A Hölder estimate for non-uniform elliptic equations in a random medium

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Uniform regularity for second order elliptic equations in a highly heterogeneous random medium is concerned. The medium is separated by a random ensemble of simply closed interfaces into a connected sub-region with high conductivity and a disconnected subset with low conductivity. The elliptic equations, whose diffusion coefficients depend on the conductivity, have fast diffusion in the connected sub-region and slow diffusion in the disconnected subset. Without a stationary-ergodic assumption, a uniform Hölder estimate in $\omega, \epsilon, \lambda$ for the elliptic solutions is derived, where ω is a realization of the random ensemble, $\epsilon \in (0,1]$ is the length scale of the interfaces, and $\lambda^2 \in (0,1]$ is the conductivity ratio of the disconnected subset to the connected sub-region. Results show that if external sources are small enough in the disconnected subset, the uniform Hölder estimate in $\omega, \epsilon, \lambda$ holds in the whole domain. If not, it holds only in the connected sub-region. Meanwhile, the elliptic solutions change rapidly in the disconnected subset.

Keywords: random media, conductivity, stationary-ergodic, realization, diffeomorphism

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1. Introduction

In this article, we study the uniform regularity for second order elliptic equations in a highly heterogeneous random medium. The medium is separated by a random ensemble of simply closed surfaces of codimension one with jump type conductivity across the surfaces. This problem has many applications in physics and engineering, for example, contaminant transport in the subsurface, heat flow in random media, the stress in composite materials, and so on (see [11, 18, 19, 30]).

Let $\epsilon \in (0,1]$ represent the length scale of the surfaces separating the spatial domain, $Y \equiv [0,1)^n$ be the unit cube in $\mathbb{R}^n$ for $n \geq 2$, Y_m be a smooth simply-connected sub-domain of Y with boundary ∂Y_m, $Y_f \equiv Y \setminus \overline{Y_m}$, $\mathbf{d}_1 \equiv \mathrm{dist}(\partial Y, Y_m) > 0$, $(\Omega, \mathcal{F}, \mathcal{P})$ mean a probability space, $\Phi(\cdot,\omega) : \mathbb{R}^n \to \mathbb{R}^n$ denote a

C^1-diffeomorphism for any $\omega \in \Omega$, $\mathcal{D}$ be a bounded Lipschitz domain in $\mathbb{R}^n$, and

$$\begin{cases} \Upsilon(\cdot,\omega) \text{ denote the inverse mapping of } \Phi(\cdot,\omega), \\ \mathcal{D}(\epsilon) \equiv \{x \in \mathcal{D} | \operatorname{dist}(x, \partial\mathcal{D}) \geq \epsilon\}, \\ \mathcal{I}_{\omega,\epsilon} \equiv \{j \in \mathbb{Z}^n | \ \epsilon\Phi(Y + j, \omega) \subset \mathcal{D}(\epsilon)\}, \\ \Gamma_\omega^\epsilon \equiv \bigcup_{j\in\mathcal{I}_{\omega,\epsilon}} \epsilon\Phi(\partial Y_m + j, \omega), \\ \mathcal{D}_{\omega,m}^\epsilon \equiv \bigcup_{j\in\mathcal{I}_{\omega,\epsilon}} \epsilon\Phi(Y_m + j, \omega), \\ \mathcal{D}_{\omega,f}^\epsilon \equiv \mathcal{D} \setminus \overline{\mathcal{D}_{\omega,m}^\epsilon}. \end{cases} \tag{1.1}$$

From (1.1), we see that $\mathcal{D} = \mathcal{D}_{\omega,f}^\epsilon \cup \mathcal{D}_{\omega,m}^\epsilon \cup \Gamma_\omega^\epsilon$ is obtained from the image of a periodic medium under a random deformation followed by a rescaling, $\mathcal{D}_{\omega,f}^\epsilon$ is a connected sub-region of $\mathcal{D}$, and $\mathcal{D}_{\omega,m}^\epsilon$ is a disconnected subset of $\mathcal{D}$ for each $\omega \in \Omega$.

Let $\lambda^2 \in (0, 1]$ represent the conductivity ratio of the subset $\mathcal{D}_{\omega,m}^\epsilon$ to the sub-region $\mathcal{D}_{\omega,f}^\epsilon$. So the connected region $\mathcal{D}_{\omega,f}^\epsilon$ has high conductivity and the disconnected set $\mathcal{D}_{\omega,m}^\epsilon$ has low conductivity. Let $\mathbf{K}$ be a positive definite piecewise continuous matrix in $\mathbb{R}^n$. The problem that we consider is

$$\begin{cases} -\nabla \cdot (\mathbf{K}_{\omega,\lambda^2,\epsilon} \nabla U + Q) = F & \text{in } \mathcal{D}, \\ U = 0 & \text{on } \partial\mathcal{D}, \end{cases} \tag{1.2}$$

where Q, F are known functions and $\mathbf{K}_{\omega,\lambda^2,\epsilon}$ is the conductivity with the expression

$$\mathbf{K}_{\omega,\lambda^2,\epsilon}(x) \equiv \begin{cases} \mathbf{K}(\Upsilon(\frac{x}{\epsilon},\omega)) & \text{if } x \in \mathcal{D}_{\omega,f}^\epsilon \\ \lambda^2\mathbf{K}(\Upsilon(\frac{x}{\epsilon},\omega)) & \text{if } x \in \mathcal{D}_{\omega,m}^\epsilon \end{cases} \quad \text{for } \omega \in \Omega, \epsilon \in (0,1], \lambda > 0.$$

Problem (1.2) is a non-uniform elliptic equation with random coefficients and random interfaces.

This setting of the random medium is introduced in [21, 22], is a random perturbation of periodic structures, but is not a special case of the existing theories. Under a stationary-ergodic assumption, some new homogenization results can be obtained (see [21, 22]). For more results on the medium of above setting, please see [3, 5, 10, 15, 21, 22, 28] and references therein. Our concern is the uniform regularity for second order elliptic equations on the medium of above setting. Different from the usual homogenization results on random media, our main result shows that, without a stationary-ergodic assumption, a Hölder estimate independent of the realization ω for the solution of (1.2) can be proved.

Whenever Q and F are smooth in $\mathcal{D}$ and small in $\mathcal{D}_{\omega,m}^\epsilon$, a piecewise smooth solution U of (1.2) exists uniquely and the H^1 norm of the solution in the connected sub-region $\mathcal{D}_{\omega,f}^\epsilon$ is bounded uniformly in $\epsilon, \lambda, \omega$ (see [20]). However, that is not the case for the solution U in the disconnected subset $\mathcal{D}_{\omega,m}^\epsilon$ and also the second order derivatives of U may not be bounded uniformly in $\epsilon, \lambda, \omega$ in the connected sub-region $\mathcal{D}_{\omega,f}^\epsilon$.

For periodic domain case (a special case of random media [18, 25]), uniform regularity of the elliptic equations had been studied extensively. For example, uniform Hölder, $W^{1,p}$, and Lipschitz estimates in ϵ for uniform elliptic case of (1.2) (i.e., $\lambda = 1$) with Hölder coefficients were proved in [8, 9]. Uniform $W^{1,p}$ estimate in ϵ for uniform elliptic case of (1.2) with continuous coefficients was shown in [13] and the same problem with VMO coefficients could be found in [27]. Uniform $W^{1,p}$ estimate for the Laplace equation in periodic perforated domains was considered in [23] and the same problem in Lipschitz estimate was studied in [26]. Uniform Hölder and Lipschitz estimates in λ, ϵ for non-uniform elliptic equations with discontinuous periodic coefficients were shown in [32].

In the modelling of random media, a stationary-ergodic assumption plays an important role because, after the homogenization process, the averaged model is deterministic (see [4, 18, 21, 22]). In [6], a class of non-uniform nonlinear elliptic equations was considered. Under a general stationary-ergodic setting, uniform L^∞ regularity of the solutions was proved for almost each realization. On the other hand, it is known that, without a stationary-ergodic assumption, uniform Hölder estimate in ω, ϵ for uniform elliptic case of (1.2) can be shown (see [17, 20]).

In this work, we consider non-uniform elliptic equations with discontinuous coefficients in a random medium. As mentioned above, without stationary-ergodic assumption, a Hölder estimate uniform in $\epsilon, \lambda, \omega$ for the solution of (1.2) is proved. The idea of reasoning is as follows. Since the domain $\mathcal{D}^\epsilon_{\omega,f} \cup \mathcal{D}^\epsilon_{\omega,m} \cup \Gamma^\epsilon_\omega$ for each $\omega \in \Omega$ is obtained from the image of a periodic medium under a random deformation followed by a rescaling, a cell-like problem can be derived, the local regularity of the solutions of the cell-like problem can be obtained, and a three-step compactness argument (see [8, 9]) can be applied to the problem (1.2) to get a uniform Hölder estimate in $\epsilon, \lambda, \omega$. Let us mention that the derivation of the uniform Hölder estimate of the solutions of (1.2) strongly relies on the fact: the uniform Hölder estimate for the uniform elliptic equations can be obtained without a stationary-ergodic assumption [17, 20]. In order to apply the three-step compactness argument, we consider a H-convergence problem for non-uniform elliptic equations and obtain some estimates for the homogenized matrix (see [14, 18] for uniform elliptic equation case). To obtain the local regularity for the solutions of the cell-like problem, an interface problem for non-uniform elliptic equations is considered. Meanwhile, the Morrey space as the function space is used to downplay the coefficient smoothness requirements.

The rest of this paper is organized as follows. In section 2, we recall some notations and state the main result. In section 3, we study the H-convergence of non-uniform elliptic equations. In section 4, we consider an interface problem for non-uniform elliptic equations. The proof of the main result is given in section 5.

2. Notation and main result

For any measurable subset $\mathbf{S}$ of $\mathbb{R}^n$, $\partial\mathbf{S}$ is the boundary of $\mathbf{S}$, $|\mathbf{S}|$ denotes its Lebesgue measure, $\overline{\mathbf{S}}$ is the closure of $\mathbf{S}$, $\mathbf{S}^\circ$ is the interior of $\mathbf{S}$, $\mathcal{X}_\mathbf{S}$ is the characteristic function on $\mathbf{S}$, and $\mathbf{S}/r = \{x | rx \in \mathbf{S}\}$ for $r > 0$. $B_r(x)$ denotes the ball centered at x with radius $r > 0$. $(p-2)_+ \equiv \max\{0, p-2\}$.

If $\mathbf{B}_1, \mathbf{B}_2$ are Banach spaces, $\|\varphi_1, \cdots, \varphi_i\|_{\mathbf{B}_1} \equiv \max\{\|\varphi_1\|_{\mathbf{B}_1}, \cdots, \|\varphi_i\|_{\mathbf{B}_1}\}$ and $\|\varphi\|_{\mathbf{B}_1 \cap \mathbf{B}_2} = \|\varphi\|_{\mathbf{B}_1} + \|\varphi\|_{\mathbf{B}_2}$. $C^{k,\alpha}$, L^p, $W^{k,p}$, $L^{k,p}$, and $\mathcal{L}^{k,p}$ are used for the Hölder space, Lebesgue space, the Sobolev space, Morrey space, and Campanato space respectively (see [2, 17, 31]). For $p \geq 2$, $W_0^{1,p}(\mathbf{S}) \equiv \{\varphi \in W^{1,p}(\mathbf{S}) |\ \varphi = 0 \text{ on } \partial \mathbf{S}\}$ and $H_0^1(\mathbf{S}) = W_0^{1,2}(\mathbf{S})$. $H^k = W^{k,2}$ and $[\varphi]_{C^{0,\alpha}}$ is the Hölder semi-norm of φ.

$$\begin{cases} (\varphi)_{\mathbf{S}} \equiv \fint_{\mathbf{S}} \varphi(y)dy \equiv \dfrac{1}{|\mathbf{S}|} \displaystyle\int_{\mathbf{S}} \varphi(y)dy & \text{if } \varphi \in L^1(\mathbf{S}), \\ (\varphi)_{x,r} \equiv (\varphi)_{B_r(x) \cap \mathcal{D}} & \text{if } \varphi \in L^1(B_r(x) \cap \mathcal{D}). \end{cases} \tag{2.1}$$

Denote by $\mathbf{A}^{-1}$ the inverse matrix of $\mathbf{A}$ and define

$$\begin{cases} \mathcal{M}_1(c_1, c_2) \equiv \big\{\mathbf{A}|\ \text{a } n \times n \text{ matrix with } 0 < c_1 \leq \mathbf{A}, \frac{1}{c_2} \leq \mathbf{A}^{-1}, c_1 < c_2\big\}, \\ \mathcal{M}_2(c_1, c_2) \equiv \big\{\mathbf{A}|\ \text{a } n \times n \text{ matrix with } 0 < c_1 \leq \mathbf{A} \leq c_2\big\}, \\ \mathcal{T}(c_1, c_2) \equiv \big\{\mathbf{A}|\text{a bounded linear operator with } 0 < c_1 \leq \mathbf{A} \leq c_2\big\}. \end{cases}$$

It is easy to see that $\mathcal{M}_1(c_1, c_2) \subset \mathcal{M}_2(c_1, c_2)$.

The main result is

Theorem 2.1. *Under the following assumptions:*

- *(A1)* $\mathcal{D} \subset \mathbb{R}^n$ *is a bounded Lipschitz domain for* $n \geq 2$, Y_m *is a* C^2 *simply-connected sub-domain of* Y, $\mathbf{d}_1 \equiv \operatorname{dist}(\partial Y, Y_m) > 0$,
- *(A2)* $\Phi(\cdot, \omega)$ *is a* C^1*-diffeomorphism in* $\mathbb{R}^n$ *for any fixed* $\omega \in \Omega$ *and there are constants* $\mathbf{d}_2, \mathbf{d}_3$ *such that*

$$\begin{cases} \inf\limits_{\omega \in \Omega, y \in \mathbb{R}^n} \det(\nabla \Phi(y, \omega)) \geq \mathbf{d}_2 > 0, \\ \sup\limits_{\omega \in \Omega, y \in \mathbb{R}^n} |\nabla \Phi(y, \omega)| \leq \mathbf{d}_3 < \infty, \end{cases}$$

- *(A3)* $\mathbf{K} \in \mathcal{M}_1(\beta, \gamma)$ *is continuous on* $\cup_{j \in \mathbb{Z}^n}(Y_f + j)$ *and* $\cup_{j \in \mathbb{Z}^n}(Y_m + j)$,
- *(A4)* $Q \in L^{2,q}(\mathcal{D})$, $F \in L^{2,(q-2)_+}(\mathcal{D})$, $q = n - 2 + 2\mu$, $\mu \in (0, 1)$, $\epsilon, \lambda \in (0, 1]$, $\mathbf{e} \in [0, 1]$, $\omega \in \Omega$,

a $H_0^1(\mathcal{D})$ *solution of (1.2) exists uniquely and the solution satisfies*

$$\begin{aligned} &[U]_{C^{0,\mu}(\mathcal{D}^\epsilon_{\omega,f})} + \lambda^{\mathbf{e}}[U]_{C^{0,\mu}(\mathcal{D}^\epsilon_{\omega,m})} \\ &\quad \leq c\big(\|\mathbf{K}_{\omega,\lambda^{\mathbf{e}-2},\epsilon} Q\|_{L^{2,q}(\mathcal{D})} + \|\mathbf{K}_{\omega,\lambda^{\mathbf{e}-2},\epsilon} F\|_{L^{2,(q-2)_+}(\mathcal{D})}\big), \end{aligned} \tag{2.2}$$

where c *is a positive constant independent of* $\epsilon, \lambda, \omega, \mathbf{e}$.

Remark 2.1. The inverse matrix of $\nabla \Phi(\cdot, \omega)$ is $\frac{1}{\det(\nabla \Phi(\cdot, \omega))}(\operatorname{cof} \nabla \Phi(\cdot, \omega)_{i,j})^T$, where $\operatorname{cof} \nabla \Phi(\cdot, \omega)_{i,j}$ is the cofactor of the (i, j)-component of $\nabla \Phi(\cdot, \omega)$. By $(1.1)_1$ and (A2), we see that

$$\begin{cases} \inf\limits_{\omega \in \Omega, y \in \mathbb{R}^n} \det(\nabla \Upsilon(y, \omega)) \geq c_3(\mathbf{d}_3) > 0, \\ \sup\limits_{\omega \in \Omega, y \in \mathbb{R}^n} |\nabla \Upsilon(y, \omega)| \leq c_4(\mathbf{d}_2, \mathbf{d}_3) < \infty. \end{cases}$$

Therefore, in the reference random medium $\Phi(\mathbb{R}^n, \omega)$, the neighbor interfaces are separated at least by a distance of $c_5(\mathbf{d}_2, \mathbf{d}_3)$ and at most by a distance of $c_6(\mathbf{d}_2, \mathbf{d}_3)$.

Suppose the right hand side of (2.2) is bounded from above. By Theorem 2.1, we know that if $\mathbf{e} = 0$ (that is, external sources are small in the disconnected subset $\mathcal{D}^\epsilon_{\omega,m}$), the Hölder norm of the solution of (1.2) is bounded uniformly in $\epsilon, \lambda, \omega$ in the whole domain $\mathcal{D}$. If $\mathbf{e} > 0$ (that is, external sources are not small in $\mathcal{D}^\epsilon_{\omega,m}$), the Hölder norm of the solution of (1.2) is bounded uniformly in $\epsilon, \lambda, \omega$ only in the connected high conductivity region $\mathcal{D}^\epsilon_{\omega,f}$. Meanwhile, in the disconnected subset $\mathcal{D}^\epsilon_{\omega,m}$, the elliptic solution changes rapidly when λ is small. This is different from uniform elliptic equation case, where the Hölder norm of the elliptic solution is always bounded uniformly in the whole domain.

We should remark that in [21, 22], $\Phi(\cdot, \omega)$ is assumed to be a C^1-diffeomorphism in $\mathbb{R}^n$ for almost every $\omega \in \Omega$ and is slightly different from (A2) above. As far as the derivation of uniform Hölder estimate is concerned, the argument in this paper can be applied their case without any change and the same conclusion can be obtained as long as we take out a set of measure 0. So for simplicity of presentation, we assume (A2).

3. *H*-convergence

The H-convergence for uniform elliptic equations is well-studied (see [14, 18]). In this section, (A1)–(A3) are assumed and we consider the H-convergence for non-uniform elliptic equations in random media. We present two convergence results: Lemma 3.3 and Lemma 3.5. Lemma 3.3 is used in the interior estimate in subsection 5.1 and Lemma 3.5 is used in the boundary estimate in subsection 5.2.

Define

$$\begin{cases} \mathbb{D}^\epsilon_\omega \equiv \epsilon\Upsilon(\mathcal{D}/\epsilon, \omega), \\ \mathbb{D}^\epsilon_{\omega,m} \equiv \bigcup_{j\in\mathcal{I}_{\omega,\epsilon}} \epsilon(Y_m + j), \\ \mathbb{D}^\epsilon_{\omega,f} \equiv \mathbb{D}^\epsilon_\omega \setminus \overline{\mathbb{D}^\epsilon_{\omega,m}}. \end{cases} \tag{3.1}$$

See (1.1) for $\mathcal{I}_{\omega,\epsilon}$. By (A1)–(A2), $\mathbb{D}^\epsilon_\omega$ is a bounded Lipschitz domain in $\mathbb{R}^n$ with periodic microstructure. By extension theorem [1], we know

Remark 3.1. There is a constant $\mathbf{d}_4(n, Y_m)$ and a linear continuous extension operator $\Pi_\epsilon : H^1(\mathbb{D}^\epsilon_{\omega,f}) \to H^1(\mathbb{D}^\epsilon_\omega)$ such that
(1) if $\varphi \in H^1(\mathbb{D}^\epsilon_{\omega,f})$, then

$$\begin{cases} \Pi_\epsilon\varphi = \varphi \quad \text{in } \mathbb{D}^\epsilon_{\omega,f} \text{ almost everywhere}, \\ \|\Pi_\epsilon\varphi\|_{L^2(\mathbb{D}^\epsilon_\omega)} \le \mathbf{d}_4\|\varphi\|_{L^2(\mathbb{D}^\epsilon_{\omega,f})}, \\ \|\nabla\Pi_\epsilon\varphi\|_{L^2(\mathbb{D}^\epsilon_\omega)} \le \mathbf{d}_4\|\nabla\varphi\|_{L^2(\mathbb{D}^\epsilon_{\omega,f})}, \end{cases}$$

(2) if $0 < \epsilon < r$ and $\zeta(x) \equiv \varphi(rx)$, then $\Pi_{\epsilon/r}\zeta(x) = \Pi_\epsilon\varphi(rx)$.

