Uniform bound and convergence for elliptic homogenization problems

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Uniform bound and convergence for the solutions of elliptic homogenization problems are concerned. The problem domain has periodic microstructure as well as consists of a connected subregion with high permeability and a disconnected matrix block subset with low permeability. Let $\omega \in (0,1)$ denote the size ratio of the period to the whole domain and let $\epsilon^2 \in (0,1)$ denote the permeability ratio of the disconnected matrix block subset to the connected subregion. For elliptic equations in this domain, the elliptic solutions are smooth in the connected subregion but change rapidly in the disconnected matrix block subset. Indeed, the elliptic solutions in the connected subregion can be bounded uniformly in ϵ, ω in Hölder norm, but not for the solutions in the matrix block subset. It is known that the elliptic solutions converge to a solution of some homogenized elliptic equation as ϵ, ω converge to 0. Here the L^p convergence rate for $p \in (2, \infty]$ is derived. Depending on strongly coupled or weakly coupled case, the convergence rate of the elliptic solutions is related to the convergence rate of $\epsilon, \omega, \epsilon/\omega$ for the former and to the convergence rate of ϵ, ω for the latter as ϵ, ω converge to 0.

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1. Introduction

Uniform bound and convergence for the solutions of elliptic homogenization problems are presented. The problems have applications in contaminant transport in the subsurface, heat transfer in two-phase media, the stress in composite materials, and so on (see [3, 7, 12, 13]). The problem domain $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ contains two subsets, a periodic connected subregion with high permeability and a periodic disconnected matrix block subset with low permeability. Let $\omega \in (0,1)$ be a parameter, $Y \equiv (0,1)^n$ consist of a sub-domain Y_m completely surrounded by another connected sub-domain Y_f $(\equiv Y \setminus \overline{Y_m})$, $\Omega(\omega) \equiv \{x \in \Omega | dist(x, \partial\Omega) > \omega\}$, $\Omega_m^\omega \equiv \{x | x \in \omega(Y_m + j) \subset \Omega(\omega)$ for some $j \in \mathbb{Z}^n\}$ be the disconnected matrix block subset of Ω, $\Omega_f^\omega \equiv \Omega \setminus \overline{\Omega_m^\omega}$ be the connected subregion of Ω, and $\partial\Omega$ (resp. $\partial\Omega_m^\omega$) be the boundary of Ω (resp. Ω_m^ω). The problem that we consider is

$$\begin{cases} -\nabla \cdot (\mathbf{E}_{\epsilon^2,\omega} \mathbf{K}_\omega \nabla \Psi_{\epsilon,\omega} + V_{\epsilon,\omega}) + \mathbf{T}_{\epsilon,\omega} \Psi_{\epsilon,\omega} = G_{\epsilon,\omega} & \text{in } \Omega, \\ \Psi_{\epsilon,\omega} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\epsilon, \omega \in (0,1)$, $\mathbf{E}_{\nu,\omega} \equiv \begin{cases} 1 & \text{in } \Omega_f^\omega \\ \nu & \text{in } \Omega_m^\omega \end{cases}$ for any $\nu > 0$, $\mathbf{K}_\omega(x) \equiv \mathbf{K}(\frac{x}{\omega})$, $\mathbf{K}$ is a positive periodic function in $\mathbb{R}^n$ with period Y, $\mathbf{T}_{\epsilon,\omega}$ is a non-negative function, and $V_{\epsilon,\omega}, G_{\epsilon,\omega}$ are given functions. It is known that if $\mathbf{K}_\omega, \mathbf{T}_{\epsilon,\omega}, V_{\epsilon,\omega}, G_{\epsilon,\omega}$ are smooth in $\Omega_f^\omega \cup \Omega_m^\omega$, a piecewise smooth solution of (1.1) exists uniquely [14]. The H^1 norm of the solution in the high permeability subregion Ω_f^ω is bounded uniformly in ϵ, ω when $V_{\epsilon,\omega}, G_{\epsilon,\omega}$ are small in Ω_m^ω. However, that may not be the case for the solution in the low permeability subset Ω_m^ω (see Remark 2.2). Also the second order derivatives of the solution of (1.1) may not be bounded uniformly in ϵ, ω in the high permeability subregion Ω_f^ω even when $V_{\epsilon,\omega}, G_{\epsilon,\omega}$ are bounded uniformly in ϵ, ω and are small in Ω_m^ω (see Remark 3.1). By homogenization theory (see [6, 12, 18]), if ϵ, ω become small, the solution of (1.1) approaches to a solution of some homogenized elliptic differential equation. It seems that, if ϵ, ω are small, the solution of the homogenized elliptic differential equation is a good approximation of the solution of (1.1). We shall see in section 2 that the solution of (1.1) can be approximated by the solution of the homogenized elliptic differential equation plus the solutions of some mutually independent local problems.

Lipschitz estimate and $W^{2,p}$ estimate for the solutions of the uniform elliptic equations with discontinuous coefficients had been considered in [15, 16]. For the uniform elliptic case of (1.1) (that is, $\epsilon = 1$), uniform bound and convergence results were also studied. For example, uniform Hölder, $W^{1,p}$, and Lipschitz estimates in ω for uniform elliptic case of (1.1) with Hölder coefficients were proved in [4, 5]. Uniform $W^{1,p}$ estimate in ω for uniform elliptic case of (1.1) with continuous coefficients was shown in [8] and the same problem with VMO coefficients could be found in [20]. Uniform Lipschitz estimate in ω for the Laplace equation in periodic perforated domains was studied in [19]. By [6, 12, 18], the solution of uniform elliptic case of (1.1) with Dirichlet boundary condition converges to a solution of some homogenized elliptic equation with convergence rate ω in L^2 norm and with convergence rate $\omega^{1/2}$ in H^1 norm as ω closes to 0. In this work, we consider the non-uniform elliptic case of (1.1) with discontinuous coefficients. We derive uniform Hölder estimates in ϵ, ω for the solution of (1.1) as well as derive some L^p convergence estimates for $p \in (2, \infty]$ for the approximation of the solution of (1.1).

The rest of the work is organized as follows: Notation and main results are stated in section 2. In section 3, we derive a priori uniform estimates for interface problems. Uniform Hölder estimates for the non-uniform elliptic solutions in heterogeneous media are considered in section 4. L^p convergence estimates for elliptic homogenization problems are presented in section 5.

2. Notation and main result

If $D \subset \mathbb{R}^n$ is a set, $\overline{D}$ denotes the closure of the set D, $\mathcal{X}_D$ is the characteristic function on D, $|D|$ is the volume of D, ∂D is the boundary of D, and $D/r = \{x | rx \in D\}$ for $r > 0$. $B_r(x)$ denotes a ball centered at x with radius $r > 0$. If $\mathbf{B}_1, \mathbf{B}_2$

are Banach spaces, $\|\varphi_1, \cdots, \varphi_m\|_{\mathbf{B}_1} \equiv \|\varphi_1\|_{\mathbf{B}_1} + \cdots + \|\varphi_m\|_{\mathbf{B}_1}$ and $\|\varphi\|_{\mathbf{B}_1 \cap \mathbf{B}_2} \equiv \|\varphi\|_{\mathbf{B}_1} + \|\varphi\|_{\mathbf{B}_2}$. Let $C^{k,\alpha}$ denote the Hölder space with norm $\|\cdot\|_{C^{k,\alpha}}$, $W^{s,p}$ denote the Sobolev space with norm $\|\cdot\|_{W^{s,p}}$, and $[\varphi]_{C^{0,\alpha}}$ be the Hölder semi-norm of φ for $k \geq 0$, $\alpha \in (0,1], s \geq -1, p \in [1,\infty]$ (see [2, 11]). $L^p(D) \equiv W^{0,p}(D), H^s(D) \equiv W^{s,2}(D)$ for $s \geq -1, p \in [1,\infty]$. $C_0^\infty(D)$ is the space of infinitely differentiable functions with support in D and $C_{per}^\infty(\mathbb{R}^n)$ is the space of infinitely differentiable Y-periodic functions in $\mathbb{R}^n$. $W_{per}^{s,p}(D)$ is the closure of $C_{per}^\infty(\mathbb{R}^n)$ under the $W^{s,p}$ norm and $\|\varphi\|_{W_{per}^{s,p}(D)} \equiv \|\varphi\|_{W^{s,p}(D\cap Y)}$ for $s \geq 0, p \in [1,\infty]$. $H_{per}^1(D) \equiv W_{per}^{1,2}(D)$ and $L_{per}^\infty(D) \equiv W_{per}^{0,\infty}(D)$. For $p \geq 2$, $W_0^{1,p}(D) \equiv \{\varphi \in W^{1,p}(D) |\ \varphi = 0 \text{ on } \partial D\}$ and $H_0^1(D) = W_0^{1,2}(D)$. For any $\varphi \in L^1(D)$ and $r > 0$,

$$(\varphi)_{x,r} \equiv \fint_{D\cap B_r(x)} \varphi(y)dy \equiv \frac{1}{|D \cap B_r(x)|} \int_{D\cap B_r(x)} \varphi(y)dy.$$

If $\vec{\mathbf{n}}$ is an outward normal vector on ∂Y_m, we define, for any function φ and $x \in \partial Y_m$,

$$\varphi_{,\pm}(x) \equiv \lim_{t\to 0^+} \varphi(x \pm t\vec{\mathbf{n}}), \qquad \lfloor \varphi \rfloor(x) = \varphi_{,+}(x) - \varphi_{,-}(x). \tag{2.1}$$

Similarly if $\vec{\mathbf{n}}^\omega$ is an outward normal vector on $\partial\Omega_m^\omega$, we define, for any $x \in \partial\Omega_m^\omega$,

$$\varphi_{,\pm}(x) \equiv \lim_{t\to 0^+} \varphi(x \pm t\vec{\mathbf{n}}^\omega), \qquad \lfloor \varphi \rfloor(x) = \varphi_{,+}(x) - \varphi_{,-}(x). \tag{2.2}$$

Next we recall an extension result in [1].

Remark 2.1. For any $\omega \in (0,1)$ and $p \in [1,\infty)$, there are a constant $\ell_1(Y_f, p)$ and a linear continuous extension operator $\Pi_\omega : W^{1,p}(\Omega_f^\omega) \to W^{1,p}(\Omega)$ such that
(1) if $\varphi \in W^{1,p}(\Omega_f^\omega)$, then

$$\begin{cases} \Pi_\omega\varphi = \varphi \quad \text{in } \Omega_f^\omega \text{ almost everywhere,} \\ \|\Pi_\omega\varphi\|_{L^p(\Omega)} \leq \ell_1\|\varphi\|_{L^p(\Omega_f^\omega)}, \\ \|\nabla\Pi_\omega\varphi\|_{L^p(\Omega)} \leq \ell_1\|\nabla\varphi\|_{L^p(\Omega_f^\omega)}, \\ \ell_2 \leq \Pi_\omega\varphi \leq \ell_3 \qquad \text{if } \varphi \in L^\infty(\Omega_f^\omega) \text{ and } \ell_2 \leq \varphi \leq \ell_3, \\ \Pi_\omega\varphi = \zeta \quad \text{in } \Omega \text{ if } \varphi = \zeta|_{\Omega_f^\omega} \text{ for some linear function } \zeta \text{ in } \Omega, \end{cases}$$

(2) if $r > 0$, $\omega/r < 1$, and $\zeta(x) \equiv \varphi(rx)$, then $\Pi_{\omega/r}\zeta(x) = (\Pi_\omega\varphi)(rx)$.

If $\varphi \in W^{1,p}(\Omega)$ for any $p \geq 1$, then $\Pi_\omega\varphi|_{\Omega_f^\omega} \in W^{1,p}(\Omega)$ denotes the extension function of $\varphi|_{\Omega_f^\omega} \in W^{1,p}(\Omega_f^\omega)$ in Ω. First we give a uniform Hölder estimate for the solution of (1.1) in strongly coupled case.

Theorem 2.1. *Suppose*

A1. $\Omega \subset \mathbb{R}^n$ *is a* $C^{2,1}$ *domain for* $n \in \{2,3\}$, Y_m *is a smooth simply-connected subdomain of* Y, $\overline{Y_m} \subset Y$,
A2. $\mathbf{K} \in H_{per}^1(\mathbb{R}^n)$ *is a positive function,* $\|\nabla\mathbf{K}\|_{L^\infty(Y)}$ *is small compared with* $\min_{x\in Y} \mathbf{K}(x)$, *and* $\mathbf{K} \in C^{1,\alpha}(\overline{Y_f}) \cap C^{1,\alpha}(\overline{Y_m})$ *for some* $\alpha \in (0,1)$,
A3. $\epsilon, \omega \in (0,1)$, $\delta \in (0,3)$, $V_{\epsilon,\omega}, G_{\epsilon,\omega} \in L^{n+\delta}(\Omega)$,
A4. $\mathbf{M}_0, \mathbf{M} > 0$, $\mathbf{T}_{\epsilon,\omega}(x) \in [\mathbf{M}_0, \mathbf{M}]$ *for all* $x \in \Omega$,

then a $H^1(\Omega)$ solution of (1.1) exists uniquely and there is a constant $\epsilon_0 \in (0,1)$ (depending on $\delta, \mathbf{K}, \mathbf{M}, Y_f, \Omega$) such that, for $\epsilon < \epsilon_0$ and $\frac{\epsilon}{\omega} > \ell_4 > 0$,

$$[\Psi_{\epsilon,\omega}]_{C^{0,\mu}(\overline{\Omega_f^\omega})} + \sup_{\substack{j\in\mathbb{Z}^n \\ \omega(Y_m+j)\subset\Omega_m^\omega}} \epsilon[\Psi_{\epsilon,\omega}]_{C^{0,\mu}(\omega(\overline{Y_m}+j))} \le c\|\mathbf{E}_{1/\epsilon,\omega}V_{\epsilon,\omega}, G_{\epsilon,\omega}\|_{L^{n+\delta}(\Omega)}, \tag{2.3}$$

where ℓ_4 is any number, $\mu \equiv \frac{\delta}{n+\delta}$, and c is a positive constant independent of ϵ, ω.

Next we give a uniform Hölder estimate for the solution of (1.1) in weakly coupled case.

Theorem 2.2. *Besides A1–A3, if*

A4'. $\mathbf{E}_{1/\epsilon,\omega}\mathbf{T}_{\epsilon,\omega}(x) \in [0,\mathbf{M}]$ *for all* $x \in \Omega$,

then a $H^1(\Omega)$ solution of (1.1) exists uniquely and there is a constant $\epsilon_0 \in (0,1)$ (depending on $\delta, \mathbf{K}, \mathbf{M}, Y_f, \Omega$) such that, for any $\epsilon < \epsilon_0$,

$$\begin{aligned}[][\Psi_{\epsilon,\omega}]_{C^{0,\mu}(\overline{\Omega_f^\omega})} &+ \sup_{\substack{j\in\mathbb{Z}^n \\ \omega(Y_m+j)\subset\Omega_m^\omega}} \epsilon^\lambda[\Psi_{\epsilon,\omega}]_{C^{0,\mu}(\omega(\overline{Y_m}+j))} \\ &\le c(\|V_{\epsilon,\omega}, G_{\epsilon,\omega}\|_{L^{n+\delta}(\Omega_f^\omega)} + \epsilon^{-1}\|V_{\epsilon,\omega}, \max\{\epsilon,\omega\}G_{\epsilon,\omega}\|_{L^{n+\delta}(\Omega_m^\omega)}),\end{aligned} \tag{2.4}$$

where $\mu \equiv \frac{\delta}{n+\delta}$ and c is a positive constant independent of ϵ, ω. In (2.4), λ is $\frac{3}{2}$ if $\mathbf{T}_{\epsilon,\omega} \neq 0$ and 1 if $\mathbf{T}_{\epsilon,\omega} = 0$.

Proofs of Theorem 2.1 and Theorem 2.2 are given in section 4. From Theorem 2.1 and Theorem 2.2, we know that if the right hand side of (2.3) or (2.4) is bounded, the Hölder norm of the solution of (1.1) in the connected high permeability region Ω_f^ω is bounded uniformly in ϵ, ω, but the solution in Ω_m^ω may change rapidly when ϵ, ω are small. This is different from uniform elliptic equation case, where the solution is bounded uniformly in the whole domain. Below is one example to show that the Hölder norm and the H^1 norm of the solution of (1.1) in Ω_m^ω in general are not be bounded uniformly in ϵ, ω.

Remark 2.2. Suppose $\varphi \in C^\infty_{per}(\mathbb{R}^n)$ and φ in the cell $Y \equiv (0,1)^n$ has support in Y_m, we define, for any $\omega \in (0,1)$,

$$\Psi_{\epsilon,\omega}(x) \equiv \begin{cases} 0 & \text{if } x \in \Omega_f^\omega, \\ \varphi(\frac{x}{\omega}) & \text{if } x \in \Omega_m^\omega. \end{cases}$$

Then $\Psi_{\epsilon,\omega}$ satisfies

$$\begin{cases} -\nabla\cdot(\mathbf{E}_{\epsilon^2,\omega}\nabla\Psi_{\epsilon,\omega}) = G_{\epsilon,\omega} & \text{in } \Omega, \\ \Psi_{\epsilon,\omega} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $G_{\epsilon,\omega}(x) = -\epsilon^2\omega^{-2}\Delta\varphi(\frac{x}{\omega})\mathcal{X}_{\Omega_m^\omega}$. Note $[\Psi_{\epsilon,\omega}]_{C^{0,\mu}(\omega(\overline{Y_m}+j))} = \omega^{-\mu}[\varphi]_{C^{0,\mu}(\overline{Y_m}+j)}$, $\|\nabla\Psi_{\epsilon,\omega}\|_{L^2(\Omega_m^\omega)} \approx \omega^{-1}\|\nabla\varphi\|_{L^2(Y_m)}$, and $\|G_{\epsilon,\omega}\|_{L^{n+\delta}(\Omega_m^\omega)} \approx \epsilon^2\omega^{-2}\|\Delta\varphi\|_{L^{n+\delta}(Y_m)}$ where $\delta > 0, \mu \equiv \frac{\delta}{n+\delta}$. Here $A \approx B$ means that A is almost like B times a constant when ω is small. If $\epsilon \le \omega < 1$, then the right hand side of (2.4) is finite and (2.4) holds for $\Psi_{\epsilon,\omega}$. But the $C^{0,\mu}$ norm and the H^1 norm of $\Psi_{\epsilon,\omega}$ in Ω_m^ω are not bounded uniformly in $\omega \in (0,1)$.

Set $\mathbb{E}_\nu \equiv \begin{cases} 1 & \text{in } Y_f \\ \nu & \text{in } Y_m \end{cases}$ for any $\nu > 0$, $\mathbb{A}_m \equiv \{x \in \mathbb{R}^n | x \in \cup_{j \in \mathbb{Z}^n}(Y_m + j)\}$, and $\mathbb{A}_f \equiv \mathbb{R}^n \setminus \overline{\mathbb{A}_m}$. For each $i \in \{1, 2, \cdots, n\}$, $\nu \in (0,1]$, and $\beta \in (0,\infty)$, we find $\mathbb{X}_\nu^{(i)} \in H^1_{per}(\mathbb{R}^n)$ satisfying

$$\begin{cases} -\nabla \cdot (\mathbb{E}_{\nu^2} \mathbf{K}(\nabla \mathbb{X}_\nu^{(i)} + \vec{e}_i)) = 0 & \text{in } Y, \\ \int_{Y_f} \mathbb{X}_\nu^{(i)}(y) dy = 0, \end{cases} \tag{2.5}$$

find $\mathbb{X}_0^{(i)} \in H^1_{per}(\mathbb{A}_f) \cap H^1_{per}(\mathbb{A}_m)$ satisfying

$$\begin{cases} -\nabla \cdot (\mathbf{K}(\nabla \mathbb{X}_0^{(i)} + \vec{e}_i)) = 0 & \text{in } Y_f, \\ \mathbb{X}_0^{(i)} = 0 & \text{in } Y_m, \\ \mathbf{K}(\nabla \mathbb{X}_0^{(i)} + \vec{e}_i)_{,+} \cdot \vec{\mathbf{n}} = 0 & \text{on } \partial Y_m, \\ \int_{Y_f} \mathbb{X}_0^{(i)}(y) dy = 0, \end{cases} \tag{2.6}$$

and find $\mathbb{W}_\beta \in H^1_{per}(\mathbb{A}_f) \cap H^1_{per}(\mathbb{A}_m)$ satisfying

$$\begin{cases} \beta^2 \nabla \cdot (\mathbf{K} \nabla \mathbb{W}_\beta) - \mathbf{T} \mathbb{W}_\beta = 0 & \text{in } Y_m, \\ \mathbb{W}_\beta = 1 & \text{on } \partial Y_m, \end{cases} \tag{2.7}$$

$$\begin{cases} \nabla \cdot (\mathbf{K} \nabla \mathbb{W}_\beta) - \mathbf{T} = \frac{-1}{|Y_f|} \left(\int_{Y_f} \mathbf{T} dy + \int_{Y_m} \mathbf{T} \mathbb{W}_\beta dy \right) & \text{in } Y_f, \\ \lfloor \mathbb{E}_{\beta^2} \mathbf{K} \nabla \mathbb{W}_\beta \rfloor \cdot \vec{\mathbf{n}} = 0 & \text{on } \partial Y_m, \\ \int_{Y_f} \mathbb{W}_\beta(y) dy = 0, \end{cases} \tag{2.8}$$

where $\vec{e}_i$ is a unit vector in the i-th coordinate direction, $|Y_f|$ is the volume of Y_f, and $\vec{\mathbf{n}}$ is an outward normal vector on ∂Y_m. See (2.1) for $(2.6)_3$ and $(2.8)_2$. Let $\mathbb{X}_{\nu,s}^{(i)}(x) \equiv s\mathbb{X}_\nu^{(i)}(\frac{x}{s})$ $\mathbb{X}_{\nu,s} \equiv (\mathbb{X}_{\nu,s}^{(1)}, \cdots, \mathbb{X}_{\nu,s}^{(n)})$, and $\mathbb{W}_{\beta,s,i}(x) \equiv s^i \mathbb{W}_\beta(\frac{x}{s})$ for any $\nu \in [0,1]$, $s \in (0,1)$, $\beta \in (0,\infty)$, $i \in \mathbb{Z}$. By Lax-Milgram Theorem [11], (2.5)–(2.8) are uniquely solvable. Denote by Ξ_ν for $\nu \in [0,1]$ a $n \times n$ matrix function whose (i,j)-component is $\partial_i \mathbb{X}_\nu^{(j)}$. By remark in pages 17-19, 94-95 [12],

$$\mathcal{K}_\nu \equiv \int_{Y_f \cup Y_m} \mathbb{E}_{\nu^2} \mathbf{K}(I + \Xi_\nu(y)) dy \qquad \text{for } \nu \in [0,1] \tag{2.9}$$

is a constant symmetric positive definite matrix. Here I is the identity matrix.

