Pointwise estimate for elliptic equations in periodic perforated domains

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Pointwise estimate for the solutions of elliptic equations in periodic perforated domains is concerned. Let ϵ denote the size ratio of the period of a periodic perforated domain to the whole domain. It is known that even if the given functions of the elliptic equations are bounded uniformly in ϵ, the $C^{1,\alpha}$ norm and the $W^{2,p}$ norm of the elliptic solutions may not be bounded uniformly in ϵ. It is also known that when ϵ closes to 0, the elliptic solutions in the periodic perforated domains approach a solution of some homogenized elliptic equation. In this work, the Hölder uniform bound in ϵ and the Lipschitz uniform bound in ϵ for the elliptic solutions in perforated domains are proved. The L^∞ and the Lipschitz convergence estimates for the difference between the elliptic solutions in the perforated domains and the solution of the homogenized elliptic equation are derived.

Keywords: periodic perforated domains, homogenized elliptic equation, two-phase media

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1. Introduction

Pointwise estimate for the solutions of elliptic equations in perforated domains with periodic microstructure is presented. This problem arises from contaminant transport in the subsurface, heat transfer in two-phase media, the stress in composite materials, and so on (see [3, 6, 14, 15]). Let $\Omega \subset \mathbb{R}^n$ ($n = 2$ or 3) be a bounded smooth simply-connected domain with boundary $\partial\Omega$, $Y \equiv (0,1)^n$ consist of a smooth sub-domain Y_m completely surrounded by another connected sub-domain Y_f ($\equiv Y \setminus \overline{Y_m}$), $\epsilon \in (0,1)$, $\Omega(\epsilon) \equiv \{x \in \Omega | dist(x, \partial\Omega) > \epsilon\}$, $\Omega_m^\epsilon \equiv \{x | x \in \epsilon(Y_m + j) \subset \Omega(\epsilon)$ for some $j \in \mathbb{Z}^n\}$ with boundary $\partial\Omega_m^\epsilon$, and $\Omega_f^\epsilon \equiv \Omega \setminus \overline{\Omega_m^\epsilon}$ be a connected region. The problem that we consider is

$$\begin{cases} -\nabla \cdot (\mathbf{K}_\epsilon \nabla U_\epsilon + \mathbf{T}_\epsilon U_\epsilon + Q_\epsilon) + \mathbf{P}_\epsilon \nabla U_\epsilon + \mathbf{E}_\epsilon U_\epsilon = F_\epsilon & \text{in } \Omega_f^\epsilon, \\ (\mathbf{K}_\epsilon \nabla U_\epsilon + \mathbf{T}_\epsilon U_\epsilon + Q_\epsilon) \cdot \vec{\mathbf{n}}_\epsilon = 0 & \text{on } \partial\Omega_m^\epsilon, \\ U_\epsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\mathbf{K}_\epsilon(x) = \mathbf{K}(\frac{x}{\epsilon})$, $\mathbf{T}_\epsilon(x) = \mathbf{T}(\frac{x}{\epsilon})$, $\mathbf{P}_\epsilon(x) = \mathbf{P}(\frac{x}{\epsilon})$, $\mathbf{E}_\epsilon(x) = \mathbf{E}(\frac{x}{\epsilon})$, $\mathbf{K}, \mathbf{T}, \mathbf{P}, \mathbf{E}$ are smooth periodic functions in $\mathbb{R}^n$ with period Y, $\mathbf{K}$ is positive, $\mathbf{E}$ is non-negative, Q_ϵ, F_ϵ are given functions, and $\vec{\mathbf{n}}_\epsilon$ is a unit normal vector on $\partial\Omega_m^\epsilon$.

Problem (1.1) consists of an elliptic equation with oscillatory periodic coefficients. One example shows that even the given functions Q_ϵ, F_ϵ of the elliptic equation are bounded uniformly in ϵ, the $C^{1,\alpha}$ norm and the $W^{2,p}$ norm of the elliptic solution of (1.1) may not be bounded uniformly in ϵ (see **Remark** 3.1). It is also known that, to obtain an accurate approximation of the solution of (1.1) by classical numerical methods (see [7, 10]), the computational mesh size in each space direction should be less than ϵ [13]. So a direct numerical simulation of the solution of (1.1) requires large computational memory and computing time when ϵ is small.

If $\mathbf{T}_\epsilon, \mathbf{P}_\epsilon$ are small and if Q_ϵ, F_ϵ are bounded in $L^2(\Omega_f^\epsilon)$, the H^1 solution of (1.1) exists uniquely and satisfies

$$\|U_\epsilon\|_{H^1(\Omega_f^\epsilon)} \leq c(\|Q_\epsilon\|_{L^2(\Omega_f^\epsilon)} + \|F_\epsilon\|_{L^2(\Omega_f^\epsilon)}), \tag{1.2}$$

where c is a constant independent of ϵ [12]. In addition that Q_ϵ is bounded in $H^1(\Omega_f^\epsilon)$, by tracing the proof of Theorem 2.7 [3], there exists a subsequence of $\{U_\epsilon, Q_\epsilon, F_\epsilon\}$ (same notation for subsequence) satisfying