Let $\mathcal{V}_{\omega,\epsilon} \equiv \{\varphi \in H^1(\mathcal{D}^\epsilon_{\omega,f}) |\ \varphi|_{\partial\mathcal{D}} = 0\}$ and let $\mathcal{V}'_{\omega,\epsilon}$ be the dual space of $\mathcal{V}_{\omega,\epsilon}$. If $\zeta \in \mathcal{V}'_{\omega,\epsilon}$ and $\varphi \in \mathcal{V}_{\omega,\epsilon}$, then $\langle \zeta, \varphi\rangle_{\mathcal{V}'_{\omega,\epsilon},\mathcal{V}_{\omega,\epsilon}}$ denotes the value of the functional ζ applied to the element φ. By (A1)–(A2), (3.1), and Remarks 2.1, 3.1, we also have

Remark 3.2. There is a constant $\mathbf{d}_5$ (depending on $\mathbf{d}_2, \mathbf{d}_3, \mathbf{d}_4$ only) and there is a linear continuous extension operator $\Pi_{\omega,\epsilon} : \mathcal{V}_{\omega,\epsilon} \to H^1_0(\mathcal{D})$ such that
(1) if $\varphi \in \mathcal{V}_{\omega,\epsilon}$, then

$$\begin{cases} \Pi_{\omega,\epsilon}\varphi = \varphi \quad \text{in } \mathcal{D}^\epsilon_{\omega,f} \text{ almost everywhere}, \\ \|\Pi_{\omega,\epsilon}\varphi\|_{L^2(\mathcal{D})} \le \mathbf{d}_5 \|\varphi\|_{L^2(\mathcal{D}^\epsilon_{\omega,f})}, \\ \|\nabla\Pi_{\omega,\epsilon}\varphi\|_{L^2(\mathcal{D})} \le \mathbf{d}_5 \|\nabla\varphi\|_{L^2(\mathcal{D}^\epsilon_{\omega,f})}, \end{cases}$$

(2) if $0 < \epsilon < r$ and $\zeta(x) \equiv \varphi(rx)$, then $\Pi_{\omega,\epsilon/r}\zeta(x) = \Pi_{\omega,\epsilon}\varphi(rx)$.

If $\varphi \in H^1(\mathcal{D})$, we denote the extension function of $\varphi|_{\mathcal{D}^\epsilon_{\omega,f}} \in H^1(\mathcal{D}^\epsilon_{\omega,f})$ in $\mathcal{D}$ by $\Pi_{\omega,\epsilon}\varphi|_{\mathcal{D}^\epsilon_{\omega,f}} \in H^1(\mathcal{D})$. Let $\Pi'_{\omega,\epsilon} : H^{-1}(\mathcal{D}) \to \mathcal{V}'_{\omega,\epsilon}$ denote the adjoint of $\Pi_{\omega,\epsilon}$ and $\mathcal{X}'_{\mathcal{D}^\epsilon_{\omega,f}} : \mathcal{V}'_{\omega,\epsilon} \to H^{-1}(\mathcal{D})$ denote the adjoint of $\mathcal{X}_{\mathcal{D}^\epsilon_{\omega,f}}$. So $\Pi'_{\omega,\epsilon}$ and $\mathcal{X}'_{\mathcal{D}^\epsilon_{\omega,f}}$ are bounded linear operators satisfying

$$\begin{cases} \langle \Pi'_{\omega,\epsilon}\zeta, \varphi\rangle_{\mathcal{V}'_{\omega,\epsilon},\mathcal{V}_{\omega,\epsilon}} = \langle \zeta, \Pi_{\omega,\epsilon}\varphi\rangle_{H^{-1}(\mathcal{D}),H^1_0(\mathcal{D})} & \text{for } \zeta \in H^{-1}(\mathcal{D}), \varphi \in \mathcal{V}_{\omega,\epsilon}, \\ \langle \mathcal{X}'_{\mathcal{D}^\epsilon_{\omega,f}}\zeta, \varphi\rangle_{H^{-1}(\mathcal{D}),H^1_0(\mathcal{D})} = \langle \zeta, \mathcal{X}_{\mathcal{D}^\epsilon_{\omega,f}}\varphi\rangle_{\mathcal{V}'_{\omega,\epsilon},\mathcal{V}_{\omega,\epsilon}} & \text{for } \zeta \in \mathcal{V}'_{\omega,\epsilon}, \varphi \in H^1_0(\mathcal{D}). \end{cases} \tag{3.2}$$

3.1. *Interior estimate*

In this subsection we assume $\overline{B_1(0)} \subset \mathcal{D}$.

Lemma 3.1. *Consider the following problem*

$$-\nabla\cdot(\mathbf{K}_{\omega,\lambda^2,\epsilon}\nabla P_\epsilon + V_\epsilon) = G_\epsilon \qquad \text{in } B_1(0), \tag{3.3}$$

where

$$\begin{cases} \lambda, \epsilon \in (0,1], \quad \omega \in \Omega, \\ \left\| P_\epsilon \mathcal{X}_{\mathcal{D}^\epsilon_{\omega,f}}, \lambda P_\epsilon \mathcal{X}_{\mathcal{D}^\epsilon_{\omega,m}},\ \mathbf{K}_{\omega,\lambda^{-1},\epsilon}V_\epsilon,\ \mathbf{K}_{\omega,\lambda^{-1},\epsilon}G_\epsilon \right\|_{L^2(B_1(0))} \le 1, \\ \lim_{\epsilon\to 0} \|V_\epsilon, G_\epsilon\|_{L^2(B_1(0))} = 0. \end{cases} \tag{3.4}$$

Then

(S1) $\|\mathbf{K}_{\omega,\lambda,\epsilon}\nabla P_\epsilon\|_{L^2(B_{3/4}(0))}$ *is bounded independent of* $\lambda, \epsilon, \omega$,
(S2) there is a subsequence $\{\epsilon, \omega_\epsilon, \lambda_\epsilon\}$ *(same notation for subsequence) such that*

$$\begin{cases} \epsilon, \lambda_\epsilon \to 0, \lambda_*, \\ \mathbf{K}_{\omega_\epsilon,\lambda^2_\epsilon,\epsilon}\nabla P_\epsilon \xrightarrow{L^2(B_{3/4}(0))\ weakly} \xi_1, \\ \nabla\cdot(\mathbf{K}_{\omega_\epsilon,\lambda^2_\epsilon,\epsilon}\nabla P_\epsilon) \xrightarrow{H^{-1}(B_{1/2}(0))} \nabla\cdot\xi_1, \end{cases} \tag{3.5}$$

(S3) $-\nabla\cdot\xi_1 = 0$ *in* $B_{3/4}(0)$.

Proof. (S1) follows from energy method and $(3.4)_2$. $(3.5)_1$ is easily obtained. Then $(3.5)_2$ follows from (S1) and compactness principle. Multiply (3.3) by a function $\zeta \in H_0^1(B_{3/4}(0))$ to see

$$\int_{B_{3/4}(0)} (\mathbf{K}_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon}\nabla P_\epsilon + V_\epsilon)\nabla\zeta dx = \int_{B_{3/4}(0)} G_\epsilon\, \zeta dx.$$

As $\epsilon \to 0$, we obtain (S3) by $(3.4)_3$, $(3.5)_2$.

Let $\eta \in C_0^\infty(B_{3/4}(0))$ be a bell-shaped function satisfying $\eta \in [0,1]$ and $\eta = 1$ in $B_{1/2}(0)$. From (3.3), we have

$$\begin{cases} -\nabla \cdot (\eta\mathbf{K}_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon}\nabla P_\epsilon + \eta V_\epsilon) = \eta G_\epsilon - (\mathbf{K}_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon}\nabla P_\epsilon + V_\epsilon)\nabla\eta & \text{in } B_{3/4}(0), \\ \eta P_\epsilon = 0 & \text{on } \partial B_{3/4}(0). \end{cases} \tag{3.6}$$

Claim: $\nabla \cdot (\eta\mathbf{K}_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon}\nabla P_\epsilon)$ is in a compact subset of $H^{-1}(B_{3/4}(0))$.

When the claim is true, by $(3.5)_2$, a subsequence of $\nabla\cdot(\eta\mathbf{K}_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon}\nabla P_\epsilon)$ converges to $\nabla\cdot\xi_1$ in $H^{-1}(B_{1/2}(0))$ and so $(3.5)_3$ follows.

Proof of the claim: Multiply (3.6) by $\zeta_\epsilon \in H_0^1(B_{3/4}(0))$ to obtain

$$\langle -\nabla \cdot (\eta\mathbf{K}_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon}\nabla P_\epsilon), \zeta_\epsilon\rangle_{H^{-1}(B_{3/4}(0)),H_0^1(B_{3/4}(0))} = \int_{B_{3/4}(0)} \eta\mathbf{K}_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon}\nabla P_\epsilon\nabla\zeta_\epsilon dx$$

$$= \int_{B_{3/4}(0)} \left(\eta G_\epsilon\zeta_\epsilon - V_\epsilon\nabla(\eta\zeta_\epsilon) - \nabla\eta\ \mathbf{K}_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon}\nabla P_\epsilon\ \zeta_\epsilon\right) dx. \tag{3.7}$$

We choose ζ_ϵ in (3.7) such that

$$\begin{cases} \Delta\zeta_\epsilon = \nabla \cdot (\eta\mathbf{K}_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon}\nabla P_\epsilon) & \text{in } B_{3/4}(0), \\ \zeta_\epsilon = 0 & \text{on } \partial B_{3/4}(0). \end{cases} \tag{3.8}$$

(3.8) is solvable uniquely by Lax-Milgram Theorem [17] and $\|\zeta_\epsilon\|_{H^1(B_{3/4}(0))}$ is bounded by a constant independent of $\lambda, \epsilon, \omega$ by (S1). By compactness principle, ζ_ϵ converges weakly to ζ in $H_0^1(B_{3/4}(0))$ as $\epsilon \to 0$. Moreover, ζ satisfies, by $(3.5)_2$,

$$\begin{cases} \Delta\zeta = \nabla \cdot (\eta\xi_1) & \text{in } B_{3/4}(0), \\ \zeta = 0 & \text{on } \partial B_{3/4}(0). \end{cases}$$

By $(3.4)_3$, $(3.5)_2$, (3.7)–(3.8), and Lemma 6.1 [29],

$$\|\nabla \cdot (\eta\mathbf{K}_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon}\nabla P_\epsilon)\|^2_{H^{-1}(B_{3/4}(0))} \xrightarrow[\epsilon\to 0]{} \langle -\xi_1\nabla\eta, \zeta\rangle_{L^2(B_{3/4}(0)),L^2(B_{3/4}(0))}.$$

By (S3),

$$\langle -\xi_1\nabla\eta, \zeta\rangle_{L^2(B_{3/4}(0)),L^2(B_{3/4}(0))} = \|\nabla \cdot (\xi_1\eta)\|^2_{H^{-1}(B_{3/4}(0))}.$$

Since $\nabla \cdot (\eta\mathbf{K}_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon}\nabla P_\epsilon)$ converges weakly to $\nabla \cdot (\eta\xi_1)$ in $H^{-1}(B_{3/4}(0))$ as $\epsilon \to 0$, we see that $\nabla \cdot (\eta\mathbf{K}_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon}\nabla P_\epsilon)$ converges to $\nabla \cdot (\eta\xi_1)$ in $H^{-1}(B_{3/4}(0))$ by Remark 1.16 and Proposition 1.17 [14]. Hence the claim is true. □

To find an explicit expression of ξ_1 in (3.5), we consider an adjoint problem of (3.3) as follows.

Lemma 3.2. *Let $\mathbf{K}'_{\omega,\lambda^2,\epsilon}$ denote the adjoint of $\mathbf{K}_{\omega,\lambda^2,\epsilon}$. Consider the following problem*

$$\begin{cases} -\nabla\cdot(\mathbf{K}'_{\omega,\lambda^2,\epsilon}\nabla\phi_\epsilon) = \mathcal{X}'_{\mathcal{D}^\epsilon_{\omega,f}}\Pi'_{\omega,\epsilon}G & \text{in } \mathcal{D},\\ \phi_\epsilon = 0 & \text{on } \partial\mathcal{D}, \end{cases} \tag{3.9}$$

where $\lambda,\epsilon\in(0,1]$, $\omega\in\Omega$, $G\in H^{-1}(\mathcal{D})$. Then

(S4) there is a sequence $\{\epsilon,\omega_\epsilon,\lambda_\epsilon\}$ independent of G so that

$$\begin{cases} \epsilon,\lambda_\epsilon\to 0,\lambda_*, \\ \Pi_{\omega_\epsilon,\epsilon}\phi_\epsilon|_{\mathcal{D}^\epsilon_{\omega_\epsilon,f}} \xrightarrow{H^1(\mathcal{D})\ weakly} \phi, \\ \phi_\epsilon \xrightarrow{H^1(\mathcal{D})\ weakly} \phi & \text{if } \lambda_*>0, \\ \mathbf{K}'_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon}\nabla\phi_\epsilon \xrightarrow{L^2(\mathcal{D})\ weakly} \xi_3, \\ \nabla\cdot(\mathbf{K}'_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon}\nabla\phi_\epsilon) \xrightarrow{H^{-1}(\mathcal{D})} \nabla\cdot\xi_3, \\ \nabla\cdot(\mathbf{K}'_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon}\nabla\phi_\epsilon\mathcal{X}_{\mathcal{D}^\epsilon_{\omega_\epsilon,f}}) \xrightarrow{H^{-1}(\mathcal{D})} \nabla\cdot\xi_3 & \text{if } \lambda_*=0, \\ \mathbf{K}'_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon}\nabla\phi_\epsilon\mathcal{X}_{\mathcal{D}^\epsilon_{\omega_\epsilon,f}} \xrightarrow{L^2(\mathcal{D})\ weakly} \xi_3 & \text{if } \lambda_*=0, \end{cases}$$

(S5) $-\nabla\cdot\xi_3 = G$ in $\mathcal{D}$,

(S6) $\xi_3 = \mathcal{K}'_{\lambda_*}\nabla\phi$ in $\mathcal{D}$ and $\mathcal{K}'_{\lambda_*}\in\mathcal{M}_2(\frac{\beta}{\mathbf{d}_5^2},\frac{\gamma^2}{\beta|Y_f|})$.

See Remark 3.2 for $\mathbf{d}_5$, (3.2) for $\Pi'_{\omega,\epsilon}$ and $\mathcal{X}'_{\mathcal{D}^\epsilon_{\omega,f}}$, and (A3) for β,γ. Note that the $\mathcal{K}'_{\lambda_*}$ in (S6) depends on $\mathbf{K}, Y_f, \mathcal{D}$ and the sequence $\{\epsilon,\omega_\epsilon,\lambda_\epsilon\}$.

Proof. Step 1. For any $G\in H^{-1}(\mathcal{D})$, the H^1 solution of (3.9) exists uniquely by Lax-Milgram Theorem [17]. We define an operator $\mathcal{B}^\epsilon_{\omega,\lambda}$ from $H^{-1}(\mathcal{D})$ to $H^1_0(\mathcal{D})$ by

$$\mathcal{B}^\epsilon_{\omega,\lambda}G = \Pi_{\omega,\epsilon}\phi_\epsilon|_{\mathcal{D}^\epsilon_{\omega,f}},$$

where ϕ_ϵ is the H^1 solution of (3.9). By Remark 3.2,

$$\begin{aligned} \frac{\beta}{\mathbf{d}_5^2}\|\mathcal{B}^\epsilon_{\omega,\lambda}G\|^2_{H^1(\mathcal{D})} &= \frac{\beta}{\mathbf{d}_5^2}\|\nabla\Pi_{\omega,\epsilon}\phi_\epsilon|_{\mathcal{D}^\epsilon_{\omega,f}}\|^2_{L^2(\mathcal{D})} \le \beta\int_{\mathcal{D}^\epsilon_{\omega,f}}|\nabla\phi_\epsilon|^2dx \\ &\le \int_{\mathcal{D}}\mathbf{K}'_{\omega,\lambda^2,\epsilon}\nabla\phi_\epsilon\nabla\phi_\epsilon dx = \langle G,\mathcal{B}^\epsilon_{\omega,\lambda}G\rangle_{H^{-1}(\mathcal{D}),H^1_0(\mathcal{D})}. \end{aligned} \tag{3.10}$$

Thus $\|\mathcal{B}^\epsilon_{\omega,\lambda}G\|_{H^1(\mathcal{D})}\le\frac{\mathbf{d}_5^2}{\beta}\|G\|_{H^{-1}(\mathcal{D})}$. By Proposition 3 [24], there is a sequence $\{\epsilon,\omega_\epsilon,\lambda_\epsilon\}$ and an operator $\mathcal{B}_0$ defined from $H^{-1}(\mathcal{D})$ to $H^1_0(\mathcal{D})$ so that, as $\epsilon\to 0$,

$$\mathcal{B}^\epsilon_{\omega_\epsilon,\lambda_\epsilon}G \xrightarrow{H^1(\mathcal{D})\ weakly} \mathcal{B}_0G \quad \text{for any } G\in H^{-1}(\mathcal{D}), \tag{3.11}$$

with the estimate

$$\frac{\beta}{\mathbf{d}_5^2}\|\mathcal{B}_0G\|^2_{H^1_0(\mathcal{D})} \le \langle G,\mathcal{B}_0G\rangle_{H^{-1}(\mathcal{D}),H^1_0(\mathcal{D})} \tag{3.12}$$

due to the weak lower semi-continuity of norm.

For any $G \in H^{-1}(\mathcal{D})$, let ϕ_ϵ be the corresponding solution of (3.9). By tracing the argument of Lemma 3.1, there is a subsequence of the above sequence $\{\epsilon, \omega_\epsilon, \lambda_\epsilon\}$ (same notation for subsequence) satisfying (S4) and (S5). If we define $\xi_\epsilon \equiv \mathbf{K}'_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon}\nabla\phi_\epsilon \mathcal{X}_{\mathcal{D}^\epsilon_{\omega_\epsilon,f}}$, then (3.10) implies, by (A3),

$$\frac{\beta}{\gamma^2}\|\xi_\epsilon\|^2_{L^2(\mathcal{D})} \le \beta\|\nabla\phi_\epsilon \mathcal{X}_{\mathcal{D}^\epsilon_{\omega_\epsilon,f}}\|^2_{L^2(\mathcal{D})} \le \langle G, \mathcal{B}^\epsilon_{\omega_\epsilon,\lambda_\epsilon} G\rangle_{H^{-1}(\mathcal{D}),H^1_0(\mathcal{D})}. \tag{3.13}$$

By (3.11)–(3.13) and weak lower semi-continuity,

$$\frac{\beta|Y_f|}{\gamma^2}\|G\|^2_{H^{-1}(\mathcal{D})} \le \frac{\beta|Y_f|}{\gamma^2}\|\xi_3\|^2_{L^2(\mathcal{D})} \le \liminf_{\epsilon\to 0}\frac{\beta}{\gamma^2}\|\xi_\epsilon\|^2_{L^2(\mathcal{D})} \le \langle G, \mathcal{B}_0 G\rangle_{H^{-1}(\mathcal{D}),H^1_0(\mathcal{D})}.$$

Together with (3.12), we see that $\mathcal{B}_0 \in \mathcal{T}(\frac{\beta|Y_f|}{\gamma^2}, \frac{\mathbf{d}_5^2}{\beta})$ is invertible. Denote by $\mathcal{B}_0^{-1}$ the inverse of $\mathcal{B}_0$.