Next we state convergence results. If, in addition to A1–A4,

A5. $\mathbf{T}_{\epsilon,\omega}(x) = \mathbf{T}(\frac{x}{\omega})$ and $\mathbf{T} \in C^{0,\alpha}_{per}(\mathbb{R}^n)$ for some $\alpha > 0$,
A6. $\|\mathbf{E}_{1/\epsilon,\omega} G_{\epsilon,\omega}\|_{L^{n+\delta}(\Omega)} + \|G_{\epsilon,\omega}\|_{W^{1,n+\delta}(\Omega_f^\omega)}$ is bounded independent of ϵ, ω,

the solution of (1.1) with $V_{\epsilon,\omega}=0$ satisfies $\|\mathbf{E}_{\epsilon,\omega}\nabla\Psi_{\epsilon,\omega},\Psi_{\epsilon,\omega}\|_{L^2(\Omega)}\le c$ (independent of ϵ, ω). Suppose $\epsilon,\omega\to 0$ and $\frac{\epsilon}{\omega}\to\sigma\in[0,\infty]$, by compactness principle and by tracing the proof of Theorem 2.3 [3], we can extract a subsequence (same notation for subsequence) such that

$$\begin{cases}\mathbf{E}_{\epsilon^2,\omega}\mathbf{K}_\omega\nabla\Psi_{\epsilon,\omega}\to\mathcal{K}_0\nabla\Psi & \\ \mathbf{T}_{\epsilon,\omega}\Psi_{\epsilon,\omega}\to|Y_f|\mathcal{T}_\sigma\Psi & \text{in } L^2(\Omega)\text{ weakly,} \\ G_{\epsilon,\omega}\to|Y_f|\mathcal{G} & \end{cases}\tag{2.10}$$

where $\mathcal{K}_0$ is defined in (2.9) with $\nu=0$, $|Y_f|$ is the volume of Y_f, and

$$\mathcal{T}_\sigma=\begin{cases}\dfrac{1}{|Y_f|}\displaystyle\int_Y\mathbf{T}dy & \text{if } \sigma=\infty,\\ \dfrac{1}{|Y_f|}\left(\displaystyle\int_{Y_f}\mathbf{T}dy+\int_{Y_m}\mathbf{T}\mathbb{W}_\sigma dy\right) & \text{if } \sigma\in(0,\infty),\\ \dfrac{1}{|Y_f|}\displaystyle\int_{Y_f}\mathbf{T}dy & \text{if } \sigma=0.\end{cases}\tag{2.11}$$

See (2.7) for $\mathbb{W}_\sigma$. The Ψ in (2.10) satisfies

$$\begin{cases}-\nabla\cdot(\mathcal{K}_0\nabla\Psi)+|Y_f|\mathcal{T}_\sigma\Psi=|Y_f|\mathcal{G} & \text{in } \Omega,\\ \Psi=0 & \text{on } \partial\Omega.\end{cases}\tag{2.12}$$

By Theorem 9.19 [11] and A6,

$$\|\Psi\|_{W^{3,n+\delta}(\Omega)}\le c\|\mathcal{G}\|_{W^{1,n+\delta}(\Omega)},\tag{2.13}$$

where c is a constant depending on $\mathcal{K}_0,\mathbf{M},|Y_f|,\Omega$. Now for any $\epsilon,\omega\in(0,1)$ and on any $\omega(Y_m+j)\subset\Omega_m^\omega$ for some $j\in\mathbb{Z}^n$, we consider

$$\begin{cases}-\nabla\cdot\left(\epsilon^2\mathbf{K}_\omega\nabla\phi_{\epsilon,\omega}^{(j)}\right)+\mathbf{T}_{\epsilon,\omega}\phi_{\epsilon,\omega}^{(j)}=G_{\epsilon,\omega} & \text{in } \omega(Y_m+j),\\ \phi_{\epsilon,\omega}^{(j)}=\Psi & \text{on } \omega(\partial Y_m+j),\end{cases}\tag{2.14}$$

where Ψ is the solution of (2.12). By Lax-Milgram Theorem [11], A5–A6, and (2.13), the $\phi_{\epsilon,\omega}^{(j)}$ of (2.14) is solvable uniquely in $H^1(\omega(Y_m+j))$. By Theorem 8.24 and Theorem 8.29 [11], $\phi_{\epsilon,\omega}^{(j)}\in L^\infty(\omega(Y_m+j))$. Moreover,

Theorem 2.3. *Suppose A1–A6 and* $V_{\epsilon,\omega}=0$ *in (1.1). There is a constant* $\epsilon_0\in(0,1)$ *such that, for any* $\epsilon\in(0,\epsilon_0)$ *and* $\omega\in(0,1)$*,*

1) *if* $\epsilon,\omega\to 0$ *and* $\frac{\epsilon}{\omega}\to\infty$*, the solutions of (1.1), (2.12), and (2.14) satisfy*

$$\|\mathbf{E}_{\epsilon,\omega}(\Psi_{\epsilon,\omega}-\Psi)\|_{L^\infty(\Omega)}+\|\Psi_{\epsilon,\omega}-\sum_{\substack{j\in\mathbb{Z}^n\\ \omega(Y_m+j)\subset\Omega_m^\omega}}\phi_{\epsilon,\omega}^{(j)}\|_{L^\infty(\Omega_m^\omega)}$$
$$\le c(\|G_{\epsilon,\omega}-\mathcal{G}\|_{L^{n+\delta}(\Omega_f^\omega)}+\max\{\epsilon,\omega/\epsilon\}),$$

2) if $\epsilon, \omega \to 0$ and $\frac{\epsilon}{\omega} \to \sigma \in (0,\infty)$, the solutions of (1.1), (2.12), and (2.14) satisfy

$$\|\mathbf{E}_{\epsilon,\omega}\big(\Psi_{\epsilon,\omega} - (\mathcal{X}_{\Omega_f^\omega} + \mathbb{W}_{\frac{\epsilon}{\omega},\omega,0}\mathcal{X}_{\Omega_m^\omega})\Psi\big)\|_{L^\infty(\Omega)} + \|\Psi_{\epsilon,\omega} - \sum_{\substack{j\in\mathbb{Z}^n \\ \omega(Y_m+j)\subset\Omega_m^\omega}} \phi_{\epsilon,\omega}^{(j)}\|_{L^\infty(\Omega_m^\omega)}$$
$$\le c(\|G_{\epsilon,\omega} - \mathcal{G}\|_{L^{n+\delta}(\Omega_f^\omega)} + \max\{\epsilon, \omega, |\epsilon^2/(\sigma\omega)^2 - 1|\}),$$

3) if $\epsilon, \omega \to 0$ and $\frac{\epsilon}{\omega} \to 0$, the solutions of (1.1) and (2.12) satisfy

$$\|\Psi_{\epsilon,\omega} - (\mathcal{X}_{\Omega_f^\omega} + \mathbb{W}_{\frac{\epsilon}{\omega},\omega,0}\mathcal{X}_{\Omega_m^\omega})\Psi\|_{L^{n+\delta}(\Omega)}$$
$$\le c(\|G_{\epsilon,\omega} - \mathcal{G}\|_{L^{n+\delta}(\Omega_f^\omega)} + \max\{\epsilon, \omega, |\epsilon\omega^{-1}\ln(\epsilon\omega^{-1})|^{\frac{1}{2}}\}),$$

where c is a constant independent of ϵ, ω.

Proof of Theorem 2.3 is a convergence result for the solution of (1.1) in strongly coupled case and is proved in subsection 5.1. Besides A1–A3, A4', and A6, if

A7. $\mathbf{E}_{1/\epsilon,\omega}\mathbf{T}_{\epsilon,\omega}(x) = \mathbf{P}(\frac{x}{\omega})$ and $\mathbf{P} \in L^\infty_{per}(\mathbb{R}^n) \cap C^{0,\alpha}(\overline{\mathbb{A}_f})$ for some $\alpha > 0$,

the solution of (1.1) with $V_{\epsilon,\omega} = 0$ satisfies $\|\mathbf{E}_{\epsilon,\omega}\nabla\Psi_{\epsilon,\omega}, \mathbf{T}^{1/2}_{\epsilon,\omega}\Psi_{\epsilon,\omega}\|_{L^2(\Omega)} \le c$ (independent of ϵ, ω). By compactness principle [3, 12],

$$\begin{cases} \mathbf{E}_{\epsilon^2,\omega}\mathbf{K}_\omega\nabla\Psi_{\epsilon,\omega} \to \mathcal{K}_0\nabla\Psi \\ \mathbf{T}_{\epsilon,\omega}\Psi_{\epsilon,\omega} \to |Y_f|\breve{\mathcal{T}}\Psi & \text{in } L^2(\Omega) \text{ weakly as } \epsilon,\omega \to 0, \\ G_{\epsilon,\omega} \to |Y_f|\mathcal{G} \end{cases} \tag{2.15}$$

where $\breve{\mathcal{T}}\left(= \dfrac{1}{|Y_f|}\displaystyle\int_{Y_f} \mathbf{P}(y)dy\right)$ is a constant vector and $\mathcal{K}_0$ is defined in (2.9) with $\nu = 0$. Similar to (2.12)–(2.13), the Ψ in (2.15) satisfies

$$\begin{cases} -\nabla\cdot(\mathcal{K}_0\nabla\Psi) + |Y_f|\breve{\mathcal{T}}\Psi = |Y_f|\mathcal{G} & \text{in } \Omega, \\ \Psi = 0 & \text{on } \partial\Omega, \\ \|\Psi\|_{W^{3,n+\delta}(\Omega)} \le c\|\mathcal{G}\|_{W^{1,n+\delta}(\Omega)}, \end{cases} \tag{2.16}$$

where $|Y_f|$ is the volume of Y_f and c is a constant depending on $\mathcal{K}_0, \mathbf{M}, |Y_f|, \Omega$. We have the following result:

Theorem 2.4. *Assume A1–A3, A4', A6–A7, and $V_{\epsilon,\omega} = 0$ in (1.1). There is a $\epsilon_0 \in (0,1)$ such that if $\epsilon < \epsilon_0$, then the solutions of (1.1), (2.16), and (2.14) with Ψ obtained from (2.16) satisfy*

$$\|\mathbf{E}_{\epsilon^{3/2},\omega}\Psi_{\epsilon,\omega} - \Psi\|_{L^\infty(\Omega)} + \|\Psi_{\epsilon,\omega} - \sum_{\substack{j\in\mathbb{Z}^n \\ \omega(Y_m+j)\subset\Omega_m^\omega}} \phi_{\epsilon,\omega}^{(j)}\|_{L^\infty(\Omega_m^\omega)}$$
$$\le c(\|G_{\epsilon,\omega} - \mathcal{G}\|_{L^{n+\delta}(\Omega_f^\omega)} + \max\{\epsilon, \omega\}),$$

where c is a constant independent of ϵ, ω. See (2.16) for $\mathcal{G}$.

Proof of Theorem 2.4 is in subsection 5.2. Under A1–A3 and A6, the solution of (1.1) with $V_{\epsilon,\omega} = \mathbf{T}_{\epsilon,\omega} = 0$ satisfies $\|\mathbf{E}_{\epsilon,\omega}\nabla\Psi_{\epsilon,\omega}\|_{L^2(\Omega)} \le c$ (independent of ϵ, ω). By compactness principle [3, 12],

$$\begin{cases} \mathbf{E}_{\epsilon^2,\omega}\mathbf{K}_\omega\nabla\Psi_{\epsilon,\omega} \to \mathcal{K}_0\nabla\Psi \\ G_{\epsilon,\omega} \to |Y_f|\mathcal{G} \end{cases} \quad \text{in } L^2(\Omega) \text{ weakly as } \epsilon,\omega \to 0, \tag{2.17}$$

where $\mathcal{K}_0$ is defined in (2.9) with $\nu = 0$. The Ψ in (2.17) satisfies

$$\begin{cases} -\nabla\cdot(\mathcal{K}_0\nabla\Psi) = |Y_f|\mathcal{G} & \text{in } \Omega, \\ \Psi = 0 & \text{on } \partial\Omega, \\ \|\Psi\|_{W^{3,n+\delta}(\Omega)} \le c\|\mathcal{G}\|_{W^{1,n+\delta}(\Omega)}, \end{cases} \tag{2.18}$$

where $|Y_f|$ is the volume of Y_f and c is a constant depending on $\mathcal{K}_0, |Y_f|, \Omega$. We also have

Theorem 2.5. *Assume A1–A3, A6, and* $V_{\epsilon,\omega} = \mathbf{T}_{\epsilon,\omega} = 0$ *in (1.1). There is a* $\epsilon_0 \in (0,1)$ *such that if* $\epsilon < \epsilon_0$*, the solutions of (1.1), (2.18), and (2.14) with* $\mathbf{T}_{\epsilon,\omega} = 0$ *and* Ψ *from (2.18) satisfy*

$$\begin{aligned} &\|\mathbf{E}_{\epsilon,\omega}\Psi_{\epsilon,\omega} - \Psi\|_{L^\infty(\Omega)} + \|\Psi_{\epsilon,\omega} - \sum_{\substack{j\in\mathbb{Z}^n \\ \omega(Y_m+j)\subset\Omega_m^\omega}} \phi_{\epsilon,\omega}^{(j)}\|_{L^\infty(\Omega_m^\omega)} \\ &\quad \le c(\|G_{\epsilon,\omega} - \mathcal{G}\|_{L^{n+\delta}(\Omega_f^\omega)} + \max\{\epsilon,\omega\}), \end{aligned}$$

where c *is a constant independent of* ϵ,ω*. See (2.17) for* $\mathcal{G}$*.*

Proof of Theorem 2.5 is given in subsection 5.3. Theorems 2.4, 2.5 are convergence results for the solution of (1.1) for weakly coupled case. Theorems 2.3, 2.4, 2.5 imply if $\epsilon, \omega, \|G_{\epsilon,\omega} - \mathcal{G}\|_{L^{n+\delta}(\Omega_f^\omega)}$ are small enough, the homogenized solution Ψ of (2.12) or (2.16) or (2.18) is a good approximation of the solution of (1.1) in the connected subregion Ω_f^ω, but the Ψ may not be a good approximation of the solution of (1.1) in the disconnected subset Ω_m^ω. In the disconnected subset Ω_m^ω, the solution of (1.1) can be approximated by the solution of (2.14). One also notes that $\sum_{\substack{j\in\mathbb{Z}^n \\ \omega(Y_m+j)\subset\Omega_m^\omega}} \phi_{\epsilon,\omega}^{(j)}$ can be obtained by solving mutually independent local problems.

3. A priori uniform estimates for interface problems

Let $\Gamma(x-y)$ denote the fundamental solution of the Laplace equation in $\mathbb{R}^n$, see §6.2 [9]. Define a single-layer and a double-layer potentials as, for any smooth function φ on the boundary ∂Y_m of Y_m,

$$\begin{cases} \mathcal{S}_{\partial Y_m}(\varphi)(x) \equiv \displaystyle\int_{\partial Y_m} \Gamma(x-y)\varphi(y)dy \\ \mathcal{L}_{\partial Y_m}(\varphi)(x) \equiv \displaystyle\int_{\partial Y_m} \nabla_y\Gamma(x-y)\vec{\mathbf{n}}_y\ \varphi(y)dy \end{cases} \quad \text{for } x \in \partial Y_m,$$

where $\vec{\mathbf{n}}_y$ is the unit vector outward normal to ∂Y_m. By tracing the argument of Lemma 4.1 [21], we know

Lemma 3.1. *For any* $p \in (1,\infty)$, $i \in \{0,1\}$, *and* $\alpha \in (0,1)$, *the linear operators*

$$\begin{cases} \mathcal{S}_{\partial Y_m} : W^{i-\frac{1}{p},p}(\partial Y_m) \to W^{i+1-\frac{1}{p},p}(\partial Y_m) \\ \mathcal{L}_{\partial Y_m} : W^{i+1-\frac{1}{p},p}(\partial Y_m) \to W^{i+2-\frac{1}{p},p}(\partial Y_m) \\ \mathcal{S}_{\partial Y_m} : C^{1,\alpha}(\partial Y_m) \to C^{2,\alpha}(\partial Y_m) \\ \mathcal{L}_{\partial Y_m} : C^{1,\alpha}(\partial Y_m) \to C^{2,\alpha}(\partial Y_m) \end{cases}$$

are bounded; the operator $I - \ell\mathcal{L}_{\partial Y_m}$ *is continuously invertible in* $W^{i+1-\frac{1}{p},p}(\partial Y_m)$ *and in* $C^{2,\alpha}(\partial Y_m)$ *for* $\ell \in [-2,2]$; *and there is a constant* c *independent of* ℓ *so that*

$$\begin{cases} \|\varphi\|_{W^{i+1-\frac{1}{p},p}(\partial Y_m)} \le c\|(I-\ell\mathcal{L}_{\partial Y_m})(\varphi)\|_{W^{i+1-\frac{1}{p},p}(\partial Y_m)} & \text{for } \varphi \in W^{i+1-\frac{1}{p},p}(\partial Y_m), \\ \|\varphi\|_{C^{2,\alpha}(\partial Y_m)} \le c\|(I-\ell\mathcal{L}_{\partial Y_m})(\varphi)\|_{C^{2,\alpha}(\partial Y_m)} & \text{for } \varphi \in C^{2,\alpha}(\partial Y_m), \end{cases}$$

where I *is the identity operator.*

By A1, let us assume

$$\begin{cases} Y_m \subset \mathbf{D}_1 \subset \mathbf{D}_2 \subset Y, \\ \min\{dist(Y_m, \partial\mathbf{D}_1), dist(\mathbf{D}_1, \partial\mathbf{D}_2), dist(\mathbf{D}_2, \partial Y)\} > 0. \end{cases} \tag{3.1}$$

Lemma 3.2. *Under A1–A2,* $\epsilon \in (0,1]$, $p \in (n,6)$, $\mathbf{M} > 0$, *and* $\mathbb{P}_\epsilon(x) \in [0,\mathbf{M}]$ *for all* $x \in Y$, *any solution of*

$$-\nabla\cdot(\mathbb{E}_{\epsilon^2}\mathbf{K}\nabla U_\epsilon + Q_\epsilon) + \mathbb{E}_\epsilon\mathbb{P}_\epsilon U_\epsilon = F_\epsilon \qquad \text{in } Y \tag{3.2}$$

satisfies

$$\begin{aligned} &\|\mathbb{E}_{\epsilon^{3/2}}U_\epsilon\|_{W^{1,p}(\mathbf{D}_1\setminus\overline{Y_m})\cap W^{1,p}(Y_m)} \le c(\|U_\epsilon\|_{L^2(Y_f)} + \|\mathbb{E}_{1/\epsilon}Q_\epsilon\|_{L^2(Y)} \\ &\quad + \|\mathbb{E}_{1/\epsilon}F_\epsilon\|_{H^{-1}(Y)} + \|\mathbb{E}_{1/\sqrt{\epsilon}}Q_\epsilon\|_{L^p(Y)} + \|\mathbb{E}_{1/\sqrt{\epsilon}}F_\epsilon\|_{W^{-1,p}(Y)}), \end{aligned} \tag{3.3}$$

where c *is a constant independent of* ϵ. *See section 2 for* $\mathbb{E}_\nu$.

Proof. Let $p \in (n,6)$ and c denote a constant independent of ϵ.