$$\begin{cases} \mathbf{K}_\epsilon \nabla U_\epsilon \mathcal{X}_{\Omega_f^\epsilon} + \mathbf{T}_\epsilon U_\epsilon \mathcal{X}_{\Omega_f^\epsilon} + Q_\epsilon \mathcal{X}_{\Omega_f^\epsilon} \\ \qquad \to \mathcal{K}\nabla U + \mathcal{T}_1 U + |Y_f|\mathcal{Q} \\ \mathbf{P}_\epsilon \nabla U_\epsilon \mathcal{X}_{\Omega_f^\epsilon} + \mathbf{E}_\epsilon U_\epsilon \mathcal{X}_{\Omega_f^\epsilon} - F_\epsilon \mathcal{X}_{\Omega_f^\epsilon} \\ \qquad \to \mathcal{P}\nabla U + \mathcal{E}_{1,1} U - |Y_f|\mathcal{F} \\ Q_\epsilon \mathcal{X}_{\Omega_f^\epsilon} \to |Y_f|\mathcal{Q}_* \\ F_\epsilon \mathcal{X}_{\Omega_f^\epsilon} \to |Y_f|\mathcal{F}_* \end{cases} \quad \text{in } L^2(\Omega) \text{ weakly as } \epsilon \to 0, \tag{1.3}$$

where $\mathcal{X}_{\Omega_f^\epsilon}$ is the characteristic function on Ω_f^ϵ, $\mathcal{K}$ is a symmetric positive definite matrix depending on $\mathbf{K}$ and Y_f, $\mathcal{T}_1$ and $\mathcal{P}$ are constant vectors, $\mathcal{E}_{1,1}$ is a constant function, $|Y_f|$ is the volume of Y_f, $U \in H^1(\Omega)$, and $\mathcal{Q}, \mathcal{F}, \mathcal{Q}_*, \mathcal{F}_* \in L^2(\Omega)$. $\mathcal{Q}$ depends on $\mathbf{K}$, $\mathcal{Q}_*$ and $\mathcal{F}$ depends on $\mathbf{K}, \mathbf{P}, \mathcal{Q}_*, \mathcal{F}_*$. Explicit form of $\mathcal{K}, \mathcal{T}_1, \mathcal{P}, \mathcal{E}_{1,1}, \mathcal{Q}, \mathcal{F}$ can be found in (2.4) below. The function U in (1.3) satisfies

$$\begin{cases} -\nabla \cdot (\mathcal{K}\nabla U + \mathcal{T}_1 U + |Y_f|\mathcal{Q}) + \mathcal{P}\nabla U + \mathcal{E}_{1,1} U = |Y_f|\mathcal{F} & \text{in } \Omega, \\ U = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

So it seems that the solution of (1.4) is a good approximation of the solution of (1.1), especially when ϵ is small. Indeed in those works [4, 13, 17], the solution of (1.1) was regarded as the perturbation of the homogenized solution of (1.4). Furthermore, the homogenized solution of (1.4) was used to construct the solution of (1.1). It is also known that the solution of (1.1) converges to the solution of (1.4) with convergence rate ϵ in L^2 norm and with convergence rate $\sqrt{\epsilon}$ in H^1 norm as ϵ closes to 0 (see [5, 14, 18] and references therein). In [8, 16], higher order asymptotic expansion for the solutions of elliptic equations in perforated domains was given. Higher order convergence estimate for the solution of (1.1) for $\mathbf{T}_\epsilon = \mathbf{P}_\epsilon = \mathbf{E}_\epsilon = 0$ case was also derived in Hilbert spaces (see [5, 9, 18]).

In this work, we show that although the $C^{1,\alpha}$ norm and the $W^{2,p}$ norm of the elliptic solution of (1.1) may not be bounded uniformly in ϵ, the Hölder uniform bound in ϵ and the Lipschitz uniform bound in ϵ for the elliptic solution can be obtained. We also present pointwise convergence estimate for the solution of (1.1). Precisely, the L^∞ convergence estimate with convergence rate ϵ for the solution of (1.1) is derived. For problem (1.1) with periodic boundary conditions, Lipschitz convergence estimate with higher order convergence rate ϵ^k for $k \geq 1$ is also proved.

The rest of the work is organized as follows: Notation and main results are stated in section 2. In section 3, we derive a priori $W^{2,p}$ estimates for the solutions of elliptic equations around the Neumann boundary. In section 4, a uniform Hölder estimate in ϵ for the elliptic solution of (1.1) is proved. In section 5, L^∞ convergence estimate for the solution of (1.1) is given. Uniform Lipschitz bound in ϵ and Lipschitz convergence estimate for the solution of (1.1) with periodic boundary conditions are shown in section 6.

2. Notation and main results

Denote by $C^{k,\alpha}$ the Hölder space with norm $\|\cdot\|_{C^{k,\alpha}}$, by $[\cdot]_{C^{k,\alpha}}$ the Hölder semi-norm, and by $W^{s,p}$ the Sobolev space with norm $\|\cdot\|_{W^{s,p}}$ for $k \geq 0, \alpha \in [0,1], s \in \mathbb{Z}$, and $p \in [1,\infty]$ (see [2]). $L^p \equiv W^{0,p}$ and $H^s \equiv W^{s,2}$. For any Banach space $\mathbf{B}$, we define $\|g_1, \cdots, g_k\|_{\mathbf{B}} \equiv \|g_1\|_{\mathbf{B}} + \cdots + \|g_k\|_{\mathbf{B}}$. $B_r(x)$ is a ball centered at x with radius r. For any domain D and $r > 0$, $\overline{D}$ is the closure of D, ∂D is the boundary of D, $D/r \equiv \{x | rx \in D\}$, $|D|$ is the volume of D, $\mathcal{X}_D$ is the characteristic function on D, $C_0^\infty(D)$ contains infinitely differentiable functions with compact support in D, and