Step 2. Let $\widetilde{\mathcal{D}} \supset \mathcal{D}$ be a bounded Lipschitz domain satisfying $\mathrm{dist}(\partial\widetilde{\mathcal{D}}, \mathcal{D}) > 0$, let γ be the constant in A3, and let

$$\widetilde{\mathbf{K}}_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon}(x) \equiv \begin{cases}\mathbf{K}_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon} & \text{if } x \in \mathcal{D},\\ \gamma & \text{if } x \in \widetilde{\mathcal{D}}\setminus\mathcal{D},\end{cases}$$
$$\widetilde{\mathcal{D}}^\epsilon_{\omega,f} \equiv \widetilde{\mathcal{D}} \setminus \overline{\mathcal{D}^\epsilon_{\omega,m}}.$$

We define $\mathcal{X}'_{\widetilde{\mathcal{D}}^\epsilon_{\omega,f}}, \widetilde{\Pi}_{\omega,\epsilon}, \widetilde{\Pi}'_{\omega,\epsilon}$ in a similar way as $\mathcal{X}'_{\mathcal{D}^\epsilon_{\omega,f}}, \Pi_{\omega,\epsilon}, \Pi'_{\omega,\epsilon}$ in (3.2) with $\mathcal{D}, \mathcal{D}^\epsilon_{\omega,f}$ replaced by $\widetilde{\mathcal{D}}, \widetilde{\mathcal{D}}^\epsilon_{\omega,f}$. We also define an operator $\widetilde{\mathcal{B}}^\epsilon_{\omega,\lambda}$ from $H^{-1}(\widetilde{\mathcal{D}})$ to $H^1_0(\widetilde{\mathcal{D}})$ by

$$\widetilde{\mathcal{B}}^\epsilon_{\omega,\lambda} G = \widetilde{\Pi}_{\omega,\epsilon}\psi_\epsilon|_{\widetilde{\mathcal{D}}^\epsilon_{\omega,f}},$$

where ψ_ϵ is the H^1 solution of

$$\begin{cases}-\nabla\cdot(\widetilde{\mathbf{K}}_{\omega,\lambda^2,\epsilon}\nabla\psi_\epsilon) = \mathcal{X}'_{\widetilde{\mathcal{D}}^\epsilon_{\omega,f}}\widetilde{\Pi}'_{\omega,\epsilon} G & \text{in } \widetilde{\mathcal{D}},\\ \psi_\epsilon = 0 & \text{on } \partial\widetilde{\mathcal{D}}.\end{cases} \tag{3.14}$$

By modifying the argument of **Step 1**, there is a sequence $\{\epsilon, \omega_\epsilon, \lambda_\epsilon\}$ and an operator $\widetilde{\mathcal{B}}_0$ defined from $H^{-1}(\widetilde{\mathcal{D}})$ to $H^1_0(\widetilde{\mathcal{D}})$ so that, as $\epsilon \to 0$,

$$\widetilde{\mathcal{B}}^\epsilon_{\omega_\epsilon,\lambda_\epsilon} G \xrightarrow{H^1(\widetilde{\mathcal{D}})\ weakly} \widetilde{\mathcal{B}}_0 G \quad \text{for any } G \in H^{-1}(\widetilde{\mathcal{D}}).$$

Moreover, $\widetilde{\mathcal{B}}_0 \in \mathcal{T}(\frac{\beta|Y_f|}{\gamma^2}, \frac{\mathbf{d}_5^2}{\beta})$ is invertible. Denote by $\widetilde{\mathcal{B}}_0^{-1}$ the inverse of $\widetilde{\mathcal{B}}_0$.

Step 3. Let $\rho \in C_0^\infty(\widetilde{\mathcal{D}})$ such that $\rho = 1$ on $\mathcal{D}$, set x_i is the i-component of $x \in \mathbb{R}^n$, and let $G_i = \widetilde{\mathcal{B}}_0^{-1}(\rho\ x_i) \in H^{-1}(\widetilde{\mathcal{D}})$ for any $i \in \{1, 2, \cdots, n\}$. We define $\widetilde{\mathcal{B}}^\epsilon_{\omega,\lambda} G_i = \widetilde{\Pi}_{\omega,\epsilon}\psi_{i,\epsilon}|_{\widetilde{\mathcal{D}}^\epsilon_{\omega,f}}$, where $\psi_{i,\epsilon}$ is a solution of (3.14) with G replaced by G_i.

From **Step 2**, there is a sequence $\{\epsilon, \omega_\epsilon, \lambda_\epsilon\}$ such that

$$\begin{cases} \epsilon, \lambda_\epsilon \to 0, \lambda_*, \\ \widetilde{\Pi}_{\omega_\epsilon,\epsilon}\psi_{i,\epsilon}|_{\widetilde{\mathcal{D}}^\epsilon_{\omega_\epsilon,f}} \xrightarrow{H^1(\widetilde{\mathcal{D}})\ weakly} \rho\ x_i, \\ \psi_{i,\epsilon} \xrightarrow{H^1(\widetilde{\mathcal{D}})\ weakly} \rho\ x_i & \text{if } \lambda_* > 0, \\ \widetilde{\mathbf{K}}_{\omega_\epsilon,\lambda^2_\epsilon,\epsilon}\nabla\psi_{i,\epsilon} \xrightarrow{L^2(\widetilde{\mathcal{D}})\ weakly} \eta_i, \\ \nabla\cdot(\widetilde{\mathbf{K}}_{\omega_\epsilon,\lambda^2_\epsilon,\epsilon}\nabla\psi_{i,\epsilon}) \xrightarrow{H^{-1}(\widetilde{\mathcal{D}})} \nabla\cdot\eta_i, \\ \nabla\cdot(\widetilde{\mathbf{K}}_{\omega_\epsilon,\lambda^2_\epsilon,\epsilon}\nabla\psi_{i,\epsilon}\mathcal{X}_{\widetilde{\mathcal{D}}^\epsilon_{\omega_\epsilon,f}}) \xrightarrow{H^{-1}(\widetilde{\mathcal{D}})} \nabla\cdot\eta_i & \text{if } \lambda_* = 0, \\ \widetilde{\mathbf{K}}_{\omega_\epsilon,\lambda^2_\epsilon,\epsilon}\nabla\psi_{i,\epsilon}\mathcal{X}_{\widetilde{\mathcal{D}}^\epsilon_{\omega_\epsilon,f}} \xrightarrow{L^2(\widetilde{\mathcal{D}})\ weakly} \eta_i & \text{if } \lambda_* = 0. \end{cases} \tag{3.15}$$

Step 4. Let $\{\epsilon, \omega_\epsilon, \lambda_\epsilon\}$ be the sequence from **Step 3**. For any $G \in H^{-1}(\mathcal{D})$, let ϕ_ϵ be the solution of (3.9) corresponding to G. Tracing the proof in Lemma 3.1, there is a subsequence of the above sequence $\{\epsilon, \omega_\epsilon, \lambda_\epsilon\}$ (same notation for subsequence) such that (S4) and (S5) hold. Let $\{\psi_{i,\epsilon}\}$ be the sequence from (3.15). Then, in $\mathcal{D}$,

$$\begin{cases} \nabla\widetilde{\Pi}_{\omega_\epsilon,\epsilon}\psi_{i,\epsilon}|_{\widetilde{\mathcal{D}}^\epsilon_{\omega_\epsilon,f}}\mathbf{K}'_{\omega_\epsilon,\lambda^2_\epsilon,\epsilon}\nabla\phi_\epsilon\mathcal{X}_{\mathcal{D}^\epsilon_{\omega_\epsilon,f}} = \nabla\Pi_{\omega_\epsilon,\epsilon}\phi_\epsilon|_{\mathcal{D}^\epsilon_{\omega_\epsilon,f}}\widetilde{\mathbf{K}}_{\omega_\epsilon,\lambda^2_\epsilon,\epsilon}\nabla\psi_{i,\epsilon}\mathcal{X}_{\widetilde{\mathcal{D}}^\epsilon_{\omega_\epsilon,f}}, \\ \nabla\psi_{i,\epsilon}\mathbf{K}'_{\omega_\epsilon,\lambda^2_\epsilon,\epsilon}\nabla\phi_\epsilon = \nabla\phi_\epsilon\widetilde{\mathbf{K}}_{\omega_\epsilon,\lambda^2_\epsilon,\epsilon}\nabla\psi_{i,\epsilon}. \end{cases} \tag{3.16}$$

If $\lambda_\epsilon, \epsilon \to 0$ (resp. $\epsilon \to 0, \lambda_\epsilon \to \lambda_* \in (0,1]$), equation $(3.16)_1$ (resp. $(3.16)_2$) implies

$$\vec{e_i}\cdot\xi_3 = \nabla\phi\cdot\eta_i \qquad \text{almost everywhere in } \mathcal{D}$$

by (S4)–(S5), (3.15), and Lemma 1.1 [18]. Here $\vec{e_i}$ is the unit vector in the i-th coordinate direction. Define the (i,j)-component of a matrix $\mathcal{K}'_{\lambda_*} \in [L^2(\mathcal{D})]^{n^2}$ as

$$\mathcal{K}'_{\lambda_*}(i,j) = (\eta_i)_j \in L^2(\mathcal{D}), \qquad i,j \in \{1,2,\cdots,n\},$$

where $(\eta_i)_j$ is the j-th component of η_i. So we have $\xi_3 = \mathcal{K}'_{\lambda_*}\nabla\phi$ in $\mathcal{D}$.

It is noted that the expression $''\ \xi_3 = \mathcal{K}'_{\lambda_*}\nabla\phi$ in $\mathcal{D}\ ''$ is independent of any subsequence of the sequence $\{\epsilon, \omega_\epsilon, \lambda_\epsilon\}$.

Step 5. For any $\phi \in C_0^\infty(\mathcal{D})$, we set $G \equiv \mathcal{B}_0^{-1}\phi \in H^{-1}(\mathcal{D})$ and use the G to obtain ϕ_ϵ by solving (3.9). See **Step 1** for $\mathcal{B}_0^{-1}$. From **Step 4**, there is a sequence $\{\epsilon, \omega_\epsilon, \lambda_\epsilon\}$ such that (S4)–(S5) hold and $\Pi_{\omega_\epsilon,\epsilon}\phi_\epsilon|_{\mathcal{D}^\epsilon_{\omega_\epsilon,f}}$ converges weakly to ϕ weakly in $H^1(\mathcal{D})$ as $\epsilon \to 0$. By Remark 3.2, weak lower semi-continuity of norm, Lemma 1.1 [18], and $\xi_\epsilon \equiv \mathbf{K}'_{\omega_\epsilon,\lambda^2_\epsilon,\epsilon}\nabla\phi_\epsilon\mathcal{X}_{\mathcal{D}^\epsilon_{\omega_\epsilon,f}}$, we see

$$\begin{aligned} \frac{\beta}{\mathbf{d}_5^2}\|\nabla\phi\|^2_{L^2(\mathcal{D})} &\le \frac{\beta}{\mathbf{d}_5^2}\liminf_{\epsilon\to0}\|\nabla\Pi_{\omega_\epsilon,\epsilon}\phi_\epsilon|_{\mathcal{D}^\epsilon_{\omega_\epsilon,f}}\|^2_{L^2(\mathcal{D})} \le \beta\liminf_{\epsilon\to0}\int_{\mathcal{D}^\epsilon_{\omega_\epsilon,f}}|\nabla\phi_\epsilon|^2dx \\ &\le \liminf_{\epsilon\to0}\int_{\mathcal{D}}\mathbf{K}'_{\omega_\epsilon,\lambda^2_\epsilon,\epsilon}\nabla\phi_\epsilon\nabla\phi_\epsilon dx = \int_{\mathcal{D}}\mathcal{K}'_{\lambda_*}\nabla\phi\nabla\phi dx, \\ \frac{\beta|Y_f|}{\gamma^2}\|\mathcal{K}'_{\lambda_*}\nabla\phi\|^2_{L^2(\mathcal{D})} &\le \frac{\beta}{\gamma^2}\liminf_{\epsilon\to0}\|\xi_\epsilon\|^2_{L^2(\mathcal{D})} \le \int_{\mathcal{D}}\mathcal{K}'_{\lambda_*}\nabla\phi\nabla\phi dx. \end{aligned}$$

So $\mathcal{K}'_{\lambda_*} \in \mathcal{M}_2(\frac{\beta}{\mathbf{d}_5^2}, \frac{\gamma^2}{\beta|Y_f|})$. Together with the result in **Step 4**, we prove (S6). □

Lemma 3.3. *Under the same assumptions in Lemma 3.1, there exists a matrix* $\mathcal{K}_{\lambda_*} \in \mathcal{M}_2(\frac{\beta}{\mathbf{d}_5^2}, \frac{\gamma^2}{\beta|Y_f|})$ *and a sequence* $\{\epsilon, \omega_\epsilon, \lambda_\epsilon\}$ *such that*

$$(S7) \begin{cases} \epsilon, \lambda_\epsilon \to 0, \lambda_*, \\ \Pi_{\omega_\epsilon,\epsilon} P_\epsilon|_{\mathcal{D}^\epsilon_{\omega_\epsilon,f}} \xrightarrow{H^1(B_{3/4}(0)) \ weakly} P, \\ \mathbf{K}_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon} \nabla P_\epsilon \xrightarrow{L^2(B_{3/4}(0)) \ weakly} \mathcal{K}_{\lambda_*} \nabla P, \end{cases}$$

$$(S8) \ -\nabla \cdot (\mathcal{K}_{\lambda_*} \nabla P) = 0 \ in \ B_{1/2}(0).$$

Proof. Let $\{\epsilon, \omega_\epsilon, \lambda_\epsilon\}$ and $\mathcal{K}'_{\lambda_*}$ be the sequence and its corresponding positive definite matrix respectively from Lemma 3.2. Given any $\phi \in H^1_0(\mathcal{D})$, we set $G \equiv -\nabla \cdot (\mathcal{K}'_{\lambda_*} \nabla \phi) \in H^{-1}(\mathcal{D})$ and use this G to obtain ϕ_ϵ by solving (3.9). By Lemma 3.2, function $\Pi_{\omega_\epsilon,\epsilon}\phi_\epsilon|_{\mathcal{D}^\epsilon_{\omega_\epsilon,f}}$ converges weakly to ϕ in $H^1(\mathcal{D})$ as $\epsilon \to 0$.

By Lemma 3.1, there is a subsequence of $\{\epsilon, \omega_\epsilon, \lambda_\epsilon\}$ (same notation for subsequence) so that

$$\bullet \begin{cases} \epsilon, \lambda_\epsilon \to 0, \lambda_*, \\ \Pi_{\omega_\epsilon,\epsilon} P_\epsilon|_{\mathcal{D}^\epsilon_{\omega_\epsilon,f}} \xrightarrow{H^1(B_{3/4}(0)) \ weakly} P, \\ P_\epsilon \xrightarrow{H^1(B_{3/4}(0)) \ weakly} P & \text{if } \lambda_* > 0, \\ \mathbf{K}_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon} \nabla P_\epsilon \xrightarrow{L^2(B_{3/4}(0)) \ weakly} \xi_1, \\ \nabla \cdot (\mathbf{K}_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon} \nabla P_\epsilon) \xrightarrow{H^{-1}(B_{1/2}(0))} 0, \\ \nabla \cdot (\mathbf{K}_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon} \nabla P_\epsilon \mathcal{X}_{\mathcal{D}^\epsilon_{\omega_\epsilon,f}}) \xrightarrow{H^{-1}(B_{1/2}(0))} 0 & \text{if } \lambda_* = 0, \\ \mathbf{K}_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon} \nabla P_\epsilon \mathcal{X}_{\mathcal{D}^\epsilon_{\omega_\epsilon,f}} \xrightarrow{L^2(B_{3/4}(0)) \ weakly} \xi_1 & \text{if } \lambda_* = 0, \end{cases}$$

- $-\nabla \cdot \xi_1 = 0$ in $B_{3/4}(0)$.

Note, in $B_{1/2}(0)$,

$$\begin{cases} \nabla \Pi_{\omega_\epsilon,\epsilon}\phi_\epsilon|_{\mathcal{D}^\epsilon_{\omega_\epsilon,f}} \mathbf{K}_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon} \nabla P_\epsilon \mathcal{X}_{\mathcal{D}^\epsilon_{\omega_\epsilon,f}} = \nabla \Pi_{\omega_\epsilon,\epsilon} P_\epsilon|_{\mathcal{D}^\epsilon_{\omega_\epsilon,f}} \mathbf{K}'_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon} \nabla \phi_\epsilon \mathcal{X}_{\mathcal{D}^\epsilon_{\omega_\epsilon,f}}, \\ \nabla \phi_\epsilon \mathbf{K}_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon} \nabla P_\epsilon = \nabla P_\epsilon \mathbf{K}'_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon} \nabla \phi_\epsilon. \end{cases} \tag{3.17}$$

If $\epsilon, \lambda_\epsilon \to 0$ (resp. $\epsilon \to 0, \lambda_\epsilon \to \lambda_* \in (0,1]$), equation $(3.17)_1$ (resp. $(3.17)_2$) implies

$$\nabla \phi \cdot \xi_1 = \nabla P \cdot \mathcal{K}'_{\lambda_*} \nabla \phi \qquad \text{almost everywhere in } B_{1/2}(0)$$

by Lemma 1.1 [18] and Lemma 3.2. Since $\phi \in H^1_0(\mathcal{D})$ is arbitrary, we obtain that $\xi_1 = \mathcal{K}_{\lambda_*} \nabla P$ in $B_{3/4}(0)$, where $\mathcal{K}_{\lambda_*}$ is the transpose of $\mathcal{K}'_{\lambda_*}$. So (S7)–(S8) hold. □

3.2. *Boundary estimate*

Assume $0 \in \partial\mathcal{D}$. By (A1), there is a Lipschitz function $\Psi : \mathbb{R}^{n-1} \to \mathbb{R}$ satisfying

$$\begin{cases} \Psi(0) = 0, \\ B_1(0) \cap \mathcal{D}/s \equiv B_1(0) \cap \{(\tilde{x}, x_n) \in \mathbb{R}^n | \ sx_n > \Psi(s\tilde{x})\} & \text{if } s \in (0,1]. \end{cases} \tag{3.18}$$

Define $B_1(0)\cap\mathcal{D}/s \equiv B_1(0)\cap\{(\tilde{x},x_n)\in\mathbb{R}^n|\ x_n>0\}$ if $s=0$. For any $\omega\in\Omega$, $\lambda>0$, and $\epsilon,s\in(0,1]$, we set

$$\mathbf{K}_{\omega,\lambda,\epsilon,s}(x)\equiv\begin{cases}\mathbf{K}(\Upsilon(\frac{sx}{\epsilon},\omega)) & \text{if } x\in\mathcal{D}^{\epsilon}_{\omega,f}/s,\\ \lambda\mathbf{K}(\Upsilon(\frac{sx}{\epsilon},\omega)) & \text{if } x\in\mathcal{D}^{\epsilon}_{\omega,m}/s.\end{cases}\tag{3.19}$$

Lemma 3.4. *Consider the following problem*

$$\begin{cases}-\nabla\cdot(\mathbf{K}_{\omega,\lambda^2,\epsilon,s}\nabla P_\epsilon+V_\epsilon)=G_\epsilon & \text{in } B_1(0)\cap\mathcal{D}/s,\\ P_\epsilon=0 & \text{on } B_1(0)\cap\partial\mathcal{D}/s,\end{cases}\tag{3.20}$$

where

$$\begin{cases}\lambda,\epsilon,s\in(0,1],\quad \omega\in\Omega,\\ \left\|P_\epsilon\mathcal{X}_{\mathcal{D}^{\epsilon}_{\omega,f}/s},\lambda P_\epsilon\mathcal{X}_{\mathcal{D}^{\epsilon}_{\omega,m}/s},\ \mathbf{K}_{\omega,\lambda^{-1},\epsilon,s}V_\epsilon,\ \mathbf{K}_{\omega,\lambda^{-1},\epsilon,s}G_\epsilon\right\|_{L^2(B_1(0)\cap\mathcal{D}/s)}\le 1,\\ \lim_{\epsilon/s\to 0}\|V_\epsilon,G_\epsilon\|_{L^2(B_1(0)\cap\mathcal{D}/s)}=0.\end{cases}\tag{3.21}$$

Then the following statements are true:

(S9) $\|\mathbf{K}_{\omega,\lambda,\epsilon,s}\nabla P_\epsilon\|_{L^2(B_{3/4}(0)\cap\mathcal{D}/s)}$ *is bounded independent of* $\lambda,\epsilon,s,\omega$,

(S10) there is a subsequence $\{\epsilon,\omega_\epsilon,\lambda_\epsilon,s_\epsilon\}$ *(same notation for subsequence) such that*

$$\begin{cases}\epsilon,\epsilon/s_\epsilon,\lambda_\epsilon,s_\epsilon\to 0,0,\lambda_*,s_*,\\ \mathbf{K}_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon,s_\epsilon}\nabla P_\epsilon\mathcal{X}_{\mathcal{D}/s_\epsilon}\xrightarrow{L^2(B_{3/4}(0)\cap\mathcal{D}/s_*)\ weakly}\xi_1,\\ \nabla\cdot(\mathbf{K}_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon,s_\epsilon}\nabla P_\epsilon\mathcal{X}_{\mathcal{D}/s_\epsilon})\xrightarrow{H^{-1}(B_{1/2}(0)\cap\mathcal{D}/s_*)}0,\end{cases}\tag{3.22}$$

(S11) $-\nabla\cdot\xi_1=0$ *in* $B_{3/4}(0)\cap\mathcal{D}/s_*$.

Proof. By energy method and $(3.21)_2$, we obtain (S9). $(3.22)_1$ is easily obtained. $(3.22)_2$ is a direct result of (S9) and compactness principle. Let $\mathbf{S}$ be any compact subset in $B_{3/4}(0)\cap\mathcal{D}/s_*$. So if ϵ/s_ϵ is close to 0, then $\mathbf{S}\subset B_{3/4}(0)\cap\mathcal{D}/s_\epsilon$. Multiply (3.20) by any function $\zeta\in C_0^\infty(\mathbf{S})$ to see, for ϵ/s_ϵ close to 0,

$$\int_{\mathbf{S}}(\mathbf{K}_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon,s_\epsilon}\nabla P_\epsilon+V_\epsilon)\nabla\zeta dx=\int_{\mathbf{S}}G_\epsilon\,\zeta dx.$$

As $\epsilon/s_\epsilon\to 0$, we see, by $(3.21)_3$, $(3.22)_2$, and (S9),

$$-\nabla\cdot\xi_1=0\qquad\text{in } \mathbf{S}.$$

Since $\mathbf{S}$ is any compact subset in $B_{3/4}(0)\cap\mathcal{D}/s_*$, we prove (S11).