Step 1: Assume $Q_\epsilon \in W_0^{1,p}(Y_f) \cap W_0^{1,p}(Y_m)$, $F_\epsilon \in L^p(Y)$, and consider

$$\begin{cases} -\nabla\cdot(\mathbb{E}_{\epsilon^2}\mathbf{K}\nabla\varphi_\epsilon + Q_\epsilon) + \mathbb{E}_\epsilon\mathbb{P}_\epsilon\varphi_\epsilon = F_\epsilon & \text{in } \mathbf{D}_2, \\ \varphi_\epsilon = 0 & \text{on } \partial\mathbf{D}_2. \end{cases} \tag{3.4}$$

The unique existence of a solution of (3.4) in $H^1(\mathbf{D}_2)$ is known by Lax-Milgram Theorem [11]. By Theorem 7.26 and Poincaré inequality [11],

$$\begin{aligned} &\|\varphi_\epsilon\|_{L^p(\mathbf{D}_2\setminus\overline{Y_m})} + \epsilon^{1/2}\|\mathbb{P}_\epsilon^{1/2}\varphi_\epsilon\|_{W^{-1,p}(Y_m)} \le c\|\mathbb{E}_\epsilon\nabla\varphi_\epsilon, \mathbb{E}_{\epsilon^{1/2}}\mathbb{P}_\epsilon^{1/2}\varphi_\epsilon\|_{L^2(\mathbf{D}_2)} \\ &\quad \le c(\|\mathbb{E}_{1/\epsilon}Q_\epsilon\|_{L^2(\mathbf{D}_2)} + \|\mathbb{E}_{1/\epsilon}F_\epsilon\|_{H^{-1}(\mathbf{D}_2)} \\ &\qquad + \|\mathbb{E}_{1/\sqrt{\epsilon}}Q_\epsilon\|_{L^p(\mathbf{D}_2)} + \|\mathbb{E}_{1/\sqrt{\epsilon}}F_\epsilon\|_{W^{-1,p}(\mathbf{D}_2)}) \equiv c\mathcal{I}_\epsilon. \end{aligned} \tag{3.5}$$

By (3.4)–(3.5) and [17], we have

$$\|\varphi_\epsilon\|_{W^{1,p}(\mathbf{D}_2\setminus\mathbf{D}_1)} \le c\mathcal{I}_\epsilon. \tag{3.6}$$

See (3.1) for $\mathbf{D}_1$. Let $\widehat{\varphi}$ in Y_m be the solution of

$$\begin{cases} -\nabla\cdot(\epsilon^2\widehat{\mathbf{k}}\nabla\widehat{\varphi}+\epsilon^2(\mathbf{K}-\widehat{\mathbf{k}})\nabla\varphi_\epsilon+Q_\epsilon)=F_\epsilon-\epsilon\mathbb{P}_\epsilon\varphi_\epsilon & \text{in } Y_m,\\ \widehat{\varphi}|_{\partial Y_m}=0, \end{cases} \tag{3.7}$$

and $\widehat{\varphi}$ in $\mathbf{D}_2\setminus\overline{Y_m}$ be the solution of

$$\begin{cases} -\nabla\cdot(\widehat{\mathbf{K}}\nabla\widehat{\varphi}+(\mathbf{K}-\widehat{\mathbf{K}})\nabla\varphi_\epsilon+Q_\epsilon)=F_\epsilon-\mathbb{P}_\epsilon\varphi_\epsilon & \text{in } \mathbf{D}_2\setminus\overline{Y_m},\\ \widehat{\varphi}|_{\partial(\mathbf{D}_2\setminus\overline{Y_m})}=0, \end{cases} \tag{3.8}$$

where $\widehat{\mathbf{K}},\widehat{\mathbf{k}}$ are two constants in the interval $(\min_Y\mathbf{K},\max_Y\mathbf{K})$. By [17] and (3.5),

$$\begin{cases} \epsilon^2\widehat{\mathbf{k}}\|\widehat{\varphi}\|_{W^{1,p}(Y_m)}\le c\big(\epsilon^{1/2}\mathcal{I}_\epsilon+\epsilon^2\|(\mathbf{K}-\widehat{\mathbf{k}})\nabla\varphi_\epsilon\|_{L^p(Y_m)}\big),\\ \widehat{\mathbf{K}}\|\widehat{\varphi}\|_{W^{1,p}(\mathbf{D}_2\setminus\overline{Y_m})}\le c\big(\mathcal{I}_\epsilon+\|(\mathbf{K}-\widehat{\mathbf{K}})\nabla\varphi_\epsilon\|_{L^p(\mathbf{D}_2\setminus\overline{Y_m})}\big). \end{cases} \tag{3.9}$$

If we define $\check{\varphi}\equiv\varphi_\epsilon-\widehat{\varphi}$ in $\mathbf{D}_2$, then (3.4) and (3.7)–(3.8) imply

$$\begin{cases} \Delta\check{\varphi}=0 & \text{in } \mathbf{D}_2\setminus\partial Y_m,\\ \lfloor\check{\varphi}\rfloor=0 & \text{on } \partial Y_m,\\ \lfloor\check{\mathbb{E}}\nabla\check{\varphi}\rfloor\cdot\vec{\mathbf{n}}_y=\mathcal{F}/\widehat{\mathbf{K}} & \text{on } \partial Y_m,\\ \check{\varphi}=0 & \text{on } \partial\mathbf{D}_2, \end{cases} \tag{3.10}$$

where $\check{\mathbb{E}}\equiv\begin{cases}\epsilon^2\widehat{\mathbf{k}}/\widehat{\mathbf{K}} & \text{in } Y_m\\ 1 & \text{in } Y_f\end{cases}$ and $\vec{\mathbf{n}}_y$ is the unit vector outward normal to ∂Y_m. See (2.1) for $(3.10)_{2,3}$. Since $Q_\epsilon\in W_0^{1,p}(Y_f)\cap W_0^{1,p}(Y_m)$,

$$\mathcal{F}\equiv\big(\epsilon^2\widehat{\mathbf{k}}\nabla\widehat{\varphi}_{,-}-\widehat{\mathbf{K}}\nabla\widehat{\varphi}_{,+}+\epsilon^2(\mathbf{K}-\widehat{\mathbf{k}})\nabla\varphi_{\epsilon,-}-(\mathbf{K}-\widehat{\mathbf{K}})\nabla\varphi_{\epsilon,+}\big)\cdot\vec{\mathbf{n}}_y|_{\partial Y_m}.$$

By (3.9),

$$\begin{aligned}\|\mathcal{F}\|_{W^{-\frac{1}{p},p}(\partial Y_m)}\le{}& c\big(\mathcal{I}_\epsilon+\epsilon^2\|\mathbf{K}-\widehat{\mathbf{k}}\|_{L^\infty(Y_m)}\|\varphi_\epsilon\|_{W^{1,p}(Y_m)}\\ &+\|\mathbf{K}-\widehat{\mathbf{K}}\|_{L^\infty(\mathbf{D}_2\setminus\overline{Y_m})}\|\varphi_\epsilon\|_{W^{1,p}(\mathbf{D}_2\setminus\overline{Y_m})}\big).\end{aligned} \tag{3.11}$$

By Green's formula, (3.10), and Theorem 6.5.1 [9], we see that

$$\begin{cases} \check{\varphi}/2+\mathcal{L}_{\partial Y_m}(\check{\varphi})=\mathcal{S}_{\partial Y_m}(\partial_{\mathbf{n}_y}\check{\varphi}_{,-}|_{\partial Y_m})\\ \check{\varphi}/2-\mathcal{L}_{\partial Y_m}(\check{\varphi})=-\mathcal{S}_{\partial Y_m}(\partial_{\mathbf{n}_y}\check{\varphi}_{,+}|_{\partial Y_m})+\mathcal{S}_{\partial\mathbf{D}_2}(\partial_{\mathbf{n}_y}\check{\varphi}|_{\partial\mathbf{D}_2}) \end{cases} \quad \text{on } \partial Y_m,$$

where $\partial_{\mathbf{n}_y}\check{\varphi}|_{\partial\mathbf{D}_2}$ is the normal derivative of $\check{\varphi}$ on $\partial\mathbf{D}_2$. So we have

$$\left(I-\frac{2(1-\check{\epsilon})}{1+\check{\epsilon}}\mathcal{L}_{\partial Y_m}\right)\check{\varphi}=\frac{2\,\mathcal{S}_{\partial\mathbf{D}_2}(\partial_{\mathbf{n}_y}\check{\varphi}|_{\partial\mathbf{D}_2})}{1+\check{\epsilon}}-\frac{2\,\mathcal{S}_{\partial Y_m}(\mathcal{F})}{(1+\check{\epsilon})\widehat{\mathbf{K}}} \quad \text{on } \partial Y_m, \tag{3.12}$$

where $\check{\epsilon}\equiv\epsilon^2\widehat{\mathbf{k}}/\widehat{\mathbf{K}}$. Then (3.6), (3.9), (3.12), and Lemma 3.1 imply

$$\begin{cases} \|\check{\varphi}\|_{W^{1-\frac{1}{p},p}(\partial Y_m)}\le c(\|\mathcal{F}\|_{W^{-\frac{1}{p},p}(\partial Y_m)}+\|\partial_{\mathbf{n}_y}\check{\varphi}\|_{W^{-\frac{1}{p},p}(\partial\mathbf{D}_2)}),\\ \|\partial_{\mathbf{n}_y}\check{\varphi}\|_{W^{-\frac{1}{p},p}(\partial\mathbf{D}_2)}\le c\big(\mathcal{I}_\epsilon+\|(\mathbf{K}-\widehat{\mathbf{K}})\nabla\varphi_\epsilon\|_{L^p(\mathbf{D}_2\setminus\overline{Y_m})}\big). \end{cases} \tag{3.13}$$

(3.10)–(3.11) and (3.13) imply

$$\|\check{\varphi}\|_{W^{1,p}(\mathbf{D}_2\setminus\overline{Y_m})\cap W^{1,p}(Y_m)} \le c\big(\mathcal{I}_\epsilon + \epsilon^2\|\mathbf{K}-\widehat{\mathbf{k}}\|_{L^\infty(Y_m)}\|\varphi_\epsilon\|_{W^{1,p}(Y_m)} + \|\mathbf{K}-\widehat{\mathbf{K}}\|_{L^\infty(\mathbf{D}_2\setminus\overline{Y_m})}\|\varphi_\epsilon\|_{W^{1,p}(\mathbf{D}_2\setminus\overline{Y_m})}\big).$$

Together with (3.9), we obtain

$$\|\mathbb{E}_{\epsilon^{3/2}}\varphi_\epsilon\|_{W^{1,p}(\mathbf{D}_2\setminus\overline{Y_m})\cap W^{1,p}(Y_m)} \le c\big(\mathcal{I}_\epsilon + \epsilon^{3/2}\|\mathbf{K}-\widehat{\mathbf{k}}\|_{L^\infty(Y_m)}\|\varphi_\epsilon\|_{W^{1,p}(Y_m)} + \|\mathbf{K}-\widehat{\mathbf{K}}\|_{L^\infty(\mathbf{D}_2\setminus\overline{Y_m})}\|\varphi_\epsilon\|_{W^{1,p}(\mathbf{D}_2\setminus\overline{Y_m})}\big).$$

By A2, we obtain

$$\|\mathbb{E}_{\epsilon^{3/2}}\varphi_\epsilon\|_{W^{1,p}(\mathbf{D}_2\setminus\overline{Y_m})\cap W^{1,p}(Y_m)} \le c\mathcal{I}_\epsilon. \tag{3.14}$$

Step 2: Note $W_0^{1,p}(Y_f)$ (resp. $W_0^{1,p}(Y_m)$) is dense in $L^p(Y_f)$ (resp. $L^p(Y_m)$) as well as $L^p(Y)$ is dense in $W^{-1,p}(Y)$. By a limiting argument, we see that if $Q_\epsilon \in L^p(Y)$ and $F_\epsilon \in W^{-1,p}(Y)$, the solution of (3.4) still satisfies (3.14).

Step 3: Let η be a smooth function satisfying $\eta \in C_0^\infty(\mathbf{D}_2)$, $\eta \in [0,1]$, $\eta = 1$ in $\mathbf{D}_1$, $\|\nabla\eta\|_{W^{1,\infty}(\mathbf{D}_2)} \le c$. Multiply (3.2) by η to obtain

$$\begin{cases} -\nabla\cdot(\mathbb{E}_{\epsilon^2}\mathbf{K}\nabla(U_\epsilon\eta) - \mathbf{K}U_\epsilon\nabla\eta + Q_\epsilon\eta) + \mathbb{E}_\epsilon\mathbb{P}_\epsilon U_\epsilon\eta \\ \qquad = F_\epsilon\eta - (\mathbf{K}\nabla U_\epsilon + Q_\epsilon)\nabla\eta & \text{in } \mathbf{D}_2, \\ U_\epsilon\eta = 0 & \text{on } \partial\mathbf{D}_2. \end{cases}$$

By the result of Step 2, we have

$$\|\mathbb{E}_{\epsilon^{3/2}}U_\epsilon\|_{W^{1,p}(\mathbf{D}_1\setminus\overline{Y_m})\cap W^{1,p}(Y_m)} \le c(\|U_\epsilon\|_{L^p(\mathbf{D}_2\setminus\mathbf{D}_1)} + \|\mathbb{E}_{1/\sqrt{\epsilon}}Q_\epsilon\|_{L^p(Y)} + \|\mathbb{E}_{1/\sqrt{\epsilon}}F_\epsilon\|_{W^{-1,p}(Y)} + \|\mathbb{E}_{1/\epsilon}Q_\epsilon\|_{L^2(Y)} + \|\mathbb{E}_{1/\epsilon}F_\epsilon\|_{H^{-1}(Y)}). \tag{3.15}$$

Let $\tilde{\eta}$ be another smooth function satisfying $\tilde{\eta} \in C_0^\infty(Y)$, $\tilde{\eta} \in [0,1]$, $\tilde{\eta} = 1$ in $\mathbf{D}_2$, $\|\nabla\tilde{\eta}\|_{W^{1,\infty}(Y)} \le c$. Multiply (3.2) by $\tilde{\eta}$ and then use energy method to get

$$\|U_\epsilon\|_{L^p(\mathbf{D}_2\setminus\mathbf{D}_1)} \le c(\|U_\epsilon\|_{L^2(Y_f)} + \|\mathbb{E}_{1/\epsilon}Q_\epsilon\|_{L^2(Y)} + \|\mathbb{E}_{1/\epsilon}F_\epsilon\|_{H^{-1}(Y)}).$$

Together with (3.15), we obtain (3.3). □

Modifying the argument for Lemma 3.2 and employing Lemma 3.1, we see

Lemma 3.3. *Under A1–A2, $\epsilon \in (0,1]$, and $p \in (n,\infty)$, any solution of*

$$-\nabla\cdot(\mathbb{E}_{\epsilon^2}\mathbf{K}\nabla U_\epsilon + Q_\epsilon) = F_\epsilon \qquad \text{in } Y$$

satisfies

$$\|\mathbb{E}_{\epsilon^i}U_\epsilon\|_{W^{1,p}(\mathbf{D}_1\setminus\overline{Y_m})\cap W^{1,p}(Y_m)} \le c(\|U_\epsilon\|_{L^2(Y_f)} + \|\mathbb{E}_{\epsilon^{i-2}}Q_\epsilon\|_{L^p(Y)} + \|\mathbb{E}_{\epsilon^{i-2}}F_\epsilon\|_{W^{-1,p}(Y)}), \tag{3.16}$$

$$\|U_\epsilon\|_{W^{2,p}(\mathbf{D}_1\setminus\overline{Y_m})\cap W^{2,p}(Y_m)} \le c(\|U_\epsilon\|_{L^2(Y_f)} + \|\mathbb{E}_{\epsilon^{-2}}Q_\epsilon\|_{W^{1,p}(Y_f)\cap W^{1,p}(Y_m)} + \|\mathbb{E}_{\epsilon^{-2}}F_\epsilon\|_{L^p(Y)}),$$

$$\|\mathbb{E}_{\epsilon^i}U_\epsilon\|_{C^{2,\alpha}(\overline{\mathbf{D}_1\setminus Y_m})\cap C^{2,\alpha}(\overline{Y_m})} \le c(\|U_\epsilon\|_{L^2(Y_f)} + \|\mathbb{E}_{\epsilon^{i-2}}Q_\epsilon\|_{C^{1,\alpha}(\overline{Y_f})\cap C^{1,\alpha}(\overline{Y_m})} + \|\mathbb{E}_{\epsilon^{i-2}}F_\epsilon\|_{C^{0,\alpha}(\overline{Y_f})\cap C^{0,\alpha}(\overline{Y_m})}),$$

where $i \in \{0,1\}$, $\alpha \in (0,1)$, and c is a constant independent of ϵ. See (3.1) for $\mathbf{D}_1$.

Under A1–A2 and $\nu \in (0,1]$, the solution of (2.5) satisfies, by Lemma 3.3,

$$\|\mathbb{X}_\nu^{(i)}\|_{C^2(\overline{Y_f})\cap C^2(\overline{Y_m})} \le c, \tag{3.17}$$

where c is independent of ν. Under A1–A2, the solution of (2.6) satisfies, by Theorem 6.30 [11],

$$\|\mathbb{X}_0^{(i)}\|_{C^2(\overline{Y_f})} \le c, \tag{3.18}$$

where c is a constant. By (3.17) and (3.18), it is not difficult to see that there are positive constants $\epsilon_0, \ell_5, \ell_6$ such that the symmetric positive definite matrix $\mathcal{K}_\nu$ for $\nu \in [0,1]$ in (2.9) satisfies

$$\begin{cases} \ell_5 I \le \mathcal{K}_\nu \le \ell_6 I, \\ |\mathcal{K}_\nu - \mathcal{K}_0| \le c\nu \quad \text{where } c \text{ is independent of } \nu. \end{cases} \tag{3.19}$$

Define a part of boundary of Y by $\partial\widetilde{Y}_n \equiv \{y \in \partial Y | y = (y_1, y_2, \cdots, y_{n-1}, 0)\}$ and consider the following problem

$$\begin{cases} -\nabla\cdot(\mathbb{E}_{\epsilon^2}\mathbf{K}\nabla U_\epsilon + Q_\epsilon) + \mathbb{E}_\epsilon\mathbb{P}_\epsilon U_\epsilon = F_\epsilon & \text{in } Y, \\ U_\epsilon = U_{b_\epsilon} & \text{on } \partial\widetilde{Y}_n. \end{cases} \tag{3.20}$$

Let $Y_m \subset \mathbf{D}_3 \subset Y$ satisfy $\min\{dist(Y_m, \partial\mathbf{D}_3), dist(\mathbf{D}_3, \partial Y \setminus \partial\widetilde{Y}_n)\} > 0$. By an analogous argument as that for Lemma 3.2, we see

Lemma 3.4. *Under A1–A2, $\epsilon \in (0,1]$, $p \in (n,6)$, $\mathbf{M} > 0$, and $\mathbb{P}_\epsilon(x) \in [0,\mathbf{M}]$ for all $x \in Y$, any solution of (3.20) satisfies*

$$\begin{aligned} &\|\mathbb{E}_{\epsilon^{3/2}}U_\epsilon\|_{W^{1,p}(\mathbf{D}_3\setminus\overline{Y_m})\cap W^{1,p}(Y_m)} \le c(\|U_{b_\epsilon}\|_{W^{1,p}(Y_f)} + \|U_\epsilon\|_{L^2(Y_f)} \\ &+\|\mathbb{E}_{1/\epsilon}Q_\epsilon\|_{L^2(Y)} + \|\mathbb{E}_{1/\epsilon}F_\epsilon\|_{H^{-1}(Y)} + \|\mathbb{E}_{1/\sqrt{\epsilon}}Q_\epsilon\|_{L^p(Y)} + \|\mathbb{E}_{1/\sqrt{\epsilon}}F_\epsilon\|_{W^{-1,p}(Y)}), \end{aligned}$$

where c is a constant independent of ϵ.

Under A1–A2, $\epsilon \in (0,1]$, and $p \in (n,\infty)$, any solution of (3.20) with $\mathbb{P}_\epsilon = 0$ satisfies

$$\begin{aligned} \|\mathbb{E}_\epsilon U_\epsilon\|_{W^{1,p}(\mathbf{D}_3\setminus\overline{Y_m})\cap W^{1,p}(Y_m)} &\le c(\|U_{b_\epsilon}\|_{W^{1,p}(Y_f)} + \|U_\epsilon\|_{L^2(Y_f)} \\ &+\|\mathbb{E}_{1/\epsilon}Q_\epsilon\|_{L^p(Y)} + \|\mathbb{E}_{1/\epsilon}F_\epsilon\|_{W^{-1,p}(Y)}), \end{aligned}$$

where c is a constant independent of ϵ.

One example below shows that the second order derivatives of the solution of (1.1) may not be bounded uniformly in ϵ, ω in the high permeability subregion Ω_f^ω.

Remark 3.1. Assume that $B_1(0) \subset \Omega(\epsilon)$ and η is a bell-shaped smooth function satisfying $\eta \in C_0^\infty(B_1(0))$, $\eta \in [0,1]$, and $\eta(x) = 1$ in $B_{1/2}(0)$. Employ (2.5), η, and $\mathbb{X}_{\epsilon,\epsilon}^{(1)}$ for $\epsilon \in (0,1)$ to obtain

$$\begin{cases} -\nabla\cdot\big(\mathbf{E}_{\epsilon^2,\epsilon}\mathbf{K}_\epsilon(\nabla(\eta\mathbb{X}_{\epsilon,\epsilon}^{(1)}) - \mathbb{X}_{\epsilon,\epsilon}^{(1)}\nabla\eta + \eta\vec{e}_1)\big) = -\mathbf{E}_{\epsilon^2,\epsilon}\mathbf{K}_\epsilon(\nabla\mathbb{X}_{\epsilon,\epsilon}^{(1)} + \vec{e}_1)\nabla\eta & \text{in } \Omega, \\ \eta\mathbb{X}_{\epsilon,\epsilon}^{(1)} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\vec{e}_1$ is a unit vector in the first coordinate direction. By (3.17), we see that

$$\|\mathbb{X}^{(1)}_{\epsilon,\epsilon}\nabla\eta - \eta\vec{e}_1\|_{W^{1,\infty}(B_1(0))} + \|(\nabla\mathbb{X}^{(1)}_{\epsilon,\epsilon} + \vec{e}_1)\nabla\eta\|_{L^\infty(B_1(0))}$$

is bounded uniformly in ϵ, but $\|\eta\mathbb{X}^{(1)}_{\epsilon,\epsilon}\|_{W^{2,p}(B_1(0)\cap\Omega^\epsilon_f)}$ for $p \in [1,\infty]$ is not bounded uniformly in ϵ.

4. Uniform Hölder estimate

A1–A2 are assumed in this section. We shall derive uniform Hölder estimates for non-uniform elliptic equations, that is, Theorem 2.1 and Theorem 2.2. The Hölder estimate in the interior region is considered in subsection 4.1, and the estimate around the boundary is in subsection 4.2.

4.1. *Interior estimate*

For convenience we let $\overline{B_1(0)} \subset \Omega$.