$$\fint_D g(y)dy \equiv \frac{1}{|D|}\int_D g(y)dy.$$

For any $g \in L^1(\Omega)$,

$$(g)_{x,r} \equiv \fint_{B_r(x)\cap\Omega} g(y)dy.$$

Define $\mathcal{A}_m \equiv \cup_{j\in\mathbb{Z}^n}(Y_m + j)$, $\mathcal{A}_f \equiv \mathbb{R}^n \setminus \overline{\mathcal{A}_m}$, $\mathcal{A}_m^\epsilon \equiv \epsilon\mathcal{A}_m$, and $\mathcal{A}_f^\epsilon \equiv \epsilon\mathcal{A}_f$. $\mathbb{D} \equiv (-L,L)^n$ for $L \in \mathbb{N}$, $\mathbb{D}_m^\epsilon \equiv \mathcal{A}_m^\epsilon \cap \mathbb{D}$ and $\mathbb{D}_f^\epsilon \equiv \mathcal{A}_f^\epsilon \cap \mathbb{D}$. Let $C_{per}^\infty(\mathbb{R}^n)$ denote the space of infinitely differentiable Y-periodic functions in $\mathbb{R}^n$. If D is $\mathbb{R}^n$ or $\mathcal{A}_f$, $W_{per}^{s,p}(D)$ (resp. $C_{per}^{k,\alpha}(D)$) is the closure of $C_{per}^\infty(\mathbb{R}^n)$ under the $W^{s,p}$ (resp. $C^{k,\alpha}$) norm and $\|g\|_{W_{per}^{s,p}(D)} \equiv \|g\|_{W^{s,p}(D\cap Y)}$ (resp. $\|g\|_{C_{per}^{k,\alpha}(D)} \equiv \|g\|_{C^{k,\alpha}(D\cap Y)}$) for $s \geq 1$, $p \in [1,\infty]$, $k \geq 0$, and $\alpha \in [0,1]$. $L_{per}^p \equiv W_{per}^{0,p}$ and $\mathcal{H}_{per}^1(\mathcal{A}_f) \equiv \{g \in W_{per}^{1,2}(\mathcal{A}_f) | \int_{Y_f} g(y)dy = 0\}$. If $g(x) = G(x/\epsilon)$ and $G \in W_{loc}^{1,p}(\mathcal{A}_f)$, we define $\|g\|_{W_{loc}^{1,p}(\mathbb{D}_f^\epsilon)} \equiv \sup_{j\in\mathbb{Z}^n}\|G\|_{W^{1,p}((Y_f+j)\cap\mathbb{D}/\epsilon)}$.

For any $\lambda \in [0,1]$ and $i \in \{1,\cdots,n\}$, we find $\mathbb{S}^{(\lambda)}, \mathbb{Y}^{(i)} \in \mathcal{H}_{per}^1(\mathcal{A}_f)$ satisfying

$$\begin{cases} \nabla\cdot(\mathbf{K}\nabla\mathbb{S}^{(\lambda)} + \lambda\mathbf{T}) = 0 & \text{in } Y_f, \\ (\mathbf{K}\nabla\mathbb{S}^{(\lambda)} + \lambda\mathbf{T})\cdot\vec{\mathbf{n}} = 0 & \text{on } \partial Y_m, \end{cases} \tag{2.1}$$

$$\begin{cases} \nabla\cdot(\mathbf{K}\nabla\mathbb{Y}^{(i)} + \vec{e}_i) = 0 & \text{in } Y_f, \\ (\mathbf{K}\nabla\mathbb{Y}^{(i)} + \vec{e}_i)\cdot\vec{\mathbf{n}} = 0 & \text{on } \partial Y_m, \end{cases} \tag{2.2}$$

where $\vec{\mathbf{n}}$ denotes a unit normal vector on ∂Y_m and $\vec{e}_i$ is the unit vector in the i-th coordinate direction. For each $i \in \{1, \cdots, n\}$, we find $\mathbb{X}_1^{(0)}, \mathbb{X}_1^{(0,i)} \in \mathcal{H}^1_{per}(\mathcal{A}_f)$ satisfying

$$\mathbb{X}_1^{(0)}(y) \equiv \mathbb{S}^{(1)}(y),$$
$$\begin{cases} \nabla \cdot (\mathbf{K}(\nabla \mathbb{X}_1^{(0,i)} + \vec{e}_i)) = 0 & \text{in } Y_f, \\ \mathbf{K}(\nabla \mathbb{X}_1^{(0,i)} + \vec{e}_i) \cdot \vec{\mathbf{n}} = 0 & \text{on } \partial Y_m. \end{cases} \tag{2.3}$$

Denote by Ξ_1 (resp. Ξ_2) a $n \times n$ matrix function whose (i,j)-component is $\partial_i \mathbb{Y}^{(j)}$ (resp. $\partial_i \mathbb{X}_1^{(0,j)}$) for $i, j \in \{1, \cdots, n\}$ and define, for $\lambda, \omega \in [0,1]$,

$$\begin{cases} \mathcal{K} \equiv \displaystyle\int_{Y_f} \mathbf{K}(I + \Xi_2)(y)dy, \\ \mathcal{P} \equiv \displaystyle\int_{Y_f} \mathbf{P}(I + \Xi_2)(y)dy, \\ \mathcal{T}_\lambda \equiv \displaystyle\int_{Y_f} \big(\mathbf{K}\nabla \mathbb{S}^{(\lambda)} + \lambda \mathbf{T}\big)(y)dy, \\ \mathcal{E}_{\lambda,\omega} \equiv \displaystyle\int_{Y_f} \big(\lambda \mathbf{P}\nabla \mathbb{S}^{(\lambda)} + \omega \mathbf{E}\big)(y)dy, \\ \mathcal{Q} \equiv \dfrac{1}{|Y_f|}\displaystyle\int_{Y_f} (I + \mathbf{K}\Xi_1)(y)dy \ \mathcal{Q}_*, \\ \mathcal{F} \equiv \mathcal{F}_* - \dfrac{1}{|Y_f|}\displaystyle\int_{Y_f} \mathbf{P}\Xi_1(y)dy \ \mathcal{Q}_*, \end{cases} \tag{2.4}$$

where I is the identity matrix and $|Y_f|, \mathcal{Q}_*$ and $\mathcal{F}_*$ are from (1.3). Following the remark in page 90 [14], $\mathcal{K}$ is a constant symmetric positive definite matrix.