Let $\eta\in C_0^\infty(B_{3/4}(0))$ be a bell-shaped function satisfying $\eta\in[0,1]$ and $\eta=1$ in $B_{1/2}(0)$. From (3.20), we have

$$\begin{cases}-\nabla\cdot(\eta\mathbf{K}_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon,s_\epsilon}\nabla P_\epsilon+\eta V_\epsilon)\\ \qquad=\eta G_\epsilon-(\mathbf{K}_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon,s_\epsilon}\nabla P_\epsilon+V_\epsilon)\nabla\eta & \text{in } B_{3/4}(0)\cap\mathcal{D}/s_\epsilon,\\ \eta P_\epsilon=0 & \text{on } \partial(B_{3/4}(0)\cap\mathcal{D}/s_\epsilon).\end{cases}\tag{3.23}$$

Claim: $\nabla\cdot(\eta\mathbf{K}_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon,s_\epsilon}\nabla P_\epsilon\mathcal{X}_{\mathcal{D}/s_\epsilon})$ is in a compact subset of $H^{-1}(\mathbf{S})$, where $\mathbf{S}$ is any compact subset in $B_{3/4}(0)\cap\mathcal{D}/s_*$.

Proof of the claim: Multiply (3.23), for ϵ/s_ϵ close to 0, by any $\zeta_\epsilon\in H_0^1(\mathbf{S})$ to obtain

$$\begin{aligned}&\langle-\nabla\cdot(\eta\mathbf{K}_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon,s_\epsilon}\nabla P_\epsilon\mathcal{X}_{\mathcal{D}/s_\epsilon}),\zeta_\epsilon\rangle_{H^{-1}(B_{3/4}(0)),H_0^1(B_{3/4}(0))}\\&=\int_{B_{3/4}(0)\cap\mathcal{D}/s_\epsilon}\eta\mathbf{K}_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon,s_\epsilon}\nabla P_\epsilon\nabla\zeta_\epsilon dx\\&=\int_{\mathbf{S}}\big(\eta G_\epsilon-\mathbf{K}_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon,s_\epsilon}\nabla P_\epsilon\nabla\eta\big)\zeta_\epsilon-V_\epsilon\nabla(\eta\zeta_\epsilon)dx.\end{aligned}\tag{3.24}$$

We choose $\zeta_\epsilon\in H_0^1(\mathbf{S})$ in (3.24) such that

$$\begin{cases}\Delta\zeta_\epsilon=\nabla\cdot(\eta\mathbf{K}_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon,s_\epsilon}\nabla P_\epsilon\mathcal{X}_{\mathcal{D}/s_\epsilon}) & \text{in } \mathbf{S},\\ \zeta_\epsilon=0 & \text{on } \partial\mathbf{S}.\end{cases}\tag{3.25}$$

(3.25) is uniquely solvable by Lax-Milgram theorem [17] and $\|\zeta_\epsilon\|_{H^1(\mathbf{S})}$ is bounded by a constant independent of $\lambda_\epsilon,\epsilon,s_\epsilon,\omega_\epsilon$ by (S9). By compactness principle, ζ_ϵ converges weakly to ζ in $H_0^1(\mathbf{S})$ as $\epsilon/s_\epsilon\to0$, and ζ satisfies, by $(3.22)_2$,

$$\begin{cases}\Delta\zeta=\nabla\cdot(\eta\xi_1) & \text{in } \mathbf{S},\\ \zeta=0 & \text{on } \partial\mathbf{S}.\end{cases}$$

By (S9), $(3.22)_2$, (3.24)–(3.25), and Lemma 6.1 [29],

$$\|\nabla\cdot(\eta\mathbf{K}_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon,s_\epsilon}\nabla P_\epsilon\mathcal{X}_{\mathcal{D}/s_\epsilon})\|^2_{H^{-1}(\mathbf{S})}\xrightarrow[\epsilon/s_\epsilon\to0]{}\langle-\xi_1\nabla\eta,\zeta\rangle_{L^2(\mathbf{S}),L^2(\mathbf{S})}.$$

By (S11),

$$\langle-\xi_1\nabla\eta,\zeta\rangle_{L^2(\mathbf{S}),L^2(\mathbf{S})}=\|\nabla\cdot(\xi_1\eta)\|^2_{H^{-1}(\mathbf{S})}.$$

Since $\nabla\cdot(\eta\mathbf{K}_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon,s_\epsilon}\nabla P_\epsilon\mathcal{X}_{\mathcal{D}/s_\epsilon})$ converges weakly to $\nabla\cdot(\xi_1\eta)$ in $H^{-1}(\mathbf{S})$ as $\epsilon/s_\epsilon\to0$, we know that $\nabla\cdot(\eta\mathbf{K}_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon,s_\epsilon}\nabla P_\epsilon\mathcal{X}_{\mathcal{D}/s_\epsilon})$ converges to $\nabla\cdot(\xi_1\eta)$ in $H^{-1}(\mathbf{S})$ by Remark 1.16 and Proposition 1.17 [14]. Above convergence is true for any compact subset $\mathbf{S}$ in $B_{3/4}(0)\cap\mathcal{D}/s_*$. Take a sequence of compact subsets $\{\mathbf{S}_i\}_{i=1}^\infty$ such that $\mathbf{S}_i\subset\mathbf{S}_{i+1}^\circ$ and $\mathbf{S}_i\to B_{3/4}(0)\cap\mathcal{D}/s_*$. For each $\mathbf{S}_i$, we obtain a convergent sequence. By a diagonal process, we can find a fixed subsequence such that the convergence result holds for all $\mathbf{S}_i$. So the claim is true.

Also note that, by $(3.22)_2$ and (S11), $\nabla\cdot(\mathbf{K}_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon,s_\epsilon}\nabla P_\epsilon\mathcal{X}_{\mathcal{D}/s_\epsilon})$ converges to $\nabla\cdot\xi_1$ in $H^{-1}(B_{1/2}(0)\cap\mathbf{S}_i)$ as $\epsilon/s_\epsilon\to0$ for any $\mathbf{S}_i$. So we prove $(3.22)_3$. □

Lemma 3.5. *Under the same assumptions as in Lemma 3.4, there exists a matrix* $\mathcal{K}_{\lambda_*}\in\mathcal{M}_2(\frac{\beta}{\mathbf{d}_5^2},\frac{\gamma^2}{\beta|Y_f|})$ *and a sequence* $\{\epsilon,\omega_\epsilon,\lambda_\epsilon,s_\epsilon\}$ *such that*

$$\text{(S12)}\quad\begin{cases}\epsilon,\epsilon/s_\epsilon,\lambda_\epsilon,s_\epsilon\to0,0,\lambda_*,s_*,\\ \Pi_{\omega_\epsilon,\epsilon/s_\epsilon}P_\epsilon|_{\mathcal{D}^\epsilon_{\omega_\epsilon,f}/s_\epsilon}\xrightarrow{H^1(B_{1/2}(0)\cap\mathcal{D}/s_*)\ weakly}P,\\ \mathbf{K}_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon,s_\epsilon}\nabla P_\epsilon\mathcal{X}_{\mathcal{D}/s_\epsilon}\xrightarrow{L^2(B_{1/2}(0)\cap\mathcal{D}/s_*)\ weakly}\mathcal{K}_{\lambda_*}\nabla P,\end{cases}$$

$$\text{(S13)}\ \begin{cases} -\nabla\cdot(\mathcal{K}_{\lambda_*}\nabla P)=0 & \textit{in } B_{1/2}(0)\cap\mathcal{D}/s_*,\\ P=0 & \textit{on } B_{1/2}(0)\cap\partial\mathcal{D}/s_*.\end{cases}$$

Proof. Step 1. Let $\mathbf{S}$ be a bounded Lipschitz subset of $B_1(0)\cap\mathcal{D}/s_*$ and let

$$\begin{cases} \mathbb{S}^{\nu}_{\omega,m}\equiv\bigcup_j \nu\Phi(Y_m+j,\omega) \quad \text{where } j \text{ satisfies } \nu\Phi(Y+j,\omega)\subset\mathbf{S},\\ \mathbb{S}^{\nu}_{\omega,f}\equiv\mathbf{S}\setminus\overline{\mathbb{S}^{\nu}_{\omega,m}},\\ \mathbb{K}_{\omega,\lambda^2,\epsilon,s}(x)\equiv\begin{cases}\mathbf{K}(\Upsilon(\frac{sx}{\epsilon},\omega)) & \text{if } x\in\mathbb{S}^{\epsilon/s}_{\omega,f}\\ \lambda^2\mathbf{K}(\Upsilon(\frac{sx}{\epsilon},\omega)) & \text{if } x\in\mathbb{S}^{\epsilon/s}_{\omega,m}\end{cases},\\ \mathbb{K}'_{\omega,\lambda^2,\epsilon,s} \text{ denote the adjoint of } \mathbb{K}_{\omega,\lambda^2,\epsilon,s},\end{cases}$$

where $\nu>0$. We define $\mathcal{X}'_{\mathbb{S}^{\nu}_{\omega,f}},\widetilde{\Pi}_{\omega,\nu},\widetilde{\Pi}'_{\omega,\nu}$ in a similar way as $\mathcal{X}'_{\mathcal{D}^{\nu}_{\omega,f}},\Pi_{\omega,\nu},\Pi'_{\omega,\nu}$ in (3.2) with $\mathcal{D},\mathcal{D}^{\nu}_{\omega,f}$ replaced by $\mathbf{S},\mathbb{S}^{\nu}_{\omega,f}$ respectively. For any given $G\in H^{-1}(\mathbf{S})$, we get ϕ_ϵ by solving

$$\begin{cases} -\nabla\cdot(\mathbb{K}'_{\omega,\lambda^2,\epsilon,s}\nabla\phi_\epsilon)=\mathcal{X}'_{\mathbb{S}^{\epsilon/s}_{\omega,f}}\widetilde{\Pi}'_{\omega,\epsilon/s}G & \text{in } \mathbf{S},\\ \phi_\epsilon=0 & \text{on } \partial\mathbf{S}.\end{cases} \tag{3.26}$$

By Lemma 3.2, there is a sequence $\{\epsilon,\omega_\epsilon,\lambda_\epsilon,s_\epsilon\}$ (independent of G) such that

$$\begin{cases} \widetilde{\Pi}_{\omega_\epsilon,\epsilon/s_\epsilon}\phi_\epsilon|_{\mathbb{S}^{\epsilon/s_\epsilon}_{\omega_\epsilon,f}} \xrightarrow{H^1(\mathbf{S})\ weakly} \phi, & \\ \phi_\epsilon \xrightarrow{H^1(\mathbf{S})\ weakly} \phi & \text{if } \lambda_*>0,\\ \mathbb{K}'_{\omega_\epsilon,\lambda^2_\epsilon,\epsilon,s_\epsilon}\nabla\phi_\epsilon \xrightarrow{L^2(\mathbf{S})\ weakly} \widehat{\mathcal{K}}'_{\mathbf{S}}\nabla\phi, & \\ \nabla\cdot(\mathbb{K}'_{\omega_\epsilon,\lambda^2_\epsilon,\epsilon,s_\epsilon}\nabla\phi_\epsilon) \xrightarrow{H^{-1}(\mathbf{S})} \nabla\cdot(\widehat{\mathcal{K}}'_{\mathbf{S}}\nabla\phi), & \\ \nabla\cdot(\mathbb{K}'_{\omega_\epsilon,\lambda^2_\epsilon,\epsilon,s_\epsilon}\nabla\phi_\epsilon\mathcal{X}_{\mathbb{S}^{\epsilon/s_\epsilon}_{\omega_\epsilon,f}}) \xrightarrow{H^{-1}(\mathbf{S})} \nabla\cdot(\widehat{\mathcal{K}}'_{\mathbf{S}}\nabla\phi) & \text{if } \lambda_*=0,\\ \mathbb{K}'_{\omega_\epsilon,\lambda^2_\epsilon,\epsilon,s_\epsilon}\nabla\phi_\epsilon\mathcal{X}_{\mathbb{S}^{\epsilon/s_\epsilon}_{\omega_\epsilon,f}} \xrightarrow{L^2(\mathbf{S})\ weakly} \widehat{\mathcal{K}}'_{\mathbf{S}}\nabla\phi & \text{if } \lambda_*=0,\end{cases} \tag{3.27}$$

and

$$\begin{cases} -\nabla\cdot(\widehat{\mathcal{K}}'_{\mathbf{S}}\nabla\phi)=G & \text{in } \mathbf{S},\\ \phi=0 & \text{on } \partial\mathbf{S},\end{cases} \tag{3.28}$$

where $\widehat{\mathcal{K}}'_{\mathbf{S}}\in\mathcal{M}_2(\frac{\beta}{\mathbf{d}_5^2},\frac{\gamma^2}{\beta|Y_f|})$ depends on $\mathbf{S},\mathbf{K},Y_f$ and the sequence $\{\epsilon,\omega_\epsilon,\lambda_\epsilon,s_\epsilon\}$.

Step 2. We take a sequence of compact subsets $\{\mathbf{S}_i\}_{i=1}^{\infty}$ such that $\overline{\mathbf{S}_i}\subset\mathbf{S}^{\circ}_{i+1}$ and $\mathbf{S}_i\to B_1(0)\cap\mathcal{D}/s_*$. For each $\mathbf{S}_i$, we consider problem (3.26) and obtain a convergent sequence satisfying (3.27)–(3.28). By a diagonal process, we can find a fixed sequence $\{\epsilon,\omega_\epsilon,\lambda_\epsilon,s_\epsilon\}$ such that the solution of (3.26) satisfies (3.27)–(3.28) for all $\mathbf{S}_i$. Moreover, when ϵ/s_ϵ is small enough, $\mathbf{K}'_{\omega_\epsilon,\lambda^2_\epsilon,\epsilon,s_\epsilon}=\mathbb{K}'_{\omega_\epsilon,\lambda^2_\epsilon,\epsilon,s_\epsilon}$ in $\mathbf{S}_i$ for any i.

Step 3. Let $\{\mathbf{S}_i\}_{i=1}^{\infty}$ and $\{\epsilon,\omega_\epsilon,\lambda_\epsilon,s_\epsilon\}$ be the sequences from **Step 2**. From **Step 1**, each $\mathbf{S}_i$ is associated with an invertible matrix $\widehat{\mathcal{K}}'_{\mathbf{S}_i}\in\mathcal{M}_2(\frac{\beta}{\mathbf{d}_5^2},\frac{\gamma^2}{\beta|Y_f|})$. Given any $\phi \in H_0^1(\mathbf{S}_i)$, we set $G \equiv -\nabla\cdot(\widehat{\mathcal{K}}'_{\mathbf{S}_i}\nabla\phi) \in H^{-1}(\mathbf{S}_i)$ and use the G to obtain ϕ_ϵ by solving (3.26). By **Steps 1,2**, $\widetilde{\Pi}_{\omega_\epsilon,\epsilon/s_\epsilon}\phi_\epsilon|_{\mathbb{S}^{\epsilon/s_\epsilon}_{\omega,f}}$ converges weakly to ϕ in $H^1(\mathbf{S}_i)$ as $\epsilon \to 0$ for any $\mathbf{S}_i$.

By Lemma 3.4, there is a subsequence of $\{\epsilon, \omega_\epsilon, \lambda_\epsilon\}$ (same notation for subsequence) so that

$$\begin{cases} \Pi_{\omega_\epsilon,\epsilon/s_\epsilon} P_\epsilon|_{\mathcal{D}^\epsilon_{\omega_\epsilon,f}/s_\epsilon} \xrightarrow{H^1(B_{3/4}(0)\cap\mathcal{D}/s_*)\ weakly} P, & \\ P_\epsilon \xrightarrow{H^1(B_{3/4}(0)\cap\mathcal{D}/s_*)\ weakly} P & \text{if } \lambda_* > 0, \\ \mathbf{K}_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon,s_\epsilon}\nabla P_\epsilon \mathcal{X}_{\mathcal{D}/s_\epsilon} \xrightarrow{L^2(B_{3/4}(0)\cap\mathcal{D}/s_*)\ weakly} \xi_1, & \\ \nabla\cdot(\mathbf{K}_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon,s_\epsilon}\nabla P_\epsilon \mathcal{X}_{\mathcal{D}/s_\epsilon}) \xrightarrow{H^{-1}(B_{1/2}(0)\cap\mathcal{D}/s_*)} 0, & \\ \nabla\cdot(\mathbf{K}_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon,s_\epsilon}\nabla P_\epsilon \mathcal{X}_{\mathcal{D}^\epsilon_{\omega_\epsilon,f}/s_\epsilon}) \xrightarrow{H^{-1}(B_{1/2}(0)\cap\mathcal{D}/s_*)} 0 & \text{if } \lambda_* = 0, \\ \mathbf{K}_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon,s_\epsilon}\nabla P_\epsilon \mathcal{X}_{\mathcal{D}^\epsilon_{\omega_\epsilon,f}/s_\epsilon} \xrightarrow{L^2(B_{3/4}(0)\cap\mathcal{D}/s_*)\ weakly} \xi_1 & \text{if } \lambda_* = 0. \end{cases}$$

By the results in **Step 2**, in $\mathbf{S}_{i-1}$,

$$\begin{cases} \nabla\Pi_{\omega_\epsilon,\epsilon/s_\epsilon}\phi_\epsilon|_{\mathcal{D}^\epsilon_{\omega_\epsilon,f}/s_\epsilon}\mathbf{K}_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon,s_\epsilon}\nabla P_\epsilon\mathcal{X}_{\mathcal{D}^\epsilon_{\omega_\epsilon,f}/s_\epsilon} \\ \qquad = \nabla\Pi_{\omega_\epsilon,\epsilon/s_\epsilon}P_\epsilon|_{\mathcal{D}^\epsilon_{\omega_\epsilon,f}/s_\epsilon}\mathbb{K}'_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon,s_\epsilon}\nabla\phi_\epsilon\mathcal{X}_{\mathcal{D}^\epsilon_{\omega_\epsilon,f}/s_\epsilon}, \\ \nabla\phi_\epsilon\ \mathbf{K}_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon,s_\epsilon}\nabla P_\epsilon = \nabla P_\epsilon\ \mathbb{K}'_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon,s_\epsilon}\nabla\phi_\epsilon. \end{cases} \tag{3.29}$$

If $\epsilon, \lambda_\epsilon \to 0$ (resp. $\epsilon \to 0, \lambda_\epsilon \to \lambda_* \in (0,1]$), equation $(3.29)_1$ (resp. $(3.29)_2$) implies

$$\nabla\phi\cdot\xi_1 = \nabla P\cdot\widehat{\mathcal{K}}'_{\mathbf{S}_i}\nabla\phi \qquad \text{almost everywhere in } \mathbf{S}_{i-1}$$

by Lemma 1.1 [18]. Since $\phi \in H_0^1(\mathbf{S}_i)$ is arbitrary, we obtain that $\xi_1 = \widehat{\mathcal{K}}_{\mathbf{S}_i}\nabla P$ in $\mathbf{S}_{i-1}$, where $\widehat{\mathcal{K}}_{\mathbf{S}_i}$ is the transpose of $\widehat{\mathcal{K}}'_{\mathbf{S}_i}$. So (S12)–(S13) hold.

It is noted that the conclusions (S12)–(S13) are independent of any subsequence of the sequence $\{\epsilon, \omega_\epsilon, \lambda_\epsilon\}$. □

4. Interface problem

Let $x \equiv (\tilde{x}, x_n)$, $B_r^+(x_*) \equiv B_r(x_*)\cap\{x|x_n > 0\}$, $B_r^-(x_*) \equiv B_r(x_*)\cap\{x|x_n < 0\}$, $\mathbf{I}_r(x_*) \equiv B_r(x_*)\cap\{x|x_n = 0\}$ for some $r > 0$, and

$$\mathbf{E}_\nu \equiv \begin{cases} 1 & \text{if } x_n \ge 0 \\ \nu & \text{if } x_n < 0 \end{cases} \qquad \text{for any } \nu \in (0,1]. \tag{4.1}$$

Consider the following problem

$$-\nabla\cdot(\mathbf{T}_{\lambda^2}\nabla P) = 0 \quad \text{in } B_1(0), \tag{4.2}$$

where $\lambda \in (0,1]$, $\mathbf{T}_{\lambda^2} \equiv \mathbf{E}_{\lambda^2}\left(T_1\mathcal{X}_{\{x_n\ge 0\}} + T_2\mathcal{X}_{\{x_n<0\}}\right)$, and T_1, T_2 are positive definite constant matrices.