Lemma 4.1. *For any $\delta, \mathbf{M} > 0$, there are $\theta_1, \theta_2 \in (0,1)$ (depending on $\delta, \mathbf{M}, \mathbf{K}, Y_f$) with $\theta_1 < \theta_2^2$ and there is a $\epsilon_0 \in (0,1)$ (depending on $\theta_1, \theta_2, \delta, \mathbf{M}, \mathbf{K}$) such that if*

$$-\nabla\cdot(\mathbf{E}_{\epsilon^2,\nu}\mathbf{K}_\nu\nabla\mathbb{U}_{\epsilon,\nu} + \mathbb{Q}_{\epsilon,\nu}) + \mathbf{E}_{\epsilon,\nu}\mathbb{P}_{\epsilon,\nu}\mathbb{U}_{\epsilon,\nu} = \mathbb{F}_{\epsilon,\nu} \qquad \text{in } B_1(0), \tag{4.1}$$

and if

$$\begin{cases} \epsilon,\nu \in (0,\epsilon_0), \quad \theta \in [\theta_1,\theta_2], \quad \mathbb{P}_{\epsilon,\nu}(x) \in [0,\mathbf{M}] \text{ for all } x \in B_1(0), \\ \max\{\|\mathbf{E}_{\epsilon,\nu}\mathbb{U}_{\epsilon,\nu}\|_{L^2(B_1(0))}, \|\epsilon_0^{-1}\mathbb{Q}_{\epsilon,\nu}\mathcal{X}_{\Omega^\nu_f} + \epsilon^{-1}\mathbb{Q}_{\epsilon,\nu}\mathcal{X}_{\Omega^\nu_m}\|_{L^{n+\delta}(B_1(0))}, \\ \qquad \epsilon_0^{-1}\|\mathbb{F}_{\epsilon,\nu}\mathcal{X}_{\Omega^\nu_f} + \epsilon^{-1}\max\{\epsilon,\nu\}\mathbb{F}_{\epsilon,\nu}\mathcal{X}_{\Omega^\nu_m}\|_{L^{n+\delta}(B_1(0))}\} \le 1, \end{cases} \tag{4.2}$$

then

$$\begin{cases} \displaystyle\fint_{B_\theta(0)} \left|\Pi_\nu\mathbb{U}_{\epsilon,\nu}|_{\Omega^\nu_f} - (\Pi_\nu\mathbb{U}_{\epsilon,\nu}|_{\Omega^\nu_f})_{0,\theta}\right|^2 dx \le \theta^{2\mu}, \\ \displaystyle\fint_{B_\theta(0)\cap\Omega^\nu_m} \epsilon^2\left|\mathbb{U}_{\epsilon,\nu} - (\Pi_\nu\mathbb{U}_{\epsilon,\nu}|_{\Omega^\nu_f})_{0,\theta}\right|^2 dx \le \theta^{2\mu}, \end{cases} \tag{4.3}$$

where $\mu \equiv \frac{\delta}{n+\delta}$. See section 2 for $(\Pi_\nu\mathbb{U}_{\epsilon,\nu}|_{\Omega^\nu_f})_{0,\theta}$, the average value of the extension function $\Pi_\nu\mathbb{U}_{\epsilon,\nu}|_{\Omega^\nu_f}$ in $B_\theta(0)$.

Proof. Consider the following problem

$$-\nabla\cdot(\mathcal{K}_0\nabla\mathbb{U}) + \mathbb{P}\mathbb{U} = 0 \qquad \text{in } B_{3/4}(0), \tag{4.4}$$

where $\mathcal{K}_0$ is defined in (2.9) and $\mathbb{P}(x) \in [0,\mathbf{M}]$ for $x \in B_{3/4}(0)$. Any solution $\mathbb{U}$ of (4.4) satisfies, by Theorem 9.11 [11] and (3.19),

$$\|\mathbb{U}\|_{C^{1,\alpha}(\overline{B_{1/2}(0)})} \le c\|\mathbb{U}\|_{L^2(B_{3/4}(0))},$$

where $\alpha \in (0,1)$ and c only depends on $\mathcal{K}_0, \mathbf{M}$. If $\check{\mu}$ satisfies $\mu < \check{\mu} < 1$, then, by Theorem 1.2 in page 70 [10],

$$\fint_{B_\theta(0)} |\mathbb{U} - (\mathbb{U})_{0,\theta}|^2 dx \le \theta^{2\check{\mu}} \fint_{B_{3/4}(0)} \mathbb{U}^2 dx \tag{4.5}$$

for θ (depending on $\delta, \mathcal{K}_0, \mathbf{M}$) sufficiently small. Let us fix $\theta_1, \theta_2 \in (0, \frac{1}{2})$ so that $\theta_1 < \theta_2^2$ and (4.5) holds for any $\theta \in [\theta_1, \theta_2]$.

Now we claim $(4.3)_1$. If not, there is a sequence $\{\theta_{\epsilon,\nu}, \mathbb{P}_{\epsilon,\nu}, \mathbb{U}_{\epsilon,\nu}, \mathbb{Q}_{\epsilon,\nu}, \mathbb{F}_{\epsilon,\nu}\}$ satisfying (4.1) and, as $\epsilon, \nu \to 0$,

$$\begin{cases} \theta_{\epsilon,\nu} \to \theta \in [\theta_1, \theta_2], \\ \mathbb{P}_{\epsilon,\nu}(x) \in [0, \mathbf{M}] \text{ for all } x \in B_1(0), \\ \max\{\|\mathbf{E}_{\epsilon,\nu}\mathbb{U}_{\epsilon,\nu}\|_{L^2(B_1(0))}, \epsilon^{-1}\|\mathbb{Q}_{\epsilon,\nu}\|_{L^{n+\delta}(B_1(0)\cap\Omega_m^\nu)}\} \le 1, \\ \lim\limits_{\epsilon,\nu\to 0} \|\mathbb{Q}_{\epsilon,\nu}, \mathbb{F}_{\epsilon,\nu}\|_{L^{n+\delta}(B_1(0)\cap\Omega_f^\nu)} + \epsilon^{-1}\max\{\epsilon,\nu\}\|\mathbb{F}_{\epsilon,\nu}\|_{L^{n+\delta}(B_1(0)\cap\Omega_m^\nu)} = 0, \\ \fint_{B_{\theta_{\epsilon,\nu}}(0)} \left|\Pi_\nu \mathbb{U}_{\epsilon,\nu}|_{\Omega_f^\nu} - (\Pi_\nu \mathbb{U}_{\epsilon,\nu}|_{\Omega_f^\nu})_{0,\theta_{\epsilon,\nu}}\right|^2 dx > |\theta_{\epsilon,\nu}|^{2\mu}. \end{cases} \tag{4.6}$$

By energy method and A2, there is a constant c independent of ϵ, ν such that

$$\|\mathbb{U}_{\epsilon,\nu}\|_{H^1(B_{4/5}(0)\cap\Omega_f^\nu)} + \|\epsilon\nabla\mathbb{U}_{\epsilon,\nu}, \epsilon^{1/2}\mathbb{P}_{\epsilon,\nu}^{1/2}\mathbb{U}_{\epsilon,\nu}\|_{L^2(B_{4/5}(0)\cap\Omega_m^\nu)} \le c.$$

By compactness principle and by tracing the proof of Theorem 2.3 [3], we can extract a subsequence (same notation for subsequence) such that

$$\begin{cases} \Pi_\nu \mathbb{U}_{\epsilon,\nu}|_{\Omega_f^\nu} \to \mathbb{U} & \text{in } L^2(B_{3/4}(0)) \text{ strongly} \\ \mathbf{E}_{\epsilon^2,\nu}\mathbf{K}_\nu \nabla\mathbb{U}_{\epsilon,\nu} \to \mathcal{K}_0\nabla\mathbb{U} & \text{in } L^2(B_{3/4}(0)) \text{ weakly} \\ \mathbf{E}_{\epsilon,\nu}\mathbb{P}_{\epsilon,\nu}\mathbb{U}_{\epsilon,\nu} \to \mathbb{P}\mathbb{U} & \text{in } L^2(B_{3/4}(0)) \text{ weakly} \end{cases} \quad \text{as } \epsilon, \nu \to 0, \tag{4.7}$$

where $\mathbb{P}(x) \in [0, \mathbf{M}]$ for all $x \in B_{3/4}(0)$, $\mathcal{K}_0$ is a constant symmetric positive definite matrix, $\ell_5 \le \mathcal{K}_0 \le \ell_6$, and ℓ_5, ℓ_6 are positive constants (see (2.9) and (3.19)). The $\mathbb{U}$ in (4.7) satisfies (4.4). Equations (4.5)–(4.7) then imply

$$\begin{aligned} \theta^{2\mu} &= \lim_{\epsilon,\nu\to 0} |\theta_{\epsilon,\nu}|^{2\mu} \le \lim_{\epsilon,\nu\to 0} \fint_{B_{\theta_{\epsilon,\nu}}(0)} \left|\Pi_\nu \mathbb{U}_{\epsilon,\nu}|_{\Omega_f^\nu} - (\Pi_\nu \mathbb{U}_{\epsilon,\nu}|_{\Omega_f^\nu})_{0,\theta_{\epsilon,\nu}}\right|^2 \\ &= \fint_{B_\theta(0)} \mathbb{U}^2 - \left|\fint_{B_\theta(0)} \mathbb{U}\right|^2 = \fint_{B_\theta(0)} |\mathbb{U} - (\mathbb{U})_{0,\theta}|^2 \le \theta^{2\check{\mu}} \fint_{B_{3/4}(0)} \mathbb{U}^2 dx. \end{aligned}$$

If θ_2 is small enough, then we get contradiction. Therefore we prove $(4.3)_1$.

Define $\zeta \equiv \theta^{-\mu}(\Pi_\nu \mathbb{U}_{\epsilon,\nu}|_{\Omega_f^\nu} - (\Pi_\nu \mathbb{U}_{\epsilon,\nu}|_{\Omega_f^\nu})_{0,\theta})$ and $\eta \equiv \theta^{-\mu}(\mathbb{U}_{\epsilon,\nu} - (\Pi_\nu \mathbb{U}_{\epsilon,\nu}|_{\Omega_f^\nu})_{0,\theta})$. Then (4.1) implies, for any smooth function φ with support in $\nu(Y_m + j) \subset B_\theta(0) \cap \Omega_m^\nu$ for some $j \in \mathbb{Z}^n$,

$$\begin{aligned} &\epsilon^2 \int_{\nu(Y_m+j)} (\eta - \zeta)\nabla\cdot(\mathbf{K}_\nu\nabla\varphi) - \epsilon\int_{\nu(Y_m+j)} \mathbb{P}_{\epsilon,\nu}(\eta-\zeta)\varphi \\ &\quad = \int_{\nu(Y_m+j)} \left(\epsilon^2\mathbf{K}_\nu\nabla\zeta + \theta^{-\mu}\mathbb{Q}_{\epsilon,\nu}\right)\nabla\varphi + \theta^{-\mu}(\epsilon\mathbb{P}_{\epsilon,\nu}\Pi_\nu\mathbb{U}_{\epsilon,\nu}|_{\Omega_f^\nu} - \mathbb{F}_{\epsilon,\nu})\varphi. \end{aligned} \tag{4.8}$$

If φ is the solution of

$$\begin{cases} -\epsilon^2 \nabla \cdot (\mathbf{K}_\nu \nabla \varphi) + \epsilon \mathbb{P}_{\epsilon,\nu} \varphi = \zeta - \eta & \text{in } \nu(Y_m + j), \\ \varphi = 0 & \text{on } \nu(\partial Y_m + j), \end{cases} \tag{4.9}$$

then $c_1 \nu^{-1} \epsilon^2 \|\varphi\|_{L^2(\nu(Y_m+j))} \le \epsilon^2 \|\nabla \varphi\|_{L^2(\nu(Y_m+j))} \le c_2 \nu \|\eta - \zeta\|_{L^2(\nu(Y_m+j))}$, where c_1, c_2 are independent of ν. (4.8) and (4.9) imply

$$\begin{aligned} \int_{\nu(Y_m+j)} \epsilon^2 |\eta - \zeta|^2 &\le c \int_{\nu(Y_m+j)} \epsilon^2 \nu^2 |\nabla \zeta|^2 \\ &+ c \int_{\nu(Y_m+j)} \nu^2 \theta^{-2\mu} (|\epsilon^{-1} \mathbb{Q}_{\epsilon,\nu}|^2 + \nu^2 \mathbb{P}^2_{\epsilon,\nu} \left| \Pi_\nu \mathbb{U}_{\epsilon,\nu}|_{\Omega_f^\nu} \right|^2 + |\nu \epsilon^{-1} \mathbb{F}_{\epsilon,\nu}|^2). \end{aligned} \tag{4.10}$$

Summing (4.10) over all $\nu(Y_m + j) \subset B_\theta(0) \cap \Omega_m^\nu$ for $j \in \mathbb{Z}^n$, we obtain $(4.3)_2$ if ϵ_0 is small enough. □

Lemma 4.2. *For any $\delta \in (0,3)$ and $\mathbf{M} > 0$, there are $\theta_1, \theta_2 \in (0,1)$ (depending on $\delta, \mathbf{M}, \mathbf{K}, Y_f$) with $\theta_1 < \theta_2^2$ and there is a $\epsilon_0 > 0$ (depending on $\theta_1, \theta_2, \delta, \mathbf{M}, \mathbf{K}$) such that if*

$$-\nabla \cdot (\mathbf{E}_{\epsilon^2,\omega} \mathbf{K}_\omega \nabla U_{\epsilon,\omega} + Q_{\epsilon,\omega}) + \gamma \mathbf{E}_{\epsilon,\omega} \mathbf{P}_{\epsilon,\omega} U_{\epsilon,\omega} = F_{\epsilon,\omega} \qquad \text{in } B_1(0), \tag{4.11}$$

and if $\epsilon, \omega \in (0, \epsilon_0)$, $\theta \in [\theta_1, \theta_2]$, $\gamma \in [0,1]$, $\mathbf{P}_{\epsilon,\omega}(x) \in [0, \mathbf{M}]$ for all $x \in B_1(0)$, and k satisfying $\omega/\theta^k \le \epsilon_0$, then

$$\begin{cases} \fint_{B_{\theta^k}(0)} \left| \Pi_\omega U_{\epsilon,\omega}|_{\Omega_f^\omega} - (\Pi_\omega U_{\epsilon,\omega}|_{\Omega_f^\omega})_{0,\theta^k} \right|^2 dx \le \theta^{2k\mu} |J_{\epsilon,\omega}|^2, \\ \fint_{B_{\theta^k}(0) \cap \Omega_m^\omega} \epsilon^2 \left| U_{\epsilon,\omega} - (\Pi_\omega U_{\epsilon,\omega}|_{\Omega_f^\omega})_{0,\theta^k} \right|^2 dx \le \theta^{2k\mu} |J_{\epsilon,\omega}|^2, \end{cases} \tag{4.12}$$

where $\mu \equiv \frac{\delta}{n+\delta}$ and

$$\begin{aligned} J_{\epsilon,\omega} \equiv 3\epsilon_0^{-1} (&\|\mathbf{E}_{\epsilon,\omega} U_{\epsilon,\omega}\|_{L^2(B_1(0))} + \epsilon^{-1} \|Q_{\epsilon,\omega}, \max\{\epsilon,\omega\} F_{\epsilon,\omega}\|_{L^{n+\delta}(B_1(0) \cap \Omega_m^\omega)} \\ &+ \|Q_{\epsilon,\omega}, F_{\epsilon,\omega}\|_{L^{n+\delta}(B_1(0) \cap \Omega_f^\omega)}). \end{aligned}$$

Proof. Let c denote a constant independent of ϵ, ω, γ. This proof is done by induction. For $k = 1$, we define $\mathbb{U}_{\epsilon,\omega} \equiv \frac{U_{\epsilon,\omega}}{J_{\epsilon,\omega}}$, $\mathbb{Q}_{\epsilon,\omega} \equiv \frac{Q_{\epsilon,\omega}}{J_{\epsilon,\omega}}$, $\mathbb{F}_{\epsilon,\omega} \equiv \frac{F_{\epsilon,\omega}}{J_{\epsilon,\omega}}$, $\mathbb{P}_{\epsilon,\omega} \equiv \gamma \mathbf{P}_{\epsilon,\omega}$. Then they satisfy (4.1) and (4.2) with $\nu = \omega$. By Lemma 4.1,

$$\begin{cases} \fint_{B_\theta(0)} \left| \Pi_\omega \mathbb{U}_{\epsilon,\omega}|_{\Omega_f^\omega} - (\Pi_\omega \mathbb{U}_{\epsilon,\omega}|_{\Omega_f^\omega})_{0,\theta} \right|^2 dx \le \theta^{2\mu}, \\ \fint_{B_\theta(0) \cap \Omega_m^\omega} \epsilon^2 \left| \mathbb{U}_{\epsilon,\omega} - (\Pi_\omega \mathbb{U}_{\epsilon,\omega}|_{\Omega_f^\omega})_{0,\theta} \right|^2 dx \le \theta^{2\mu}. \end{cases}$$

This implies (4.12) for $k = 1$. By energy method and A2, any solution of (4.11) satisfies

$$\begin{aligned} \|U_{\epsilon,\omega}\|_{H^1(B_{4/5}(0) \cap \Omega_f^\omega)} \le c(&\|\mathbf{E}_{\epsilon,\omega} U_{\epsilon,\omega}\|_{L^2(B_1(0))} + \|\mathbf{E}_{1/\epsilon,\omega} Q_{\epsilon,\omega}\|_{L^2(B_1(0))} \\ &+ \|F_{\epsilon,\omega}\|_{L^2(B_1(0))} + \omega \epsilon^{-1} \|F_{\epsilon,\omega}\|_{L^2(B_1(0) \cap \Omega_m^\omega)}) \equiv c\hat{\mathcal{I}}. \end{aligned}$$

By Theorem 7.26 [11] and Remark 2.1,

$$\left\|\Pi_\omega U_{\epsilon,\omega}|_{\Omega_f^\omega}\right\|_{L^{\frac{2n}{n-2}}(B_{4/5}(0))} \leq c\hat{\mathcal{I}}. \tag{4.13}$$

Suppose (4.12) holds for some k satisfying $\omega/\theta^k \leq \epsilon_0$, we define, in $B_1(0)\setminus\partial\Omega_m^\omega/\theta^k$,

$$\begin{cases} \mathbb{U}_{\epsilon,\omega/\theta^k}(x) \equiv J_{\epsilon,\omega}^{-1}\theta^{-k\mu}\big(U_{\epsilon,\omega}(\theta^k x) - (\Pi_\omega U_{\epsilon,\omega}|_{\Omega_f^\omega})_{0,\theta^k}\big), \\ \mathbb{Q}_{\epsilon,\omega/\theta^k}(x) \equiv J_{\epsilon,\omega}^{-1}\theta^{k(1-\mu)}Q_{\epsilon,\omega}(\theta^k x), \\ \mathbb{F}_{\epsilon,\omega/\theta^k}(x) \equiv J_{\epsilon,\omega}^{-1}\theta^{k(2-\mu)}\big(F_{\epsilon,\omega}(\theta^k x) - \gamma\mathbf{E}_{\epsilon,\omega/\theta^k}(x)\mathbf{P}_{\epsilon,\omega}(\theta^k x)(\Pi_\omega U_{\epsilon,\omega}|_{\Omega_f^\omega})_{0,\theta^k}\big), \\ \mathbb{P}_{\epsilon,\omega/\theta^k}(x) \equiv \theta^{2k}\gamma\mathbf{P}_{\epsilon,\omega}(\theta^k x). \end{cases}$$

Then they satisfy

$$-\nabla\cdot(\mathbf{E}_{\epsilon^2,\omega/\theta^k}\mathbf{K}_{\omega/\theta^k}\nabla\mathbb{U}_{\epsilon,\omega/\theta^k}+\mathbb{Q}_{\epsilon,\omega/\theta^k})+\mathbf{E}_{\epsilon,\omega/\theta^k}\mathbb{P}_{\epsilon,\omega/\theta^k}\mathbb{U}_{\epsilon,\omega/\theta^k} = \mathbb{F}_{\epsilon,\omega/\theta^k} \quad \text{in } B_1(0).$$

By triangle inequality,

$$\begin{cases} \|\mathbb{F}_{\epsilon,\omega/\theta^k}\|_{L^{n+\delta}(B_1(0)\cap\Omega_f^\omega/\theta^k)} \\ \quad \leq \frac{\theta^k}{J_{\epsilon,\omega}}\|F_{\epsilon,\omega}\|_{L^{n+\delta}(B_{\theta^k}(0)\cap\Omega_f^\omega)} + \frac{\theta^{k(3-\mu-n/2)}\mathbf{M}}{J_{\epsilon,\omega}}\left\|\Pi_\omega U_{\epsilon,\omega}|_{\Omega_f^\omega}\right\|_{L^{2n/(n-2)}(B_{\theta^k}(0))}, \\ \|\mathbb{F}_{\epsilon,\omega/\theta^k}\|_{L^{n+\delta}(B_1(0)\cap\Omega_m^\omega/\theta^k)} \\ \quad \leq \frac{\theta^k}{J_{\epsilon,\omega}}\|F_{\epsilon,\omega}\|_{L^{n+\delta}(B_{\theta^k}(0)\cap\Omega_m^\omega)} + \frac{\epsilon\theta^{k(3-\mu-n/2)}\mathbf{M}}{J_{\epsilon,\omega}}\left\|\Pi_\omega U_{\epsilon,\omega}|_{\Omega_f^\omega}\right\|_{L^{2n/(n-2)}(B_{\theta^k}(0))}. \end{cases}$$