For $i_1, i_2 \in \{1, \cdots, n\}$, we find $\mathbb{X}_2^{(0)}, \mathbb{X}_2^{(0,i_1)}, \mathbb{X}_2^{(0,i_1,i_2)} \in \mathcal{H}^1_{per}(\mathcal{A}_f)$ satisfying

$$\begin{cases} \nabla \cdot (\mathbf{K}\nabla \mathbb{X}_2^{(0)} + \mathbf{T}\mathbb{X}_1^{(0)}) = \mathbf{P}\nabla \mathbb{X}_1^{(0)} + \mathbf{E} + \mathcal{N}_2^{(0)} & \text{in } Y_f, \\ (\mathbf{K}\nabla \mathbb{X}_2^{(0)} + \mathbf{T}\mathbb{X}_1^{(0)}) \cdot \vec{\mathbf{n}} = 0 & \text{on } \partial Y_m, \end{cases} \tag{2.5}$$

$$\begin{cases} \nabla \cdot (\mathbf{K}(\nabla \mathbb{X}_2^{(0,i_1)} + \mathbb{X}_1^{(0)}\vec{e}_{i_1}) + \mathbf{T}\mathbb{X}_1^{(0,i_1)}) & \\ \qquad = -\mathbf{K}\partial_{i_1}\mathbb{X}_1^{(0)} - \mathbf{T}^{(i_1)} + \mathbf{P}^{(i_1)} + \mathbf{P}\nabla \mathbb{X}_1^{(0,i_1)} + \mathcal{N}_2^{(0,i_1)} & \text{in } Y_f, \\ (\mathbf{K}(\nabla \mathbb{X}_2^{(0,i_1)} + \mathbb{X}_1^{(0)}\vec{e}_{i_1}) + \mathbf{T}\mathbb{X}_1^{(0,i_1)}) \cdot \vec{\mathbf{n}} = 0 & \text{on } \partial Y_m, \end{cases} \tag{2.6}$$

$$\begin{cases} \nabla \cdot (\mathbf{K}(\nabla \mathbb{X}_2^{(0,i_1,i_2)} + \mathbb{X}_1^{(0,i_1)}\vec{e}_{i_2})) & \\ \qquad = -\mathbf{K}(\partial_{i_2}\mathbb{X}_1^{(0,i_1)} + \delta_{i_1,i_2}) + \mathcal{N}_2^{(0,i_1,i_2)} & \text{in } Y_f, \\ \mathbf{K}(\nabla \mathbb{X}_2^{(0,i_1,i_2)} + \mathbb{X}_1^{(0,i_1)}\vec{e}_{i_2}) \cdot \vec{\mathbf{n}} = 0 & \text{on } \partial Y_m, \end{cases} \tag{2.7}$$

where $\mathcal{N}_2^{(0)} \equiv \frac{-\mathcal{E}_{1,1}}{|Y_f|}, \mathcal{N}_2^{(0,i_1)} \equiv \frac{\mathcal{T}_1^{(i_1)} - \mathcal{P}^{(i_1)}}{|Y_f|}, \mathcal{N}_2^{(0,i_1,i_2)} \equiv \frac{\mathcal{K}^{(i_1,i_2)}}{|Y_f|}$, $\mathcal{K}^{(i_1,i_2)}$ is the (i_1,i_2)-component of $\mathcal{K}$, $\delta_{i_1,i_2} \equiv \begin{cases} 1 & \text{if } i_1 = i_2 \\ 0 & \text{if } i_1 \neq i_2 \end{cases}$, and $\mathbf{P}^{(i_1)}$ (resp. $\mathbf{T}^{(i_1)}, \mathcal{T}_1^{(i_1)}, \mathcal{P}^{(i_1)}$) is the i_1-th component of $\mathbf{P}$ (resp. $\mathbf{T}, \mathcal{T}_1, \mathcal{P}$). See (2.4) for $\mathcal{K}, \mathcal{T}_1, \mathcal{P}, \mathcal{E}_{1,1}$.