Lemma 4.1. *Let $i \in \mathbb{N}$, $\mathbf{e} \in [0,1]$, $0 < r < \frac{1}{3}$, $x_* \in \mathbf{I}_{1/3}(0)$. Any solution P of (4.2) satisfies*

$$\begin{cases} \|\mathbf{E}_{\lambda^{\mathbf{e}}} \nabla P\|_{L^2(B_r(x_*))} \leq \frac{c}{r} \|\mathbf{E}_{\lambda^{\mathbf{e}}} P\|_{L^2(B_{2r}(x_*))}, \\ \|\mathbf{E}_{\lambda^{\mathbf{e}}} \nabla^{i+1} P\|_{L^2(B_r^+(x_*)) \cap L^2(B_r^-(x_*))} \leq \frac{c}{r^i} \|\mathbf{E}_{\lambda^{\mathbf{e}}} \partial_{x_\tau} P\|_{L^2(B_{2r}(x_*))}, \end{cases} \tag{4.3}$$

where $\tau \in \{1, \cdots, n-1\}$ and c is independent of $\lambda, r, \mathbf{e}, x_*$.

Proof. Let c denote a constant independent of $\lambda, r, \mathbf{e}, x_*$. $(4.3)_1$ is obtained by employing energy method. Differentiate (4.2) with respect to x_τ for $\tau \in \{1, \cdots, n-1\}$ and employ $(4.3)_1$ to get

$$\|\mathbf{E}_{\lambda^{\mathbf{e}}} \nabla \partial_{x_\tau} P\|_{L^2(B_r(x_*))} \leq \frac{c}{r} \|\mathbf{E}_{\lambda^{\mathbf{e}}} \partial_{x_\tau} P\|_{L^2(B_{2r}(x_*))}. \tag{4.4}$$

By (4.2) and (4.4),

$$\|\mathbf{E}_{\lambda^{\mathbf{e}}} \partial_{x_n}^2 P\|_{L^2(B_r^+(x_*)) \cap L^2(B_r^-(x_*))} \leq \frac{c}{r} \|\mathbf{E}_{\lambda^{\mathbf{e}}} \partial_{x_\tau} P\|_{L^2(B_{2r}(x_*))}.$$

So we obtain $(4.3)_2$ for $i = 1$ case. By induction, we obtain $(4.3)_2$ for $i \in \mathbb{N}$. □

Lemma 4.2. *Let $i \in \mathbb{N} \cup \{0\}$, $P_{\mathbf{I}} \equiv P|_{\mathbf{I}_1(0)}$, $0 < r < \frac{1}{3}$, and $x_* \in \mathbf{I}_{1/3}(0)$. Any solution P of (4.2) satisfies*

$$\theta \int_{\mathbf{I}_\theta(x_*)} |\partial_{x_\tau}^i P_{\mathbf{I}}|^2 d\tilde{x} \leq c \Big|\frac{\theta}{r}\Big|^n \int_{B_r(x_*)} |\mathbf{E}_\lambda \partial_{x_\tau}^i P|^2 dx \qquad \text{for } 0 < \theta \leq \tfrac{r}{2}, \tag{4.5}$$

where $\tau \in \{1, \cdots, n-1\}$ and c is independent of λ, θ, r, x_*.

Proof. We fix $0 < \theta \leq \frac{r}{2}$ and k is the smallest integer satisfying $2k > n-1$. By Lemma 4.1,

$$\begin{aligned} \theta \int_{\mathbf{I}_\theta(x_*)} |P_{\mathbf{I}}|^2 d\tilde{x} &\leq c(n) \theta^n \max_{\mathbf{I}_{\frac{r}{2}}(x_*)} |P_{\mathbf{I}}|^2 \leq c(n,r) \theta^n \|P_{\mathbf{I}}\|^2_{H^k(\mathbf{I}_{\frac{r}{2}}(x_*))} \\ &\leq c(n,r) \theta^n \sum_{\ell \leq k} \|\partial_{x_\tau}^\ell P\|^2_{L^2(\mathbf{I}_{\frac{r}{2}}(x_*))} \leq c(n,r) \theta^n \sum_{\ell \leq k} \|\partial_{x_\tau}^\ell P\|^2_{H^1(B^+_{r/2}(x_*))} \\ &\leq c(n,r,k) \theta^n \|\mathbf{E}_\lambda P\|^2_{L^2(B_r(x_*))}. \end{aligned}$$

By a similarity transformation, we find $c(n,r,k) \leq \frac{c}{r^n}$. So we prove (4.5) for $i = 0$ case. For $i > 0$ case, (4.5) is proved in a similar way. □

Lemma 4.3. *Let $P_{\mathbf{I}} \equiv P|_{\mathbf{I}_1(0)}$, $\mathbf{e} \in [0,1]$, $\delta \in (0,1)$, and $x_* \in \mathbf{I}_{1-\delta}(0)$. Any solution P of (4.2) satisfies, for all $0 < \theta \leq r < \frac{\delta}{2}$,*

$$\int_{B_\theta^\pm(x_*)} |\mathbf{E}_{\lambda^{\mathbf{e}}} \nabla P|^2 dx \leq c_1 \Big|\frac{\theta}{r}\Big|^n \int_{B_r^\pm(x_*)} |\mathbf{E}_{\lambda^{\mathbf{e}}} \nabla P|^2 dx + c_2 \int_{B_r^\pm(x_*)} |\mathbf{E}_{\lambda^{\mathbf{e}}} \nabla P_{\mathbf{I}}|^2 dx \tag{4.6}$$

where c_1, c_2 are independent of $\lambda, \theta, r, \mathbf{e}, x_*$.

Proof. Let c be a constant independent of $\lambda, \theta, r, \mathbf{e}, x_*$. Then $\psi(x) \equiv P(x) - P_{\mathbf{I}}(\tilde{x}, 0)$ satisfies, for any $\zeta \in H_0^1(B_1(0))$,

$$\int_{B_1(0)} \mathbf{T}_{\lambda^2} \nabla\psi \nabla\zeta dx = -\int_{B_1(0)} \mathbf{T}_{\lambda^2} \nabla P_{\mathbf{I}} \nabla\zeta dx. \tag{4.7}$$

We find a $\phi \in H^1(B_r^+(x_*))$ such that

$$\begin{cases} -\nabla \cdot (\mathbf{T}_{\lambda^2} \nabla\phi) = 0 & \text{in } B_r^+(x_*), \\ \phi = \psi & \text{on } \partial B_r^+(x_*). \end{cases} \tag{4.8}$$

Take $\zeta = \psi - \phi \in H_0^1(B_r^+(x_*))$ in (4.7) and multiply $(4.8)_1$ by ζ to see

$$\int_{B_r^+(x_*)} |\nabla\zeta|^2 dx \le c \int_{B_r^+(x_*)} |\nabla P_{\mathbf{I}}|^2 dx. \tag{4.9}$$

By Theorem 6.2.4 [31], the solution of (4.8) satisfies

$$\int_{B_\theta^+(x_*)} |\nabla\phi|^2 dx \le c\Big|\frac{\theta}{r}\Big|^n \int_{B_r^+(x_*)} |\nabla\phi|^2 dx \qquad \text{for } 0 < \theta \le r. \tag{4.10}$$

Multiply (4.8) by $\zeta = \psi - \phi$ to see

$$\int_{B_r^+(x_*)} |\nabla\phi|^2 dx \le c \int_{B_r^+(x_*)} |\nabla\psi|^2 dx. \tag{4.11}$$

Equations (4.9)–(4.11) imply, for $0 < \theta \le r$,

$$\begin{aligned} \int_{B_\theta^+(x_*)} |\nabla\psi|^2 dx &\le \int_{B_\theta^+(x_*)} (|\nabla\phi|^2 + |\nabla\zeta|^2) dx \\ &\le c\Big|\frac{\theta}{r}\Big|^n \int_{B_r^+(x_*)} |\nabla\psi|^2 dx + c\int_{B_r^+(x_*)} |\nabla P_{\mathbf{I}}|^2 dx. \end{aligned}$$

So we obtain the estimate (4.6) in the upper domain. The estimate (4.6) in the lower domain is proved in a similar way as above. □

Next is a Campanato-type estimate for the solution P of (4.2).

Lemma 4.4. *Suppose* $\mathbf{e} \in [0,1]$, $\delta \in (0,1)$, *and* $x_* \in \mathbf{I}_{1-\delta}(0)$, *any solution* P *of (4.2) satisfies, for any* $0 < \theta \le r < \frac{\delta}{2}$ *and* $0 < \sigma \ll 1$,

$$\|\mathbf{E}_{\lambda^{\mathbf{e}}} \nabla P\|_{L^2(B_\theta(x_*))} \le c\Big|\frac{\theta}{r}\Big|^{\frac{n-\sigma}{2}} \|\mathbf{E}_{\lambda^{\mathbf{e}}} \nabla P\|_{L^2(B_r(x_*))}, \tag{4.12}$$

where c *is independent of* $\lambda, \theta, r, \mathbf{e}, \sigma, x_*$.

Proof. We let $0 < \theta \le \ell \le \frac{r}{2}$. By Lemma 4.2,

$$\int_{B_\ell(x_*)} |\partial_{x_\tau} P_{\mathbf{I}}|^2 dx \le c_0\Big|\frac{\ell}{r}\Big|^n \int_{B_r(x_*)} |\mathbf{E}_\lambda \partial_{x_\tau} P|^2 dx \quad \text{for } \ell < \tfrac{r}{2}, \tag{4.13}$$

where c_0 is independent of λ, ℓ, r, x_*. We introduce the notation

$$\mathbf{M}_1 \equiv \frac{c_0}{r^{n-\sigma}} \|\mathbf{E}_{\lambda^{\mathbf{e}}} \partial_{x_\tau} P\|^2_{L^2(B_r(x_*))} \qquad \text{for } 0 < \sigma \ll 1.$$

Then (4.13) and Lemma 4.3 imply that

$$\begin{aligned}\mathcal{G}(\theta,x_*) &\equiv \|\mathbf{E}_{\lambda^{\mathbf{e}}}\nabla P\|^2_{L^2(B_\theta(x_*))}\\ &\le c_1\Big|\frac{\theta}{\ell}\Big|^n\|\mathbf{E}_{\lambda^{\mathbf{e}}}\nabla P\|^2_{L^2(B_\ell(x_*))}+c_2c_0\frac{\ell^{n-\sigma}\ell^\sigma}{r^n}\|\mathbf{E}_{\lambda^{\mathbf{e}}}\partial_{x_\tau}P\|^2_{L^2(B_r(x_*))}\\ &\le c_1\Big|\frac{\theta}{\ell}\Big|^n\mathcal{G}(\ell,x_*)+c_2\mathbf{M}_1\ell^{n-\sigma},\end{aligned}$$

where c_1,c_2 are independent of $\lambda,\theta,\ell,r,\mathbf{e},\sigma,x_*$. By Lemma 0.1 [7], we have

$$\mathcal{G}(\theta,x_*)\le c_3\left(\Big|\frac{\theta}{\ell}\Big|^{n-\sigma}\mathcal{G}(\ell,x_*)+\mathbf{M}_1\theta^{n-\sigma}\right)\qquad\text{for } 0<\theta\le\ell\le\tfrac{r}{2}. \tag{4.14}$$

As $\ell\nearrow\frac{r}{2}$, we have, by (4.14),

$$\mathcal{G}(\theta,x_*)\le c_4\Big|\frac{\theta}{r}\Big|^{n-\sigma}\mathcal{G}(r,x_*)\qquad\text{for } 0<\theta\le\tfrac{r}{2},\ 0<\sigma\ll 1. \tag{4.15}$$

An inequality of the form (4.15) for $\theta\in(\frac{r}{2},r)$ is obvious. So we obtain (4.12). □

Let $\mathbb{K}_{\lambda^2}\equiv\mathbf{E}_{\lambda^2}\left(K_1\mathcal{X}_{\{x_n\ge 0\}}+K_2\mathcal{X}_{\{x_n<0\}}\right)$ where $K_1,K_2\in\mathcal{M}_1(\beta,\gamma)$ are continuous in $\overline{B_1(0)}$ and $\lambda\in(0,1]$. See (4.1) for $\mathbf{E}_{\lambda^2}$. We consider the following problem

$$-\nabla\cdot(\mathbb{K}_{\lambda^2}\nabla P+V)=G\quad\text{in } B_1(0). \tag{4.16}$$

Lemma 4.5. *Let $\delta\in(0,1)$, $V\in L^{2,q}(B_1(0))$, $G\in L^{2,(q-2)_+}(B_1(0))$, $\mathbf{e}\in[0,1]$, and $q\in(0,n)$. Any solution P of (4.16) satisfies*

$$\begin{aligned}\|\mathbf{E}_{\lambda^{\mathbf{e}}}\nabla P\|_{L^{2,q}(B_{1-\delta}(0))}&\le c\big(\|\mathbf{E}_{\lambda^{\mathbf{e}}}\nabla P\|_{L^2(B_1(0))}\\ &+\|\mathbf{E}_{\lambda^{\mathbf{e}-2}}V\|_{L^{2,q}(B_1(0))}+\|\mathbf{E}_{\lambda^{\mathbf{e}-2}}G\|_{L^{2,(q-2)_+}(B_1(0))}\big),\end{aligned} \tag{4.17}$$

where c is independent of $\lambda,\mathbf{e}$. If $q\in(n-2,n)$, any solution P of (4.16) satisfies

$$\begin{aligned}[P]_{C^{0,\mu}(B^+_{1-\delta}(0))}+\lambda^{\mathbf{e}}[P]_{C^{0,\mu}(B^-_{1-\delta}(0))}&\le c\big(\|\mathbf{E}_{\lambda^{\mathbf{e}}}\nabla P\|_{L^2(B_1(0))}\\ &+\|\mathbf{E}_{\lambda^{\mathbf{e}-2}}V\|_{L^{2,q}(B_1(0))}+\|\mathbf{E}_{\lambda^{\mathbf{e}-2}}G\|_{L^{2,(q-2)_+}(B_1(0))}\big),\end{aligned} \tag{4.18}$$

where $\mu\equiv\frac{q-n+2}{2}$ and c is independent of $\lambda,\mathbf{e}$.

Proof. Step 1: Assume $0<\theta<r<\frac{\delta}{2}$ and $x_*\in\mathbf{I}_{1-\delta}(0)$, and set

$$\mathbf{T}_{\lambda^2}(x)\equiv\mathbf{E}_{\lambda^2}(x)\left(K_1(x_*)\mathcal{X}_{\{x_n\ge 0\}}+K_2(x_*)\mathcal{X}_{\{x_n<0\}}\right).$$

Let $\phi\in H^1(B_r(0))$ be the weak solution of

$$\begin{cases}-\nabla\cdot(\mathbf{T}_{\lambda^2}\nabla\phi)=0 & \text{in } B_r(x_*),\\ \phi=P & \text{on } \partial B_r(x_*).\end{cases} \tag{4.19}$$

By energy method and Theorem 7.25 [17], we see

$$\int_{B_r(x_*)}|\mathbf{E}_{\lambda^{\mathbf{e}}}\nabla\phi|^2dx\le c\int_{B_r(x_*)}|\mathbf{E}_{\lambda^{\mathbf{e}}}\nabla P|^2dx, \tag{4.20}$$

where c is independent of $\lambda, r, \mathbf{e}, x_*$. Let us define $\zeta \equiv P - \phi$. (4.16) and (4.19) imply

$$\begin{cases} -\nabla\cdot(\mathbf{T}_{\lambda^2}\nabla\zeta + (\mathbb{K}_{\lambda^2} - \mathbf{T}_{\lambda^2})\nabla P + V) = G & \text{in } B_r(x_*), \\ \zeta = 0 & \text{on } \partial B_r(x_*). \end{cases}$$

By energy method and Theorem 7.25 [17], we see

$$\begin{aligned} &\int_{B_r(x_*)} |\mathbf{E}_{\lambda^{\mathbf{e}}}\nabla\zeta|^2 dx \\ &\le c\bigg(\|\widehat{\Delta\mathbb{K}}\|^2_{L^\infty(B_r(x_*))} \int_{B_r(x_*)} |\mathbf{E}_{\lambda^{\mathbf{e}}}\nabla P|^2 dx + \int_{B_r(x_*)} (|\mathbf{E}_{\lambda^{\mathbf{e}-2}} V|^2 + r^2|\mathbf{E}_{\lambda^{\mathbf{e}-2}} G|^2) dx \bigg) \\ &\le c\bigg(\|\widehat{\Delta\mathbb{K}}\|^2_{L^\infty(B_r(x_*))} \int_{B_r(x_*)} |\mathbf{E}_{\lambda^{\mathbf{e}}}\nabla P|^2 dx + \mathbf{M}_2 r^q \bigg), \end{aligned} \tag{4.21}$$

where $\widehat{\Delta\mathbb{K}} \equiv \mathbf{T}_1 - \mathbb{K}_1$, $\mathbf{M}_2 \equiv \|\mathbf{E}_{\lambda^{\mathbf{e}-2}} V\|^2_{L^{2,q}(B_1(0))} + \|\mathbf{E}_{\lambda^{\mathbf{e}-2}} G\|^2_{L^{2,(q-2)_+}(B_1(0))}$, and c is independent of $\lambda, r, \mathbf{e}, x_*$. By Lemma 4.4, the solution ϕ of (4.19) satisfies, for $0 < \sigma \ll 1$ and $0 < \theta < r < \frac{\delta}{2}$,

$$\int_{B_\theta(x_*)} |\mathbf{E}_{\lambda^{\mathbf{e}}}\nabla\phi|^2 dx \le c\Big|\frac{\theta}{r}\Big|^{n-\sigma} \int_{B_r(x_*)} |\mathbf{E}_{\lambda^{\mathbf{e}}}\nabla\phi|^2 dx, \tag{4.22}$$

where c is independent of $\lambda, \theta, r, \mathbf{e}, \sigma, \delta, x_*$. Then (4.20)–(4.22) imply, for $0 < \sigma \ll 1$ and $0 < \theta < r < \frac{\delta}{2}$,

$$\mathcal{G}(\theta, x_*) \equiv \int_{B_\theta(x_*)} |\mathbf{E}_{\lambda^{\mathbf{e}}}\nabla P|^2 dx \le c\left(\Big|\frac{\theta}{r}\Big|^{n-\sigma} + \|\widehat{\Delta\mathbb{K}}\|^2_{L^\infty(B_r(x_*))} \right) \mathcal{G}(r, x_*) + c\mathbf{M}_2 r^q,$$

where c is independent of $\lambda, \theta, r, \mathbf{e}, \sigma, x_*$. Let us fix $\sigma < n - q$. By Lemma 0.1 [7], there exists $\ell(c, n-\sigma, q)$ such that if r is small so that $\|\widehat{\Delta\mathbb{K}}\|_{L^\infty(B_r(x_*))} \le \ell$, then

$$\mathcal{G}(\theta, x_*) \le c\left(\Big|\frac{\theta}{r}\Big|^q \mathcal{G}(r, x_*) + \mathbf{M}_2\theta^q \right), \tag{4.23}$$

for $0 < \sigma \ll 1$ and $0 < \theta < r < \frac{\delta}{2}$, where c is independent of $\lambda, \theta, \mathbf{e}, \sigma, r, x_*$.

Since K_1, K_2 are continuous in $\overline{B_1(0)}$, there is a $r^* < \delta/2$ such that we have the inequality (4.23) for all $x_* \in \mathbf{I}_{1-\delta}(0)$ and $0 < \theta < r \le r^* < \frac{\delta}{2}$. So

$$\sup_{\substack{\theta \le r^* < \delta/2 \\ x_* \in \mathbf{I}_{1-\delta}(0)}} \frac{1}{\theta^q} \int_{B_\theta(x_*)} |\mathbf{E}_{\lambda^{\mathbf{e}}}\nabla P|^2 dx \le c\left(\|\mathbf{E}_{\lambda^{\mathbf{e}}}\nabla P\|^2_{L^2(B_1(0))} + \mathbf{M}_2 \right). \tag{4.24}$$

In the following argument of this proof, r^* is fixed.

Step 2: Suppose $x \in \mathbf{I}_{1-\delta}(0) \times [-\frac{r^*}{2}, \frac{r^*}{2}]$, $0 < \theta < \frac{r^*}{2}$, $B_\theta(x) \cap \mathbf{I}_{1-\delta}(0) \ne \emptyset$. Suppose $x_* \in B_\theta(x) \cap \mathbf{I}_{1-\delta}(0)$, by (4.24),

$$\frac{1}{\theta^q} \int_{B_\theta(x)} |\mathbf{E}_{\lambda^{\mathbf{e}}}\nabla P|^2 \le \frac{c}{|2\theta|^q} \int_{B_{2\theta}(x_*)} |\mathbf{E}_{\lambda^{\mathbf{e}}}\nabla P|^2 \le c\left(\|\mathbf{E}_{\lambda^{\mathbf{e}}}\nabla P\|^2_{L^2(B_1(0))} + \mathbf{M}_2 \right),$$

where c is independent of $x, x_*, \lambda, \theta, \mathbf{e}$.