By induction, (4.13), and small θ,

$$\begin{cases} \mathbb{P}_{\epsilon,\omega/\theta^k}(x) \in [0,\mathbf{M}] \quad \text{for all } x \in B_1(0), \\ \|\mathbf{E}_{\epsilon,\omega/\theta^k}\mathbb{U}_{\epsilon,\omega/\theta^k}\|_{L^2(B_1(0))} \leq 1, \\ \|\epsilon_0^{-1}\mathbb{Q}_{\epsilon,\omega/\theta^k}\mathcal{X}_{\Omega_f^\omega/\theta^k} + \epsilon^{-1}\mathbb{Q}_{\epsilon,\omega/\theta^k}\mathcal{X}_{\Omega_m^\omega/\theta^k}\|_{L^{n+\delta}(B_1(0))} \leq 1, \\ \epsilon_0^{-1}\|\mathbb{F}_{\epsilon,\omega/\theta^k}\mathcal{X}_{\Omega_f^\omega/\theta^k} + \epsilon^{-1}\max\{\epsilon,\omega/\theta^k\}\mathbb{F}_{\epsilon,\omega/\theta^k}\mathcal{X}_{\Omega_m^\omega/\theta^k}\|_{L^{n+\delta}(B_1(0))} \leq 1. \end{cases}$$

By Lemma 4.1 (take $\nu = \omega/\theta^k$), we obtain

$$\begin{cases} \fint_{B_\theta(0)} \left|\Pi_{\omega/\theta^k}\mathbb{U}_{\epsilon,\omega/\theta^k}|_{\Omega_f^\omega/\theta^k} - (\Pi_{\omega/\theta^k}\mathbb{U}_{\epsilon,\omega/\theta^k}|_{\Omega_f^\omega/\theta^k})_{0,\theta}\right|^2 dx \leq \theta^{2\mu}, \\ \fint_{B_\theta(0)\cap\Omega_m^\omega/\theta^k} \epsilon^2\left|\mathbb{U}_{\epsilon,\omega/\theta^k} - (\Pi_{\omega/\theta^k}\mathbb{U}_{\epsilon,\omega/\theta^k}|_{\Omega_f^\omega/\theta^k})_{0,\theta}\right|^2 dx \leq \theta^{2\mu}. \end{cases} \tag{4.14}$$

Note, by Remark 2.1,

$$\begin{cases} \fint_{B_\theta(0)} \left|\Pi_{\omega/\theta^k}\mathbb{U}_{\epsilon,\omega/\theta^k}|_{\Omega_f^\omega/\theta^k} - (\Pi_{\omega/\theta^k}\mathbb{U}_{\epsilon,\omega/\theta^k}|_{\Omega_f^\omega/\theta^k})_{0,\theta}\right|^2 dx \\ \qquad = \fint_{B_{\theta^{k+1}}(0)} \frac{\left|\Pi_\omega U_{\epsilon,\omega}|_{\Omega_f^\omega} - (\Pi_\omega U_{\epsilon,\omega}|_{\Omega_f^\omega})_{0,\theta^{k+1}}\right|^2}{|J_{\epsilon,\omega}|^2\ \theta^{2k\mu}}dx, \\ \fint_{B_\theta(0)\cap\Omega_m^\omega/\theta^k} \left|\mathbb{U}_{\epsilon,\omega/\theta^k} - (\Pi_{\omega/\theta^k}\mathbb{U}_{\epsilon,\omega/\theta^k}|_{\Omega_f^\omega/\theta^k})_{0,\theta}\right|^2 dx \\ \qquad = \fint_{B_{\theta^{k+1}}(0)\cap\Omega_m^\omega} \frac{\left|U_{\epsilon,\omega} - (\Pi_\omega U_{\epsilon,\omega}|_{\Omega_f^\omega})_{0,\theta^{k+1}}\right|^2}{|J_{\epsilon,\omega}|^2\ \theta^{2k\mu}}dx. \end{cases} \tag{4.15}$$

Equations (4.14)–(4.15) imply the inequality (4.12) for $k+1$ case. □

Lemma 4.3. *For any $\delta \in (0,3)$ and $\mathbf{M} > 0$, there is a $\epsilon_* \in (0,1)$ (depending on $\delta, \mathbf{M}, \mathbf{K}, Y_f$) such that if $\epsilon, \omega \in (0, \epsilon_*)$, $\gamma \in [0,1]$, and $\mathbf{P}_{\epsilon,\omega}(x) \in [0, \mathbf{M}]$ for all $x \in B_1(0)$, then any solution of (4.11) satisfies*

$$[U_{\epsilon,\omega}]_{C^{0,\mu}(\overline{B_{1/2}(0)\cap\Omega_f^\omega})} + \sup_{\substack{j\in\mathbb{Z}^n \\ \omega(Y_m+j)\subset B_{1/2}(0)\cap\Omega_m^\omega}} \epsilon^{3/2}[U_{\epsilon,\omega}]_{C^{0,\mu}(\omega(\overline{Y_m}+j))} \leq cJ_{\epsilon,\omega}, \quad (4.16)$$

where c is a constant independent of ϵ, ω, γ. See Lemma 4.2 for $\mu, J_{\epsilon,\omega}$.

Proof. Let $\theta_1, \theta_2, \epsilon_0$ be same as those in Lemma 4.2, define $\epsilon_* \equiv \epsilon_0\theta_2/2$, and let $\epsilon, \omega \leq \epsilon_*$. Denote by c a constant independent of ϵ, ω, γ. Because of $\theta_1 < \theta_2^2$, for any $r \in [\omega/\epsilon_0, \theta_2]$, there are $\theta \in [\theta_1, \theta_2]$ and $k \in \mathbb{N}$ satisfying $r = \theta^k$. Lemma 4.2 implies, for any $r \in [\omega/\epsilon_0, \theta_2]$,

$$\begin{cases} \displaystyle\fint_{B_r(0)} \left|\Pi_\omega U_{\epsilon,\omega}|_{\Omega_f^\omega} - (\Pi_\omega U_{\epsilon,\omega}|_{\Omega_f^\omega})_{0,r}\right|^2 dx \leq r^{2\mu}|J_{\epsilon,\omega}|^2, \\ \displaystyle\fint_{B_r(0)\cap\Omega_m^\omega} \epsilon^2 \left|U_{\epsilon,\omega} - (\Pi_\omega U_{\epsilon,\omega}|_{\Omega_f^\omega})_{0,r}\right|^2 dx \leq r^{2\mu}|J_{\epsilon,\omega}|^2. \end{cases} \quad (4.17)$$

Now we define, in $B_{2/\epsilon_0}(0) \setminus \partial\Omega_m^\omega/\omega$,

$$\begin{cases} \mathbb{U}_{\epsilon,1}(x) \equiv J_{\epsilon,\omega}^{-1}\omega^{-\mu}\big(U_{\epsilon,\omega}(\omega x) - (\Pi_\omega U_{\epsilon,\omega}|_{\Omega_f^\omega})_{0,2\omega/\epsilon_0}\big), \\ \mathbb{Q}_{\epsilon,1}(x) \equiv J_{\epsilon,\omega}^{-1}\omega^{1-\mu}Q_{\epsilon,\omega}(\omega x), \\ \mathbb{F}_{\epsilon,1}(x) \equiv J_{\epsilon,\omega}^{-1}\omega^{2-\mu}\big(F_{\epsilon,\omega}(\omega x) - \gamma\mathbf{E}_{\epsilon,1}(x)\mathbf{P}_{\epsilon,\omega}(\omega x)(\Pi_\omega U_{\epsilon,\omega}|_{\Omega_f^\omega})_{0,2\omega/\epsilon_0}\big), \\ \mathbb{P}_{\epsilon,1} \equiv \omega^2\gamma\mathbf{P}_{\epsilon,\omega}(\omega x). \end{cases}$$

Then they satisfy

$$-\nabla\cdot(\mathbf{E}_{\epsilon^2,1}\mathbf{K}\nabla\mathbb{U}_{\epsilon,1} + \mathbb{Q}_{\epsilon,1}) + \mathbf{E}_{\epsilon,1}\mathbb{P}_{\epsilon,1}\mathbb{U}_{\epsilon,1} = \mathbb{F}_{\epsilon,1} \quad \text{in } B_{\frac{2}{\epsilon_0}}(0).$$

Take $r = \frac{2\omega}{\epsilon_0}$ in (4.17) to get

$$\begin{cases} \mathbb{P}_{\epsilon,1} \in [0, \mathbf{M}] \quad \text{for all } x \in B_{2/\epsilon_0}(0), \\ \|\mathbf{E}_{\epsilon,1}\mathbb{U}_{\epsilon,1}\|_{L^2(B_{2/\epsilon_0}(0))} + \|\mathbf{E}_{1/\epsilon,1}\mathbb{Q}_{\epsilon,1}, \mathbf{E}_{1/\epsilon,1}\mathbb{F}_{\epsilon,1}\|_{L^{n+\delta}(B_{2/\epsilon_0}(0))} \leq c. \end{cases}$$

By (3.3) of Lemma 3.2,

$$[\mathbb{U}_{\epsilon,1}]_{C^{0,\mu}(\overline{B_{1/\epsilon_0}(0)\cap\Omega_f^\omega/\omega})} + \epsilon^{3/2}[\mathbb{U}_{\epsilon,1}]_{C^{0,\mu}(\overline{B_{1/\epsilon_0}(0)\cap\Omega_m^\omega/\omega})} \leq c. \quad (4.18)$$

(4.18) implies that $(4.17)_1$ also holds for $r \leq \omega/\epsilon_0$. So $(4.17)_1$ holds for $r \leq \theta_2$. Next we shift the origin of the coordinate system to any point $z \in B_{1/2}(0)$ and repeat above argument to see that $(4.17)_1$ with 0 replaced by any $z \in B_{1/2}(0)$ also holds for $r \in (0, \theta_2)$. Together with Theorem 1.2 in page 70 [10], we obtain the Hölder estimate of $\Pi_\omega U_{\epsilon,\omega}$ in $B_{1/2}(0)$. Hölder estimate of $U_{\epsilon,\omega}$ in $\omega(\overline{Y_m}+j) \subset B_{1/2}(0)\cap\overline{\Omega_m^\omega}$ is from (4.18). So (4.16) is proved. □

Remark 4.1. Let ϵ_* be same as that in Lemma 4.3. By (3.3) of Lemma 3.2 with $p = n+\delta$, we know that if $\delta \in (0,3)$, $\mathbf{M} > 0$, $\omega \in [\epsilon_*, 1]$, $\epsilon \in (0, \epsilon_*)$, $\gamma \in [0,1]$, and

$\mathbf{P}_{\epsilon,\omega}(x) \in [0, \mathbf{M}]$ for all $x \in B_1(0)$, any solution of (4.11) satisfies (4.16). Together with Lemma 4.3, we know that any solution of (4.11) satisfies (4.16) if $\delta \in (0,3)$, $\mathbf{M} > 0$, $\omega \in (0,1)$, $\epsilon \in (0, \epsilon_*)$, $\gamma \in [0,1]$, and $\mathbf{P}_{\epsilon,\omega}(x) \in [0, \mathbf{M}]$ for all $x \in B_1(0)$.

Let us consider the solutions of (4.11) with $\mathbf{P}_{\epsilon,\omega} = 0$. By tracing the arguments of Lemma 4.2, Lemma 4.3, and Remark 4.1 as well as employing (3.16) of Lemma 3.3, then we have

Lemma 4.4. *For any $\delta > 0$, there is a $\epsilon_* > 0$ (depending on $\delta, \mathbf{K}, Y_f$) such that if $\epsilon \in (0, \epsilon_*)$ and $\omega \in (0,1)$, then any solution of (4.11) with $\mathbf{P}_{\epsilon,\omega} = 0$ satisfies*

$$[U_{\epsilon,\omega}]_{C^{0,\mu}(\overline{B_{1/2}(0)\cap\Omega_f^\omega})} + \sup_{\substack{j\in\mathbb{Z}^n \\ \omega(Y_m+j)\subset B_{1/2}(0)\cap\Omega_m^\omega}} \epsilon[U_{\epsilon,\omega}]_{C^{0,\mu}(\omega(\overline{Y_m}+j))} \le cJ_{\epsilon,\omega},$$

where c is a constant independent of ϵ, ω. See Lemma 4.2 for $\mu, J_{\epsilon,\omega}$.

4.2. *Boundary estimate*

In this subsection, we assume $0 \in \partial\Omega$. By A1, there is a $C^{2,1}$ function $\rho : \mathbb{R}^{n-1} \to \mathbb{R}$ satisfying

$$\begin{cases} \rho(0) = |\nabla\rho(0)| = 0, \\ B_1(0) \cap \Omega/t = B_1(0) \cap \{(x', x_n) \in \mathbb{R}^n |\ tx_n > \rho(tx')\} \quad \text{if } t \in (0,1]. \end{cases}$$

If $t = 0$, we define $B_1(0) \cap \Omega/t \equiv B_1(0) \cap \{(x', x_n) \in \mathbb{R}^n |\ x_n > 0\}$. Set

$$\mathcal{E}_{\nu,\omega,t} \equiv \begin{cases} 1 & \text{in } \Omega_f^\omega/t \\ \nu & \text{in } \Omega_m^\omega/t \end{cases} \qquad \text{for } t \in (0,1]. \tag{4.19}$$

Lemma 4.5. *For any $\delta, \mathbf{M} > 0$, there are $\tilde{\theta}_1, \tilde{\theta}_2 \in (0,1)$ (depending on $\delta, \mathbf{M}, \mathbf{K}, Y_f$, Ω) satisfying $\tilde{\theta}_1 < \tilde{\theta}_2^2$ and there is a $\tilde{\epsilon}_0 > 0$ (depending on $\tilde{\theta}_1, \tilde{\theta}_2, \delta, \mathbf{M}, \mathbf{K}, \Omega$) satisfying $\tilde{\epsilon}_0 < \epsilon_0$ (ϵ_0 is that in Lemma 4.1) such that if*

$$\begin{cases} -\nabla \cdot (\mathcal{E}_{\epsilon^2,\omega,s}\mathbf{K}_{\omega/s}\nabla\mathbb{U}_{\epsilon,\omega,s} + \mathbb{Q}_{\epsilon,\omega,s}) \\ \qquad +\mathcal{E}_{\epsilon,\omega,s}\mathbb{P}_{\epsilon,\omega,s}\mathbb{U}_{\epsilon,\omega,s} = \mathbb{F}_{\epsilon,\omega,s} & \text{in } B_1(0)\cap\Omega/s, \\ \mathbb{U}_{\epsilon,\omega,s} = 0 & \text{on } B_1(0)\cap\partial\Omega/s, \end{cases} \tag{4.20}$$

and if

$$\begin{cases} \epsilon, \frac{\omega}{s} \in (0, \tilde{\epsilon}_0),\ \ s \in (0,1],\ \ \tilde{\theta} \in [\tilde{\theta}_1, \tilde{\theta}_2],\ \ \mathbb{P}_{\epsilon,\omega,s}(x) \in [0, \mathbf{M}] \text{ for } x \in B_1(0)\cap\Omega/s, \\ \max\{\|\mathcal{E}_{\epsilon,\omega,s}\mathbb{U}_{\epsilon,\omega,s}\|_{L^2(B_1(0)\cap\Omega/s)}, \|\frac{1}{\tilde{\epsilon}_0}\mathbb{Q}_{\epsilon,\omega,s}\mathcal{X}_{\Omega_f^\omega/s} + \frac{1}{\epsilon}\mathbb{Q}_{\epsilon,\omega,s}\mathcal{X}_{\Omega_m^\omega/s}\|_{L^{n+\delta}(B_1(0))}, \\ \qquad \frac{1}{\tilde{\epsilon}_0}\|\mathbb{F}_{\epsilon,\omega,s}\mathcal{X}_{\Omega_f^\omega/s} + \frac{1}{\epsilon}\max\{\epsilon, \frac{\omega}{s}\}\mathbb{F}_{\epsilon,\omega,s}\mathcal{X}_{\Omega_m^\omega/s}\|_{L^{n+\delta}(B_1(0))}\} \le 1, \end{cases}$$

then

$$\begin{cases} \displaystyle\fint_{B_{\tilde{\theta}}(0)\cap\Omega/s} \left|\Pi_{\omega/s}\mathbb{U}_{\epsilon,\omega,s}|_{\Omega_f^\omega/s}\right|^2 dx \le \tilde{\theta}^{2\mu}, \\ \displaystyle\fint_{B_{\tilde{\theta}}(0)\cap\Omega_m^\omega/s} \epsilon^2|\mathbb{U}_{\epsilon,\omega,s}|^2 dx \le \tilde{\theta}^{2\mu}, \end{cases} \tag{4.21}$$

where $\mu \equiv \frac{\delta}{n+\delta}$.

Proof. Consider the following problem

$$\begin{cases} -\nabla \cdot (\mathcal{K}_0 \nabla \mathbb{U}) + \mathbb{P}\mathbb{U} = 0 & \text{in } B_{3/4}(0) \cap \Omega/t, \\ \mathbb{U} = 0 & \text{on } B_{3/4}(0) \cap \partial\Omega/t, \end{cases} \tag{4.22}$$

where $t \in [0,1]$, $\mathbb{P}(x) \in [0, \mathbf{M}]$ for $x \in B_{3/4}(0) \cap \Omega/t$, and $\mathcal{K}_0$ is defined in (2.9). Any solution $\mathbb{U}$ of (4.22) satisfies, by Theorem 9.13 [11] and (3.19),

$$\|\mathbb{U}\|_{C^{1,\alpha}(\overline{B_{1/2}(0)\cap\Omega/t})} \le c\|\mathbb{U}\|_{L^2(B_{3/4}(0)\cap\Omega/t)}, \tag{4.23}$$

where $\alpha \in (0,1)$ and c is a constant depending on $\mathbf{M}, \mathcal{K}_0, \Omega$ but independent of t. If $\breve{\mu}$ satisfies $\mu < \breve{\mu} < 1$, by (4.23),

$$\fint_{B_{\tilde{\theta}}(0)\cap\Omega/t} \mathbb{U}^2 dx \le \tilde{\theta}^{2\breve{\mu}} \fint_{B_{3/4}(0)\cap\Omega/t} \mathbb{U}^2 dx \tag{4.24}$$

for small $\tilde{\theta}$ (depending on $\delta, \mathbf{M}, \mathcal{K}_0, \Omega$). Fix $\tilde{\theta}_1, \tilde{\theta}_2 \in (0, \frac{1}{2})$ such that $\tilde{\theta}_1 < \tilde{\theta}_2^2$ and (4.24) holds for any $\tilde{\theta} \in [\tilde{\theta}_1, \tilde{\theta}_2]$.

We claim $(4.21)_1$. If not, there is a sequence $\{s_{\epsilon,\omega}, \tilde{\theta}_{\epsilon,\omega}, \mathbb{P}_{\epsilon,\omega,s_{\epsilon,\omega}}, \mathbb{U}_{\epsilon,\omega,s_{\epsilon,\omega}}, \mathbb{Q}_{\epsilon,\omega,s_{\epsilon,\omega}}, \mathbb{F}_{\epsilon,\omega,s_{\epsilon,\omega}}\}$ satisfying (4.20) and, as $\epsilon, \omega \to 0$,

$$\begin{cases} \omega/s_{\epsilon,\omega} \to 0, \;\; s_{\epsilon,\omega} \to s_* \in [0,1], \;\; \tilde{\theta}_{\epsilon,\omega} \to \tilde{\theta}_* \in [\tilde{\theta}_1, \tilde{\theta}_2], \\ \mathbb{P}_{\epsilon,\omega,s_{\epsilon,\omega}}(x) \in [0,\mathbf{M}] \text{ for } x \in B_1(0) \cap \Omega/s_{\epsilon,\omega}, \\ \max\{\|\mathcal{E}_{\epsilon,\omega,s_{\epsilon,\omega}} \mathbb{U}_{\epsilon,\omega,s_{\epsilon,\omega}}\|_{L^2(B_1(0)\cap\Omega/s_{\epsilon,\omega})}, \\ \qquad \epsilon^{-1}\|\mathbb{Q}_{\epsilon,\omega,s_{\epsilon,\omega}}\|_{L^{n+\delta}(B_1(0)\cap\Omega_m^\omega/s_{\epsilon,\omega})}\} \le 1, \\ \lim\limits_{\epsilon,\omega/s_{\epsilon,\omega}\to 0} \|\mathbb{Q}_{\epsilon,\omega,s_{\epsilon,\omega}}, \mathbb{F}_{\epsilon,\omega,s_{\epsilon,\omega}}\|_{L^{n+\delta}(B_1(0)\cap\Omega_f^\omega/s_{\epsilon,\omega})} \\ \qquad +\epsilon^{-1}\max\{\epsilon, \omega/s_{\epsilon,\omega}\}\|\mathbb{F}_{\epsilon,\omega,s_{\epsilon,\omega}}\|_{L^{n+\delta}(B_1(0)\cap\Omega_m^\omega/s_{\epsilon,\omega})} = 0, \\ \fint_{B_{\tilde{\theta}_{\epsilon,\omega}}(0)\cap\Omega/s_{\epsilon,\omega}} \left| \Pi_{\omega/s_{\epsilon,\omega}} \mathbb{U}_{\epsilon,\omega,s_{\epsilon,\omega}}|_{\Omega_f^\omega/s_{\epsilon,\omega}} \right|^2 dx > |\tilde{\theta}_{\epsilon,\omega}|^{2\mu}. \end{cases} \tag{4.25}$$

By energy method and A2, there is a constant c independent of $\epsilon, \omega, s_{\epsilon,\omega}$ such that

$$\|\mathbb{U}_{\epsilon,\omega,s_{\epsilon,\omega}}\|_{H^1(B_{4/5}(0)\cap\Omega_f^\omega/s_{\epsilon,\omega})} + \|\epsilon\nabla\mathbb{U}_{\epsilon,\omega,s_{\epsilon,\omega}}, \epsilon^{1/2}\mathbb{P}^{1/2}_{\epsilon,\omega,s_{\epsilon,\omega}}\mathbb{U}_{\epsilon,\omega,s_{\epsilon,\omega}}\|_{L^2(B_{4/5}(0)\cap\Omega_m^\omega/s_{\epsilon,\omega})} \le c.$$

