^*-frame for $\mathcal{B}^*$. Since $S^*=U^*V$ and S^* is bijective as S is, U^* is surjective from X_d^* to $\mathcal{B}^*$. Again, by Proposition 2.9, f_j constitute an X_d-frame for $\mathcal{B}$. As a result, $\mathcal{R}(U)$ is a closed subspace of X_d. For the third condition, let us study the operator V^*. By $S=V^*U$, V^* is surjective from $\mathcal{R}(U)$ to $\mathcal{B}$. Also, by the injectivity of S and U, V^* must be injective on $\mathcal{R}(U)$. We hence obtain that V^* is a bounded bijective linear operator from the Banach space $\mathcal{R}(U)$ to $\mathcal{B}$. By the open mapping theorem, $\{f_j\}$ is an $\mathcal{R}(U)$-Riesz basis for $\mathcal{B}$.

On the other hand, suppose that $\{f_j^*\}$ is an X_d^*-frame for $\mathcal{B}^*$, $\{f_j\}$ is an X_d-frame for $\mathcal{B}$, and $\{f_j\}$ is an $\mathcal{R}(U)$-Riesz basis for $\mathcal{B}$. Then U is injective and V^* is injective on $\mathcal{R}(U)$. It follows that S is injective as well. Finally, S is surjective as V^* is surjective from $\mathcal{R}(U)$ to $\mathcal{B}$. □

When S is bijective, one immediately has the reconstruction formula (3.2). This fact is stated in the following proposition.

Proposition 3.2. *If the bounded linear operator S given by* (3.1) *is bijective then there holds* (3.2) *and*

$$f^*=\sum_{j\in\mathbb{I}}[f_j,f]\big(S^{-1}\big)^*f_j^*=\sum_{j\in\mathbb{I}}\big[S^{-1}f_j,f\big]f_j^*,\quad f\in\mathcal{B}. \tag{3.3}$$

Proof. Suppose that S is bijective. We get by the definition of S and the continuity of S^{-1} that

$$f=S^{-1}(Sf)=S^{-1}\Bigg(\sum_{j\in\mathbb{I}}[f,f_j]f_j\Bigg)=\sum_{j\in\mathbb{I}}[f,f_j]S^{-1}f_j,\quad f\in\mathcal{B}.$$

As the adjoint of S, S^* of the following form is also bijective:

$$S^*f^*=\sum_{j\in\mathbb{I}}[f_j,f]f_j^*,\quad f\in\mathcal{B}.$$

Applying $(S^*)^{-1}=(S^{-1})^*$ to both sides of the above equation yields the first equality in (3.3). By (3.2), there holds for all $f,g\in\mathcal{B}$ that

$$[g,f]=\sum_{j\in\mathbb{I}}[g,f_j]\big[S^{-1}f_j,f\big],$$

which implies that

$$f^*=\sum_{j\in\mathbb{I}}\big[S^{-1}f_j,f\big]f_j^*,\quad f\in\mathcal{B}.$$

The proof is complete. □

A remarkable property of an arbitrary $\ell^2(\mathbb{I})$-frame $\{f_j\}$ for a Hilbert space $\mathcal{B}$ is that $[f,f_j]$ are the most economical coefficients for a decomposition of $f\in\mathcal{B}$ into $S^{-1}f_j$. Specifically, if $c\neq\{[f,f_j]\}\in\ell^2(\mathbb{I})$ satisfies that

$$f=\sum_{j\in\mathbb{I}}c_jS^{-1}f_j \tag{3.4}$$

then

$$\|c\|_{\ell^2(\mathbb{I})} > \big\|\{[f, f_j]\}\big\|_{\ell^2(\mathbb{I})}.$$

We shall prove a similar property for frames in a separable Banach space $\mathcal{B}$. The following fact was proved in [41] for semi-inner products and extended to generalized semi-inner products in [46].

Lemma 3.3. *Let $[\cdot,\cdot]_{X_d}$ be a compatible semi-inner product on X_d. Then X_d is strictly convex if and only if whenever $[c_1, c_2]_{X_d} = 0$ and $c_1 \neq 0$ then $\|c_1 + c_2\|_{X_d} > \|c_2\|_{X_d}$.*

We shall use $[\cdot,\cdot]_{X_d}$ to denote a compatible semi-inner product on X_d. Recall that for each $c \in X_d$, c^* denotes its dual element in X_d^* determined by

$$c^*(c') = [c', c]_{X_d}, \quad c' \in X_d.$$

Proposition 3.4. *Suppose that the bounded linear operator S defined by* (3.1) *is bijective. Let $f \in \mathcal{B}$. If X_d is strictly convex and $\{[f, f_j]\}^* \in \mathcal{R}(V)$ then*

$$\|c\|_{X_d} > \big\|\{[f, f_j]\}\big\|_{X_d} \tag{3.5}$$

for any $c \neq \{[f, f_j]\} \in X_d$ satisfying (3.4).

Proof. Suppose that X_d is strictly convex, $\{[f, f_j]\}^* \in \mathcal{R}(V)$, and $c \neq \{[f, f_j]\} \in X_d$ satisfies (3.4). By (3.2),

$$\sum_{j\in\mathbb{I}}\big(c_j - [f, f_j]\big) f_j = S\bigg(\sum_{j\in\mathbb{I}}\big(c_j - [f, f_j]\big)S^{-1} f_j\bigg) = S(f - f) = 0.$$

In other words, $c - Uf \in \mathcal{K}(V^*)$, the kernel of V^*. Since each element in $\mathcal{K}(V^*)$ vanishes on $\mathcal{R}(V)$ (see [5, p. 168]) and $(Uf)^* \in \mathcal{R}(V)$, we get that

$$[c - Uf, Uf]_{X_d} = \big((Uf)^*\big)(c - Uf) = 0.$$

By Lemma 3.3,

$$\|c\|_{X_d} = \big\|(c - Uf) + Uf\big\|_{X_d} > \|Uf\|_{X_d},$$

which is (3.5). □

Back to the discussion of the conditions ensuring the bijectivity of S. In the most convenient case when $\mathcal{R}(U) = X_d$, we observe by Theorem 3.1 that S is bijective if and only if $\{f_j\}$ and $\{f_j^*\}$ are respectively an X_d-Riesz basis for $\mathcal{B}$ and an X_d^*-Riesz basis for $\mathcal{B}^*$. One has the following conclusion under there two conditions.

Corollary 3.5. *If $\{f_j\}$ and $\{f_j^*\}$ are respectively an X_d-Riesz basis for $\mathcal{B}$ and an X_d^*-Riesz basis for $\mathcal{B}^*$ then S defined by* (3.1) *is bijective and bounded. Furthermore, there hold* (3.2), (3.3), *and*

$$\big[S^{-1} f_j, f_k\big] = \delta_{j,k}, \quad j, k \in \mathbb{I}.$$

4. Complete sampling expansions in Banach spaces

Let $\mathcal{B}$ be a separable Banach space of complex-valued functions defined on a prescribed set X. Based on the results established in the previous sections, we shall consider the complete reconstruction of a function $f \in \mathcal{B}$ from its sampled data

$$\mathcal{I}_{\mathcal{Z}} f := \big\{f(x_j)\colon\ j \in \mathbb{I}\big\},$$

where $\mathcal{Z} := \{x_j\colon\ j \in \mathbb{I}\} \subseteq X$ is a sampling set and $\mathbb{I}$ is a countable index set as before. Our study of such reconstruction from sampling in Banach spaces will be confined to an ideal framework that satisfies the following requirements:

(i) Only finite amount of data can be handled in practice. Thus, for each $f \in \mathcal{B}$, $\mathcal{I}_{\mathcal{Z}} f$ should be of finite "energy" so that it is approximable from its finite subsets. For this reason, we shall require that $\mathcal{I}_{\mathcal{Z}} f$ belong to some *BK*-space X_d for all $f \in \mathcal{B}$.

(ii) The sampling process $\mathcal{I}_{\mathcal{Z}}:\mathcal{B}\to X_d$ should be stable. This implies that if there is a small perturbation $\tilde f$ of f in $\mathcal{B}$ with $\|\tilde f\|_{\mathcal{B}}\leqslant\delta$, where δ measures the noise level, and we end up sampling $f_\delta:=f+\tilde f$, then $\mathcal{I}_{\mathcal{Z}}(f_\delta)$ is expected to be close to $\mathcal{I}_{\mathcal{Z}}f$ in X_d. In other words, the sampling operator $\mathcal{I}_{\mathcal{Z}}:\mathcal{B}\to X_d$ should be bounded.
(iii) We aim at recovering every $f\in\mathcal{B}$ from its sampled data $\mathcal{I}_{\mathcal{Z}}f$. The recovery process should be stable as well. Therefore, $\mathcal{I}_{\mathcal{Z}}$ should possess a bounded inverse on its range.
(iv) Sampling should not be redundant, which implies that there should not exist some $j\in\mathbb{I}$ so that for each $f\in\mathcal{B}$, $f(x_j)$ can be obtained from $\{f(x_k):\ k\in\mathbb{I},\ k\neq j\}$. We shall elaborate on this requirement later.