Suppose $x \in \mathbf{I}_{1-\delta}(0) \times [-\frac{r^*}{2}, \frac{r^*}{2}]$, $0 < \theta < \frac{r^*}{2}$, $B_\theta(x) \cap \mathbf{I}_{1-\delta}(0) = \emptyset$ or suppose $x \in \mathbf{I}_{1-\delta}(0) \times ([-r^*, r^*] \setminus [-\frac{r^*}{2}, \frac{r^*}{2}])$, $0 < \theta < \frac{r^*}{2}$. Then (4.24) holds by following the arguments in Step 1 and Step 2.

Step 3: By the estimates in Step 1 and Step 2 for $\mathcal{G}(\theta, x_*)$ and by partition of unity, we obtain the inequality (4.24) for all $x \in \overline{B_{1-\delta}(0)}$ and $0 < \theta < r^* < \frac{\delta}{2}$. Consequence,

$$\sup_{\substack{\theta < r^* \\ x \in \overline{B_{1-\delta}(0)}}} \frac{1}{\theta^q} \int_{B_\theta(x)} |\mathbf{E}_{\lambda^{\mathbf{e}}} \nabla P|^2 dx \le c \left(\|\mathbf{E}_{\lambda^{\mathbf{e}}} \nabla P\|^2_{L^2(B_1(0))} + \mathbf{M}_2 \right). \tag{4.25}$$

Which guarantee the estimate (4.17).

If $q \in (n-2, n)$, then (4.18) follows from (4.17) and Morrey's Theorem [31]. □

By (A1), there is a constant $\mathbf{d}_6$ and a set $\mathbf{D}$ such that

$$\begin{cases} Y_m \subset \mathbf{D} \subset Y, \\ \text{dist}(Y_m, \partial \mathbf{D}),\ \text{dist}(\mathbf{D}, \partial Y) > \mathbf{d}_6 > 0. \end{cases} \tag{4.26}$$

Lemma 4.6. *Assume (A1)–(A3), $\omega \in \Omega$, $\lambda \in (0,1]$, $\mathbf{e} \in [0,1]$, $q \in (0,n)$, $j \in \mathbb{Z}^n$, $V \in L^{2,q}(\Phi(Y+j,\omega))$, and $G \in L^{2,(q-2)_+}(\Phi(Y+j,\omega))$. Any solution of*

$$-\nabla \cdot (\mathbf{K}_{\omega,\lambda^2,1} \nabla P + V) = G \quad \text{in } \Phi(Y+j,\omega) \tag{4.27}$$

satisfies

$$\|\mathbf{K}_{\omega,\lambda^{\mathbf{e}},1} \nabla P\|_{L^{2,q}(\Phi(\mathbf{D}+j,\omega))} \le c\bigg(\|\mathbf{K}_{\omega,\lambda^{\mathbf{e}},1} P\|_{L^2(\Phi(Y+j,\omega))} + \|\mathbf{K}_{\omega,\lambda^{\mathbf{e}-2},1} V\|_{L^{2,q}(\Phi(Y+j,\omega))} + \|\mathbf{K}_{\omega,\lambda^{\mathbf{e}-2},1} G\|_{L^{2,(q-2)_+}(\Phi(Y+j,\omega))} \bigg),$$

where c is independent of $\omega, \lambda, \mathbf{e}, j$. In case of $q \in (n-2, n)$, any solution of (4.27) satisfies

$$[P]_{C^{0,\mu}(\Phi(\mathbf{D} \setminus Y_m + j,\omega))} + \lambda^{\mathbf{e}} [P]_{C^{0,\mu}(\Phi(Y_m+j,\omega))} \le c\bigg(\|\mathbf{K}_{\omega,\lambda^{\mathbf{e}},1} P\|_{L^2(\Phi(Y+j,\omega))} + \|\mathbf{K}_{\omega,\lambda^{\mathbf{e}-2},1} V\|_{L^{2,q}(\Phi(Y+j,\omega))} + \|\mathbf{K}_{\omega,\lambda^{\mathbf{e}-2},1} G\|_{L^{2,(q-2)_+}(\Phi(Y+j,\omega))} \bigg),$$

where $\mu \equiv \frac{q-n+2}{2}$ and c is independent of $\omega, \lambda, \mathbf{e}, j$. See (4.26) for $\mathbf{D}$.

Proof. For any $x_* \in \partial Y_m$, there exists a small neighborhood $\mathcal{N}_{x_*}$ of $\Phi(x_* + j, \omega)$ and a C^1-mapping $\widetilde{\Upsilon}$ so that $\mathcal{N}_{x_*}$ under the mapping $\widetilde{\Upsilon}$ can be transformed to a ball $B_1(0)$ by (A1)–(A2). Moreover,

$$\begin{cases} \widetilde{\Upsilon}(\mathcal{N}_{x_*} \cap \Phi(Y_f + j, \omega)) = B_1^+(0), \\ \widetilde{\Upsilon}(\mathcal{N}_{x_*} \cap \Phi(Y_m + j, \omega)) = B_1^-(0), \\ \widetilde{\Upsilon}(\mathcal{N}_{x_*} \cap \Phi(\partial Y_m + j, \omega)) = \mathbf{I}_1(0). \end{cases}$$

Under the mapping, the equation (4.27) is transformed into the equation (4.16) and the conditions in Lemma 4.5 are satisfied under the assumptions of Lemma 4.6. One may see pages 3964–3965 of [7] for the expression of the transformed coefficients. So the local estimate of the solution of (4.27) can be obtained from Lemma 4.5. Then by partition of unity, we derive the results of Lemma 4.6. □

Tracing the arguments of Lemmas 4.3, 4.5, 4.6, we easily obtain the following estimate around the Dirichlet boundary.

Lemma 4.7. *Assume (A1)-(A3),* $\omega \in \Omega$, $\epsilon, \lambda \in (0,1]$, $0 \in \partial\mathcal{D}$, $V \in L^{2,q}(B_1(0) \cap \mathcal{D}/\epsilon)$, $G \in L^{2,(q-2)_+}(B_1(0) \cap \mathcal{D}/\epsilon)$, *and* $q \in (0,n)$. *Any solution of*

$$\begin{cases} -\nabla \cdot (\mathbf{K}_{\omega,\lambda^2,\epsilon,\epsilon} \nabla P + V) = G & \text{in } B_1(0) \cap \mathcal{D}/\epsilon \\ P = 0 & \text{on } B_1(0) \cap \partial\mathcal{D}/\epsilon \end{cases}$$

satisfies

$$\begin{aligned} \|\nabla P\|_{L^{2,q}(B_{1/2}(0)\cap\mathcal{D}/\epsilon)} \leq c\big(&\|P\|_{L^2(B_1(0)\cap\mathcal{D}/\epsilon)} \\ &+\|V\|_{L^{2,q}(B_1(0)\cap\mathcal{D}/\epsilon)} + \|G\|_{L^{2,(q-2)_+}(B_1(0)\cap\mathcal{D}/\epsilon)}\big), \end{aligned}$$

where c *is independent of* $\omega, \epsilon, \lambda$. *In case of* $q \in (n-2, n)$,

$$\begin{aligned} [P]_{C^{0,\mu}(B_{1/2}(0)\cap\mathcal{D}/\epsilon)} \leq c\big(&\|P\|_{L^2(B_1(0)\cap\mathcal{D}/\epsilon)} \\ &+\|V\|_{L^{2,q}(B_1(0)\cap\mathcal{D}/\epsilon)} + \|G\|_{L^{2,(q-2)_+}(B_1(0)\cap\mathcal{D}/\epsilon)}\big), \end{aligned}$$

where $\mu \equiv \frac{q-n+2}{2}$ *and* c *is independent of* $\omega, \epsilon, \lambda$. *See (3.19) for* $\mathbf{K}_{\omega,\lambda^2,\epsilon,\epsilon}$.

5. Uniform Hölder estimate

Now we prove Theorem 2.1. The estimate in the interior region is in subsection 5.1, and the estimate around the boundary is in subsection 5.2. Note that the uniform Hölder estimate of the solutions of uniform elliptic equations can be shown without a stationary-ergodic assumption [17, 20].

5.1. *Interior estimate*

Let $\overline{B_1(0)} \subset \mathcal{D}$ in this subsection.

Lemma 5.1. *There are constants* $\mu, \theta_1, \theta_2 \in (0,1)$ *(depending on* $\beta, \gamma, \mathbf{d}_5$*) satisfying* $\theta_1 < \theta_2^2$ *and there is a* $\epsilon_0 \in (0,1)$ *(depending on* $\theta_1, \mu, \beta, \gamma, \mathbf{d}_5$*) so that if*

$$-\nabla \cdot (\mathbf{K}_{\omega,\lambda^2,\nu} \nabla P + V) = G \qquad \text{in } B_1(0), \tag{5.1}$$

and if

$$\begin{cases} \omega \in \Omega, \quad \lambda \in (0,1], \quad \nu \in (0,\epsilon_0), \quad \theta \in [\theta_1, \theta_2], \quad \mathbf{e} \in [0,1], \\ \|P\mathcal{X}_{\mathcal{D}^\nu_{\omega,f}}, \lambda^{\mathbf{e}} P\mathcal{X}_{\mathcal{D}^\nu_{\omega,m}}\|_{L^2(B_1(0))}, \ \|\frac{1}{\epsilon_0}V, \ \mathbf{K}_{\omega,\lambda^{\mathbf{e}-2},\nu} V\|_{L^{2,q}(B_1(0))} \leq 1, \\ \|\frac{1}{\epsilon_0}G, \ \mathbf{K}_{\omega,\lambda^{\mathbf{e}-2},\nu} G\|_{L^{2,(q-2)_+}(B_1(0))} \leq 1, \end{cases} \tag{5.2}$$

then

$$\begin{cases} \fint_{B_\theta(0)} \left|\Pi_{\omega,\nu} P|_{\mathcal{D}^\nu_{\omega,f}} - \left(\Pi_{\omega,\nu} P|_{\mathcal{D}^\nu_{\omega,f}}\right)_{0,\theta}\right|^2 dx \le \theta^{2\mu}, \\ \fint_{B_\theta(0)\cap\mathcal{D}^\nu_{\omega,m}} \lambda^{2\mathbf{e}} \left|P - \left(\Pi_{\omega,\nu} P|_{\mathcal{D}^\nu_{\omega,f}}\right)_{0,\theta}\right|^2 dx \le \theta^{2\mu}, \end{cases} \tag{5.3}$$

where $q = n - 2 + 2\mu$*. See (2.1) for* $\left(\Pi_{\omega,\nu} P|_{\mathcal{D}^\nu_{\omega,f}}\right)_{0,\theta}$*, Remark 3.2 for* $\mathbf{d}_5$*, and (A3) for* β, γ*.*

Proof. Consider the following uniform elliptic equation

$$-\nabla\cdot(\mathcal{K}_{\lambda_*}\nabla P) = 0 \qquad \text{in } B_{4/5}(0), \tag{5.4}$$

where $\mathcal{K}_{\lambda_*} \in \mathcal{M}_2(\frac{\beta}{\mathbf{d}_5^2}, \frac{\gamma^2}{\beta|Y_f|})$. By Theorem 8.24 [17], there is a constant $\sigma(<\frac{1}{2})$ depending on $n, \beta, \gamma, \mathbf{d}_5$ such that any solution P of (5.4) satisfies

$$\|P\|_{C^{0,2\sigma}(B_{1/2}(0))} \le c\|P\|_{L^2(B_{4/5}(0))},$$

where c only depends on $n, \beta, \gamma, \mathbf{d}_5$. There is a sufficiently small number $\hat{\theta}$ (depending on $n, \beta, \gamma, \mathbf{d}_5$) such that, by Theorem 1.2 in page 70 [16],

$$\fint_{B_\theta(0)} |P - (P)_{0,\theta}|^2 dx \le \theta^{2\sigma} \fint_{B_{4/5}(0)} P^2 dx \qquad \text{for any } \theta < \hat{\theta}. \tag{5.5}$$

Let us fix a $\mu < \sigma$ and $\theta_1, \theta_2 \in (0, \hat{\theta})$ so that $\theta_1 < \theta_2^2$.

Now we claim $(5.3)_1$. If not, there is a sequence $\{\omega_\nu, \lambda_\nu, \theta_\nu, P_\nu, V_\nu, G_\nu\}$ satisfying (5.1) and

$$\begin{cases} \omega_\nu \in \Omega, \quad \nu \to 0, \quad \lambda_\nu \to \lambda_* \in [0,1], \quad \theta_\nu \to \theta_* \in [\theta_1, \theta_2], \\ \|P_\nu \mathcal{X}_{\mathcal{D}^\nu_{\omega_\nu,f}}, \lambda_\nu^{\mathbf{e}} P_\nu \mathcal{X}_{\mathcal{D}^\nu_{\omega_\nu,m}}\|_{L^2(B_1(0))} \le 1, \\ \|\mathbf{K}_{\omega_\nu,\lambda_\nu^{\mathbf{e}-2},\nu} V_\nu\|_{L^{2,q}(B_1(0))}, \ \|\mathbf{K}_{\omega_\nu,\lambda_\nu^{\mathbf{e}-2},\nu} G_\nu\|_{L^{2,(q-2)+}(B_1(0))} \le 1, \\ \lim_{\nu\to 0} \|V_\nu\|_{L^{2,q}(B_1(0))} + \|G_\nu\|_{L^{2,(q-2)+}(B_1(0))} = 0, \\ \fint_{B_{\theta_\nu}(0)} \left|\Pi_{\omega_\nu,\nu} P_\nu|_{\mathcal{D}^\nu_{\omega_\nu,f}} - \left(\Pi_{\omega_\nu,\nu} P_\nu|_{\mathcal{D}^\nu_{\omega_\nu,f}}\right)_{0,\theta_\nu}\right|^2 dx > |\theta_\nu|^{2\mu}. \end{cases} \tag{5.6}$$

By Lemma 3.3, there is a subsequence (same notation for subsequence) such that

$$\begin{cases} \Pi_{\omega_\nu,\nu} P_\nu|_{\mathcal{D}^\nu_{\omega_\nu,f}} \xrightarrow{H^1(B_{4/5}(0)) \ weakly} P \\ \mathbf{K}_{\omega_\nu,\lambda_\nu^2,\nu} \nabla P_\nu \xrightarrow{L^2(B_{4/5}(0)) \ weakly} \mathcal{K}_{\lambda_*}\nabla P \end{cases} \quad \text{as } \nu \to 0, \tag{5.7}$$

where $\mathcal{K}_{\lambda_*} \in \mathcal{M}_2(\frac{\beta}{\mathbf{d}_5^2}, \frac{\gamma^2}{\beta|Y_f|})$. Also the P in (5.7) satisfies (5.4). (5.5)–(5.7) imply

$$\theta_*^{2\mu} = \lim_{\nu\to 0} |\theta_\nu|^{2\mu} \le \lim_{\nu\to 0} \fint_{B_{\theta_\nu}(0)} \left|\Pi_{\omega_\nu,\nu} P_\nu|_{\mathcal{D}^\nu_{\omega_\nu,f}} - \left(\Pi_{\omega_\nu,\nu} P_\nu|_{\mathcal{D}^\nu_{\omega_\nu,f}}\right)_{0,\theta_\nu}\right|^2 dx$$

$$= \fint_{B_{\theta_*}(0)} P^2 dx - \left|\fint_{B_{\theta_*}(0)} P dx\right|^2 = \fint_{B_{\theta_*}(0)} |P - (P)_{0,\theta_*}|^2 dx \le \theta_*^{2\sigma} \fint_{B_{4/5}(0)} P^2 dx.$$

If θ_2 is small enough, then we get contradiction. Therefore we proved $(5.3)_1$.

Write

$$\begin{cases}\zeta \equiv \frac{1}{\theta^\mu}\big(\Pi_{\omega,\nu}P|_{\mathcal{D}^\nu_{\omega,f}} - (\Pi_{\omega,\nu}P|_{\mathcal{D}^\nu_{\omega,f}})_{0,\theta}\big),\\ \eta \equiv \frac{1}{\theta^\mu}\big(P - (\Pi_{\omega,\nu}P|_{\mathcal{D}^\nu_{\omega,f}})_{0,\theta}\big).\end{cases}$$

Then (5.1) implies, for any smooth function φ with support in $\nu\Phi(Y_m+j,\omega)\subset B_\theta(0)$ for some $j\in\mathbb{Z}^n$,

$$\begin{aligned}\int_{\nu\Phi(Y_m+j,\omega)}&(\eta-\zeta)\nabla\cdot\big(\mathbf{K}_{\omega,\lambda^2,\nu}\nabla\varphi\big)\\ &=\int_{\nu\Phi(Y_m+j,\omega)}\big(\mathbf{K}_{\omega,\lambda^2,\nu}\nabla\zeta+\frac{1}{\theta^\mu}V\big)\nabla\varphi-\frac{1}{\theta^\mu}G\varphi.\end{aligned}\tag{5.8}$$

If φ is the solution of

$$\begin{cases}-\nabla\cdot\big(\mathbf{K}_{\omega,\lambda^2,\nu}\nabla\varphi\big)=\zeta-\eta & \text{in } \nu\Phi(Y_m+j,\omega),\\ \varphi=0 & \text{on } \nu\Phi(\partial Y_m+j,\omega),\end{cases}\tag{5.9}$$

then

$$\frac{c_1\lambda^2}{\nu}\|\varphi\|_{L^2(\nu\Phi(Y_m+j,\omega))}\le\lambda^2\|\nabla\varphi\|_{L^2(\nu\Phi(Y_m+j,\omega))}\le c_2\nu\|\eta-\zeta\|_{L^2(\nu\Phi(Y_m+j,\omega))},$$

where c_1, c_2 are independent of λ,ν,ω. (5.8)–(5.9) and Remark 3.2 imply

$$\int_{\nu\Phi(Y_m+j,\omega)}\lambda^2|\eta-\zeta|^2\le c\int_{\nu\Phi(Y_m+j,\omega)}\lambda^2\nu^2|\nabla\zeta|^2+\frac{\nu^2}{\theta^{2\mu}}\Big(\big|\frac{1}{\lambda}V\big|^2+\big|\frac{\nu}{\lambda}G\big|^2\Big).\tag{5.10}$$

Summing (5.10) over all $\nu\Phi(Y_m+j,\omega)\subset B_\theta(0)$ for $j\in\mathbb{Z}^n$, we obtain $(5.3)_2$ if ϵ_0 is small enough. □

Lemma 5.2. *There exist $\mu,\theta_1,\theta_2\in(0,1)$ (depending on $\beta,\gamma,\mathbf{d}_5$) satisfying $\theta_1<\theta_2^2$ and there is a $\epsilon_0\in(0,1)$ (depending on $\theta_1,\mu,\beta,\gamma,\mathbf{d}_5$) so that if $\omega\in\Omega$, $\lambda\in(0,1]$, $\epsilon\in(0,\epsilon_0)$, $\theta\in[\theta_1,\theta_2]$, $\mathbf{e}\in[0,1]$, and k satisfying $\epsilon/\theta^k\le\epsilon_0$, then any solution of*

$$-\nabla\cdot(\mathbf{K}_{\omega,\lambda^2,\epsilon}\nabla U+Q)=F\qquad \text{in } B_1(0)\tag{5.11}$$

satisfies

$$\begin{cases}\displaystyle\fint_{B_{\theta^k}(0)}\Big|\Pi_{\omega,\epsilon}U|_{\mathcal{D}^\epsilon_{\omega,f}}-\big(\Pi_{\omega,\epsilon}U|_{\mathcal{D}^\epsilon_{\omega,f}}\big)_{0,\theta^k}\Big|^2dx\le\theta^{2k\mu}|J_{\lambda,\epsilon,\omega,\mathbf{e}}|^2,\\ \displaystyle\fint_{B_{\theta^k}(0)\cap\mathcal{D}^\epsilon_{\omega,m}}\lambda^{2\mathbf{e}}\Big|U-\big(\Pi_{\omega,\epsilon}U|_{\mathcal{D}^\epsilon_{\omega,f}}\big)_{0,\theta^k}\Big|^2dx\le\theta^{2k\mu}|J_{\lambda,\epsilon,\omega,\mathbf{e}}|^2,\end{cases}\tag{5.12}$$

where $J_{\lambda,\epsilon,\omega,\mathbf{e}}\equiv\big\|U\mathcal{X}_{\mathcal{D}^\epsilon_{\omega,f}},\lambda^{\mathbf{e}}U\mathcal{X}_{\mathcal{D}^\epsilon_{\omega,m}}\big\|_{L^2(B_1(0))}+\|\frac{1}{\epsilon_0}F,\mathbf{K}_{\omega,\lambda^{\mathbf{e}-2},\epsilon}F\|_{L^{2,(q-2)_+}(B_1(0))}+\|\frac{1}{\epsilon_0}Q,\mathbf{K}_{\omega,\lambda^{\mathbf{e}-2},\epsilon}Q\|_{L^{2,q}(B_1(0))}$ and $q=n-2+2\mu$.