By compactness principle and by tracing the proof of Theorem 2.3 [3], we can extract a subsequence (same notation for subsequence) such that, as $\epsilon, \omega/s_{\epsilon,\omega} \to 0$,

$$\begin{cases} \Pi_{\omega/s_{\epsilon,\omega}} \mathbb{U}_{\epsilon,\omega,s_{\epsilon,\omega}}|_{\Omega_f^\omega/s_{\epsilon,\omega}} \to \mathbb{U} & \text{in } L^2(B_{3/4}(0)\cap\Omega/s_*) \text{ strongly}, \\ \mathcal{E}_{\epsilon^2,\omega,s_{\epsilon,\omega}} \mathbf{K}_{\omega/s_{\epsilon,\omega}} \nabla\mathbb{U}_{\epsilon,\omega,s_{\epsilon,\omega}} \to \mathcal{K}_0\nabla\mathbb{U} & \text{in } L^2(B_{3/4}(0)\cap\Omega/s_*) \text{ weakly}, \\ \mathcal{E}_{\epsilon,\omega,s_{\epsilon,\omega}} \mathbb{P}_{\epsilon,\omega,s_{\epsilon,\omega}} \mathbb{U}_{\epsilon,\omega,s_{\epsilon,\omega}} \to \mathbb{P}\mathbb{U} & \text{in } L^2(B_{3/4}(0)\cap\Omega/s_*) \text{ weakly}, \end{cases} \tag{4.26}$$

where $\mathcal{K}_0$ is a constant symmetric positive definite matrix, $\ell_5 \le \mathcal{K}_0 \le \ell_6$ (see (2.9) and (3.19)), and $\mathbb{P}(x) \in [0, \mathbf{M}]$ for $x \in B_{3/4}(0) \cap \Omega/s_*$. In (4.26), function $\mathbb{U}$ satisfies

(4.22) with $t = s_*$. By (4.24)–(4.25), we conclude

$$|\tilde{\theta}_*|^{2\mu} \le \lim_{\epsilon,\omega/s_{\epsilon,\omega}\to 0} \fint_{B_{\tilde{\theta}_{\epsilon,\omega}}(0)\cap\Omega/s_{\epsilon,\omega}} \left|\Pi_{\omega/s_{\epsilon,\omega}} \mathbb{U}_{\epsilon,\omega,s_{\epsilon,\omega}}|_{\Omega_f^\omega/s_{\epsilon,\omega}}\right|^2 dx$$
$$= \fint_{B_{\tilde{\theta}_*}(0)\cap\Omega/s_*} \mathbb{U}^2 dx \le |\tilde{\theta}_*|^{2\tilde{\mu}} \fint_{B_{3/4}(0)\cap\Omega/s_*} \mathbb{U}^2 dx. \tag{4.27}$$

But (4.27) is impossible if we take $\tilde{\theta}_2$ small enough. Therefore, there is a $\tilde{\epsilon}_0$ such that $(4.21)_1$ holds for $\epsilon, \omega/s \le \tilde{\epsilon}_0$. Clearly, $\tilde{\epsilon}_0$ can be chosen so that $\tilde{\epsilon}_0 < \epsilon_0$ (see Lemma 4.1 for ϵ_0). The proof of $(4.21)_2$ is similar to that of $(4.3)_2$, so we skip it. □

Lemma 4.6. *For any $\delta, \mathbf{M} > 0$, there are $\tilde{\theta}_1, \tilde{\theta}_2 \in (0,1)$ (depending on $\delta, \mathbf{M}, \mathbf{K}, Y_f$, Ω) satisfying $\tilde{\theta}_1 < \tilde{\theta}_2^2$ and there is a $\tilde{\epsilon}_0 > 0$ (depending on $\tilde{\theta}_1, \tilde{\theta}_2, \delta, \mathbf{M}, \mathbf{K}, \Omega$) satisfying $\tilde{\epsilon}_0 < \epsilon_0$ (ϵ_0 is that in Lemma 4.2) such that if*

$$\begin{cases} -\nabla\cdot(\mathbf{E}_{\epsilon^2,\omega}\mathbf{K}_\omega \nabla U_{\epsilon,\omega} + Q_{\epsilon,\omega}) + \mathbf{E}_{\epsilon,\omega}\mathbf{P}_{\epsilon,\omega}U_{\epsilon,\omega} = F_{\epsilon,\omega} & \text{in } B_1(0)\cap\Omega, \\ U_{\epsilon,\omega} = 0 & \text{on } B_1(0)\cap\partial\Omega, \end{cases} \tag{4.28}$$

and if $\epsilon, \omega \in (0, \tilde{\epsilon}_0)$, $\tilde{\theta} \in [\tilde{\theta}_1, \tilde{\theta}_2]$, $\mathbf{P}_{\epsilon,\omega}(x) \in [0, \mathbf{M}]$ for $x \in B_1(0)\cap\Omega$, and k satisfying $\omega/\tilde{\theta}^k \le \tilde{\epsilon}_0$, then

$$\begin{cases} \fint_{B_{\tilde{\theta}^k}(0)\cap\Omega} \left|\Pi_\omega U_{\epsilon,\omega}|_{\Omega_f^\omega}\right|^2 dx \le \tilde{\theta}^{2k\mu}|\tilde{J}_{\epsilon,\omega}|^2, \\ \fint_{B_{\tilde{\theta}^k}(0)\cap\Omega_m^\omega} \epsilon^2 \left|U_{\epsilon,\omega}\right|^2 dx \le \tilde{\theta}^{2k\mu}|\tilde{J}_{\epsilon,\omega}|^2, \end{cases} \tag{4.29}$$

where $\tilde{J}_{\epsilon,\omega} \equiv \|\mathbf{E}_{\epsilon,\omega}U_{\epsilon,\omega}\|_{L^2(B_1(0)\cap\Omega)} + \epsilon^{-1}\|Q_{\epsilon,\omega}, \frac{1}{\tilde{\epsilon}_0}\max\{\epsilon,\omega\}F_{\epsilon,\omega}\|_{L^{n+\delta}(B_1(0)\cap\Omega_m^\omega)} + \frac{1}{\tilde{\epsilon}_0}\|Q_{\epsilon,\omega}, F_{\epsilon,\omega}\|_{L^{n+\delta}(B_1(0)\cap\Omega_f^\omega)}$ and $\mu \equiv \frac{\delta}{n+\delta}$.

Proof. The proof is similar to that of Lemma 4.2 and is done by induction on k. For $k = 1$, (4.29) is deduced from Lemma 4.5 with $s = 1$. Suppose (4.29) holds for some k with $\omega/\tilde{\theta}^k \le \tilde{\epsilon}_0$, we define

$$\begin{cases} \mathbb{U}_{\epsilon,\omega,\tilde{\theta}^k}(x) \equiv \tilde{J}_{\epsilon,\omega}^{-1}\tilde{\theta}^{-k\mu}U_{\epsilon,\omega}(\tilde{\theta}^k x) \\ \mathbb{Q}_{\epsilon,\omega,\tilde{\theta}^k}(x) \equiv \tilde{J}_{\epsilon,\omega}^{-1}\tilde{\theta}^{k(1-\mu)}Q_{\epsilon,\omega}(\tilde{\theta}^k x) \\ \mathbb{F}_{\epsilon,\omega,\tilde{\theta}^k}(x) \equiv \tilde{J}_{\epsilon,\omega}^{-1}\tilde{\theta}^{k(2-\mu)}F_{\epsilon,\omega}(\tilde{\theta}^k x) \\ \mathbb{P}_{\epsilon,\omega,\tilde{\theta}^k}(x) \equiv \tilde{\theta}^{2k}\mathbf{P}_{\epsilon,\omega}(\tilde{\theta}^k x) \end{cases} \quad \text{in } B_1(0)\setminus\partial\Omega_m^\omega/\tilde{\theta}^k.$$

Then they satisfies

$$\begin{cases} -\nabla\cdot(\mathcal{E}_{\epsilon^2,\omega,\tilde{\theta}^k}\mathbf{K}_{\omega/\tilde{\theta}^k}\nabla\mathbb{U}_{\epsilon,\omega,\tilde{\theta}^k} + \mathbb{Q}_{\epsilon,\omega,\tilde{\theta}^k}) \\ \qquad +\mathcal{E}_{\epsilon,\omega,\tilde{\theta}^k}\mathbb{P}_{\epsilon,\omega,\tilde{\theta}^k}\mathbb{U}_{\epsilon,\omega,\tilde{\theta}^k} = \mathbb{F}_{\epsilon,\omega,\tilde{\theta}^k} & \text{in } B_1(0)\cap\Omega/\tilde{\theta}^k, \\ \mathbb{U}_{\epsilon,\omega,\tilde{\theta}^k} = 0 & \text{on } B_1(0)\cap\partial\Omega/\tilde{\theta}^k. \end{cases}$$

Following the argument of Lemma 4.2 and employing Lemma 4.5 with $s = \tilde{\theta}^k$, we obtain (4.29) with $k+1$ in place of k. □

Lemma 4.7. *For any $\delta \in (0,3)$ and $\mathbf{M} > 0$, there is a $\tilde{\epsilon}_* \in (0,1)$ (depending on $\delta, \mathbf{M}, \mathbf{K}, Y_f, \Omega$) such that if $\epsilon, \omega \leq \tilde{\epsilon}_*$ and $\mathbf{P}_{\epsilon,\omega}(x) \in [0, \mathbf{M}]$ for $x \in B_1(0) \cap \Omega$, then any solution of (4.28) satisfies*

$$[U_{\epsilon,\omega}]_{C^{0,\mu}(\overline{B_{1/2}(0)\cap\Omega_f^\omega})} + \sup_{\substack{j\in\mathbb{Z}^n \\ \omega(Y_m+j)\subset B_{1/2}(0)\cap\Omega_m^\omega}} \epsilon^{3/2}[U_{\epsilon,\omega}]_{C^{0,\mu}(\omega(\overline{Y_m}+j))} \leq c\hat{J}_{\epsilon,\omega}, \quad (4.30)$$

where $\mu \equiv \frac{\delta}{n+\delta}$; c is a constant independent of ϵ, ω; and $\hat{J}_{\epsilon,\omega}$ is defined as

$$\begin{aligned}\hat{J}_{\epsilon,\omega} \equiv \frac{3}{\tilde{\epsilon}_*}(&\|\mathbf{E}_{\epsilon,\omega}U_{\epsilon,\omega}\|_{L^2(B_1(0)\cap\Omega)} + \|Q_{\epsilon,\omega}, F_{\epsilon,\omega}\|_{L^{n+\delta}(B_1(0)\cap\Omega_f^\omega)} \\ &+\epsilon^{-1}\|Q_{\epsilon,\omega}, \max\{\epsilon,\omega\}F_{\epsilon,\omega}\|_{L^{n+\delta}(B_1(0)\cap\Omega_m^\omega)}).\end{aligned} \quad (4.31)$$

Proof. Let $\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\epsilon}_0, \tilde{J}_{\epsilon,\omega}$ be same as those in Lemma 4.6, set $\tilde{\epsilon}_* \equiv \min\{\tilde{\epsilon}_0\tilde{\theta}_2/3, \epsilon_*\}$ where ϵ_* is the one in Lemma 4.3, and let $\epsilon, \omega \leq \tilde{\epsilon}_*$. Denote by c a constant independent of ϵ, ω. By energy method and A2, any solution of (4.28) satisfies

$$\begin{aligned}\|U_{\epsilon,\omega}\|_{H^1(B_{3/4}(0)\cap\Omega_f^\omega)} &\leq c(\|\mathbf{E}_{\epsilon,\omega}U_{\epsilon,\omega}\|_{L^2(B_1(0)\cap\Omega)} + \|\mathbf{E}_{1/\epsilon,\omega}Q_{\epsilon,\omega}\|_{L^2(B_1(0)\cap\Omega)} \\ &+\|F_{\epsilon,\omega}\|_{L^2(B_1(0)\cap\Omega)} + \omega\epsilon^{-1}\|F_{\epsilon,\omega}\|_{L^2(B_1(0)\cap\Omega_m^\omega)}) \equiv c\check{\mathcal{I}}.\end{aligned}$$

By Theorem 7.26 [11] and Remark 2.1,

$$\left\|\Pi_\omega U_{\epsilon,\omega}|_{\Omega_f^\omega}\right\|_{L^{\frac{2n}{n-2}}(B_{3/4}(0)\cap\Omega)} \leq c\check{\mathcal{I}}. \quad (4.32)$$

For any $x \in B_{\tilde{\theta}_2/3}(0) \cap \Omega_f^\omega$, define $\eta(x) \equiv |x - x_0|$ where $x_0 \in \partial\Omega$ satisfying $|x - x_0| = \min_{y\in\partial\Omega}|x-y|$. Then we have either case (1) $\eta(x) > \frac{2\omega}{3\tilde{\epsilon}_0}$ or case (2) $\eta(x) \leq \frac{2\omega}{3\tilde{\epsilon}_0}$.

Let us consider case (1). Because of $\tilde{\theta}_1 < \tilde{\theta}_2^2$, for any $r \in [\omega/\tilde{\epsilon}_0, \tilde{\theta}_2]$, there are $\tilde{\theta} \in [\tilde{\theta}_1, \tilde{\theta}_2]$ and $k \in \mathbb{N}$ satisfying $r = \tilde{\theta}^k$. Since $\eta(x) \in [\frac{2\omega}{3\tilde{\epsilon}_0}, \frac{\tilde{\theta}_2}{3}]$, by Lemma 4.6,

$$\begin{cases} \displaystyle\fint_{B_r(x_0)\cap\Omega} \left|\Pi_\omega U_{\epsilon,\omega}|_{\Omega_f^\omega}\right|^2 dy \leq r^{2\mu}|\tilde{J}_{\epsilon,\omega}|^2 \\ \displaystyle\fint_{B_r(x_0)\cap\Omega_m^\omega} \epsilon^2\left|U_{\epsilon,\omega}\right|^2 dy \leq r^{2\mu}|\tilde{J}_{\epsilon,\omega}|^2 \end{cases} \quad \text{for } r \in [\tfrac{3}{2}\eta(x), \tilde{\theta}_2].$$

So, for $s \in [\frac{\eta(x)}{2}, \frac{\tilde{\theta}_2}{3}]$,

$$\begin{cases} \displaystyle\fint_{B_s(x)\cap\Omega} \left|\Pi_\omega U_{\epsilon,\omega}|_{\Omega_f^\omega} - (\Pi_\omega U_{\epsilon,\omega}|_{\Omega_f^\omega})_{x,s}\right|^2 dy \leq cs^{2\mu}|\tilde{J}_{\epsilon,\omega}|^2, \\ \displaystyle\fint_{B_s(x)\cap\Omega_m^\omega} \epsilon^2\left|U_{\epsilon,\omega} - (\Pi_\omega U_{\epsilon,\omega}|_{\Omega_f^\omega})_{x,s}\right|^2 dy \leq cs^{2\mu}|\tilde{J}_{\epsilon,\omega}|^2. \end{cases} \quad (4.33)$$

Next we shift the coordinate system such that x is located at the origin and we define, in $B_1(x) \setminus \partial\Omega_m^\omega/\eta(x)$,

$$\begin{cases} \mathbb{U}_{\epsilon,\omega,\eta(x)}(y) \equiv \dfrac{U_{\epsilon,\omega}(\eta(x)y) - (\Pi_\omega U_{\epsilon,\omega}|_{\Omega_f^\omega})_{x,\eta(x)}}{\hat{J}_{\epsilon,\omega}\ \ \eta^\mu(x)}, \\ \mathbb{Q}_{\epsilon,\omega,\eta(x)}(y) \equiv \dfrac{Q_{\epsilon,\omega}(\eta(x)y)}{\hat{J}_{\epsilon,\omega}\ \ \eta^{\mu-1}(x)}, \\ \mathbb{F}_{\epsilon,\omega,\eta(x)}(y) \equiv \dfrac{F_{\epsilon,\omega}(\eta(x)y) - \mathbf{E}_{\epsilon,\omega/\eta(x)}(y)\mathbf{P}_{\epsilon,\omega}(\eta(x)y)(\Pi_\omega U_{\epsilon,\omega}|_{\Omega_f^\omega})_{x,\eta(x)}}{\hat{J}_{\epsilon,\omega}\ \ \eta^{\mu-2}(x)}, \\ \mathbb{P}_{\epsilon,\omega,\eta(x)}(y) \equiv \mathbf{P}_{\epsilon,\omega}(\eta(x)y). \end{cases}$$

See (4.31) for $\hat{J}_{\epsilon,\omega}$. Then these functions satisfy

$$\begin{aligned}-\nabla\cdot(\mathbf{E}_{\epsilon^2,\omega/\eta(x)}\mathbf{K}_{\omega/\eta(x)}\nabla\mathbb{U}_{\epsilon,\omega,\eta(x)}+\mathbb{Q}_{\epsilon,\omega,\eta(x)})\\ +|\eta(x)|^2\mathbf{E}_{\epsilon,\omega/\eta(x)}\mathbb{P}_{\epsilon,\omega,\eta(x)}\mathbb{U}_{\epsilon,\omega,\eta(x)}=\mathbb{F}_{\epsilon,\omega,\eta(x)}\quad \text{in } B_1(x).\end{aligned}\tag{4.34}$$

Take $s=\eta(x)<1$ in (4.33) to see, by (4.32),

$$\begin{cases}\mathbb{P}_{\epsilon,\omega,\eta(x)}(y)\in[0,\mathbf{M}] \text{ for all } y\in B_1(x).\\ \tilde{\epsilon}_0^{-1}(\|\mathbf{E}_{\epsilon,\omega/\eta(x)}\mathbb{U}_{\epsilon,\omega,\eta(x)}\|_{L^2(B_1(x))}+\|\mathbb{Q}_{\epsilon,\omega,\eta(x)},\mathbb{F}_{\epsilon,\omega,\eta(x)}\|_{L^{n+\delta}(B_1(x)\cap\Omega_f^\omega/\eta(x))}\\ \quad +\epsilon^{-1}\|\mathbb{Q}_{\epsilon,\omega,\eta(x)},\max\{\epsilon,\omega/\eta(x)\}\mathbb{F}_{\epsilon,\omega,\eta(x)}\|_{L^{n+\delta}(B_1(x)\cap\Omega_m^\omega/\eta(x))})\le c.\end{cases}$$

Apply Lemma 4.3 to (4.34) to obtain

$$\begin{aligned}&[\mathbb{U}_{\epsilon,\omega,\eta(x)}]_{C^{0,\mu}(\overline{B_{1/2}(x)\cap\Omega_f^\omega/\eta(x)})}\\ &\quad+\sup_{\substack{j\in\mathbb{Z}^n\\ \frac{\omega}{\eta(x)}(Y_m+j)\subset B_{1/2}(0)\cap\Omega_m^\omega/\eta(x)}}\epsilon^{3/2}[\mathbb{U}_{\epsilon,\omega,\eta(x)}]_{C^{0,\mu}(\frac{\omega}{\eta(x)}(\overline{Y_m}+j))}\le c.\end{aligned}\tag{4.35}$$

Which implies

$$⨍_{B_s(x)\cap\Omega}\left|\Pi_\omega U_{\epsilon,\omega}|_{\Omega_f^\omega}-(\Pi_\omega U_{\epsilon,\omega}|_{\Omega_f^\omega})_{x,s}\right|^2dy\le cs^{2\mu}|\hat{J}_{\epsilon,\omega}|^2\quad\text{for } s<\tfrac{\eta(x)}{2}.\tag{4.36}$$

Next we consider case (2). Because of $\tilde{\theta}_1<\tilde{\theta}_2^2$, for any $r\in[\omega/\tilde{\epsilon}_0,\tilde{\theta}_2]$, there are $\tilde{\theta}\in[\tilde{\theta}_1,\tilde{\theta}_2]$ and $k\in\mathbb{N}$ satisfying $r=\tilde{\theta}^k$. By Lemma 4.6,

$$\begin{cases}⨍_{B_r(x_0)\cap\Omega}\left|\Pi_\omega U_{\epsilon,\omega}|_{\Omega_f^\omega}\right|^2dy\le cr^{2\mu}|\tilde{J}_{\epsilon,\omega}|^2\\ ⨍_{B_r(x_0)\cap\Omega_m^\omega}\epsilon^2\left|U_{\epsilon,\omega}\right|^2dy\le cr^{2\mu}|\tilde{J}_{\epsilon,\omega}|^2\end{cases}\quad\text{for } r\in[\omega/\tilde{\epsilon}_0,\tilde{\theta}_2].\tag{4.37}$$

This implies, for $s\in[\frac{\omega}{3\tilde{\epsilon}_0},\frac{\tilde{\theta}_2}{3}]$,

$$\begin{cases}⨍_{B_s(x)\cap\Omega}\left|\Pi_\omega U_{\epsilon,\omega}|_{\Omega_f^\omega}-(\Pi_\omega U_{\epsilon,\omega}|_{\Omega_f^\omega})_{x,s}\right|^2dy\le cs^{2\mu}|\tilde{J}_{\epsilon,\omega}|^2,\\ ⨍_{B_s(x)\cap\Omega_m^\omega}\epsilon^2\left|U_{\epsilon,\omega}-(\Pi_\omega U_{\epsilon,\omega}|_{\Omega_f^\omega})_{x,s}\right|^2dy\le cs^{2\mu}|\tilde{J}_{\epsilon,\omega}|^2.\end{cases}\tag{4.38}$$