Note that (i)–(iii) can be summarized into that $\mathcal{I}_{\mathcal{Z}}(\mathcal{B})\subseteq X_d$ and

$$A\|f\|_{\mathcal{B}}\leqslant\|\mathcal{I}_{\mathcal{Z}}f\|_{X_d}\leqslant B\|f\|_{\mathcal{B}}\quad\text{for all } f\in\mathcal{B},\tag{4.1}$$

where A, B are two positive constants. The second inequality above together with that coordinate functionals are continuous on X_d implies that for every $j\in\mathbb{I}$, the point evaluation functional δ_{x_j} is continuous on $\mathcal{B}$. In the search of a suitable sampling set $\mathcal{Z}$ for $\mathcal{B}$, it would hence be convenient to assume that δ_x is continuous on $\mathcal{B}$ for all $x\in\mathcal{B}$. In a recent article [20], a Banach space $\mathcal{B}$ of functions on X where point evaluation functionals are always continuous was called a reproducing kernel Banach space. In our work [45], to ensure the existence of a reproducing kernel, an RKBS was required to have two more crucial properties: uniform Fréchet differentiability and uniform convexity. A normed vector space $\mathcal{C}$ is *uniformly Fréchet differentiable* if for all $x, y\in\mathcal{C}$ with $x\neq 0$

$$\lim_{t\in\mathbb{R},\,t\to 0}\frac{\|x+ty\|_{\mathcal{C}}-\|x\|_{\mathcal{C}}}{t}$$

exists and the limit is uniform on $\mathcal{S}(\mathcal{C})\times\mathcal{S}(\mathcal{C})$, where $S(\mathcal{C}):=\{x\in\mathcal{C}:\ \|x\|_{\mathcal{C}}=1\}$. We say that $\mathcal{C}$ is *uniformly convex* if for all $\varepsilon>0$ there exists a $\delta>0$ such that

$$\|x+y\|_{\mathcal{C}}\leqslant 2-\delta\quad\text{for all } x, y\in\mathcal{S}(\mathcal{C})\text{ with }\|x-y\|_{\mathcal{C}}\geqslant\varepsilon.$$

For simplicity, $\mathcal{C}$ is said to be *uniform* if it is both uniformly Fréchet differentiable and uniformly convex. In this paper, we call $\mathcal{B}$ a *reproducing kernel Banach space* (RKBS) on X if it is a uniform Banach space of functions on X where point evaluations are always continuous linear functionals on $\mathcal{B}$. Translation invariant RKBS on Euclidean spaces are the main subject of this section. Before introducing them, let us briefly give an explicit example of RKBS that is not an RKHS. The space $\mathbb{E}^p_\tau$, $p\in(1,+\infty)$, $\tau>0$ consisting of all entire functions f on $\mathbb{C}$ of exponential type at most τ for which

$$\|f\|_{\mathbb{E}^p_\tau}:=\left(\int_{\mathbb{R}}\left|f(t)\right|^p dt\right)^{1/p}<+\infty$$

is an RKBS. In fact, there is a constant C depending on p and τ only such that (see [44, p. 99])

$$\left|f(x+iy)\right|\leqslant Ce^{\tau|y|}\|f\|_{\mathbb{E}^p_\tau}\quad\text{for all } x, y\in\mathbb{R},\ f\in\mathbb{E}^p_\tau.$$

It follows from the above two equations that $\mathbb{E}^p_\tau$ is a Banach space isometrically isomorphic to a closed subspace of $L^p(\mathbb{R})$. Consequently, $\mathbb{E}^p_\tau$ is uniform, and is thus an RKBS on $\mathbb{C}$. When $p\neq 2$, the space is not a Hilbert space.

In this section, we shall be satisfied with the assumption that $\mathcal{B}$ is an RKBS on X. There are some useful consequences following this assumption. Firstly, $\mathcal{B}$ has a unique compatible semi-inner product $[\cdot,\cdot]$ [15]. Secondly, $\mathcal{B}$ is reflexive, strictly convex, and its dual $\mathcal{B}^*$ is also uniform [7]. Most importantly of all, by the arguments in the proof of Theorem 9 in [45], there exists a unique function $G: X\times X\to\mathbb{C}$ such that $G(x,\cdot)\in\mathcal{B}$ for all $x\in X$ and

$$f(x)=\left[f, G(x,\cdot)\right]\quad\text{for all } x\in X\text{ and } f\in\mathcal{B}.\tag{4.2}$$

By virtue of the above equation, we call G the *s.i.p. reproducing kernel* of $\mathcal{B}$. Set

$$G_{\mathcal{Z}}:=\left\{G(x_j,\cdot):\ j\in\mathbb{I}\right\}\quad\text{and}\quad G^*_{\mathcal{Z}}:=\left\{G(x_j,\cdot)^*:\ j\in\mathbb{I}\right\}.$$

By (4.1) and (4.2), the requirements (i)–(iii) are equivalent to saying that $G_{\mathcal{Z}}$ forms an X_d-frame for $\mathcal{B}$. The fourth requirement (iv) implies that $G^*_{\mathcal{Z}}$ are minimal in $\mathcal{B}^*$. Conversely, assume that

$$G(x_j,\cdot)^*\in\overline{\operatorname{span}}\left\{G(x_k,\cdot)^*:\ k\in\mathbb{I},\ k\neq j\right\}\quad\text{for some } j\in\mathbb{I}.$$

Then for all $f\in\mathcal{B}$, $f(x_j)$ could be approximated by finite linear combinations of $f(x_k)$, $k\in\mathbb{I}\setminus\{j\}$. It is hence unnecessary to sample at the point x_j from the practical point of view. We would like our ideal sampling framework to contain no redundant sampling points. Therefore, $G^*_{\mathcal{Z}}$ should be minimal in $\mathcal{B}^*$.

To conclude the above discussion, we observe from Proposition 2.11 that by an ideal sampling framework for $\mathcal{B}$, we seek a sampling set $\mathcal{Z}\subseteq X$ such that there exists some *BK*-space X_d for which $G^*_{\mathcal{Z}}$ is an X^*_d-Riesz basis for $\mathcal{B}^*$. Once such a sampling set exists, one obtains by Theorems 2.14, 2.15, and Corollary 3.5 a series of sampling expansions in the Banach space $\mathcal{B}$.

Theorem 4.1. *Let $\mathcal{B}$ be an RKBS on X and $\mathcal{Z} = \{x_j\} \subseteq X$.*

(1) *If $G_{\mathcal{Z}}$ is an X_d-frame for $\mathcal{B}$ and $\mathcal{I}_{\mathcal{Z}}(\mathcal{B})$ has an algebraic complement in X_d then there exists an X_d^*-frame $\{g_j^*\}$ for $\mathcal{B}^*$ such that*

$$f(x) = \sum_{j\in\mathbb{I}} f(x_j) g_j(x) \quad \text{for all } f \in \mathcal{B},\ x \in X. \tag{4.3}$$

(2) *If $G_{\mathcal{Z}}^*$ is an X_d^*-Riesz basis for $\mathcal{B}^*$ then the above g_j are unique and form an X_d-Riesz basis for $\mathcal{B}$. Moreover,*

$$g_j(x_k) = \delta_{j,k}, \quad j, k \in \mathbb{I}.$$

(3) *If $G_{\mathcal{Z}}$ and $G_{\mathcal{Z}}^*$ are respectively an X_d-Riesz basis and X_d^*-Riesz basis for $\mathcal{B}$ and $\mathcal{B}^*$ then the operator $\mathcal{S} : \mathcal{B} \to \mathcal{B}$ defined by*

$$(\mathcal{S} f)(x) := \sum_{j\in\mathbb{I}} f(x_j) G(x_j, x), \quad x \in X,\ f \in \mathcal{B} \tag{4.4}$$

is bijective and bounded. Furthermore,

$$f(x) = \sum_{j\in\mathbb{I}} f(x_j) \big(\mathcal{S}^{-1} G(x_j, \cdot)\big)(x) \quad \text{for all } f \in \mathcal{B},\ x \in X. \tag{4.5}$$

The sampling expansion (4.3) was formulated in [20]. When $\mathcal{B}$ is an RKHS, the formula (4.5) was first discovered in [34], and further explored in [13,17,18,21,32].

We aim at ensuring the uniqueness of $\{g_j\} \subseteq \mathcal{B}$ satisfying (4.3). For this sake, we call a subset $\mathcal{Z} \subseteq X$ an *X_d-Riesz sampling set* for $\mathcal{B}$ if $G_{\mathcal{Z}}^*$ is an X_d^*-Riesz basis for $\mathcal{B}^*$. If, in addition, $G_{\mathcal{Z}}$ is an X_d-Riesz basis for $\mathcal{B}$, then we call $\mathcal{Z}$ a *double X_d-Riesz sampling set* for $\mathcal{B}$. By Theorem 4.1, a double X_d-Riesz sampling set enables us to reconstruct functions in $\mathcal{B}$ from their sampling by the standard reconstruction operator (4.4). If $\mathcal{B}$ is a Hilbert space then a Riesz sampling set is automatically a double Riesz sampling set. In the rest of this section, we shall discuss the existence of Riesz sampling sets in translation invariant RKBS and RKHS.