For $k \geq 3$, we find $\mathbb{X}_k^{(i_0,i_1,\cdot\cdot,i_s)} \in \mathcal{H}^1_{per}(\mathcal{A}_f)$ for $s \in \{0,1,\cdot\cdot,k\}$, $i_0 = 0$, and $i_1,\cdot\cdot,i_s \in \{1,\cdot\cdot,n\}$ in such a way that

(1) $\mathbb{X}_k^{(0)}$ satisfies

$$\begin{cases} \nabla\cdot(\mathbf{K}\nabla\mathbb{X}_k^{(0)}) = \mathbf{E}\mathbb{X}_{k-2}^{(0)} + \mathcal{N}_k^{(0)} & \text{in } Y_f, \\ \mathbf{K}\nabla\mathbb{X}_k^{(0)}\cdot\vec{\mathbf{n}} = 0 & \text{on } \partial Y_m, \end{cases} \tag{2.8}$$

and (2) $\mathbb{X}_k^{(i_0,i_1,\cdot\cdot,i_s)}$ satisfies, if $s \neq 0$,

$$\begin{cases} \nabla\cdot\big(\mathbf{K}(\nabla\mathbb{X}_k^{(i_0,i_1,\cdot\cdot,i_s)} + \mathbb{X}_{k-1}^{(i_0,i_1,\cdot\cdot,i_{s-1})}\vec{e}_{i_s})\big) & \\ \quad = -\mathbf{K}\big(\partial_{i_s}\mathbb{X}_{k-1}^{(i_0,i_1,\cdot\cdot,i_{s-1})} + \delta_{i_{s-1},i_s}\mathbb{X}_{k-2}^{(i_0,i_1,\cdot\cdot,i_{s-2})}\big) & \\ \qquad +(1-\delta_{s,k-1})(1-\delta_{s,k})\mathbf{E}\mathbb{X}_{k-2}^{(i_0,i_1,\cdot\cdot,i_s)} + \mathcal{N}_k^{(i_0,i_1,\cdot\cdot,i_s)} & \text{in } Y_f, \\ \mathbf{K}(\nabla\mathbb{X}_k^{(i_0,i_1,\cdot\cdot,i_s)} + \mathbb{X}_{k-1}^{(i_0,i_1,\cdot\cdot,i_{s-1})}\vec{e}_{i_s})\cdot\vec{\mathbf{n}} = 0 & \text{on } \partial Y_m, \end{cases} \tag{2.9}$$

where $\mathcal{N}_k^{(i_0,i_1,\cdot\cdot,i_s)}$ is a constant defined as

$$\mathcal{N}_k^{(i_0,i_1,\cdot\cdot,i_s)} \equiv \begin{cases} \dfrac{-1}{|Y_f|}\displaystyle\int_{Y_f}\mathbf{E}\mathbb{X}_{k-2}^{(0)}dy & \text{if } s=0, \\ \dfrac{1}{|Y_f|}\displaystyle\int_{Y_f}\mathbf{K}\big(\partial_{i_s}\mathbb{X}_{k-1}^{(i_0,i_1,\cdot\cdot,i_{s-1})} + \delta_{i_{s-1},i_s}\mathbb{X}_{k-2}^{(i_0,i_1,\cdot\cdot,i_{s-2})}\big)dy & \\ \quad -\dfrac{1}{|Y_f|}\displaystyle\int_{Y_f}(1-\delta_{s,k-1})(1-\delta_{s,k})\mathbf{E}\mathbb{X}_{k-2}^{(i_0,i_1,\cdot\cdot,i_s)}dy & \text{if } s \neq 0. \end{cases}$$

In (2.9), we adopt the convention: If $s=1$, then $\delta_{i_{s-1},i_s}\mathbb{X}_{k-2}^{(i_0,i_1,\cdot\cdot,i_{s-2})} = 0$ and if $s \in \{k-1,k\}$, then $(1-\delta_{s,k-1})(1-\delta_{s,k})\mathbf{E}\mathbb{X}_{k-2}^{(i_0,i_1,\cdot\cdot,i_s)} = 0$. By Lax-Milgram Theorem [12], we know that if $\lambda \in [0,1]$, $\mathbf{K} \geq \tau > 0$, $\mathbf{K},\mathbf{T},\mathbf{P},\mathbf{E} \in L^\infty_{per}(\mathbb{R}^n)$, and Y_m is a $C^{1,1}$ domain, then $\mathbb{S}^{(\lambda)}, \mathbb{Y}^{(i)}$ in (2.1)–(2.2) and $\mathbb{X}_k^{(i_0,i_1,\cdots,i_s)}$ in (2.3) and (2.5)–(2.9) are solvable uniquely in $\mathcal{H}^1_{per}(\mathcal{A}_f)$. For any $\nu > 0$, $k \geq 1$, $s \in \{0,1,\cdot\cdot,k\}$, $i_0 = 0$, and $i_1,\cdots,i_s \in \{1,\cdots,n\}$, we define

$$\begin{cases} \mathbb{X} \equiv (\mathbb{X}_1^{(0,1)},\cdots,\mathbb{X}_1^{(0,n)}), \\ \mathbb{X}_{k,\nu}^{(i_0,i_1,\cdots,i_s)}(x) \equiv \nu^k\mathbb{X}_k^{(i_0,i_1,\cdots,i_s)}(\frac{x}{\nu}), \\ \mathbb{X}_\nu \equiv (\mathbb{X}_{1,\nu}^{(0,1)},\cdots,\mathbb{X}_{1,\nu}^{(0,n)}). \end{cases} \tag{2.10}$$

We recall an extension result in [1].