Proof. Let c denote a constant independent of $\lambda,\epsilon,\omega,\mathbf{e}$. This proof is done by induction on k. For $k=1$, we define $P\equiv\frac{U}{J_{\lambda,\epsilon,\omega,\mathbf{e}}}$, $V\equiv\frac{Q}{J_{\lambda,\epsilon,\omega,\mathbf{e}}}$, $G\equiv\frac{F}{J_{\lambda,\epsilon,\omega,\mathbf{e}}}$. Then they satisfy (5.1) and (5.2) with $\nu = \epsilon$. By Lemma 5.1,

$$\begin{cases} \fint_{B_\theta(0)} \left|\Pi_{\omega,\epsilon} P|_{\mathcal{D}^\epsilon_{\omega,f}} - \left(\Pi_{\omega,\epsilon} P|_{\mathcal{D}^\epsilon_{\omega,f}}\right)_{0,\theta}\right|^2 dx \le \theta^{2\mu}, \\ \fint_{B_\theta(0)\cap\mathcal{D}^\epsilon_{\omega,m}} \lambda^{2\mathbf{e}} \left|P - \left(\Pi_{\omega,\epsilon} P|_{\mathcal{D}^\epsilon_{\omega,f}}\right)_{0,\theta}\right|^2 dx \le \theta^{2\mu}. \end{cases}$$

After change of variable, we obtain (5.12) for $k = 1$. Suppose (5.12) holds for some k satisfying $\omega/\theta^k \le \epsilon_0$, we define

$$\begin{cases} P(x) \equiv \frac{1}{\theta^{k\mu}} \left(U(\theta^k x) - \left(\Pi_{\omega,\epsilon} U|_{\mathcal{D}^\epsilon_{\omega,f}}\right)_{0,\theta^k}\right)/J_{\lambda,\epsilon,\omega,\mathbf{e}} \\ V(x) \equiv \frac{1}{\theta^{k(\mu-1)}} Q(\theta^k x)/J_{\lambda,\epsilon,\omega,\mathbf{e}} \\ G(x) \equiv \frac{1}{\theta^{k(\mu-2)}} F(\theta^k x)/J_{\lambda,\epsilon,\omega,\mathbf{e}} \end{cases} \quad \text{in } B_1(0)\setminus\Gamma^\epsilon_\omega/\theta^k.$$

Then they satisfy

$$-\nabla\cdot(\mathbf{K}_{\omega,\lambda^2,\epsilon/\theta^k}\nabla P + V) = G \quad \text{in } B_1(0).$$

By induction,

$$\begin{cases} \left\|P\mathcal{X}_{\mathcal{D}^\epsilon_{\omega,f}/\theta^k}, \lambda^{\mathbf{e}} P\mathcal{X}_{\mathcal{D}^\epsilon_{\omega,m}/\theta^k}\right\|_{L^2(B_1(0))}, \|\frac{1}{\epsilon_0}V, \mathbf{K}_{\omega,\lambda^{\mathbf{e}-2},\epsilon/\theta^k}V\|_{L^{2,q}(B_1(0))} \le 1, \\ \|\frac{1}{\epsilon_0}G, \mathbf{K}_{\omega,\lambda^{\mathbf{e}-2},\epsilon/\theta^k}G\|_{L^{2,(q-2)_+}(B_1(0))} \le 1. \end{cases}$$

By Lemma 5.1 (take $\nu = \epsilon/\theta^k$), we obtain

$$\begin{cases} \fint_{B_\theta(0)} \left|\Pi_{\omega,\epsilon/\theta^k} P|_{\mathcal{D}^\epsilon_{\omega,f}/\theta^k} - \left(\Pi_{\omega,\epsilon/\theta^k} P|_{\mathcal{D}^\epsilon_{\omega,f}/\theta^k}\right)_{0,\theta}\right|^2 dx \le \theta^{2\mu}, \\ \fint_{B_\theta(0)\cap\mathcal{D}^\epsilon_{\omega,m}/\theta^k} \lambda^{2\mathbf{e}} \left|P - \left(\Pi_{\omega,\epsilon/\theta^k} P|_{\mathcal{D}^\epsilon_{\omega,f}/\theta^k}\right)_{0,\theta}\right|^2 dx \le \theta^{2\mu}. \end{cases} \tag{5.13}$$

Note, by Remark 3.2,

$$\begin{cases} \fint_{B_\theta(0)} \left|\Pi_{\omega,\epsilon/\theta^k} P|_{\mathcal{D}^\epsilon_{\omega,f}/\theta^k} - \left(\Pi_{\omega,\epsilon/\theta^k} P|_{\mathcal{D}^\epsilon_{\omega,f}/\theta^k}\right)_{0,\theta}\right|^2 dx \\ \qquad = \fint_{B_{\theta^{k+1}}(0)} \frac{\left|\Pi_{\omega,\epsilon} U|_{\mathcal{D}^\epsilon_{\omega,f}} - \left(\Pi_{\omega,\epsilon} U|_{\mathcal{D}^\epsilon_{\omega,f}}\right)_{0,\theta^{k+1}}\right|^2}{\theta^{2k\mu}\ |J_{\lambda,\epsilon,\omega,\mathbf{e}}|^2} dx, \\ \fint_{B_\theta(0)\cap\mathcal{D}^\epsilon_{\omega,m}/\theta^k} \lambda^{2\mathbf{e}} \left|P - \left(\Pi_{\omega,\epsilon/\theta^k} P|_{\mathcal{D}^\epsilon_{\omega,f}/\theta^k}\right)_{0,\theta}\right|^2 dx \\ \qquad = \fint_{B_{\theta^{k+1}}(0)\cap\mathcal{D}^\epsilon_{\omega,m}} \lambda^{2\mathbf{e}} \frac{\left|U - \left(\Pi_{\omega,\epsilon} U|_{\mathcal{D}^\epsilon_{\omega,f}}\right)_{0,\theta^{k+1}}\right|^2}{\theta^{2k\mu}\ |J_{\lambda,\epsilon,\omega,\mathbf{e}}|^2} dx. \end{cases} \tag{5.14}$$

Equations (5.13)–(5.14) imply the inequality (5.12) for $k+1$ case. □

Lemma 5.3. *Let $\mu, J_{\lambda,\epsilon,\omega,\mathbf{e}}$ be from Lemma 5.2. There is a constant $\epsilon_* \in (0,1)$ (depending on $\mu, \beta, \gamma, \mathbf{d}_5$) such that if $\omega \in \Omega$, $\lambda \in (0,1]$, $\epsilon \in (0,\epsilon_*)$, and $\mathbf{e} \in [0,1]$, then any solution of (5.11) satisfies*

$$[U]_{C^{0,\mu}(B_{1/2}(0)\cap\mathcal{D}^\epsilon_{\omega,f})} + \lambda^{\mathbf{e}}[U]_{C^{0,\mu}(B_{1/2}(0)\cap\mathcal{D}^\epsilon_{\omega,m})} \le cJ_{\lambda,\epsilon,\omega,\mathbf{e}}, \tag{5.15}$$

where c is a constant independent of $\lambda, \epsilon, \omega, \mathbf{e}$.

Proof. Let $\theta_1, \theta_2, q, \epsilon_0$ be the same as those in Lemma 5.2, define $\epsilon_* \equiv \epsilon_0\theta_2/2$, and let $\epsilon \leq \epsilon_*$. Denote by c a constant independent of $\lambda, \epsilon, \omega, \mathbf{e}$. Because of $\theta_1 < \theta_2^2$, for any $r \in [\epsilon/\epsilon_0, \theta_2]$, there are $\theta \in [\theta_1, \theta_2]$ and $k \in \mathbb{N}$ satisfying $r = \theta^k$. Lemma 5.2 implies, for any $r \in [\epsilon/\epsilon_0, \theta_2]$,

$$\begin{cases} \displaystyle\fint_{B_r(0)} \left|\Pi_{\omega,\epsilon}U|_{\mathcal{D}^\epsilon_{\omega,f}} - \left(\Pi_{\omega,\epsilon}U|_{\mathcal{D}^\epsilon_{\omega,f}}\right)_{0,r}\right|^2 dx \leq r^{2\mu}|J_{\lambda,\epsilon,\omega,\mathbf{e}}|^2, \\ \displaystyle\fint_{B_r(0)\cap\mathcal{D}^\epsilon_{\omega,m}} \lambda^{2\mathbf{e}} \left|U - \left(\Pi_{\omega,\epsilon}U|_{\mathcal{D}^\epsilon_{\omega,f}}\right)_{0,r}\right|^2 dx \leq r^{2\mu}|J_{\lambda,\epsilon,\omega,\mathbf{e}}|^2. \end{cases} \tag{5.16}$$

Now we define

$$\begin{cases} P(x) \equiv \frac{1}{\epsilon^\mu}\left(U(\epsilon x) - \left(\Pi_{\omega,\epsilon}U|_{\mathcal{D}^\epsilon_{\omega,f}}\right)_{0,2\epsilon/\epsilon_0}\right)/J_{\lambda,\epsilon,\omega,\mathbf{e}} \\ V(x) \equiv \frac{1}{\epsilon^{\mu-1}}Q(\epsilon x)/J_{\lambda,\epsilon,\omega,\mathbf{e}} \\ G(x) \equiv \frac{1}{\epsilon^{\mu-2}}F(\epsilon x)/J_{\lambda,\epsilon,\omega,\mathbf{e}} \end{cases} \quad \text{in } B_{2/\epsilon_0}(0) \setminus \Gamma^\epsilon_\omega/\epsilon.$$

Then they satisfy

$$-\nabla \cdot (\mathbf{K}_{\omega,\lambda^2,1}\nabla P + V) = G \quad \text{in } B_{\frac{2}{\epsilon_0}}(0).$$

Take $r = \frac{2\epsilon}{\epsilon_0}$ in (5.16) to get

$$\begin{aligned} &\left\|P\mathcal{X}_{\mathcal{D}^\epsilon_{\omega,f}/\epsilon}, \lambda^{\mathbf{e}}P\mathcal{X}_{\mathcal{D}^\epsilon_{\omega,m}/\epsilon}\right\|_{L^2(B_{2/\epsilon_0}(0))} + \|\mathbf{K}_{\omega,\lambda^{\mathbf{e}-2},1}V\|_{L^{2,q}(B_{2/\epsilon_0}(0))} \\ &\quad + \|\mathbf{K}_{\omega,\lambda^{\mathbf{e}-2},1}G\|_{L^{2,(q-2)+}(B_{2/\epsilon_0}(0))} \leq c. \end{aligned}$$

By Lemma 4.6, we see

$$[P]_{C^{0,\mu}(B_{1/\epsilon_0}(0)\cap\mathcal{D}^\epsilon_{\omega,f}/\epsilon)} + \lambda^{\mathbf{e}}[P]_{C^{0,\mu}(B_{1/\epsilon_0}(0)\cap\mathcal{D}^\epsilon_{\omega,m}/\epsilon)} \leq c. \tag{5.17}$$

(5.17) implies that $(5.16)_1$ also holds for $r \leq \epsilon/\epsilon_0$. So $(5.16)_1$ holds for $r \leq \theta_2$. Next we shift the origin of the coordinate system to any point $z \in B_{1/2}(0)$ and repeat above argument to see that $(5.16)_1$ with 0 replaced by any $z \in B_{1/2}(0)$ also holds for $r \in (0, \theta_2)$. Together with Theorem 1.2 in page 70 [16], we obtain the Hölder estimate of $\Pi_{\omega,\epsilon}U|_{\mathcal{D}^\epsilon_{\omega,f}}$ in $B_{1/2}(0)$. Hölder estimate of U in $B_{1/2}(0) \cap \mathcal{D}^\epsilon_{\omega,m}$ is from (5.17), triangle inequality, and the Hölder estimate of $\Pi_{\omega,\epsilon}U|_{\mathcal{D}^\epsilon_{\omega,f}}$ in $B_{1/2}(0)$. So (5.15) is proved. □

Remark 5.1. Let ϵ_* be the same as that in Lemma 5.3. By Lemma 4.6 again, we know that if $\omega \in \Omega$, $\lambda \in (0, 1]$, $\epsilon \in [\epsilon_*, 1]$, and $\mathbf{e} \in [0, 1]$, any solution of (5.11) satisfies (5.15). Together with Lemma 5.3, we know that any solution of (5.11) satisfies (5.15) if $\omega \in \Omega$, $\lambda, \epsilon \in (0, 1]$, and $\mathbf{e} \in [0, 1]$.

5.2. *Boundary estimate*

Let us assume $0 \in \partial\mathcal{D}$ and (3.18)–(3.19) in this subsection.

Lemma 5.4. *There are constants $\mu, \breve{\theta}_1, \breve{\theta}_2 \in (0,1)$ (depending on $\beta, \gamma, \mathbf{d}_5, \partial\mathcal{D}$) satisfying $\breve{\theta}_1 < \breve{\theta}_2^2$ and there is a $\breve{\epsilon}_0 > 0$ (depending on $\breve{\theta}_1, \mu, \beta, \gamma, \mathbf{d}_5, \partial\mathcal{D}$) satisfying $\breve{\epsilon}_0 < \epsilon_0$ (see Lemma 5.1 for ϵ_0) such that if*

$$\begin{cases} -\nabla\cdot(\mathbf{K}_{\omega,\lambda^2,\epsilon,s}\nabla P + V) = G & \text{in } B_1(0)\cap\mathcal{D}/s, \\ P = 0 & \text{on } B_1(0)\cap\partial\mathcal{D}/s, \end{cases} \tag{5.18}$$

and if

$$\begin{cases} \omega\in\Omega, \quad \lambda,\epsilon,s\in(0,1], \quad \frac{\epsilon}{s}\in(0,\breve{\epsilon}_0), \quad \theta\in[\breve{\theta}_1,\breve{\theta}_2], \quad \mathbf{e}\in[0,1], \\ \left\|P\mathcal{X}_{\mathcal{D}^\epsilon_{\omega,f}/s}, \lambda^{\mathbf{e}}P\mathcal{X}_{\mathcal{D}^\epsilon_{\omega,m}/s}\right\|_{L^2(B_1(0))}, \left\|\frac{1}{\breve{\epsilon}_0}V, \mathbf{K}_{\omega,\lambda^{\mathbf{e}-2},\epsilon,s}V\right\|_{L^{2,q}(B_1(0)\cap\mathcal{D}/s)} \le 1, \\ \left\|\frac{1}{\breve{\epsilon}_0}G, \mathbf{K}_{\omega,\lambda^{\mathbf{e}-2},\epsilon,s}G\right\|_{L^{2,(q-2)+}(B_1(0)\cap\mathcal{D}/s)} \le 1, \end{cases}$$

then

$$\begin{cases} \displaystyle⨍_{B_\theta(0)\cap\mathcal{D}/s}\left|\Pi_{\omega,\epsilon/s}P|_{\mathcal{D}^\epsilon_{\omega,f}/s}\right|^2 dx \le \theta^{2\mu}, \\ \displaystyle⨍_{B_\theta(0)\cap\mathcal{D}^\epsilon_{\omega,m}/s}\lambda^{2\mathbf{e}}|P|^2 dx \le \theta^{2\mu}, \end{cases} \tag{5.19}$$

where $q = n-2+2\mu$. See Remark 3.2 for $\mathbf{d}_5$ and (A3) for β,γ.

Proof. Consider the following uniform elliptic equation

$$\begin{cases} -\nabla\cdot(\mathcal{K}_{\lambda_*}\nabla P) = 0 & \text{in } B_{3/5}(0)\cap\mathcal{D}/s_*, \\ P = 0 & \text{on } B_{3/5}(0)\cap\partial\mathcal{D}/s_*, \end{cases} \tag{5.20}$$

where $\mathcal{K}_{\lambda_*}\in\mathcal{M}_2(\frac{\beta}{\mathbf{d}_5^2}, \frac{\gamma^2}{\beta|Y_f|})$ and $s_*\in[0,1]$. By Theorem 8.29 [17], there is a constant $\sigma(<1/2)$ depending on $\beta,\gamma,\mathbf{d}_5,\partial\mathcal{D}$ such that any solution P of (5.20) satisfies

$$\|P\|_{C^{0,2\sigma}(B_{1/2}(0)\cap\mathcal{D}/s_*)} \le c\|P\|_{L^2(B_{3/5}(0)\cap\mathcal{D}/s_*)}, \tag{5.21}$$

where c is a constant depending on $\beta,\gamma,\mathbf{d}_5,\partial\mathcal{D}$ but independent of s_*. There is a sufficiently small number $\hat{\theta}$ (depending on $\beta,\gamma,\mathbf{d}_5,\partial\mathcal{D}$) such that, by (5.21),

$$⨍_{B_\theta(0)\cap\mathcal{D}/s_*} P^2 dx \le \theta^{2\sigma}⨍_{B_{3/5}(0)\cap\mathcal{D}/s_*} P^2 dx \qquad \text{for } \theta\le\hat{\theta}. \tag{5.22}$$

Fix a $\mu<\sigma$ and $\breve{\theta}_1,\breve{\theta}_2\in(0,\hat{\theta})$ such that $\breve{\theta}_1<\breve{\theta}_2^2$.

We claim $(5.19)_1$. If not, there is a sequence $\{\omega_\epsilon, \lambda_\epsilon, s_\epsilon, \theta_\epsilon, P_\epsilon, V_\epsilon, G_\epsilon\}$ satisfying (5.18) and

$$\begin{cases} \omega_\epsilon \in \Omega, \quad \frac{\epsilon}{s_\epsilon} \to 0, \quad \lambda_\epsilon, s_\epsilon \to \lambda_*, s_* \in [0,1], \quad \theta_\epsilon \to \theta_* \in [\breve{\theta}_1, \breve{\theta}_2], \\ \|P_\epsilon \mathcal{X}_{\mathcal{D}^\epsilon_{\omega_\epsilon,f}/s_\epsilon}, \lambda_\epsilon^{\mathbf{e}} P_\epsilon \mathcal{X}_{\mathcal{D}^\epsilon_{\omega_\epsilon,m}/s_\epsilon}\|_{L^2(B_1(0))} \le 1, \\ \|\mathbf{K}_{\omega_\epsilon, \lambda_\epsilon^{\mathbf{e}-2}, \epsilon, s_\epsilon} V_\epsilon\|_{L^{2,q}(B_1(0)\cap\mathcal{D}/s_\epsilon)} \le 1, \\ \|\mathbf{K}_{\omega_\epsilon, \lambda_\epsilon^{\mathbf{e}-2}, \epsilon, s_\epsilon} G_\epsilon\|_{L^{2,(q-2)+}(B_1(0)\cap\mathcal{D}/s_\epsilon)} \le 1, \\ \lim\limits_{\epsilon/s_\epsilon\to 0} \|V_\epsilon\|_{L^{2,q}(B_1(0)\cap\mathcal{D}/s_\epsilon)} + \|G_\epsilon\|_{L^{2,(q-2)+}(B_1(0)\cap\mathcal{D}/s_\epsilon)} = 0, \\ ⨍_{B_{\theta_\epsilon}(0)\cap\mathcal{D}/s_\epsilon} \left|\Pi_{\omega_\epsilon,\epsilon/s_\epsilon} P_\epsilon|_{\mathcal{D}^\epsilon_{\omega_\epsilon,f}/s_\epsilon}\right|^2 dx > |\theta_\epsilon|^{2\mu}. \end{cases} \tag{5.23}$$

By Lemma 3.5, there is a subsequence (same notation for subsequence) such that

$$\begin{cases} \Pi_{\omega_\epsilon,\epsilon/s_\epsilon} P_\epsilon|_{\mathcal{D}^\epsilon_{\omega_\epsilon,f}/s_\epsilon} \xrightarrow{H^1(B_{3/4}(0))\cap\mathcal{D}/s_* \ weakly} P \\ \mathbf{K}_{\omega_\epsilon,\lambda_\epsilon^2,\epsilon,s_\epsilon} \nabla P_\epsilon \mathcal{X}_{\mathcal{D}/s_\epsilon} \xrightarrow{L^2(B_{3/4}(0))\cap\mathcal{D}/s_* \ weakly} \mathcal{K}_{\lambda_*}\nabla P \end{cases} \text{as } \epsilon/s_\epsilon \to 0, \tag{5.24}$$

where $\mathcal{K}_{\lambda_*} \in \mathcal{M}_2(\frac{\beta}{\mathbf{d}_5^2}, \frac{\gamma^2}{\beta|Y_f|})$. The P in (5.24) satisfies (5.20). By (5.22)–(5.24), we conclude

$$\begin{aligned} |\theta_*|^{2\mu} &= \lim_{\epsilon/s_\epsilon\to 0} |\theta_\epsilon|^{2\mu} \le \lim_{\epsilon/s_\epsilon\to 0} ⨍_{B_{\theta_\epsilon}(0)\cap\mathcal{D}/s_\epsilon} \left|\Pi_{\omega_\epsilon,\epsilon/s_\epsilon} P_\epsilon|_{\mathcal{D}^\epsilon_{\omega_\epsilon,f}/s_\epsilon}\right|^2 dx \\ &= ⨍_{B_{\theta_*}(0)\cap\mathcal{D}/s_*} P^2 dx \le |\theta_*|^{2\sigma} ⨍_{B_{3/5}(0)\cap\mathcal{D}/s_*} P^2 dx. \end{aligned} \tag{5.25}$$