We start with feature map representations of the s.i.p. reproducing kernel of RKBS. The following result was from [45].

Lemma 4.2. *Let $\mathcal{W}$ be a uniform Banach space and Φ a mapping from X to $\mathcal{W}$ such that*

$$\overline{\operatorname{span}} \Phi(X) = \mathcal{W}, \qquad \overline{\operatorname{span}} \big(\Phi(X)\big)^* = \mathcal{W}^*. \tag{4.6}$$

Denote by $[\cdot,\cdot]_{\mathcal{W}}$ the unique compatible semi-inner product on $\mathcal{W}$. Then $\mathcal{B} := \{[u, \Phi(\cdot)]_{\mathcal{W}}\colon u \in \mathcal{W}\}$ equipped with

$$\big[[u, \Phi(\cdot)]_{\mathcal{W}}, [v, \Phi(\cdot)]_{\mathcal{W}}\big] := [u, v]_{\mathcal{W}} \tag{4.7}$$

and $\mathcal{B}^* := \{[\Phi(\cdot), u]_{\mathcal{W}}\colon u \in \mathcal{W}\}$ with

$$\big[[\Phi(\cdot), u]_{\mathcal{W}}, [\Phi(\cdot), v]_{\mathcal{W}}\big]_{\mathcal{B}^*} := [v, u]_{\mathcal{W}} \tag{4.8}$$

are RKBS. And $\mathcal{B}^*$ is indeed the dual of $\mathcal{B}$ with

$$\big([\Phi(\cdot), v]_{\mathcal{W}}\big)\big([u, \Phi(\cdot)]_{\mathcal{W}}\big) := [u, v]_{\mathcal{W}}, \quad u, v \in \mathcal{W}. \tag{4.9}$$

Moreover, the s.i.p. reproducing kernel G of $\mathcal{B}$ is given by

$$G(x, y) = \big[\Phi(x), \Phi(y)\big]_{\mathcal{W}}, \quad x, y \in X. \tag{4.10}$$

We shall construct translation invariant RKBS on $\mathbb{R}^d$, $d \in \mathbb{N}$ according to the above lemma. Let $p, q \in (1, +\infty)$ with $1/p + 1/q = 1$, and ϕ a nonnegative Borel measurable function on $\mathbb{R}^d$ satisfying

$$\int_{\mathbb{R}^d} \phi(t)\, dt = 1. \tag{4.11}$$

The feature space will be chosen as $L_\phi^p(\mathbb{R}^d)$ consisting of Borel measurable functions u on $\mathbb{R}^d$ for which

$$\|u\|_{L_\phi^p(\mathbb{R}^d)} := \bigg(\int_{\mathbb{R}^d} \big|u(t)\big|^p \phi(t)\, dt\bigg)^{1/p} < +\infty.$$

The semi-inner product on $L_\phi^p(\mathbb{R}^d)$ has the form

$$[u,v]_{L^p_\phi(\mathbb{R}^d)} = \frac{1}{\|v\|^{p-2}_{L^p_\phi(\mathbb{R}^d)}} \int_{\mathbb{R}^d} u(t)\overline{v(t)}\big|v(t)\big|^{p-2}\phi(t)\,dt, \quad u,v \in L^p_\phi\big(\mathbb{R}^d\big). \tag{4.12}$$

The feature map $\Phi : \mathbb{R}^d \to L^p_\phi(\mathbb{R}^d)$ is given by

$$\Phi(x)(t) := e_x(t) := e^{i(x,t)}, \quad t \in \mathbb{R}^d,\ x \in \mathbb{R}^d,$$

where $(\cdot,\cdot)$ denotes the standard inner product on $\mathbb{R}^d$. The dual space of $L^p_\phi(\mathbb{R}^d)$ is $L^q_\phi(\mathbb{R}^d)$. One obtains by (4.11) and (4.12) that for each $x \in \mathbb{R}^d$, the dual function of e_x in $L^q_\phi(\mathbb{R}^d)$ is given as

$$(e_x)^* = e_{-x}.$$

Thus, it is clear that the completeness condition (4.6)

$$\overline{\operatorname{span}}\Phi\big(\mathbb{R}^d\big) = L^p_\phi\big(\mathbb{R}^d\big), \qquad \overline{\operatorname{span}}\big(\Phi\big(\mathbb{R}^d\big)\big)^* = L^q_\phi\big(\mathbb{R}^d\big)$$

is satisfied. With these choices, functions f in $\mathcal{B}$ have the form

$$f(x) = \int_{\mathbb{R}^d} u(t)e^{-i(x,t)}\phi(t)\,dt, \quad x \in \mathbb{R}^d,\ u \in L^p_\phi\big(\mathbb{R}^d\big) \tag{4.13}$$

with the norm

$$\|f\|_{\mathcal{B}} = \|u\|_{L^p_\phi(\mathbb{R}^d)}. \tag{4.14}$$

Lemma 4.2 tells us that $\mathcal{B}$ is an RKBS on $\mathbb{R}^d$. This fact can actually be verified directly without much effort. Firstly, by (4.14), $\mathcal{B}$ is isometrically isomorphic to $L^p_\phi(\mathbb{R}^d)$. Therefore, $\mathcal{B}$ is uniform as $L^p_\phi(\mathbb{R}^d)$ is. Secondly, we note by the Hölder inequality that for all $x \in \mathbb{R}^d$,

$$\begin{aligned}\big|f(x)\big| &\leqslant \int_{\mathbb{R}^d} \big|u(t)\big|\phi(t)\,dt = \int_{\mathbb{R}^d} \big|u(t)\big|\phi(t)^{1/p}\phi(t)^{1/q}\,dt \leqslant \bigg(\int_{\mathbb{R}^d} \big|u(t)\big|^p\phi(t)\,dt\bigg)^{1/p}\bigg(\int_{\mathbb{R}^d}\phi(t)\,dt\bigg)^{1/q}\\ &= \|u\|_{L^p_\phi(\mathbb{R}^d)} = \|f\|_{\mathcal{B}},\end{aligned}$$

which implies that point evaluations are bounded on $\mathcal{B}$.

Functions in $\mathcal{B}$ can be characterized in terms of their Fourier transforms. Define the Fourier transform $\hat{u}$ of $u \in L^1(\mathbb{R}^d)$ by

$$\hat{u}(\xi) := \int_{\mathbb{R}^d} u(t)e^{-i(\xi,t)}\,dt, \quad \xi \in \mathbb{R}^d.$$

The inverse Fourier transform $\check{u}$ of $u \in L^1(\mathbb{R}^d)$ is hence given by

$$\check{u}(\xi) := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} u(t)e^{i(\xi,t)}\,dt, \quad \xi \in \mathbb{R}^d.$$

Let us assume that the Fourier transform and its inverse have been extended to temperate distributions by the duality principle [14]. We also set for each function g on $\mathbb{R}^d$

$$\Omega_g := \big\{t \in \mathbb{R}^d:\ g(t) \neq 0\big\}.$$

With these definitions and notations, we observe that

$$\mathcal{B} = \bigg\{f \in C\big(\mathbb{R}^d\big):\ \Omega_{\check{f}} \subseteq \Omega_\phi,\ \frac{\check{f}}{\phi} \in L^p_\phi\big(\mathbb{R}^d\big)\bigg\} \tag{4.15}$$

with the norm

$$\|f\|_{\mathcal{B}} = \bigg\|\frac{\check{f}}{\phi}\bigg\|_{L^p_\phi(\mathbb{R}^d)}, \quad f \in \mathcal{B}.$$

The semi-inner product on $\mathcal{B}$ is of the form

$$[f,g]=\left[\frac{\check{f}}{\phi},\frac{\check{g}}{\phi}\right]_{L^p_\phi(\mathbb{R}^d)}. \tag{4.16}$$

By (4.10), we identify the s.i.p. reproducing kernel G of $\mathcal{B}$ as

$$G(x,y)=[e_x,e_y]_{L^p_\phi(\mathbb{R}^d)}=\int\limits_{\mathbb{R}^d} e^{i(x-y,t)}\phi(t)\,dt=\hat{\phi}(y-x),\quad x,y\in\mathbb{R}^d. \tag{4.17}$$

One can verify directly by (4.16) that G given above indeed is the s.i.p. reproducing kernel for $\mathcal{B}$. Another obvious fact is that $\mathcal{B}$ is *translation invariant* in the sense that for all $f\in\mathcal{B}$ and $x_0\in\mathbb{R}^d$, the function $f(\cdot-x_0)\in\mathcal{B}$ and

$$\left\|f(\cdot-x_0)\right\|_{\mathcal{B}}=\|f\|_{\mathcal{B}}.$$

As an example, we remark that when $p=2$ and ϕ is a Gaussian function

$$\phi(t):=\frac{1}{(4\pi\sigma)^{d/2}}e^{-\frac{\|t\|^2}{4\sigma}},\quad t\in\mathbb{R}^d,\ \sigma>0,$$

where $\|t\|:=(t,t)^{1/2}$, $\mathcal{B}$ is the RKHS $\mathcal{H}_{\mathcal{G}_\sigma}$ of the Gaussian kernel

$$\mathcal{G}_\sigma(x,y):=\exp\bigl(-\sigma\|x-y\|^2\bigr),\quad x,y\in\mathbb{R}^d. \tag{4.18}$$

Let $\mathcal{B}$ be an RKBS given by (4.15) with the s.i.p. reproducing kernel G of the form (4.17) and $\mathbb{I}$ an infinite countable index set. The main theme of this section is on the existence of Riesz sampling sets $\{x_j\}\subseteq\mathbb{R}^d$ for $\mathcal{B}$. We point out below that this question can be reformulated into one in the feature space of G.