Lemma 2.1. *For any $\epsilon \in (0,1)$ and $p \in [1,\infty)$, there is a constant $c(Y_f,p)$ and a linear continuous extension operator $\Pi_\epsilon : W^{1,p}(\Omega_f^\epsilon) \to W^{1,p}(\Omega)$ such that*
(1) if $g \in W^{1,p}(\Omega_f^\epsilon)$, then

$$\begin{cases} \Pi_\epsilon g = g \quad \textit{in } \Omega_f^\epsilon \textit{ almost everywhere}, \\ \|\Pi_\epsilon g\|_{L^p(\Omega)} \leq c(Y_f,p)\|g\|_{L^p(\Omega_f^\epsilon)}, \\ \|\nabla\Pi_\epsilon g\|_{L^p(\Omega)} \leq c(Y_f,p)\|\nabla g\|_{L^p(\Omega_f^\epsilon)}, \\ \|\Pi_\epsilon g\|_{C^{0,\alpha}(\overline{\Omega})} \leq c(Y_f,p)\|g\|_{C^{0,\alpha}(\overline{\Omega_f^\epsilon})} \quad \textit{if } g \in C^{0,\alpha}(\overline{\Omega_f^\epsilon}), \alpha \in (0,1), \\ \Pi_\epsilon g = G \quad \textit{in } \Omega \textit{ if } g = G|_{\Omega_f^\epsilon} \textit{ for some linear function } G \textit{ in } \Omega, \end{cases}$$

(2) if $G(x) \equiv g(rx)$ *in* $B_1(z) \cap \Omega_f^\epsilon/r$ *for any* $z \in \mathbb{R}^n$ *and any constant* $r \geq \epsilon$*, there is a* $\theta \in (0,1)$ *so that* $\Pi_{\epsilon/r}G(x) = (\Pi_\epsilon g)(rx)$ *in* $B_\theta(z) \cap \Omega/r$*.*

Lemma 2.1 also holds if $\{\Omega_f^\epsilon, \Omega\}$ is replaced by $\{\mathcal{A}_f^\epsilon, \mathbb{R}^n\}$. First, we state a uniform estimate for problem (1.1).

Theorem 2.1. *Suppose*

A1. $\epsilon \in (0,1)$*,* Ω *and* Y_m *are bounded* $C^{1,1}$ *simply-connected domains in* $\mathbb{R}^n$ *for* $n \in \{2,3\}$*,*

A2. $\mathbf{K}, \mathbf{T} \in C^{0,1}_{per}(\mathbb{R}^n)$ *and* $\mathbf{K}$ *is a positive function,*

A3. $\mathbf{P}, \mathbf{E} \in L^\infty_{per}(\mathbb{R}^n)$*,* $\|\mathbf{T}, \nabla\mathbf{T}, \mathbf{P}\|_{L^\infty(\mathbb{R}^n)} \leq \mathbf{M}$*,* $\mathbf{E}(x) - 3\frac{|\mathbf{T}|^2+|\mathbf{P}|^2}{4\mathbf{K}}(x) > 0$ *and* $\mathbf{E}(x) \in (0, \mathbf{M}]$ *for all* $x \in \mathbb{R}^n$*,*

A4. $\delta \in (0,3)$*,* $Q_\epsilon, F_\epsilon \in L^{n+\delta}(\Omega_f^\epsilon)$*,*

the solution of (1.1) exists uniquely in $H^1(\Omega_f^\epsilon)$ *and there is a* $\mu_* \in (0, \frac{\delta}{n+\delta})$ *(depending on* $\delta, Y_f, \mathbf{K}, \mathbf{M}, \Omega$*) such that*

$$\|U_\epsilon\|_{C^{0,\mu_*}(\overline{\Omega_f^\epsilon})} \leq c\|Q_\epsilon, F_\epsilon\|_{L^{n+\delta}(\Omega_f^\epsilon)},$$

where c is a constant independent of ϵ*.*

Proof of Theorem 2.1 is given in section 4. Under A1–A3 and A5 given below, it is easy to see that the solution of (1.1) satisfies (1.2) and that there is a subsequence of $\{U_\epsilon, Q_\epsilon, F_\epsilon\}$ such that (1.3), (1.4) hold. Moreover, we have the following L^∞ convergence estimate.

Theorem 2.2. *Under A1–A3 and*

A5. Ω *is a* $C^{2,1}$ *domain,* $\delta \in (0,3)$*,* $\mathbf{T} \cdot \vec{\mathbf{n}} = 0$ *on* ∂Y_m*,* Q_ϵ *(resp.* F_ϵ*) is bounded in* $W^{2,n+\delta}(\Omega)$ *(resp.* $W^{1,n+\delta}(\Omega)$*),*

the solution U_ϵ *of (1.1) and the solution* U *of (1.4) satisfy*

$$\|U_\epsilon - U\|_{L^\infty(\Omega_f^\epsilon)} \leq c(\epsilon + \|Q_\epsilon, F_\epsilon - \nabla \cdot \mathcal{Q} - \mathcal{F}\|_{L^{n+\delta}(\Omega_f^\epsilon)}), \tag{2.11}$$

where c is a constant independent of ϵ*. See (1.4) and (2.4) for* $\mathcal{Q}, \mathcal{F}$*.*

Proof of Theorem 2.2 is given in section 5. From Theorem 2.2, we note that if Q_ϵ is small and if F_ϵ is close to $\mathcal{F}$ in Ω_f^ϵ, then the right hand side of (2.11) is small. In this case, the U_ϵ of (1.1) converges uniformly to the U of (1.4) in Ω_f^ϵ.