Again we shift the coordinate system such that x is located at the origin. Define, in $(B_{1/\tilde{\epsilon}_0}(x)\cap\Omega/\omega)\setminus\partial\Omega_m^\omega/\omega$,

$$\begin{cases}\mathbb{U}_{\epsilon,1}(y)\equiv\hat{J}_{\epsilon,\omega}^{-1}\omega^{-\mu}\big(U_{\epsilon,\omega}(\omega y)-(\Pi_\omega U_{\epsilon,\omega}|_{\Omega_f^\omega})_{x,\omega/\tilde{\epsilon}_0}\big),\\ \mathbb{Q}_{\epsilon,1}(y)\equiv\hat{J}_{\epsilon,\omega}^{-1}\omega^{1-\mu}Q_{\epsilon,\omega}(\omega y),\\ \mathbb{F}_{\epsilon,1}(y)\equiv\hat{J}_{\epsilon,\omega}^{-1}\omega^{2-\mu}(F_{\epsilon,\omega}(\omega y)-\mathcal{E}_{\epsilon,\omega,\omega}(y)\mathbf{P}_{\epsilon,\omega}(\omega y)(\Pi_\omega U_{\epsilon,\omega}|_{\Omega_f^\omega})_{x,\omega/\tilde{\epsilon}_0}),\\ \mathbb{P}_{\epsilon,1}(y)\equiv\omega^2\mathbf{P}_{\epsilon,\omega}(\omega y),\end{cases}$$

and define

$$\mathbb{U}_{b_\epsilon}\equiv-\hat{J}_{\epsilon,\omega}^{-1}\omega^{-\mu}(\Pi_\omega U_{\epsilon,\omega}|_{\Omega_f^\omega})_{x,\omega/\tilde{\epsilon}_0}\qquad\text{in } B_{\frac{1}{\tilde{\epsilon}_0}}(x)\cap\Omega/\omega.$$

See (4.19) for $\mathcal{E}_{\epsilon,\omega,\omega}$. By $(4.37)_1$, $\mathbb{U}_{b_\epsilon}$ is a constant independent of ϵ,ω. Then these functions satisfy

$$\begin{cases} -\nabla\cdot(\mathcal{E}_{\epsilon^2,\omega,\omega}\mathbf{K}\nabla\mathbb{U}_{\epsilon,1}+\mathbb{Q}_{\epsilon,1})+\mathcal{E}_{\epsilon,\omega,\omega}\mathbb{P}_{\epsilon,1}\mathbb{U}_{\epsilon,1}=\mathbb{F}_{\epsilon,1} & \text{in } B_{\frac{1}{\tilde\epsilon_0}}(x)\cap\Omega/\omega,\\ \mathbb{U}_{\epsilon,1}=\mathbb{U}_{b_\epsilon} & \text{on } B_{\frac{1}{\tilde\epsilon_0}}(x)\cap\partial\Omega/\omega. \end{cases}$$

Take $s=\frac{\omega}{\tilde\epsilon_0}$ in (4.38) to see, by $(4.37)_1$,

$$\begin{cases} \mathbb{P}_{\epsilon,1}\in[0,\mathbf{M}] \text{ for all } x\in B_{1/\tilde\epsilon_0}(x)\cap\Omega/\omega,\\ \|\mathcal{E}_{\epsilon,\omega,\omega}\mathbb{U}_{\epsilon,1}\|_{L^2(B_{1/\tilde\epsilon_0}(x)\cap\Omega/\omega)}+\|\mathcal{E}_{1/\epsilon,\omega,\omega}\mathbb{Q}_{\epsilon,1},\mathcal{E}_{1/\epsilon,\omega,\omega}\mathbb{F}_{\epsilon,1}\|_{L^{n+\delta}(B_{1/\tilde\epsilon_0}(x)\cap\Omega/\omega)}\\ \qquad+\|\mathbb{U}_{b_\epsilon}\|_{W^{1,n+\delta}(B_{1/\tilde\epsilon_0}(x)\cap\Omega/\omega)}\le c. \end{cases}$$

By Lemma 3.4,

$$[\mathbb{U}_{\epsilon,1}]_{C^{0,\mu}(\overline{B_{1/2\tilde\epsilon_0}(x)\cap\Omega_f^\omega/\omega})}+\epsilon^{3/2}[\mathbb{U}_{\epsilon,1}]_{C^{0,\mu}(\overline{B_{1/2\tilde\epsilon_0}(x)\cap\Omega_m^\omega/\omega})}\le c. \tag{4.39}$$

(4.39) imply $(4.38)_1$ holds for $s\le\frac{\omega}{2\tilde\epsilon_0}$.

The Hölder estimate of $\Pi_\omega U_{\epsilon,\omega}$ follows from $(4.33)_1$, (4.36), $(4.38)_1$, (4.39), and Theorem 1.2 in page 70 [10]. The Hölder estimate of $U_{\epsilon,\omega}$ in $\omega(\overline{Y_m}+j)\subset B_{1/2}(0)\cap\overline{\Omega_m^\omega}$ is from (4.35) and (4.39). □

Remark 4.2. Let $\tilde\epsilon_*$ be same as that in Lemma 4.7. By Lemma 3.4 with $p=n+\delta$, we know that if $\delta\in(0,3)$, $\mathbf{M}>0$, $\omega\in[\tilde\epsilon_*,1]$, $\epsilon\in(0,\tilde\epsilon_*)$, and $\mathbf{P}_{\epsilon,\omega}(x)\in[0,\mathbf{M}]$ for all $x\in B_1(0)\cap\Omega$, any solution of (4.28) satisfies (4.30). Together with Lemma 4.7, any solution of (4.28) satisfies (4.30) if $\delta\in(0,3)$, $\mathbf{M}>0$, $\omega\in(0,1)$, $\epsilon\in(0,\tilde\epsilon_*)$, and $\mathbf{P}_{\epsilon,\omega}(x)\in[0,\mathbf{M}]$ for all $x\in B_1(0)\cap\Omega$.

Let us consider the solutions of (4.28) with $\mathbf{P}_{\epsilon,\omega}=0$. By tracing the arguments of Lemma 4.7 and Remark 4.2 and employing Lemma 3.4, then we have

Lemma 4.8. *For any $\delta>0$, there is a $\tilde\epsilon_*\in(0,1)$ (depending on $\delta,\mathbf{K},Y_f,\Omega$) such that, if $\epsilon\in(0,\tilde\epsilon_*)$ and $\omega\in(0,1)$, then any solution of (4.28) with $\mathbf{P}_{\epsilon,\omega}=0$ satisfies*

$$[U_{\epsilon,\omega}]_{C^{0,\mu}(\overline{B_{1/2}(0)\cap\Omega_f^\omega})}+\sup_{\substack{j\in\mathbb{Z}^n\\ \omega(Y_m+j)\subset B_{1/2}(0)\cap\Omega_m^\omega}}\epsilon[U_{\epsilon,\omega}]_{C^{0,\mu}(\omega(\overline{Y_m}+j))}\le c\hat J_{\epsilon,\omega},$$

where c is a constant independent of ϵ,ω. See Lemma 4.7 for $\mu,\hat J_{\epsilon,\omega}$.

By energy method, partition of unity, Remark 4.1, Remark 4.2, Lemma 4.4, Lemma 4.8, and Poincaré inequality [11], we conclude

Lemma 4.9. *Under A1–A2, for any $\delta\in(0,3)$ and $\mathbf{M}>0$, there is a constant $\tilde\epsilon_*\in(0,1)$ (depending on $\delta,\mathbf{M},\mathbf{K},Y_f,\Omega$) such that if*

$$\begin{cases} -\nabla\cdot(\mathbf{E}_{\epsilon^2,\omega}\mathbf{K}_\omega\nabla U_{\epsilon,\omega}+Q_{\epsilon,\omega})+\mathbf{E}_{\epsilon,\omega}\mathbf{P}_\omega U_{\epsilon,\omega}=F_{\epsilon,\omega} & \textit{in } \Omega,\\ U_{\epsilon,\omega}=0 & \textit{on } \partial\Omega, \end{cases}$$

and if $\epsilon \in (0, \tilde{\epsilon}_*)$*,* $\omega \in (0,1)$*, and* $\mathbf{P}_{\epsilon,\omega}(x) \in [0, \mathbf{M}]$ *for all* $x \in \Omega$*, then*

$$[U_{\epsilon,\omega}]_{C^{0,\mu}(\overline{\Omega^\omega_f})} + \sup_{\substack{j\in\mathbb{Z}^n \\ \omega(Y_m+j)\subset\Omega^\omega_m}} \epsilon^{\lambda}[U_{\epsilon,\omega}]_{C^{0,\mu}(\omega(\overline{Y_m}+j))}$$
$$\le c(\|Q_{\epsilon,\omega}, F_{\epsilon,\omega}\|_{L^{n+\delta}(\Omega^\omega_f)} + \epsilon^{-1}\|Q_{\epsilon,\omega}, \max\{\epsilon,\omega\}F_{\epsilon,\omega}\|_{L^{n+\delta}(\Omega^\omega_m)}),$$

where $\mu \equiv \frac{\delta}{n+\delta}$ *and* c *is a positive constant independent of* ϵ, ω*. Here* λ *is* $\frac{3}{2}$ *if* $\mathbf{P}_{\epsilon,\omega} \neq 0$ *and is* 1 *if* $\mathbf{P}_{\epsilon,\omega} = 0$*.*

Under A1–A4, we multiply (1.1) by $|\Psi_{\epsilon,\omega}|^{q-2}\Psi_{\epsilon,\omega}$ for $q > 2$ and integrate over Ω to obtain

$$\|\Psi_{\epsilon,\omega}\|_{L^q(\Omega)} \le c\|\mathbf{E}_{1/\epsilon,\omega}V_{\epsilon,\omega}, G_{\epsilon,\omega}\|_{L^q(\Omega)}. \tag{4.40}$$

where c is independent of ϵ, ω. Then we write (1.1) as

$$\begin{cases} -\nabla\cdot(\mathbf{E}_{\epsilon^2,\omega}\mathbf{K}_\omega\nabla\Psi_{\epsilon,\omega} + V_{\epsilon,\omega}) = G_{\epsilon,\omega} - \mathbf{T}_{\epsilon,\omega}\Psi_{\epsilon,\omega} & \text{in } \Omega, \\ \Psi_{\epsilon,\omega} = 0 & \text{on } \partial\Omega. \end{cases}$$

Theorem 2.1 follows by energy method, (4.40) for $q = n + \delta$, and Lemma 4.9 for $\mathbf{P}_{\epsilon,\omega} = 0$. Theorem 2.2 is a direct consequence of energy method and Lemma 4.9.

5. Convergence estimates

In this section, we prove Theorems 2.3, 2.4, 2.5. For each $\nu \in (0,1)$ and $i_1, i_2 \in \{1, 2, \cdots, n\}$, we find $\mathbb{Y}^{(i_1,i_2)}_\nu \in H^1_{per}(\mathbb{R}^n)$ satisfying

$$\begin{cases} \nabla\cdot\left(\mathbb{E}_{\nu^2}\mathbf{K}(\nabla\mathbb{Y}^{(i_1,i_2)}_\nu + \mathbb{X}^{(i_2)}_\nu\vec{e}_{i_1})\right) \\ \qquad = \frac{\mathcal{K}^{(i_1,i_2)}_\nu}{|Y_f|}\mathcal{X}_{Y_f} - \mathbb{E}_{\nu^2}\mathbf{K}(\partial_{i_1}\mathbb{X}^{(i_2)}_\nu + \delta_{i_1,i_2}) \quad \text{in } Y, \\ \int_{Y_f}\mathbb{Y}^{(i_1,i_2)}_\nu(y)dy = 0, \end{cases} \tag{5.1}$$

where $\vec{e}_{i_1}$ is a unit vector in the i_1-th coordinate direction, δ_{i_1,i_2} is 1 if $i_1 = i_2$ and is 0 if $i_1 \neq i_2$, and $\mathcal{K}^{(i_1,i_2)}_\nu$ is the (i_1, i_2)-th component of $\mathcal{K}_\nu$. See (2.5) for $\mathbb{X}^{(i)}_\nu$ and (2.9) for $\mathcal{K}_\nu$. By Lax-Milgram Theorem [11], A1–A2, and (3.17), $\mathbb{Y}^{(i_1,i_2)}_\nu$ is uniquely solvable. By Lemma 3.3,

$$\|\mathbb{Y}^{(i_1,i_2)}_\nu\|_{C^2(\overline{Y_f})\cap C^2(\overline{Y_m})} \le c, \tag{5.2}$$

where c is a constant independent of ν. Define $n\times n$ matrices $\mathbb{Y}_\nu \equiv (\mathbb{Y}^{(i_1,i_2)}_\nu)$ and $\mathbb{Y}_{\nu,s}(x) \equiv s^2\mathbb{Y}_\nu(\frac{x}{s})$ for $\nu, s \in (0,1)$.

5.1. *Proof of Theorem 2.3*

A1–A6 are assumed. This subsection consists of two parts. The first part is for $\epsilon, \omega \to 0$, $\frac{\epsilon}{\omega} \to \infty$ and the second part is for $\epsilon, \omega \to 0$, $\frac{\epsilon}{\omega} \to \sigma \in [0, \infty)$.

5.1.1. *Part 1:* $\epsilon, \omega \to 0, \frac{\epsilon}{\omega} \to \infty$.

For each $\nu \in (0,1)$, we find $\widetilde{\mathbb{W}}_\nu \in H^1_{per}(\mathbb{R}^n)$ satisfying

$$\begin{cases} \nabla \cdot (\mathbb{E}_{\nu^2} \mathbf{K} \nabla \widetilde{\mathbb{W}}_\nu) = \mathbf{T} - \mathcal{T}_\infty \mathcal{X}_{Y_f} \quad \text{in } Y, \\ \int_{Y_f} \widetilde{\mathbb{W}}_\nu(y) dy = 0. \end{cases} \tag{5.3}$$

See (2.11) for $\mathcal{T}_\infty$. By Lax-Milgram Theorem [11], $\widetilde{\mathbb{W}}_\nu$ is uniquely solvable. By Lemma 3.3 and A5,

$$\|\mathbb{E}_\nu \widetilde{\mathbb{W}}_\nu\|_{C^2(\overline{Y_f}) \cap C^2(\overline{Y_m})} \leq c/\nu, \tag{5.4}$$

where c is independent of ν. Define $\widetilde{\mathbb{W}}_{\nu,s}(x) \equiv s^2 \widetilde{\mathbb{W}}_\nu(\frac{x}{s})$ for $\nu, s \in (0,1)$.

Let $\Psi_{\epsilon,\omega}$ be the solution of (1.1) with $V_{\epsilon,\omega} = 0$, Ψ be the solution of (2.12), and

$$\varphi_{\epsilon,\omega} \equiv \Psi_{\epsilon,\omega} - \Psi - \widetilde{\mathbb{W}}_{\epsilon,\omega} \Psi - \mathbb{X}_{\epsilon,\omega} \nabla \Psi - \mathbb{Y}_{\epsilon,\omega} \nabla^2 \Psi \qquad \text{in } \Omega.$$

See (2.5) for $\mathbb{X}_{\epsilon,\omega}$ and (5.1) for $\mathbb{Y}_{\epsilon,\omega}$. By (2.12)–(2.13), (3.17), (3.19), and (5.1)–(5.4),

$$\begin{cases} -\nabla \cdot \big(\mathbf{E}_{\epsilon^2,\omega} \mathbf{K}_\omega \big(\nabla \varphi_{\epsilon,\omega} + \widetilde{\mathbb{W}}_{\epsilon,\omega} \nabla \Psi + \mathbb{Y}_{\epsilon,\omega} \nabla^3 \Psi\big)\big) + \mathbf{T}_{\epsilon,\omega} \varphi_{\epsilon,\omega} \\ \qquad = \mathcal{O}_1(\epsilon + \omega/\epsilon) + G_{\epsilon,\omega} - \mathcal{G} \mathcal{X}_{\Omega_f^\omega} & \text{in } \Omega, \\ \varphi_{\epsilon,\omega} = \mathcal{O}_2(\omega + \omega^2/\epsilon^2) & \text{on } \partial\Omega, \end{cases}$$

where $\mathcal{O}_1(\nu)$ denotes a function satisfying $\|\mathcal{O}_1(\nu)\|_{L^{n+\delta}(\Omega)} \leq c\nu$ and $\mathcal{O}_2(\nu)$ denotes a function satisfying $\|\mathcal{O}_2(\nu)\|_{L^\infty(\Omega)} \leq c\nu$ for some constant c independent of ϵ, ω. See (2.12) for $\mathcal{G}$. Decompose $\varphi_{\epsilon,\omega}$ as $\varphi_{\epsilon,\omega} = \widehat{\varphi}_{\epsilon,\omega} + \check{\varphi}_{\epsilon,\omega}$, where $\widehat{\varphi}_{\epsilon,\omega}$ satisfies

$$\begin{cases} -\nabla \cdot \big(\mathbf{E}_{\epsilon^2,\omega} \mathbf{K}_\omega \big(\nabla \widehat{\varphi}_{\epsilon,\omega} + \widetilde{\mathbb{W}}_{\epsilon,\omega} \nabla \Psi + \mathbb{Y}_{\epsilon,\omega} \nabla^3 \Psi\big)\big) + \mathbf{T}_{\epsilon,\omega} \widehat{\varphi}_{\epsilon,\omega} \\ \qquad = \mathcal{O}_1(\epsilon + \omega/\epsilon) + G_{\epsilon,\omega} - \mathcal{G} \mathcal{X}_{\Omega_f^\omega} & \text{in } \Omega, \\ \widehat{\varphi}_{\epsilon,\omega} = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.5}$$

and $\check{\varphi}_{\epsilon,\omega}$ satisfies

$$\begin{cases} -\nabla \cdot (\mathbf{E}_{\epsilon^2,\omega} \mathbf{K}_\omega \nabla \check{\varphi}_{\epsilon,\omega}) + \mathbf{T}_{\epsilon,\omega} \check{\varphi}_{\epsilon,\omega} = 0 & \text{in } \Omega, \\ \check{\varphi}_{\epsilon,\omega} = \mathcal{O}_2(\omega + \omega^2/\epsilon^2) & \text{on } \partial\Omega. \end{cases} \tag{5.6}$$

By Theorem 2.1, A6, (2.13), (5.2), and (5.4), the solution of (5.5) satisfies

$$\|\mathbf{E}_{\epsilon,\omega} \widehat{\varphi}_{\epsilon,\omega}\|_{L^\infty(\Omega)} \leq c(\max\{\epsilon, \omega/\epsilon\} + \|G_{\epsilon,\omega} - \mathcal{G}\|_{L^{n+\delta}(\Omega_f^\omega)}), \tag{5.7}$$

where c is independent of ϵ, ω. By Theorem 8.1 [11], the solution of (5.6) satisfies

$$\|\check{\varphi}_{\epsilon,\omega}\|_{L^\infty(\Omega)} = \|\check{\varphi}_{\epsilon,\omega}\|_{L^\infty(\partial\Omega)} \leq c(\omega + \omega^2/\epsilon^2), \tag{5.8}$$

where c is a constant independent of ϵ, ω.

From (5.7) and (5.8), we see that the difference between the solution of (1.1) with $V_{\epsilon,\omega} = 0$ and the solution of (2.12) satisfies

$$\|\mathbf{E}_{\epsilon,\omega}(\Psi_{\epsilon,\omega} - \Psi)\|_{L^\infty(\Omega)} \leq c(\max\{\epsilon, \omega/\epsilon\} + \|G_{\epsilon,\omega} - \mathcal{G}\|_{L^{n+\delta}(\Omega_f^\omega)}), \tag{5.9}$$

where c is a constant independent of ϵ, ω. Now let us consider (2.14). We note that the solution of (1.1) with $V_{\epsilon,\omega} = 0$ and the solution of (2.14) satisfy, for any $\omega(Y_m + j) \subset \Omega_m^\omega$ and $j \in \mathbb{Z}^n$,

$$\begin{cases} -\epsilon^2 \nabla \cdot \big(\mathbf{K}_\omega \nabla(\Psi_{\epsilon,\omega} - \phi_{\epsilon,\omega}^{(j)})\big) + \mathbf{T}_{\epsilon,\omega}(\Psi_{\epsilon,\omega} - \phi_{\epsilon,\omega}^{(j)}) = 0 & \text{in } \omega(Y_m + j), \\ \Psi_{\epsilon,\omega} - \phi_{\epsilon,\omega}^{(j)} = \Psi_{\epsilon,\omega} - \Psi & \text{on } \partial\omega(Y_m + j). \end{cases}$$

By (5.9) and Theorem 8.1 [11], we conclude

$$\|\Psi_{\epsilon,\omega} - \sum_{\substack{j \in \mathbb{Z}^n \\ \omega(Y_m + j) \subset \Omega_m^\omega}} \phi_{\epsilon,\omega}^{(j)}\|_{L^\infty(\Omega_m^\omega)} \le c(\max\{\epsilon, \omega/\epsilon\} + \|G_{\epsilon,\omega} - \mathcal{G}\|_{L^{n+\delta}(\Omega_f^\omega)}), \quad (5.10)$$

where c is independent of ϵ, ω. (5.9)–(5.10) imply Theorem 2.3 for $\epsilon, \omega \to 0, \frac{\epsilon}{\omega} \to \infty$.

5.1.2. *Part 2:* $\epsilon, \omega \to 0, \frac{\epsilon}{\omega} \to \sigma \in [0, \infty)$.