Lemma 4.3. *Let $\mathcal{B}$ and G be given as in Lemma 4.2 through a feature map $\Phi: X\to\mathcal{W}$ satisfying (4.6). Then $\mathcal{Z}\subseteq X$ is an X_d-Riesz sampling set for $\mathcal{B}$ if and only if $(\Phi(\mathcal{Z}))^*$ is an X_d^*-Riesz basis for $\mathcal{W}^*$. Consequently, if $\mathcal{B}$ and G are respectively given by (4.15) and (4.17) then $\mathcal{Z}=\{x_j\}\subseteq\mathbb{R}^d$ is an X_d-Riesz sampling set for $\mathcal{B}$ if and only if $\{e_{-x_j}\}$ forms an X_d^*-Riesz basis for $L^q_\phi(\mathbb{R}^d)$.*

We shall use the above lemma and existing research [29,30,35,44] on complete interpolating sequences in Paley–Wiener spaces to give a positive example of RKBS where Riesz sampling sets exist. Following these references, we call $\mathcal{Z}=\{x_j\}$ an *X_d-complete interpolating sequence* for $\mathcal{B}$ if there exist positive constants $A\leqslant B$ such that

$$A\|f\|_{\mathcal{B}}\leqslant\left\|\bigl\{f(x_j)\bigr\}\right\|_{X_d}\leqslant B\|f\|_{\mathcal{B}}\quad\text{for all } f\in\mathcal{B}$$

and for all $c\in X_d$ there is some $f\in\mathcal{B}$ such that $f(x_j)=c_j$, $j\in\mathbb{I}$. Note that the first conditions is equivalent to that $\{G(x_j,\cdot)\}$ is an X_d-frame for $\mathcal{B}$ while the second one implies that $\{G(x_j,\cdot)\}$ forms an X_d-Riesz–Fischer sequence for $\mathcal{B}$. Therefore, we obtain by Propositions 2.4 and 2.11 the following simple fact, which to some extent justifies the notion of X_d-Riesz sampling sets by connecting it to a known concept.

Lemma 4.4. *An indexed set $\{x_j\}$ is an X_d-complete interpolating sequence for $\mathcal{B}$ if and only if it is an X_d-Riesz sampling set for $\mathcal{B}$.*

Example 4.5. Let $p=2$, $\mathbb{I}=\mathbb{Z}$, and $\phi=\chi_{[-\pi,\pi]}$, the characteristic function of $[-\pi,\pi]$. Then $\mathcal{B}=\mathbb{E}^2_\pi$ is the Paley–Wiener space of square-integrable functions on $\mathbb{R}$ that are bandlimited to $[-\pi,\pi]$. By the well-known Kadec's $\frac{1}{4}$-theorem, if for some nonnegative constant L

$$|x_j-j|\leqslant L<\frac{1}{4},\quad j\in\mathbb{Z} \tag{4.19}$$

then e_{x_j} form a Riesz basis for $L^2([-\pi,\pi])$. By Lemma 4.3, $\mathcal{Z}=\{x_j\}$ satisfying (4.19) is a Riesz sampling set for $\mathbb{E}^2_\pi$. More Riesz sampling sets for $\mathbb{E}^2_\pi$ can be formed by the zeros of an entire function of sine type (see [44, p. 172]). For a complete characterization, see [35]. The Kadec's theorem was generalized to the space $\mathbb{E}^p_\pi$, $1<p<+\infty$, in [29]. Let q be such that $1/p+1/q=1$. It was proved there that $x_j\in\mathbb{R}$, $j\in\mathbb{Z}$ satisfying

$$|x_j-j|\leqslant L<\min\left\{\frac{1}{2p},\frac{1}{2q}\right\},\quad j\in\mathbb{Z} \tag{4.20}$$

is an $\ell^p(\mathbb{Z})$-complete interpolating sequence for $\mathbb{E}^p_\pi$. By Lemma 4.4, such sequences are $\ell^p(\mathbb{Z})$-Riesz sampling sets for $\mathbb{E}^p_\pi$. A characterization of $\ell^p(\mathbb{Z})$-complete interpolating sequences, thus of $\ell^p(\mathbb{Z})$-Riesz sampling sets was established in [29].

The next positive example is based on the study of interpolating refinable function vectors in the wavelets theory [19, 23,24].

Example 4.6. Set $\mathbb{N}_n := \{1, 2, \ldots, n\}$ for each $n \in \mathbb{N}$. Let $r \in \mathbb{N}$ and $\{\phi_j\colon\ j \in \mathbb{N}_r\}$ be a set of compactly supported continuous functions on $\mathbb{R}$. The *BK*-space X_d consists of all the sequences $c = \{c_{jk} \in \mathbb{C}\colon\ j \in \mathbb{N}_r,\ k \in \mathbb{Z}\}$ such that

$$\|c\|_{X_d} := \left(\sum_{j=1}^{r}\sum_{k\in\mathbb{Z}}|c_{jk}|^p\right)^{1/p} < +\infty.$$

We require the function vector $\{\phi_j : j \in \mathbb{N}_r\}$ be *stable* in $L^p(\mathbb{R})$ [24] in the sense that there exists $0 < A \leqslant B < +\infty$ such that for all $c \in X_d$

$$A\|c\|_{X_d} \leqslant \left\|\sum_{j=1}^{r}\sum_{k\in\mathbb{Z}} c_{jk}\phi_j(\cdot - k)\right\|_{L^p(\mathbb{R})} \leqslant B\|c\|_{X_d}. \tag{4.21}$$

It was proved in [24] that $\{\phi_j\colon\ j \in \mathbb{N}_r\}$ is stable in $L^p(\mathbb{R})$ if and only if $\{\hat{\phi}_j(\xi + 2k\pi)\colon\ k \in \mathbb{Z}\}$, $j \in \mathbb{N}_r$ are linearly independent for all $\xi \in \mathbb{R}$. We also impose the *generalized interpolation* property [19] that

$$\phi_j\left(\frac{l-1}{r} + k\right) = \delta_{0,k}\delta_{j,l}, \quad j, l \in \mathbb{N}_r,\ k \in \mathbb{Z}. \tag{4.22}$$

Let $\mathcal{B}$ be the closure in $L^p(\mathbb{R})$ of $\operatorname{span}\{\phi_j(\cdot - k)\colon\ j \in \mathbb{N}_r,\ k \in \mathbb{Z}\}$. We verify that it is an RKBS. As a closed subspace of $L^p(\mathbb{R})$, $\mathcal{B}$ is uniform. We then notice by (4.21) that functions f in $\mathcal{B}$ are of the form

$$f(x) = \sum_{j=1}^{r}\sum_{k\in\mathbb{Z}} c_{jk}\phi_j(x - k), \quad x \in \mathbb{R} \tag{4.23}$$

where $c \in X_d$. Since ϕ_j are compactly supported, for each $x \in \mathbb{R}$, $\{\phi_j(x - k)\colon\ j \in \mathbb{N}_r,\ k \in \mathbb{Z}\} \in X_d^*$. By the Hölder inequality and the stability condition (4.21), we get for functions f of the form (4.23) that

$$\left|f(x)\right| \leqslant \|c\|_{X_d}\left\|\left\{\phi_j(x - k)\colon\ j \in \mathbb{N}_r,\ k \in \mathbb{Z}\right\}\right\|_{X_d^*} \leqslant \frac{1}{A}\left\|\left\{\phi_j(x - k)\colon\ j \in \mathbb{N}_r,\ k \in \mathbb{Z}\right\}\right\|_{X_d^*}\|f\|_{\mathcal{B}}.$$

Therefore, point evaluations are continuous on $\mathcal{B}$, which proves that $\mathcal{B}$ is an RKBS.