Next we state Lipschitz uniform bound results. Consider the following problem: Find $U_\epsilon \in H^1(\mathbb{D}_f^\epsilon)$ satisfying

$$\begin{cases} -\nabla \cdot (\mathbf{K}_\epsilon \nabla U_\epsilon + Q_\epsilon) + \mathbf{E}_\epsilon U_\epsilon = F_\epsilon & \text{in } \mathbb{D}_f^\epsilon, \\ (\mathbf{K}_\epsilon \nabla U_\epsilon + Q_\epsilon) \cdot \vec{\mathbf{n}}_\epsilon = 0 & \text{on } \partial\mathbb{D}_m^\epsilon, \\ U_\epsilon \text{ satisfies periodic boundary conditions on } \partial\mathbb{D}. \end{cases} \tag{2.12}$$

Theorem 2.3. *Suppose*

A6. $\epsilon \in (0,1)$, Y_m *is a* $C^{1,1}$ *simply-connected domain in* $\mathbb{R}^n$ *for* $n \in \{2,3\}$,
A7. $\mathbf{K} \in C^{0,1}_{per}(\mathbb{R}^n)$ *is a positive function,*
A8. $\mathbf{E} \in L^\infty_{per}(\mathbb{R}^n)$, $\mathbf{E}(x) \in (0, \mathbf{M}]$ *for all* $x \in \mathbb{R}^n$,
A9. $\delta \in (0,3), F_\epsilon \in L^{n+\delta}(\mathbb{D}^\epsilon_f)$, *and* $Q_\epsilon \in W^{1,n+\delta}(\mathbb{D}^\epsilon_f)$ *is a periodic function in* $\mathcal{A}^\epsilon_f$ *with period* $\mathbb{D}$,

then there is a $\mathbf{M}_0 < 1$ *(depending on* $\delta, Y_m, \mathbf{K}$*) such that if* $\mathbf{M} \le \mathbf{M}_0$*, the solution of (2.12) exists uniquely in* $H^1(\mathbb{D}^\epsilon_f)$ *and satisfies*

$$\|\nabla U_\epsilon\|_{L^\infty(\mathbb{D}^\epsilon_f)} \le c\left(\|\epsilon^{\frac{\mu}{2}-1} Q_\epsilon, F_\epsilon\|_{L^{n+\delta}(\mathbb{D}^\epsilon_f)} + \epsilon^{-\mu/2} \|Q_\epsilon\|_{W^{1,n+\delta}_{loc}(\mathbb{D}^\epsilon_f)}\right),$$

where $\mu \equiv \frac{\delta}{n+\delta}$ *and* c *is a constant independent of* ϵ.

Consider the following problem: Find $U_\epsilon \in H^1(\mathbb{D}^\epsilon_f)$ such that

$$\begin{cases} -\nabla \cdot (\mathbf{K}_\epsilon \nabla U_\epsilon + Q_\epsilon) = F_\epsilon & \text{in } \mathbb{D}^\epsilon_f, \\ (\mathbf{K}_\epsilon \nabla U_\epsilon + Q_\epsilon) \cdot \vec{\mathbf{n}}_\epsilon = 0 & \text{on } \partial\mathbb{D}^\epsilon_m, \\ U_\epsilon \text{ satisfies periodic boundary conditions on } \partial\mathbb{D}, \\ \displaystyle\int_{\mathbb{D}} \Pi_\epsilon U_\epsilon dx = 0. \end{cases} \tag{2.13}$$

See Lemma 2.1 for Π_ϵ. Then we have

Theorem 2.4. *Besides A6–A7 and A9, if* $\displaystyle\int_{\mathbb{D}^\epsilon_f} F_\epsilon(x)dx = 0$*, the solution of (2.13) exists uniquely in* $H^1(\mathbb{D}^\epsilon_f)$ *and satisfies*

$$\|\nabla U_\epsilon\|_{L^\infty(\mathbb{D}^\epsilon_f)} \le c\left(\|\epsilon^{\mu/2-1} Q_\epsilon, F_\epsilon\|_{L^{n+\delta}(\mathbb{D}^\epsilon_f)} + \epsilon^{-\mu/2} \|Q_\epsilon\|_{W^{1,n+\delta}_{loc}(\mathbb{D}^\epsilon_f)}\right),$$

where $\mu \equiv \frac{\delta}{n+\delta}$ *and* c *is a constant independent of* ϵ.

Proofs of Theorem 2.3 and Theorem 2.4 are in subsection 6.1. Now we state Lipschitz convergence estimates. We find $U_\epsilon \in H^1(\mathbb{D}^\epsilon_f)$ satisfying

$$\begin{cases} -\nabla \cdot (\mathbf{K}_\epsilon \nabla U_\epsilon) + \mathbf{E}_\epsilon U_\epsilon = F_\epsilon & \text{in } \mathbb{D}^\epsilon_f, \\ \mathbf{K}_\epsilon \nabla U_\epsilon \cdot \vec{\mathbf{n}}_\epsilon = 0 & \text{on } \partial\mathbb{D}^\epsilon_m, \\ U_\epsilon \text{ satisfies periodic boundary conditions on } \partial\mathbb{D}. \end{cases} \tag{2.14}$$

Under A6–A8 and $F_\epsilon \in L^2(\mathbb{D}^\epsilon_f)$, (2.14) is solvable uniquely. By Lemma 2.1 and tracing the argument of Theorem 2.7 [3], there is a subsequence of $\{U_\epsilon, F_\epsilon\}$ (same notation for subsequence) satisfying

$$\begin{cases} \mathbf{K}_\epsilon \nabla U_\epsilon \mathcal{X}_{\mathbb{D}^\epsilon_f} \to \mathcal{K} \nabla U \\ \mathbf{E}_\epsilon U_\epsilon \mathcal{X}_{\mathbb{D}^\epsilon_f} \to \mathcal{E}_{0,1} U & \text{in } L^2(\mathbb{D}) \text{ weakly as } \epsilon \to 0, \\ F_\epsilon \mathcal{X}_{\mathbb{D}^\epsilon_f} \to |Y_f| \widehat{\mathcal{F}} \end{cases}$$

where $U \in H^1(\mathbb{D}), \widehat{\mathcal{F}} \in L^2(\mathbb{D})$. See (2.4) for $\mathcal{K}, \mathcal{E}_{0,1}$. The U above satisfies

$$\begin{cases} -\nabla \cdot (\mathcal{K}\nabla U) + \mathcal{E}_{0,1} U = |Y_f| \widehat{\mathcal{F}} & \text{in } \mathbb{D}, \\ U \text{ satisfies periodic boundary conditions on } \partial\mathbb{D}. \end{cases} \tag{2.15}$$