For $\nu \in (0,1)$, $\beta \in (0,\infty)$, and $i_1, i_2 \in \{1, \cdots, n\}$, we find $\widetilde{\mathbb{X}}_{\nu,\beta}^{(i_1)}, \widetilde{\mathbb{Y}}_{\nu,\beta}^{(i_1,i_2)} \in H^1_{per}(\mathbb{R}^n)$ satisfying

$$\begin{cases} \nabla \cdot \Big(\mathbb{E}_{\nu^2}\mathbf{K}\big(\nabla \widetilde{\mathbb{X}}_{\nu,\beta}^{(i_1)} + (\mathcal{X}_{Y_f} + \mathbb{W}_\beta \mathcal{X}_{Y_m})\vec{e}_{i_1}\big)\Big) = 0 \quad \text{in } Y, \\ \displaystyle\int_{Y_f} \widetilde{\mathbb{X}}_{\nu,\beta}^{(i_1)}(y) dy = 0, \end{cases} \quad (5.11)$$

$$\begin{cases} \nabla \cdot \Big(\mathbb{E}_{\nu^2}\mathbf{K}(\nabla \widetilde{\mathbb{Y}}_{\nu,\beta}^{(i_1,i_2)} + \widetilde{\mathbb{X}}_{\nu,\beta}^{(i_2)}\vec{e}_{i_1})\Big) \\ \qquad = \frac{\widetilde{\mathcal{K}}_{\nu,\beta}^{(i_1,i_2)}}{|Y_f|}\mathcal{X}_{Y_f} - \mathbb{E}_{\nu^2}\mathbf{K}\left(\partial_{i_1}\widetilde{\mathbb{X}}_{\nu,\beta}^{(i_2)} + (\mathcal{X}_{Y_f} + \mathbb{W}_\beta \mathcal{X}_{Y_m})\delta_{i_1,i_2}\right) \quad \text{in } Y, \\ \displaystyle\int_{Y_f} \widetilde{\mathbb{Y}}_{\nu,\beta}^{(i_1,i_2)}(y) dy = 0, \end{cases} \quad (5.12)$$

where $\vec{e}_{i_1}, \delta_{i_1,i_2}$ are same as those in (5.1), $\mathbb{W}_\beta$ is from (2.7)–(2.8), and $\widetilde{\mathcal{K}}_{\nu,\beta}^{(i_1,i_2)}$ is defined as

$$\widetilde{\mathcal{K}}_{\nu,\beta}^{(i_1,i_2)} \equiv \int_Y \mathbb{E}_{\nu^2}\mathbf{K}\left(\partial_{i_1}\widetilde{\mathbb{X}}_{\nu,\beta}^{(i_2)} + (\mathcal{X}_{Y_f} + \mathbb{W}_\beta \mathcal{X}_{Y_m})\delta_{i_1,i_2}\right) dy. \quad (5.13)$$

$\widetilde{\mathbb{X}}_{\nu,\beta}^{(i_1)}, \widetilde{\mathbb{Y}}_{\nu,\beta}^{(i_1,i_2)}$ in (5.11)–(5.12) are uniquely solvable by Lax-Milgram Theorem [11].

Lemma 5.1. *Under A1–A2 and A4, the solution* $\mathbb{W}_\beta$ *of (2.7)–(2.8) satisfies*

$$\begin{cases} \mathbb{W}_\beta(x) \in [0,1] & \textit{for any } x \in Y_m, \beta \in (0,\infty), \\ \|\mathbb{E}_{\beta^2}\mathbb{W}_\beta\|_{W^{2,p}(Y_f)\cap W^{2,p}(Y_m)} \le c & \textit{for any } p \in (1,\infty), \beta \in (0,\infty), \\ \displaystyle\int_{Y_m} \mathbf{T}\mathbb{W}_\beta dy \le c\sqrt{|\beta \ln \beta|} & \textit{if } \beta \in (0, \beta^*), \end{cases} \quad (5.14)$$

where β^* *is a constant depending on* $\mathbf{K}, \mathbf{T}$ *and diam*$|\Omega|$*, and* c *is independent of* β.

Proof. Corollary 3.2 [11] implies $(5.14)_1$. Theorems 9.11, 9.15 [11], extension method in Theorem 7.25 [11], and $(5.14)_1$ imply $(5.14)_2$. For any $x \in Y_m$, we set

$\eta_x \equiv \min_{z\in\partial Y_m} |z-x|$ and $\xi_x \equiv \max_{z\in\partial Y_m} |z-x|$. Next we fix $t\in\mathbb{R}$ and $x\in Y_m$ as well as define $\varphi(y)\equiv \exp((|y-x|^2-\eta_x^2)t)$ for $y\in Y_m$. Then φ satisfies

$$\beta^2\nabla\cdot(\mathbf{K}\nabla\varphi)-\mathbf{T}\varphi=\beta^2\varphi(4t^2\mathbf{K}|y-x|^2+2t(y-x)\nabla\mathbf{K}+2t\mathbf{K}n-\mathbf{T}/\beta^2).$$

We find that there are $\beta^*<1$ and $c^*>0$ (depending on $\mathbf{K},\mathbf{T}, diam|\Omega|$ but not $x\in Y_m$) such that if $\beta\in(0,\beta^*)$, $t=c^*/\beta$, and $x\in Y_m$, then

$$\begin{cases}\beta^2\nabla\cdot(\mathbf{K}\nabla\varphi)-\mathbf{T}\varphi\le 0 & \text{in } B_{\xi_x}(x)\cap Y_m,\\ \varphi\ge 1 & \text{in } Y_m\setminus B_{\eta_x}(x).\end{cases} \tag{5.15}$$

Corollary 3.2 [11], (2.7), and (5.15) imply

$$0\le \mathbb{W}_\beta\le\varphi \qquad \text{on } Y_m. \tag{5.16}$$

For any $\nu>0$, define $Y_m(\nu)\equiv\{y\in Y_m|\min_{z\in\partial Y_m}|z-y|\ge\nu\}$. By (5.16), it is easy to see that if $\beta\in(0,\beta^*)$, then, for any $x\in Y_m(\sqrt{2|\beta\ln\beta|/c^*})$,

$$0\le\mathbb{W}_\beta\le\beta \qquad \text{on } B_{\eta_x/2}(x). \tag{5.17}$$

So if $\beta\in(0,\beta^*)$, $\displaystyle\int_{Y_m}\mathbf{T}\mathbb{W}_\beta dy\le c\sqrt{|\beta\ln\beta|}$ by $(5.14)_1$ and (5.17). □

By Lemma 3.3, Lemma 5.1, and energy method, there is a $\epsilon_0\in(0,1)$ such that, for any $\nu\in(0,\epsilon_0)$ and $\beta\in(0,\infty)$,

$$\begin{cases}\|\widetilde{\mathbb{X}}^{(i_1)}_{\nu,\beta}\|_{W^{1,p}(Y_f)\cap W^{1,p}(Y_m)}, \|\widetilde{\mathbb{Y}}^{(i_1,i_2)}_{\nu,\beta}\|_{W^{2,p}(Y_f)\cap W^{2,p}(Y_m)}\le c,\\ \|\widetilde{\mathbb{X}}^{(i_1)}_{\nu,\beta},\widetilde{\mathbb{Y}}^{(i_1,i_2)}_{\nu,\beta}\|_{C^{2,\alpha}(\overline{Y_f})\cap C^{2,\alpha}(\overline{Y_m})}\le \dfrac{c}{\beta^2},\\ |\widetilde{\mathcal{K}}^{(i_1,i_2)}_{\nu,\beta}-\mathcal{K}^{(i_1,i_2)}_0|\le c(\dfrac{\nu}{\beta}+\nu^2) \qquad \text{as } \nu\to 0,\\ |\mathcal{T}_\tau-\mathcal{T}_\sigma|\le\begin{cases}c|\frac{\tau^2}{\sigma^2}-1| & \text{if } \tau\to\sigma\in(0,\infty),\\ c\sqrt{|\tau\ln\tau|} & \text{if } \tau\to\sigma=0,\end{cases}\end{cases} \tag{5.18}$$

where $p\in(n,\infty)$, $\alpha\in(0,1)$, $\mathcal{K}^{(i_1,i_2)}_0$ is the (i_1,i_2)-th component of $\mathcal{K}_0$ (see (2.9)), and c is a constant independent of ν,β,τ,σ. See (5.13) for $\widetilde{\mathcal{K}}^{(i_1,i_2)}_{\nu,\beta}$ and see (2.11) for $\mathcal{T}_\sigma$. Define $\widetilde{\mathbb{X}}_{\epsilon,\beta,\omega}(x)\equiv\omega\widetilde{\mathbb{X}}_{\epsilon,\beta}(\frac{x}{\omega})$ $\widetilde{\mathbb{Y}}_{\epsilon,\beta}\equiv(\widetilde{\mathbb{Y}}^{(i_1,i_2)}_{\epsilon,\beta})$ and $\widetilde{\mathbb{Y}}_{\epsilon,\beta,\omega}(x)\equiv\omega^2\widetilde{\mathbb{Y}}_{\epsilon,\beta}(\frac{x}{\omega})$ for $\beta\in(0,\infty),\epsilon,\omega\in(0,1),i_1,i_2\in\mathbb{Z}$.

Let $\Psi_{\epsilon,\omega}$ be the solution of (1.1) with $V_{\epsilon,\omega}=0$, Ψ be the solution of (2.12), and

$$\Phi_{\epsilon,\omega}\equiv\Psi_{\epsilon,\omega}-(\mathcal{X}_{\Omega^\omega_f}+\mathbb{W}_{\frac{\epsilon}{\omega},\omega,0}\mathcal{X}_{\Omega^\omega_m})\Psi-\mathbb{W}_{\frac{\epsilon}{\omega},\omega,2}\Psi\mathcal{X}_{\Omega^\omega_f}-\widetilde{\mathbb{X}}_{\epsilon,\frac{\epsilon}{\omega},\omega}\nabla\Psi-\widetilde{\mathbb{Y}}_{\epsilon,\frac{\epsilon}{\omega},\omega}\nabla^2\Psi$$

in Ω. See remark after (2.8) for $\mathbb{W}_{\frac{\epsilon}{\omega},\omega,i}$. By (2.7)–(2.8), (5.11)–(5.12), (5.18), and Lemma 5.1 with $\beta=\frac{\epsilon}{\omega}$, we obtain

$$\begin{cases}-\nabla\cdot\big(\mathbf{E}_{\epsilon^2,\omega}\mathbf{K}_\omega\big(\nabla\Phi_{\epsilon,\omega}+\mathbb{W}_{\frac{\epsilon}{\omega},\omega,2}\nabla\Psi\mathcal{X}_{\Omega^\omega_f}+\widetilde{\mathbb{Y}}_{\epsilon,\frac{\epsilon}{\omega},\omega}\nabla^3\Psi\big)\big)\\ \quad+\mathbf{T}_{\epsilon,\omega}\Phi_{\epsilon,\omega}=G_{\epsilon,\omega}+\frac{\widetilde{\mathcal{K}}_{\epsilon,\epsilon/\omega}}{|Y_f|}\nabla^2\Psi\mathcal{X}_{\Omega^\omega_f}-\mathcal{T}_{\frac{\epsilon}{\omega}}\Psi\mathcal{X}_{\Omega^\omega_f}+\mathcal{O}_1(\omega) & \text{in } \Omega\setminus\partial\Omega^\omega_m,\\ \lfloor\mathbf{E}_{\epsilon^2,\omega}\mathbf{K}_\omega\big(\nabla\Phi_{\epsilon,\omega}+\mathbb{W}_{\frac{\epsilon}{\omega},\omega,2}\nabla\Psi\mathcal{X}_{\Omega^\omega_f}+\widetilde{\mathbb{Y}}_{\epsilon,\frac{\epsilon}{\omega},\omega}\nabla^3\Psi\big)\rfloor\cdot\vec{\mathbf{n}}^\omega=0 & \text{on } \partial\Omega^\omega_m,\\ \lfloor\Phi_{\epsilon,\omega}\rfloor=-\mathbb{W}_{\frac{\epsilon}{\omega},\omega,2}\Psi\mathcal{X}_{\Omega^\omega_f} & \text{on } \partial\Omega^\omega_m,\\ \Phi_{\epsilon,\omega}=\mathcal{O}_2(\omega) & \text{on } \partial\Omega,\end{cases} \tag{5.19}$$

where $\mathcal{O}_1(\nu), \mathcal{O}_2(\nu)$ are same as those in Part 1. See (2.2) for $(5.19)_{2,3}$. Let us define

$$\varphi_{\epsilon,\omega} \equiv \begin{cases} \Phi_{\epsilon,\omega} + \mathbb{W}_{\frac{\epsilon}{\omega},\omega,2}\Psi\mathcal{X}_{\Omega_f^\omega} & \text{on } \Omega_f^\omega, \\ \Phi_{\epsilon,\omega} & \text{on } \Omega_m^\omega. \end{cases}$$

By (2.12)–(2.13), (5.18), and Lemma 5.1, $\varphi_{\epsilon,\omega}$ satisfies

$$\begin{cases} -\nabla\cdot\big(\mathbf{E}_{\epsilon^2,\omega}\mathbf{K}_\omega\big(\nabla\varphi_{\epsilon,\omega} + \mathcal{O}_1(\omega)\big)\big) + \mathbf{T}_{\epsilon,\omega}\varphi_{\epsilon,\omega} = G_{\epsilon,\omega} - \mathcal{G}\mathcal{X}_{\Omega_f^\omega} + \tilde{\mathcal{X}} & \text{in } \Omega, \\ \varphi_{\epsilon,\omega} = \mathcal{O}_2(\omega) & \text{on } \partial\Omega, \end{cases}$$

where

$$\tilde{\mathcal{X}} \equiv \begin{cases} \mathcal{O}_1(\omega + |\frac{\epsilon^2}{\sigma^2\omega^2} - 1|) & \text{if } \frac{\epsilon}{\omega} \to \sigma \in (0,\infty), \\ \mathcal{O}_1(\omega + |\frac{\epsilon}{\omega}\ln\frac{\epsilon}{\omega}|^{\frac{1}{2}}) & \text{if } \frac{\epsilon}{\omega} \to 0. \end{cases}$$

We write the $\varphi_{\epsilon,\omega}$ as $\varphi_{\epsilon,\omega} = \check{\varphi}_{\epsilon,\omega} + \widehat{\varphi}_{\epsilon,\omega}$, where $\check{\varphi}_{\epsilon,\omega}$ satisfies

$$\begin{cases} -\nabla\cdot(\mathbf{E}_{\epsilon^2,\omega}\mathbf{K}_\omega\nabla\check{\varphi}_{\epsilon,\omega}) + \mathbf{T}_{\epsilon,\omega}\check{\varphi}_{\epsilon,\omega} = 0 & \text{in } \Omega, \\ \check{\varphi}_{\epsilon,\omega} = \mathcal{O}_2(\omega) & \text{on } \partial\Omega, \end{cases} \tag{5.20}$$

and $\widehat{\varphi}_{\epsilon,\omega}$ satisfies

$$\begin{cases} -\nabla\cdot\big(\mathbf{E}_{\epsilon^2,\omega}\mathbf{K}_\omega\big(\nabla\widehat{\varphi}_{\epsilon,\omega} + \mathcal{O}_1(\omega)\big)\big) + \mathbf{T}_{\epsilon,\omega}\widehat{\varphi}_{\epsilon,\omega} = G_{\epsilon,\omega} - \mathcal{G}\mathcal{X}_{\Omega_f^\omega} + \tilde{\mathcal{X}} & \text{in } \Omega, \\ \widehat{\varphi}_{\epsilon,\omega} = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.21}$$

By Theorem 8.1 [11], the solution of (5.20) satisfies

$$\|\check{\varphi}_{\epsilon,\omega}\|_{L^\infty(\Omega)} = \|\check{\varphi}_{\epsilon,\omega}\|_{L^\infty(\partial\Omega)} \le c\omega, \tag{5.22}$$

where c is independent of ϵ, ω. Next we consider (5.21) for $\frac{\epsilon}{\omega} \to \sigma \in (0,\infty)$ and $\frac{\epsilon}{\omega} \to 0$ separately.

Case 1: $\frac{\epsilon}{\omega} \to \sigma \in (0,\infty)$**.** By Theorem 2.1 and A6, the solution of (5.21) satisfies

$$\|\mathbf{E}_{\epsilon,\omega}\widehat{\varphi}_{\epsilon,\omega}\|_{L^\infty(\Omega)} \le c(\max\{\epsilon,\omega,|\epsilon^2/(\sigma\omega)^2 - 1|\} + \|G_{\epsilon,\omega} - \mathcal{G}\|_{L^{n+\delta}(\Omega_f^\omega)}), \tag{5.23}$$

where c is independent of ϵ, ω. Employing (5.22)–(5.23) and modifying the argument of Part 1, we obtain Theorem 2.3 for $\epsilon, \omega \to 0, \frac{\epsilon}{\omega} \to \sigma \in (0,\infty)$ case.

Case 2: $\frac{\epsilon}{\omega} \to 0$**.** If $\widehat{\varphi}_{\epsilon,\omega}$ is the solution of (5.21), we multiply (5.21) by $|\widehat{\varphi}_{\epsilon,\omega}|^{n+\delta-2}\widehat{\varphi}_{\epsilon,\omega}$ and integrate by part to see, by A6,

$$\|\widehat{\varphi}_{\epsilon,\omega}\|_{L^{n+\delta}(\Omega)} \le c(\max\{\epsilon,\omega,|\frac{\epsilon}{\omega}\ln\frac{\epsilon}{\omega}|^{\frac{1}{2}}\} + \|G_{\epsilon,\omega} - \mathcal{G}\|_{L^{n+\delta}(\Omega_f^\omega)}), \tag{5.24}$$

where c is independent of ϵ, ω. By (5.22) and (5.24), we obtain Theorem 2.3 for $\epsilon, \omega \to 0, \frac{\epsilon}{\omega} \to 0$ case.

5.2. *Proof of Theorem 2.4*

The proof is similar to that of Theorem 2.3. Let us assume that A1–A3, A4', A6, and A7 hold. For each $\nu \in (0,1)$, we find $\breve{\mathbb{W}}_\nu \in H^1_{per}(\mathbb{R}^n)$ satisfying

$$\begin{cases} \nabla \cdot (\mathbb{E}_{\nu^2}\mathbf{K}\nabla\breve{\mathbb{W}}_\nu) = (\mathbb{E}_\nu \mathbf{P} - \breve{\mathcal{T}})\mathcal{X}_{Y_f} \quad \text{in } Y, \\ \int_{Y_f} \breve{\mathbb{W}}_\nu(y)dy = 0. \end{cases} \tag{5.25}$$

See A7 for $\mathbf{P}$, and (2.15) for constant $\breve{\mathcal{T}}$. By Lax-Milgram Theorem [11], Lemma 3.3, and A7, $\breve{\mathbb{W}}_\nu$ is uniquely solvable and

$$\|\breve{\mathbb{W}}_\nu\|_{C^2(\overline{Y_f})\cap C^2(\overline{Y_m})} \leq c, \tag{5.26}$$

where c is a constant independent of ν. Define $\breve{\mathbb{W}}_{\nu,s}(x) \equiv s^2\breve{\mathbb{W}}_\nu(\frac{x}{s})$ for $\nu, s \in (0,1)$.

Let $\Psi_{\epsilon,\omega}$ be the solution of (1.1) with $V_{\epsilon,\omega} = 0$, Ψ be the solution of (2.16), and

$$\varphi_{\epsilon,\omega} \equiv \Psi_{\epsilon,\omega} - \Psi - \breve{\mathbb{W}}_{\epsilon,\omega}\Psi - \mathbb{X}_{\epsilon,\omega}\nabla\Psi - \mathbb{Y}_{\epsilon,\omega}\nabla^2\Psi \qquad \text{in } \Omega.$$

See (2.5) for $\mathbb{X}_{\epsilon,\omega}$ and (5.1) for $\mathbb{Y}_{\epsilon,\omega}$. By (2.16), (3.17), (3.19), (5.1)–(5.2), and (5.25)–(5.26), we obtain

$$\begin{cases} -\nabla \cdot \big(\mathbf{E}_{\epsilon^2,\omega}\mathbf{K}_\omega\big(\nabla\varphi_{\epsilon,\omega} + \mathcal{O}_1(\omega^2)\big)\big) + \mathbf{T}_{\epsilon,\omega}\varphi_{\epsilon,\omega} \\ \qquad = \mathcal{O}_1(\omega+\epsilon)\mathcal{X}_{\Omega_f^\omega} + \mathcal{O}_1(\epsilon)\mathcal{X}_{\Omega_m^\omega} + (G_{\epsilon,\omega} - \mathcal{G})\mathcal{X}_{\Omega_f^\omega} \quad \text{in } \Omega, \\ \varphi_{\epsilon,\omega} = \mathcal{O}_2(\omega) \quad \text{on } \partial\Omega, \end{cases}$$

where $\mathcal{O}_1(\nu), \mathcal{O}_2(\nu)$ are defined as those in Part 1. See (2.15) for $\mathcal{G}$. Modifying the argument of Part 1 in subsection 5.1 and employing Theorem 2.2, we obtain Theorem 2.4.

5.3. *Proof of Theorem 2.5*

We assume A1–A3 and A6. Let $\Psi_{\epsilon,\omega}$ be the solution of (1.1) with $V_{\epsilon,\omega} = \mathbf{T}_{\epsilon,\omega} = 0$, Ψ be the solution of (2.18), and define

$$\varphi_{\epsilon,\omega} \equiv \Psi_{\epsilon,\omega} - \Psi - \mathbb{X}_{\epsilon,\omega}\nabla\Psi - \mathbb{Y}_{\epsilon,\omega}\nabla^2\Psi \qquad \text{in } \Omega.$$

See (2.5) for $\mathbb{X}_{\epsilon,\omega}$ and (5.1) for $\mathbb{Y}_{\epsilon,\omega}$. By (2.18), (3.17), (3.19), and (5.2), $\varphi_{\epsilon,\omega}$ satisfies

$$\begin{cases} -\nabla \cdot \big(\mathbf{E}_{\epsilon^2,\omega}\mathbf{K}_\omega\big(\nabla\varphi_{\epsilon,\omega} + \mathcal{O}_1(\omega^2)\big)\big) \\ \qquad = \mathcal{O}_1(\omega+\epsilon)\mathcal{X}_{\Omega_f^\omega} + \mathcal{O}_1(\epsilon)\mathcal{X}_{\Omega_m^\omega} + (G_{\epsilon,\omega} - \mathcal{G})\mathcal{X}_{\Omega_f^\omega} \quad \text{in } \Omega, \\ \varphi_{\epsilon,\omega} = \mathcal{O}_2(\omega) \quad \text{on } \partial\Omega. \end{cases}$$

Modifying the argument of Part 1 in subsection 5.1 and employing Theorem 2.2, we obtain Theorem 2.5.

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