We claim that $\{\frac{j-1}{r} + k\colon\ j \in \mathbb{N}_r,\ k \in \mathbb{Z}\}$ is an X_d-complete interpolating sequence for $\mathcal{B}$, thus by Lemma 4.4, an X_d-Riesz sampling set for $\mathcal{B}$. Firstly, it is clear by (4.22) that for all $c \in X_d$ the function (4.23) satisfies

$$f\left(\frac{j-1}{r} + k\right) = c_{jk}, \quad j \in \mathbb{N}_r,\ k \in \mathbb{Z}.$$

This equation also implies that

$$\frac{1}{B}\|f\|_{\mathcal{B}} \leqslant \left\|\left\{f\left(\frac{j-1}{r} + k\right)\colon\ j \in \mathbb{N}_r,\ k \in \mathbb{Z}\right\}\right\|_{X_d} \leqslant \frac{1}{A}\|f\|_{\mathcal{B}},$$

which concludes our example.

Let us turn to our main purpose of proving nonexistence of Riesz sampling sets for some common RKBS $\mathcal{B}$ of the form (4.15). Firstly, the nonexistence can result from an inappropriate choice of the *BK*-space X_d. This is explained in the following lemma.

Lemma 4.7. *It is necessary that $\mathcal{B}$ and $L_\phi^p(\mathbb{R}^d)$ are isomorphic to a closed subspace of X_d in order for $\mathcal{B}$ to have an X_d-Riesz sampling set.*

Proof. Suppose that $\mathcal{Z} = \{x_j\}$ is an X_d-Riesz sampling set for $\mathcal{B}$. Then $G(x_j, \cdot)$ constitute an X_d-frame for $\mathcal{B}$. As a result, $\mathcal{B}$ is isomorphic to a closed subspace of X_d through the operator $U = \mathcal{I}_{\mathcal{Z}}$. Since by Lemma 4.2 $\mathcal{B}$ is isometrically isomorphic to its feature space $L_\phi^p(\mathbb{R}^d)$, $L_\phi^p(\mathbb{R}^d)$ must be isomorphic to the same subspace of X_d. $\square$

Proposition 4.8. *Let $p \neq 2$. If there exists some $x_0 \in \mathbb{R}^d$ such that $\phi(x_0) > 0$ and ϕ is continuous at x_0 then $\mathcal{B}$ does not have any $\ell^{p'}(\mathbb{I})$-Riesz sampling set regardless of the choice of $p' \in (1, +\infty)$.*

Proof. Assume to the contrary that $\mathcal{B}$ has an $\ell^{p'}(\mathbb{I})$-Riesz sampling set. Since ϕ is continuous and positive at x_0, there exist some $a < b \in \mathbb{R}$ and positive constants $\alpha \leqslant \beta$ such that

$$\alpha \leqslant \phi(t) \leqslant \beta \quad \text{for all } t \in [a, b]^d.$$

Introduce a linear mapping $T : L^p([a,b]) \to L^p_\phi(\mathbb{R}^d)$ by setting

$$(Tf)(t) := \begin{cases} f(t_1), & t \in [a,b]^d, \\ 0, & \text{otherwise.} \end{cases}$$

Direct computations yield the estimate that

$$(b-a)^{(d-1)/p}\alpha^{1/p}\|f\|_{L^p([a,b])} \leqslant \|Tf\|_{L^p_\phi(\mathbb{R}^d)} \leqslant (b-a)^{(d-1)/p}\beta^{1/p}\|f\|_{L^p([a,b])}.$$

It follows that $L^p([a,b])$ is isomorphic to a subspace $T(L^p([a,b]))$ of $L^p_\phi(\mathbb{R}^d)$. By Lemma 4.7, $L^p([a,b])$ must be isomorphic to a subspace of $\ell^{p'}(\mathbb{I})$. However, this is possible if and only if $p = p' = 2$ (see [11, pp. 179–180]), contradicting $p \neq 2$. □

Secondly, Riesz sampling sets can still not exist if ϕ is continuous and positive on the whole $\mathbb{R}^d$, no matter how the *BK*-space X_d is chosen. We start the proof of this main result by revealing a general phenomenon, which when $\mathcal{B}$ is the Paley–Wiener space $\mathbb{E}^2_\pi$ has long been known (see, for example, [44, p. 179]).

Theorem 4.9. *Let X be a metric space with the distance $\mathcal{D}$, and $\mathcal{B}$ an RKBS on X with the s.i.p. reproducing kernel G. If the function $x \to G(x,\cdot)^*$ is uniformly continuous from X to $\mathcal{B}^*$ then any X_d-Riesz sampling set $\{x_j\} \subseteq X$ for $\mathcal{B}$ must be separated in the sense that*

$$\mathcal{D}(x_j, x_k) \geqslant \delta > 0 \quad \textit{for all } j \neq k \in \mathbb{I}$$

for some positive constant δ.

Proof. Suppose that $\{x_j\}$ is an X_d-Riesz sampling set for $\mathcal{B}$. In other words, $G(x_j,\cdot)^*$ form an X_d^*-Riesz basis for $\mathcal{B}^*$. By Theorem 4.1, there exists an X_d-Riesz basis $\{g_j\}$ for $\mathcal{B}$ such that

$$g_j(x_k) = \delta_{j,k}, \quad j \neq k \in \mathbb{I}. \tag{4.24}$$

Being a Riesz basis, $\{g_j\}$ satisfies

$$A \leqslant \|g_j\|_{\mathcal{B}} \leqslant B, \quad j \in \mathbb{I} \tag{4.25}$$

for some positive constants A, B. We get by (4.24) and the reproducing property (4.2) that for all $j \neq k$

$$\big(G(x_j,\cdot)^* - G(x_k,\cdot)^*\big)(g_j) = \big[g_j, G(x_j,\cdot)\big] - \big[g_j, G(x_k,\cdot)\big] = g_j(x_j) - g_j(x_k) = 1. \tag{4.26}$$

We obtain from Eqs. (4.25) and (4.26) that

$$\big\|G(x_j,\cdot)^* - G(x_k,\cdot)^*\big\|_{\mathcal{B}^*} \geqslant \frac{1}{B} \quad \text{for all } j \neq k \in \mathbb{I}.$$

Since $x \to G(x,\cdot)^*$ is uniformly continuous from X to $\mathcal{B}^*$, $\{x_j\}$ must be separated in X. □

The next lemma will pave our way to prove the main theorem. The result in the one-dimensional case $d = 1$ is well-known. We follow the idea in [44, p. 162].

Lemma 4.10. *If $\{x_j\}$ is separated in $\mathbb{R}^d$ then $\{e_{x_j}\}$ is an $\ell^2(\mathbb{I})$-Riesz–Fischer sequence for $L^2([-a,a]^d)$ provided that a is sufficiently large.*

Proof. Let $\|\cdot\|_\infty$ be the norm on $\mathbb{R}^d$ defined by

$$\|t\|_\infty := \max\big\{|t_l|\colon l \in \mathbb{N}_d\big\}.$$

Since $\{x_j\}$ is separated, there exists some $\gamma > 0$ such that

$$\|x_j - x_k\|_\infty \geqslant \gamma \quad \text{for all } j \neq k \in \mathbb{I}. \tag{4.27}$$

We shall make use of a technical lemma proved in [33] that for all $k \in \mathbb{N}$

$$\inf\big\{\big\|\psi^{(k)}\big\|_{L^1([-a,a])}\colon \psi \in C_0^k\big([-a,a]\big),\ \hat{\psi}(0) = 1\big\} = \frac{2^{k-1}}{a^k}k!,$$

where $C_0^k([-a,a])$ denotes the set of k-times continuously differentiable functions on $\mathbb{R}$ that are supported on $[-a,a]$. By this result, we may find a nonnegative $\psi \in C_0^{(d+2)}([-a,a])$ such that

$$\hat{\psi}(0) = 1 \tag{4.28}$$

and

$$\left\|\psi^{(d+2)}\right\|_{L^1([-a,a])} \leqslant 2\frac{2^{d+1}}{a^{d+2}}(d+2)!. \tag{4.29}$$

Since ψ is nonnegative, by (4.28),

$$\left|\hat{\psi}(\xi)\right| \leqslant 1, \quad \xi \in \mathbb{R}. \tag{4.30}$$

Using integration by parts, we obtain by (4.29) that

$$\left|\hat{\psi}(\xi)\right| \leqslant \frac{1}{|\xi|^{d+2}}\left\|\psi^{(d+2)}\right\|_{L^1([-a,a])} \leqslant \frac{1}{|\xi|^{d+2}}\frac{2^{d+2}}{a^{d+2}}(d+2)!, \quad \xi \neq 0. \tag{4.31}$$