But (5.25) is impossible if we take $\breve{\theta}_2$ small enough. Therefore, there is a $\breve{\epsilon}_0$ such that $(5.19)_1$ holds for $\epsilon/s \le \breve{\epsilon}_0$. Clearly, $\breve{\epsilon}_0$ can be chosen so that $\breve{\epsilon}_0 < \epsilon_0$ (see Lemma 5.1 for ϵ_0). The proof of $(5.19)_2$ is similar to that of $(5.3)_2$, so we skip it. □

Lemma 5.5. *There exist $\mu, \breve{\theta}_1, \breve{\theta}_2 \in (0,1)$ (depending on $\beta, \gamma, \mathbf{d}_5, \partial\mathcal{D}$) satisfying $\breve{\theta}_1 < \breve{\theta}_2^2$ and there is a $\breve{\epsilon}_0 > 0$ (depending on $\breve{\theta}_1, \mu, \beta, \gamma, \mathbf{d}_5, \partial\mathcal{D}$) satisfying $\breve{\epsilon}_0 < \epsilon_0$ (ϵ_0 is that in Lemma 5.2) such that if $\omega \in \Omega$, $\lambda \in (0,1]$, $\epsilon \in (0, \breve{\epsilon}_0)$, $\theta \in [\breve{\theta}_1, \breve{\theta}_2]$, $\mathbf{e} \in [0,1]$, and k satisfying $\epsilon/\theta^k \le \breve{\epsilon}_0$, then any solution of*

$$\begin{cases} -\nabla\cdot(\mathbf{K}_{\omega,\lambda^2,\epsilon}\nabla U + Q) = F & \text{in } B_1(0)\cap\mathcal{D} \\ U = 0 & \text{on } B_1(0)\cap\partial\mathcal{D} \end{cases} \tag{5.26}$$

satisfies

$$\begin{cases} ⨍_{B_{\theta^k}(0)\cap\mathcal{D}} \left|\Pi_{\omega,\epsilon} U|_{\mathcal{D}^\epsilon_{\omega,f}}\right|^2 dx \le \theta^{2k\mu} |\breve{J}_{\lambda,\epsilon,\omega,\mathbf{e}}|^2, \\ ⨍_{B_{\theta^k}(0)\cap\mathcal{D}^\epsilon_{\omega,m}} \lambda^{2\mathbf{e}} \left|U\right|^2 dx \le \theta^{2k\mu} |\breve{J}_{\lambda,\epsilon,\omega,\mathbf{e}}|^2, \end{cases} \tag{5.27}$$

where $\breve{J}_{\lambda,\epsilon,\omega,\mathbf{e}} \equiv \left\|U\mathcal{X}_{\mathcal{D}^\epsilon_{\omega,f}}, \lambda^{\mathbf{e}} U \mathcal{X}_{\mathcal{D}^\epsilon_{\omega,m}}\right\|_{L^2(B_1(0))} + \|\frac{1}{\breve{\epsilon}_0} Q, \mathbf{K}_{\omega,\lambda^{\mathbf{e}-2},\epsilon} Q\|_{L^{2,q}(B_1(0)\cap\mathcal{D})} + \|\frac{1}{\breve{\epsilon}_0} F, \mathbf{K}_{\omega,\lambda^{\mathbf{e}-2},\epsilon} F\|_{L^{2,(q-2)+}(B_1(0)\cap\mathcal{D})}$ *and* $q = n - 2 + 2\mu$.

Proof. The proof is similar to that of Lemma 5.2 and is done by induction on k. For $k=1$, (5.27) is deduced from Lemma 5.4 with $s=1$. Suppose (5.27) holds for some k with $\epsilon/\theta^k \le \breve{\epsilon}_0$, we define

$$\begin{cases} P(x) \equiv \frac{1}{\theta^{k\mu}} U(\theta^k x)/\breve{J}_{\lambda,\epsilon,\omega,\mathbf{e}} \\ V(x) \equiv \frac{1}{\theta^{k(\mu-1)}} Q(\theta^k x)/\breve{J}_{\lambda,\epsilon,\omega,\mathbf{e}} \\ G(x) \equiv \frac{1}{\theta^{k(\mu-2)}} F(\theta^k x)/\breve{J}_{\lambda,\epsilon,\omega,\mathbf{e}} \end{cases} \quad \text{in } (B_1(0)\cap\mathcal{D}/\theta^k)\setminus\Gamma^\epsilon_\omega/\theta^k.$$

Then they satisfies

$$\begin{cases} -\nabla\cdot(\mathbf{K}_{\omega,\lambda^2,\epsilon,\theta^k}\nabla P + V) = G & \text{in } B_1(0)\cap\mathcal{D}/\theta^k, \\ P = 0 & \text{on } B_1(0)\cap\partial\mathcal{D}/\theta^k. \end{cases}$$

Following the argument of Lemma 5.2 and employing Lemma 5.4 with $s=\theta^k$, we obtain (5.27) with $k+1$ in place of k. □

Lemma 5.6. *Let μ, q be from Lemma 5.5. There is a constant $\breve{\epsilon}_* \in (0,1)$ (depending on $\mu,\beta,\gamma,\mathbf{d}_5,\partial\mathcal{D}$) such that if $\omega\in\Omega$, $\lambda\in(0,1]$, $\epsilon\in(0,\breve{\epsilon}_*]$, and $\mathbf{e}\in[0,1]$, then any solution of (5.26) satisfies*

$$[U]_{C^{0,\mu}(B_{1/2}(0)\cap\mathcal{D}^\epsilon_{\omega,f})} + \lambda^{\mathbf{e}}[U]_{C^{0,\mu}(B_{1/2}(0)\cap\mathcal{D}^\epsilon_{\omega,m})} \le c\hat{J}_{\lambda,\epsilon,\omega,\mathbf{e}}, \tag{5.28}$$

where c is a constant independent of $\lambda,\epsilon,\omega,\mathbf{e}$. Here

$$\begin{aligned} \hat{J}_{\lambda,\epsilon,\omega,\mathbf{e}} \equiv & \big\|U\mathcal{X}_{\mathcal{D}^\epsilon_{\omega,f}}, \lambda^{\mathbf{e}}U\mathcal{X}_{\mathcal{D}^\epsilon_{\omega,m}}\big\|_{L^2(B_1(0))} + \big\|\frac{1}{\breve{\epsilon}_*}Q, \mathbf{K}_{\omega,\lambda^{\mathbf{e}-2},\epsilon}Q\big\|_{L^{2,q}(B_1(0)\cap\mathcal{D})} \\ & + \big\|\frac{1}{\breve{\epsilon}_*}F, \mathbf{K}_{\omega,\lambda^{\mathbf{e}-2},\epsilon}F\big\|_{L^{2,(q-2)_+}(B_1(0)\cap\mathcal{D})}. \end{aligned} \tag{5.29}$$

Proof. Let $\breve{\theta}_1, \breve{\theta}_2, \breve{\epsilon}_0, \breve{J}_{\lambda,\epsilon,\omega,\mathbf{e}}$ be same as those in Lemma 5.5, set $\breve{\epsilon}_* \equiv \min\{\breve{\epsilon}_0\breve{\theta}_2/3, \epsilon_*\}$ where ϵ_* is the one in Lemma 5.3, and let $\epsilon \le \breve{\epsilon}_*$. Denote by c a constant independent of $\lambda,\epsilon,\omega,\mathbf{e}$. For any $x\in B_{\breve{\theta}_2/3}(0)\cap\mathcal{D}$, define $\xi(x)\equiv|x-x_0|$ where $x_0\in\partial\mathcal{D}$ satisfying $|x-x_0| = \min_{y\in\partial\mathcal{D}}|x-y|$. Then we have case (1) $\xi(x) > \frac{2\epsilon}{3\breve{\epsilon}_0}$ or case (2) $\xi(x)\le\frac{2\epsilon}{3\breve{\epsilon}_0}$.

Let us consider case (1). Because of $\breve{\theta}_1 < \breve{\theta}_2^2$, for any $r\in[\epsilon/\breve{\epsilon}_0, \breve{\theta}_2]$, there are $\theta\in[\breve{\theta}_1,\breve{\theta}_2]$ and $k\in\mathbb{N}$ satisfying $r=\theta^k$. Since $\xi(x)\in[\frac{2\epsilon}{3\breve{\epsilon}_0}, \frac{\breve{\theta}_2}{3}]$, by Lemma 5.5,

$$\begin{cases} \displaystyle\fint_{B_r(x_0)\cap\mathcal{D}} \Big|\Pi_{\omega,\epsilon}U|_{\mathcal{D}^\epsilon_{\omega,f}}\Big|^2 dy \le r^{2\mu}|\breve{J}_{\lambda,\epsilon,\omega,\mathbf{e}}|^2 \\ \displaystyle\fint_{B_r(x_0)\cap\mathcal{D}^\epsilon_{\omega,m}} \lambda^{2\mathbf{e}}|U|^2 dy \le r^{2\mu}|\breve{J}_{\lambda,\epsilon,\omega,\mathbf{e}}|^2 \end{cases} \quad \text{for } r\in[\tfrac{3}{2}\xi(x),\breve{\theta}_2].$$

So, for $s\in[\frac{\xi(x)}{2}, \frac{\breve{\theta}_2}{3}]$,

$$\begin{cases} \displaystyle\fint_{B_s(x)\cap\mathcal{D}} \Big|\Pi_{\omega,\epsilon}U|_{\mathcal{D}^\epsilon_{\omega,f}} - (\Pi_{\omega,\epsilon}U|_{\mathcal{D}^\epsilon_{\omega,f}})_{x,s}\Big|^2 dy \le cs^{2\mu}|\breve{J}_{\lambda,\epsilon,\omega,\mathbf{e}}|^2, \\ \displaystyle\fint_{B_s(x)\cap\mathcal{D}^\epsilon_{\omega,m}} \lambda^{2\mathbf{e}}\Big|U - (\Pi_{\omega,\epsilon}U|_{\mathcal{D}^\epsilon_{\omega,f}})_{x,s}\Big|^2 dy \le cs^{2\mu}|\breve{J}_{\lambda,\epsilon,\omega,\mathbf{e}}|^2. \end{cases} \tag{5.30}$$

Next we shift the coordinate system such that x is located at the origin and we define

$$\begin{cases} P(y) \equiv \frac{1}{\xi^{\mu}(x)}\big(U(\xi(x)y) - (\Pi_{\omega,\epsilon}U|_{\mathcal{D}^{\epsilon}_{\omega,f}})_{x,\xi(x)}\big)/\hat{J}_{\lambda,\epsilon,\omega,\mathbf{e}} \\ V(y) \equiv \frac{1}{\xi^{\mu-1}(x)}Q(\xi(x)y)/\hat{J}_{\lambda,\epsilon,\omega,\mathbf{e}} \\ G(y) \equiv \frac{1}{\xi^{\mu-2}(x)}F(\xi(x)y)/\hat{J}_{\lambda,\epsilon,\omega,\mathbf{e}} \end{cases} \quad \text{in } B_1(x) \setminus \Gamma^{\epsilon}_{\omega}/\xi(x).$$

See (5.29) for $\hat{J}_{\lambda,\epsilon,\omega,\mathbf{e}}$. Then these functions satisfy

$$-\nabla\cdot(\mathbf{K}_{\omega,\lambda^2,\epsilon/\xi(x)}\nabla P + V) = G \quad \text{in } B_1(x). \tag{5.31}$$

Take $s = \xi(x) < 1$ in (5.30) to see

$$\begin{aligned}&\big\|P\mathcal{X}_{\mathcal{D}^{\epsilon}_{\omega,f}/\xi(x)}, \lambda^{\mathbf{e}}P\mathcal{X}_{\mathcal{D}^{\epsilon}_{\omega,m}/\xi(x)}\big\|_{L^2(B_1(x))} + \|\frac{1}{\breve{\epsilon}_0}V, \mathbf{K}_{\omega,\lambda^{\mathbf{e}-2},\epsilon/\xi(x)}V\|_{L^{2,q}(B_1(x))} \\ &+\|\frac{1}{\breve{\epsilon}_0}G, \mathbf{K}_{\omega,\lambda^{\mathbf{e}-2},\epsilon/\xi(x)}G\|_{L^{2,(q-2)+}(B_1(x))} \le c.\end{aligned}$$

Apply Lemma 5.3 to (5.31) to obtain

$$[P]_{C^{0,\mu}(B_{1/2}(x)\cap\mathcal{D}^{\epsilon}_{\omega,f}/\xi(x))} + \lambda^{\mathbf{e}}[P]_{C^{0,\mu}(B_{1/2}(x)\cap\mathcal{D}^{\epsilon}_{\omega,m}/\xi(x))} \le c. \tag{5.32}$$

Which implies, for $s < \frac{\xi(x)}{2}$,

$$\fint_{B_s(x)} \Big|\Pi_{\omega,\epsilon}U|_{\mathcal{D}^{\epsilon}_{\omega,f}} - (\Pi_{\omega,\epsilon}U|_{\mathcal{D}^{\epsilon}_{\omega,f}})_{x,s}\Big|^2 dy \le cs^{2\mu}|\hat{J}_{\lambda,\epsilon,\omega,\mathbf{e}}|^2. \tag{5.33}$$

Next we consider case (2). Because of $\breve{\theta}_1 < \breve{\theta}_2^2$, for any $r \in [\epsilon/\breve{\epsilon}_0, \breve{\theta}_2]$, there are $\theta \in [\breve{\theta}_1, \breve{\theta}_2]$ and $k \in \mathbb{N}$ satisfying $r = \theta^k$. By Lemma 5.5,

$$\begin{cases} \fint_{B_r(x_0)\cap\mathcal{D}} \Big|\Pi_{\omega,\epsilon}U|_{\mathcal{D}^{\epsilon}_{\omega,f}}\Big|^2 dy \le cr^{2\mu}|\breve{J}_{\lambda,\epsilon,\omega,\mathbf{e}}|^2 \\ \fint_{B_r(x_0)\cap\mathcal{D}^{\epsilon}_{\omega,m}} \lambda^{2\mathbf{e}}|U|^2 dy \le cr^{2\mu}|\breve{J}_{\lambda,\epsilon,\omega,\mathbf{e}}|^2 \end{cases} \quad \text{for } r \in [\epsilon/\breve{\epsilon}_0, \breve{\theta}_2]. \tag{5.34}$$

This implies, for $s \in [\frac{\epsilon}{3\breve{\epsilon}_0}, \frac{\breve{\theta}_2}{3}]$,

$$\begin{cases} \fint_{B_s(x)\cap\mathcal{D}} \Big|\Pi_{\omega,\epsilon}U|_{\mathcal{D}^{\epsilon}_{\omega,f}} - (\Pi_{\omega,\epsilon}U|_{\mathcal{D}^{\epsilon}_{\omega,f}})_{x,s}\Big|^2 dy \le cs^{2\mu}|\breve{J}_{\lambda,\epsilon,\omega,\mathbf{e}}|^2, \\ \fint_{B_s(x)\cap\mathcal{D}^{\epsilon}_{\omega,m}} \lambda^{2\mathbf{e}}\Big|U - (\Pi_{\omega,\epsilon}U|_{\mathcal{D}^{\epsilon}_{\omega,f}})_{x,s}\Big|^2 dy \le cs^{2\mu}|\breve{J}_{\lambda,\epsilon,\omega,\mathbf{e}}|^2. \end{cases} \tag{5.35}$$

Again we shift the coordinate system such that x is located at the origin. Define

$$\begin{cases} P(y) \equiv \frac{1}{\epsilon^{\mu}}\big(U(\epsilon y) - (\Pi_{\omega,\epsilon}U|_{\mathcal{D}^{\epsilon}_{\omega,m}})_{x,\epsilon/\breve{\epsilon}_0}\big)/\hat{J}_{\lambda,\epsilon,\omega,\mathbf{e}} \\ V(y) \equiv \frac{1}{\epsilon^{\mu-1}}Q(\epsilon y)/\hat{J}_{\lambda,\epsilon,\omega,\mathbf{e}} \\ G(y) \equiv \frac{1}{\epsilon^{\mu-2}}F(\epsilon y)/\hat{J}_{\lambda,\epsilon,\omega,\mathbf{e}} \end{cases} \quad \text{in } (B_{1/\breve{\epsilon}_0}(x)\cap\mathcal{D}/\epsilon)\setminus\Gamma^{\epsilon}_{\omega}/\epsilon,$$

and define

$$P_b \equiv \frac{-1}{\epsilon^{\mu}}\big(\Pi_{\omega,\epsilon}U|_{\mathcal{D}^{\epsilon}_{\omega,f}}\big)_{x,\epsilon/\breve{\epsilon}_0}/\hat{J}_{\lambda,\epsilon,\omega,\mathbf{e}} \quad \text{in } B_{\frac{1}{\breve{\epsilon}_0}}(x)\cap\mathcal{D}/\epsilon.$$

By $(5.34)_1$, P_b is a constant independent of $\lambda, \epsilon, \omega, \mathbf{e}$. Then these functions satisfy

$$\begin{cases} -\nabla \cdot (\mathbf{K}_{\omega,\lambda^2,\epsilon,\epsilon} \nabla P + V) = G & \text{in } B_{\frac{1}{\breve{\epsilon}_0}}(x) \cap \mathcal{D}/\epsilon, \\ P = P_b & \text{on } B_{\frac{1}{\breve{\epsilon}_0}}(x) \cap \partial\mathcal{D}/\epsilon. \end{cases}$$

Take $s = \frac{\epsilon}{\breve{\epsilon}_0}$ in (5.35) to see, by $(5.34)_1$,

$$\begin{aligned} &\left\| P\mathcal{X}_{\mathcal{D}^\epsilon_{\omega,f}/\epsilon}, \lambda^{\mathbf{e}} P\mathcal{X}_{\mathcal{D}^\epsilon_{\omega,m}/\epsilon} \right\|_{L^2(B_{1/\breve{\epsilon}_0}(x))} + \left\| \mathbf{K}_{\omega,\lambda^{\mathbf{e}-2},\epsilon,\epsilon} V \right\|_{L^{2,q}(B_{1/\breve{\epsilon}_0}(x)\cap\mathcal{D}/\epsilon)} \\ &\quad + \left\| \mathbf{K}_{\omega,\lambda^{\mathbf{e}-2},\epsilon,\epsilon} G \right\|_{L^{2,(q-2)+}(B_{1/\breve{\epsilon}_0}(x)\cap\mathcal{D}/\epsilon)} + \|P_b\|_{C^1(B_{1/\breve{\epsilon}_0}(x)\cap\mathcal{D}/\epsilon)} \le c. \end{aligned}$$

By Lemmas 4.6, 4.7,

$$[P]_{C^{0,\mu}(B_{1/2\breve{\epsilon}_0}(x)\cap\mathcal{D}^\epsilon_{\omega,f}/\epsilon)} + \lambda^{\mathbf{e}}[P]_{C^{0,\mu}(B_{1/2\breve{\epsilon}_0}(x)\cap\mathcal{D}^\epsilon_{\omega,m}/\epsilon)} \le c. \tag{5.36}$$

(5.36) imply $(5.35)_1$ holds for $s \le \frac{\epsilon}{2\breve{\epsilon}_0}$.

The Hölder estimate of $\Pi_{\omega,\epsilon} U|_{\mathcal{D}^\epsilon_{\omega,f}}$ in $B_{1/2}(0) \cap \mathcal{D}$ follows from $(5.30)_1$, (5.33), $(5.35)_1$, (5.36), and Theorem 1.2 in page 70 [16]. The Hölder estimate of U in $B_{1/2}(0) \cap \mathcal{D}^\epsilon_{\omega,m}$ is from (5.32), (5.36), triangle inequality, and the Hölder estimate of U in $B_{1/2}(0) \cap \mathcal{D}^\epsilon_{\omega,f}$. □

Remark 5.2. Let $\breve{\epsilon}_*$ be same as that in Lemma 5.6. By Lemmas 4.6, 4.7, we know that if $\omega \in \Omega$, $\lambda \in (0, 1]$, $\epsilon \in [\breve{\epsilon}_*, 1]$, and $\mathbf{e} \in [0, 1]$, any solution of (5.26) satisfies (5.28). Together with Lemma 5.6, any solution of (5.26) satisfies (5.28) if $\omega \in \Omega$, $\lambda, \epsilon \in (0, 1]$, and $\mathbf{e} \in [0, 1]$.

By energy method, partition of unity, Remark 5.1, Remark 5.2, and Poincaré inequality [17], we obtain Theorem 2.1.

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