We shall then rely on Proposition 2.4. Let $\{c_j\} \subseteq \mathbb{C}$ have at most finitely many nonzero components. Set

$$\Psi(t) := \prod_{l=1}^{d} \psi(t_l), \quad t \in \mathbb{R}^d,$$

where t_l is the l-th component of t. Clearly,

$$\int_{[-a,a]^d} \left|\sum_{j\in\mathbb{I}} c_j e_{x_j}\right|^2 dt \geqslant \frac{1}{\|\psi\|^d_{L^\infty([-a,a])}} \int_{[-a,a]^d} \left|\sum_{j\in\mathbb{I}} c_j e_{x_j}(t)\right|^2 \Psi(t)\,dt. \tag{4.32}$$

We further estimate by (4.28) that

$$\begin{aligned} \int_{[-a,a]^d} \left|\sum_{j\in\mathbb{I}} c_j e_{x_j}(t)\right|^2 \Psi(t)\,dt &= \sum_{j\in\mathbb{I}} |c_j|^2 \hat{\Psi}(0) + \sum_{j\in\mathbb{I}}\sum_{k\neq j} |c_j\overline{c_k}|\hat{\Psi}(x_k - x_j) \\ &\geqslant \sum_{j\in\mathbb{I}} |c_j|^2 - \sum_{j\in\mathbb{I}} |c_j|^2 \sum_{k\neq j} \left|\hat{\Psi}(x_k - x_j)\right|. \end{aligned} \tag{4.33}$$

It remains to estimate for each $j \in \mathbb{I}$

$$\sum_{k\neq j} \left|\hat{\Psi}(x_k - x_j)\right|.$$

Fix $j \in \mathbb{I}$. For the sake of simplicity, assume that $x_j = 0$. Divide the whole space $\mathbb{R}^d$ into the union of V_m, $m = 0, 1, 2, \ldots,$ where

$$V_0 := \left\{x \in \mathbb{R}^d\colon \|x\|_\infty \leqslant 2\gamma\right\}, \qquad V_m := \left\{x \in \mathbb{R}^d\colon 2^m\gamma < \|x\|_\infty \leqslant 2^{m+1}\gamma\right\}, \quad m \in \mathbb{N}.$$

Note that any two distinct x_k's in V_m are separated at least by γ under the norm $\|\cdot\|_\infty$. By estimating the volume, we obtain that there exists a positive constant α such that

$$\#\{k\colon k \in \mathbb{I},\ k \neq j,\ x_k \in V_m\} \leqslant \alpha \frac{(2^{m+1}+1)^d\gamma^d - (2^m-1)^d\gamma^d}{\gamma^d} \leqslant \alpha\left(2^{m+2}\right)^d$$

and

$$\#\{k\colon k \in \mathbb{I},\ k \neq j,\ x_k \in V_0\} \leqslant \alpha 3^d < \alpha 4^d.$$

Note that for x_k in V^m, $m \geqslant 0$,

$$\|x_k - x_j\|_\infty \geqslant 2^m\gamma,$$

implying by (4.30) and (4.31) that

$$\left|\hat{\Psi}(x_k - x_j)\right| \leqslant \frac{1}{|2^m\gamma|^{d+2}}\frac{2^{d+2}}{a^{d+2}}(d+2)!.$$

We now get by the above four equations that

$$\sum_{k\neq j}\left|\hat{\Psi}(x_k-x_j)\right|\leqslant \alpha\sum_{m=0}^{\infty}\left(2^{m+2}\right)^d\frac{1}{|2^m\gamma|^{d+2}}\frac{2^{d+2}}{a^{d+2}}(d+2)!=\frac{\alpha}{3}4^{d+1}\frac{2^{d+2}}{(\gamma a)^{d+2}}(d+2)!.$$

The above equation together with (4.32) and (4.33) yields that if a is large enough so that

$$1>\frac{\alpha}{3}4^{d+1}\frac{2^{d+2}}{(\gamma a)^{d+2}}(d+2)!,$$

then $\{e_{x_j}\}$ is an $\ell^2(\mathbb{I})$-Riesz–Fischer sequence for $L^2([-a,a]^d)$. □

We are finally in a position to prove the main result of the section.

Theorem 4.11. *Let $\mathcal{B}$ be given by (4.15). If ϕ is continuous and positive everywhere on $\mathbb{R}^d$ then regardless of the choice of X_d, $\mathcal{B}$ does not have any X_d-Riesz sampling set.*

Proof. Assume that there exists a *BK*-space X_d for which $\mathcal{B}$ has an X_d-Riesz sampling set $\{x_j\}\subseteq\mathbb{R}^d$. Since $\mathcal{B}$ is isomorphic to the feature space $L^p_\phi(\mathbb{R}^d)$ through the feature map $\Phi(x):=e_x$, $x\in\mathbb{R}^d$, we get by Lemma 4.2 that

$$\left\|G(x,\cdot)^*-G(y,\cdot)^*\right\|_{\mathcal{B}^*}=\left\|e_x^*-e_y^*\right\|_{L^q_\phi(\mathbb{R}^d)}=\int_{\mathbb{R}^d}\left|e^{-i(x,t)}-e^{-i(y,t)}\right|^q\phi(t)\,dt\leqslant 2^{q-1}\int_{\mathbb{R}^d}\left|e^{i(x-y,t)}-1\right|\phi(t)\,dt.$$

Standard arguments show that $x\to G(x,\cdot)^*$ is uniformly continuous from $\mathbb{R}^d$ to $\mathcal{B}^*$. Thus, $\{x_j\}$ must be separated by Theorem 4.9.

We then apply Lemma 4.3 to obtain that $\{e_{-x_j}\}$ forms an X_d^*-Riesz basis for $L^q_\phi(\mathbb{R}^d)$. In particular, its linear span is dense in $L^q_\phi(\mathbb{R}^d)$. We claim that $\operatorname{span}\{e_{-x_j}\}$ is dense in $L^q([-a,a]^d)$ for all $a>0$. To see this, let $f\in L^q([-a,a]^d)$. By the continuity and positivity of ϕ on $\mathbb{R}^d$, there exist positive constants α,β such that

$$\alpha\leqslant\phi(t)\leqslant\beta,\quad t\in[-a,a]^d.$$

Thus, $f\chi_{[-a,a]^d}\in L^q_\phi(\mathbb{R}^d)$. Let $\varepsilon>0$. As $\operatorname{span}\{e_{-x_j}\}$ is dense in $L^q_\phi(\mathbb{R}^d)$, we can find $\tilde{f}\in\operatorname{span}\{e_{-x_j}\}$ such that

$$\|\tilde{f}-f\chi_{[-a,a]^d}\|_{L^q_\phi(\mathbb{R}^d)}\leqslant\alpha^{1/q}\varepsilon.$$

We estimate from the above two equations that

$$\|\tilde{f}-f\|_{L^q([-a,a]^d)}=\|\tilde{f}-f\chi_{[-a,a]^d}\|_{L^q([-a,a]^d)}\leqslant\frac{1}{\alpha^{1/q}}\|\tilde{f}-f\chi_{[-a,a]^d}\|_{L^q_\phi(\mathbb{R}^d)}\leqslant\varepsilon.$$

Therefore, $\operatorname{span}\{e_{-x_j}\}$ is dense in $L^q([-a,a]^d)$ for all $a>0$. By the Hölder inequality, $\operatorname{span}\{e_{-x_j}\}$ is dense in $L^1([-a,a]^d)$ for all $a>0$.

We shall prove that $\operatorname{span}\{e_{-x_j}\}$ is then dense in $L^2([-a,a]^d)$ for all $a>0$. Assume that this is not true for some $a>0$. Then there is a nontrivial $f\in L^2([-a,a]^d)$ such that the entire function of exponential type a

$$F(z):=\int_{[-a,a]^d}e^{i(z,t)}f(t)\,dt,\quad z\in\mathbb{C}^d$$

vanishes on $\{-x_j\}$. Take any $b>0$ and any nontrivial $h\in L^2([-b,b]^d)$. Define

$$H(z):=\int_{[-b,b]^d}e^{i(z,t)}h(t)\,dt,\quad z\in\mathbb{C}^d$$

and set $L:=FH$. Then L is of exponential type $a+b$. By the Paley–Wiener theorem [14], $\hat{L}$ is supported on $[-a-b,a+b]^d$. Moreover, $\hat{L}$ is the convolution of f and h, and hence belongs to $L^\infty(\mathbb{R}^d)$. Therefore,

$$L(x)=F(x)H(x)=\frac{1}{(2\pi)^d}\int_{[-a-b,a+b]^d}e^{i(x,t)}\hat{L}(t)\,dt,\quad x\in\mathbb{R}^d$$

vanishes on $\{-x_j\}$, contradicting that $\operatorname{span}\{e_{-x_j}\}$ is dense in $L^1([-c,c]^d)$ for all $c>0$.

We now complete the proof. Add one more point y to $\{-x_j\}$ that is different from any $-x_j$, $j \in \mathbb{I}$. Then the resulting sequence is still separated. By Lemma 4.10, for large enough $a > 0$, $\{e_y, e_{-x_j}\colon j \in \mathbb{I}\}$ is a Riesz–Fischer sequence for $L^2([-a,a]^d)$. As a result, there exists some nontrivial function $f \in L^2([-a,a]^d)$ such that

$$(f, e_y)_{L^2([-a,a]^d)} = 1, \qquad (f, e_{-x_j})_{L^2([-a,a]^d)} = 0, \quad j \in \mathbb{I}.$$

This contradicts the fact established in the last paragraph that $\operatorname{span}\{e_{-x_j}\}$ is dense in $L^2([-a,a]^d)$ for all $a > 0$. □

As a corollary to the above theorem, we get that the RKHS of the Gaussian kernels (4.18) do not have a Riesz sampling set. Therefore, Shannon type sampling expansions do not exist in such spaces, despite that they all consist of entire functions of finite order.

In the search of Riesz sampling sets, the two fundamental hurdles raised in Proposition 4.8 and Theorem 4.11 should be avoided. For the first one, one might choose the feature space as a proper subspace of L^p spaces. To overcome the second one, one might consider giving up completeness and seeking Riesz bases for subspaces of the RKBS. For studies in RKHS along the latter approach, see, for example, [32,34,39,40]. Favorable properties of the original RKHS, for instance, translation invariance, are generally missing from the resulting subspaces.

5. Finite-dimensional Banach spaces

In this section, we let $\mathcal{B}$ be a normed vector space of finite dimension n and discuss results that hold true in this special case. Set $X_d := \mathbb{C}^n$ equipped with an arbitrary norm. We first examine the assumptions about $\mathcal{B}$ and X_d that were imposed at the beginning of Section 2. Thanks to the finite-dimensionality condition, we shall see that most of them become true automatically.

Note that any two norms on a finite-dimensional vector space are equivalent. As a consequence, $\mathcal{B}$ and X_d are reflexive as there is always an equivalent norm that makes them into a Hilbert space. They are complete for the same reason. The canonical unit vectors e_j, $j \in \mathbb{N}_n$ form a basis for X_d and X_d^*. A basis for a Banach space of finite dimension is of course a Schauder basis. Therefore, the assumptions on the sequence space X_d and its dual space are all satisfied.

As far as $\mathcal{B}$ is concerned, the condition we shall need is for the duality mapping from $\mathcal{B}$ to $\mathcal{B}^*$ induced by a compatible semi-inner product on $\mathcal{B}$ to be bijective. To investigate this desired property, we recall the introduction [15,27] of a compatible semi-inner product on $\mathcal{B}$. Set for each $f \in \mathcal{B}$

$$\mathcal{J}_f := \bigl\{\mu \in \mathcal{B}^*\colon \|\mu\|_{\mathcal{B}^*} = \|f\|_{\mathcal{B}},\ \mu(f) = \|f\|_{\mathcal{B}}\|\mu\|_{\mathcal{B}^*}\bigr\}.$$

By the Hahn–Banach theorem, $\mathcal{J}_f$ is nonempty for every $f \in \mathcal{B}$. A compatible semi-inner product can be defined only in the following way. Select for each $g \in \mathcal{B}$ some $\mu_g \in \mathcal{J}_g$ and set

$$[f, g] := \mu_g(f) \quad \text{for all } f, g \in \mathcal{B}.$$

The duality mapping from $\mathcal{B}$ to $\mathcal{B}^*$ induced from such a compatible semi-inner product is thus given by $f^* := \mu_f$, or in terms of the semi-inner product,

$$f^*(g) := \mu_f(g) = [g, f], \quad g \in \mathcal{B}.$$

Since $\mathcal{B}$ is reflexive, a result due to James [22] states that we are always able to find an appropriate μ_f for each $f \in \mathcal{B}$ so that the duality mapping is surjective onto $\mathcal{B}^*$. Therefore, it remains to check its injectivity. We point out that there exist finite-dimensional Banach spaces for which the duality mapping fails to be injective. Set $\mathcal{B} := \ell^1(\mathbb{N}_3)$. Then $\mathcal{B}^* = \ell^\infty(\mathbb{N}_3)$. For $f := (1,2,1)$ and $g := (2,1,1)$, we observe that

$$\mathcal{J}_f = \mathcal{J}_g = \bigl\{(4,4,4)\bigr\}.$$

Thus, the duality mapping is not injective for this space.

To ensure the injectivity of the duality mapping, we impose the requirement that $\mathcal{B}$ be strictly convex. We claim that $\mathcal{B}^*$ is then strictly convex as well. Assume that there exist $\mu, \nu \in \mathcal{B}^* \setminus \{0\}$ such that

$$\|\mu + \nu\|_{\mathcal{B}^*} = \|\mu\|_{\mathcal{B}^*} + \|\nu\|_{\mathcal{B}^*}.$$

Let $f \in \mathcal{B}$ be a nonzero element such that

$$(\mu + \nu)(f) = \|\mu + \nu\|_{\mathcal{B}^*}\|f\|_{\mathcal{B}}.$$

We observe that

$$(\mu + \nu)(f) \leqslant \bigl|\mu(f)\bigr| + \bigl|\nu(f)\bigr| \leqslant \|\mu\|_{\mathcal{B}^*}\|f\|_{\mathcal{B}} + \|\nu\|_{\mathcal{B}^*}\|f\|_{\mathcal{B}} = \bigl(\|\mu\|_{\mathcal{B}^*} + \|\nu\|_{\mathcal{B}^*}\bigr)\|f\|_{\mathcal{B}}.$$

By the above three equations,

$$\mu(f) = \|\mu\|_{\mathcal{B}^*}\|f\|_{\mathcal{B}}, \qquad \nu(f) = \|\nu\|_{\mathcal{B}^*}\|f\|_{\mathcal{B}},$$

which implies that both $\mu/\|\mu\|_{\mathcal{B}^*}$ and $\nu/\|\nu\|_{\mathcal{B}^*}$ are the image of $f/\|f\|_{\mathcal{B}}$ under the duality mapping. Since $\mathcal{B}$ is strictly convex, the duality mapping is injective. Consequently,

$$\mu = \frac{\|\mu\|_{\mathcal{B}^*}}{\|\nu\|_{\mathcal{B}^*}}\nu.$$

The claim is hence true. Therefore, $\mathcal{B}^*$ is uniformly convex. Since a normed vector space is uniformly Fréchet differentiable if and only if its dual is uniformly convex [7], $\mathcal{B}$ is a uniform Banach space.

We start with characterizing frames in $\mathcal{B}$.

Proposition 5.1. *Let $\mathcal{B}$ be of finite dimension and X_d an arbitrary BK-space. Then any finite sequence $\{f_j\} \subseteq \mathcal{B}$ is an X_d-Bessel sequence for $\mathcal{B}$. It is an X_d-frame if and only if* (2.16) *holds true.*

Proof. Since any linear operator from a finite-dimensional Banach space must be bounded, there exists some $B > 0$ such that

$$\|U(f)\|_{X_d} \leqslant B\|f\|_{\mathcal{B}}, \quad f \in \mathcal{B},$$

which implies that f_j form an X_d-Bessel sequence for $\mathcal{B}$.

Suppose that (2.16) holds true. It follows that the operator U is injective. Since the range $\mathcal{R}(U)$ of U is finite-dimensional, it is a closed subspace of X_d. This together with the boundedness of U implies by the open mapping theorem that U has a bounded inverse on $\mathcal{R}(U)$. By Lemma 2.8, $\{f_j\}$ is an X_d-frame for $\mathcal{B}$.

On the other hand, assume that (2.16) is not true. Then there exists a nontrivial $\nu \in \mathcal{B}^{**}$ such that $\nu(f_j^*) = 0$, $j \in \mathbb{I}$. Since $\mathcal{B}$ is reflexive, there exists nontrivial $f \in \mathcal{B}$ such that

$$\nu(\mu) = \mu(f) \quad \text{for all } \mu \in \mathcal{B}^*.$$

We hence reach that

$$[f, f_j] = f_j^*(f) = \nu(f_j^*) = 0, \quad j \in \mathbb{I}.$$

Consequently, f_j do not form an X_d-frame for $\mathcal{B}$. The proof is complete. □

Likewise, one obtains by (2.2) and (2.3) the following result.

Proposition 5.2. *Let $\mathcal{B}$ be finite-dimensional. Then any finite sequence $\{f_j^*\} \subseteq \mathcal{B}^*$ is an X_d^*-Bessel sequence for $\mathcal{B}^*$. It is an X_d^*-frame if and only if there holds* (2.17)*.*

We next present the promised example showing that $\{f_j\}$ being a frame for $\mathcal{B}$ does not necessarily imply that $\{f_j^*\}$ is a frame for $\mathcal{B}^*$. The sequence $\{f_j\}$ used below was constructed in [45] by the Matlab for a different purpose.

Example 5.3. We investigate $\mathcal{B} := \ell^{3/2}(\mathbb{N}_3)$ with the semi-inner product

$$[a, b] := \|b\|_{\mathcal{B}}^{1/2} \sum_{j=1}^{3} a_j \overline{b_j} |b_j|^{-1/2}, \quad a, b \in \mathcal{B}$$

and the following sequence in $\mathcal{B}$:

$$f_1 = \frac{(4, 81, 1)}{(738)^{1/3}}, \qquad f_2 = \frac{(1, 64, 0)}{(513)^{1/3}}, \qquad f_3 = \frac{(25, 25, 9)}{(277)^{1/3}}. \tag{5.1}$$

Then

$$f_1^* = (2, 9, 1), \qquad f_2^* = (1, 8, 0), \qquad f_3^* = (5, 5, 3). \tag{5.2}$$

It can be verified that $\operatorname{span}\{f_1^*, f_2^*, f_3^*\} = \mathcal{B}^*$ while $\operatorname{span}\{f_1, f_2, f_3\} \subsetneqq \mathcal{B}$. By Propositions 5.1 and 5.2, for any BK-space X_d, $\{f_1, f_2, f_3\}$ is an X_d-frame for $\mathcal{B}$ but $\{f_1^*, f_2^*, f_3^*\}$ is not an X_d^*-frame for $\mathcal{B}^*$.

Turning to Riesz bases, we have a simple observation.

Lemma 5.4. *A sequence f_j, $j \in \mathbb{I}$ forms an X_d-Riesz basis for $\mathcal{B}$ if and only if $\#\mathbb{I} = \dim \mathcal{B}$ and f_j are linearly independent.*

Proof. By Propositions 2.13 and 5.2, $\{f_j\}$ is an X_d-Riesz basis for $\mathcal{B}$ if and only if f_j are linearly independent and $\operatorname{span}\{f_j\}=\mathcal{B}$. □

Let $n=\dim\mathcal{B}$. For simplicity, suppose that $\mathbb{I}=\mathbb{N}_n$. By the above lemma, any basis $\{f_j\}$ for $\mathcal{B}$ is an X_d-Riesz basis for $\mathcal{B}$ regardless of the choice of the sequence space X_d. However, as we have seen from Example 5.3, this does not guarantee that f_j^* form an X_d^*-Riesz basis for $\mathcal{B}^*$. The reason is that f_j^* might fail to be linearly independent. We shall propose a nonlinear Gram–Schmidt algorithm of producing a sequence $\{h_j\}\in\mathcal{B}$ from an arbitrary basis $\{f_j\}$ for $\mathcal{B}$ so that $\{h_j\}$ and $\{h_j^*\}$ are an X_d-Riesz basis for $\mathcal{B}$ and X_d^*-Riesz basis for $\mathcal{B}^*$, respectively. The constructed sequence is designed to satisfy

$$[h_j,h_k]=\delta_{j,k},\quad 1\leqslant j\leqslant k\leqslant n. \tag{5.3}$$

The algorithm bases on the characterization of best approximation proved by Giles [15] that $f,g\in\mathcal{B}$ satisfies $\|f+\alpha g\|_{\mathcal{B}}\geqslant\|f\|_{\mathcal{B}}$ for all $\alpha\in\mathbb{C}$ if and only if $[g,f]=0$.

The algorithm starts by setting

$$h_1:=\frac{f_1}{\|f_1\|}.$$

Assume that h_j, $1\leqslant j\leqslant k$ have been constructed such that

$$[h_j,h_l]=\delta_{j,l},\quad 1\leqslant j\leqslant l\leqslant k \tag{5.4}$$

and

$$\operatorname{span}\{h_j\colon 1\leqslant j\leqslant k\}=\operatorname{span}\{f_j\colon 1\leqslant j\leqslant k\}. \tag{5.5}$$

We next construct h_{k+1}. Let g_k be the element in $\operatorname{span}\{f_j\colon 1\leqslant j\leqslant k\}$ such that

$$\|f_{k+1}-g_k\|_{\mathcal{B}}=\min\bigl\{\|f_{k+1}-g\|_{\mathcal{B}}\colon g\in\operatorname{span}\{f_j\colon 1\leqslant j\leqslant k\}\bigr\}.$$

According to the characterization of best approximation due to Giles,

$$[g,f_{k+1}-g_k]=0\quad\text{for all } g\in\operatorname{span}\{f_j\colon 1\leqslant j\leqslant k\}.$$

By (5.5), there exist constants $\alpha_j\in\mathbb{C}$, $1\leqslant j\leqslant k$ such that $g_k=\sum_{j=1}^k\alpha_jh_j$, where α_j are uniquely determined by

$$\left[h_l,f_{k+1}-\sum_{j=1}^k\alpha_jh_j\right]=0,\quad 1\leqslant l\leqslant k. \tag{5.6}$$

By (5.5) and the linear independence of f_j,

$$f_{k+1}-\sum_{j=1}^k\alpha_jh_j\neq 0.$$

We then set

$$h_{k+1}:=\frac{f_{k+1}-\sum_{j=1}^k\alpha_jh_j}{\|f_{k+1}-\sum_{j=1}^k\alpha_jh_j\|_{\mathcal{B}}}.$$

Clearly, (5.4) and (5.5) are preserved when k is updated to $k+1$ therein. Successively applying the construction until $k=n$, we obtain a basis $\{h_j\colon j\in\mathbb{N}_n\}$ for $\mathcal{B}$ satisfying (5.3). The algorithm is said to be nonlinear as Eqs. (5.6) are in general nonlinear with respect to α_j. This is because that a semi-inner product is nonadditive with respect to its second variable unless it reduces to an inner product [36].

We fulfill a main purpose of the section by proving that for the basis $\{h_j\colon j\in\mathbb{N}_n\}$ generated by the above algorithm, h_j^* form an X_d^*-Riesz basis for $\mathcal{B}^*$.

Proposition 5.5. *The above described Gram–Schmidt algorithm generates from an arbitrary basis $\{f_j\colon j\in\mathbb{N}_n\}$ for $\mathcal{B}$ a sequence $\{h_j\colon j\in\mathbb{N}_n\}$ such that $\{h_j\colon j\in\mathbb{N}_n\}$ and $\{h_j^*\colon j\in\mathbb{N}_n\}$ are an X_d-Riesz basis for $\mathcal{B}$ and X_d^*-Riesz basis for $\mathcal{B}^*$, respectively.*

Proof. The generated sequence $\{h_j\colon j\in\mathbb{N}_n\}$ remains a basis for $\mathcal{B}$. In particular, it is linearly independent. By Lemma 5.4, it is an X_d-Riesz basis for $\mathcal{B}$. It remains to show that h_j^* are linearly independent as well. Assume to the contrary that they are linearly dependent. Consequently, there exists some $\mu\in\mathcal{B}^{**}\setminus\{0\}$ such that

$$\mu(h_k^*) = 0, \quad k \in \mathbb{N}_n.$$

Since $\mathcal{B}$ is finite-dimensional, it is automatically reflexive. Thus, there exist constants α_j, $j \in \mathbb{N}_n$ all of whose are not zero such that

$$\left[\sum_{j\in\mathbb{N}_n} \alpha_j h_j, h_k\right] = h_k^*\left(\sum_{j\in\mathbb{N}_n} \alpha_j h_j\right) = \mu(h_k^*) = 0, \quad k \in \mathbb{N}_n.$$

Successively letting $k = n, n-1, \ldots, 1$ in the above equation yields by (5.3) that $\alpha_j = 0$ for all $j \in \mathbb{N}_n$, a contradiction. □

By contrast to the negative result in Section 4, we close the paper by showing that a finite-dimensional RKBS always has a Riesz sampling set.

Proposition 5.6. *A finite-dimensional RKBS possesses an X_d-Riesz sampling set for any BK-space X_d.*

Proof. Let $\mathcal{B}$ be an RKBS on X with finite dimension n and s.i.p. reproducing kernel G. By (4.2),

$$\operatorname{span}\{G(x,\cdot)^*\colon x \in X\} = \mathcal{B}^*.$$

Since $\dim \mathcal{B}^* = \dim \mathcal{B} = n$, the above equation implies that there exist n points $x_j \in X$, $j \in \mathbb{N}_n$ such that $G(x_j,\cdot)^*$, $j \in \mathbb{N}_n$, are linearly independent. As a result,

$$\operatorname{span}\{G(x_j,\cdot)^*\colon j \in \mathbb{N}_n\} = \mathcal{B}^*. \tag{5.7}$$

By (5.7), $|\cdot| : \mathcal{B} \to \mathbb{R}_+$ defined by

$$|f| := \left\|\{f(x_j)\}\right\|_{X_d}, \quad f \in \mathcal{B}$$

is a norm on $\mathcal{B}$. Since $\mathcal{B}$ is finite-dimensional, this norm is equivalent to the original one on $\mathcal{B}$. It implies that $G(x_j,\cdot)$, $j \in \mathbb{N}_n$ form an X_d-frame for $\mathcal{B}$. This together with the linear independence of $G(x_j,\cdot)^*$, $j \in \mathbb{N}_n$ prove that $\{x_j\colon j \in \mathbb{N}_n\}$ is an X_d-Riesz sampling set for $\mathcal{B}$. □

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