We note that if $\mathbf{T} = \mathbf{P} = 0$ in (2.1), (2.3), and (2.5)–(2.7), then the $\mathbb{X}_\ell^{(i_0,i_1,\cdot\cdot,i_s)}$, $\mathcal{N}_\ell^{(i_0,i_1,\cdot\cdot,i_s)}$ in (2.3)–(2.9) satisfy

- $\mathbb{X}_1^{(0)} = \mathbb{X}_2^{(0,i_1)} = 0$,
- $\mathbb{X}_1^{(0,i)}$ and $\mathbb{X}_2^{(0,i_1,i_2)}$ still satisfy (2.3) and (2.7),
- $\begin{cases} \nabla \cdot (\mathbf{K}\nabla \mathbb{X}_2^{(0)}) = \mathbf{E} + \mathcal{N}_2^{(0)} & \text{in } Y_f, \\ \mathbf{K}\nabla \mathbb{X}_2^{(0)} \cdot \vec{\mathbf{n}} = 0 & \text{on } \partial Y_m, \end{cases}$
- $\mathcal{N}_2^{(0)} = \frac{-\mathcal{E}_{0,1}}{|Y_f|} < 0$ and $(\mathcal{N}_2^{(0,i_1,i_2)}) = \frac{\mathcal{K}}{|Y_f|}$ is a symmetric positive definite matrix,
- $\mathbb{X}_\ell^{(i_0,i_1,\cdot\cdot,i_s)} = \mathcal{N}_\ell^{(i_0,i_1,\cdot\cdot,i_s)} = 0$ for $\ell + s = 2m+1, \ell \geq 2, 0 \leq s \leq \ell, m \geq 0$,

where $i_0 = 0, i_1, \cdots, i_s \in \{1, \cdots, n\}$. For any $k \geq 1$, let us define $\widehat{\Phi}_k \equiv \sum_{j=0}^{k-1} \epsilon^j \widehat{\varphi}_j$ in $\mathbb{D}$, where $\widehat{\varphi}_0 = U$ (i.e., the solution of (2.15)) and $\widehat{\varphi}_j \in H^1(\mathbb{D})$ for each $j = 1, \cdots, k-1$ is the solution of

$$\begin{cases} \displaystyle\sum_{s=0}^{2} \sum_{|\vec{\mathbf{i}}|=s} \mathcal{N}_2^{\vec{\mathbf{i}}} \partial_{\vec{\mathbf{i}}} \widehat{\varphi}_j = -\sum_{q=0}^{j-1} \sum_{s=0}^{j+2-q} \sum_{|\vec{\mathbf{i}}|=s} \mathcal{N}_{j+2-q}^{\vec{\mathbf{i}}} \partial_{\vec{\mathbf{i}}} \widehat{\varphi}_q & \text{in } \mathbb{D}, \\ \widehat{\varphi}_j \text{ satisfies periodic boundary condition on } \partial\mathbb{D}, \end{cases} \tag{2.16}$$

and define

$$\widehat{\mathcal{V}}_k(x) \equiv \widehat{\Phi}_k(x) + \sum_{\ell=1}^{k+1} \sum_{s=0}^{\ell} \sum_{|\vec{\mathbf{i}}|=s} \mathbb{X}_{\ell,\epsilon}^{\vec{\mathbf{i}}}(x) \partial_{\vec{\mathbf{i}}} \widehat{\Phi}_k(x) \qquad \text{in } x \in \mathbb{D}_f^\epsilon. \tag{2.17}$$

Here $\vec{\mathbf{i}} \equiv (i_0, i_1, \cdots, i_s), |\vec{\mathbf{i}}| \equiv s, i_0 = 0, i_1, \cdots, i_s \in \{1, \cdots, n\}$, $\mathcal{N}_\ell^{\vec{\mathbf{i}}} \equiv \mathcal{N}_\ell^{(i_0,i_1,\cdots,i_s)}$, $0 \leq s \leq \ell$, $\partial_{\vec{\mathbf{i}}}\varphi \equiv \partial_{i_1,\cdots,i_s}\varphi$, and $\mathbb{X}_{\ell,\epsilon}^{\vec{\mathbf{i}}} \equiv \mathbb{X}_{\ell,\epsilon}^{(i_0,i_1,\cdots,i_s)}$. See (2.3) and (2.5)–(2.10) for $\mathbb{X}_{\ell,\epsilon}^{(i_0,i_1,\cdots,i_s)}$. It is easy to see that, under A7-A8 and (2.4), problem (2.16) is a uniform elliptic equation with constant coefficients and it is uniquely solvable in $H^1(\mathbb{D})$ for each $j = 1, \cdots, k-1$ if $F_\epsilon \in H^{2k}(\mathbb{D})$ for $k \geq 2$.

Theorem 2.5. *Under A6–A8 and*

> *A10.* $\delta \in (0,3), k \geq 1, F_\epsilon$ *is bounded in* $W^{2k,n+\delta}(\mathbb{D})$ *and is a periodic function in* $\mathbb{R}^n$ *with period* $\mathbb{D}$*,*

there is a $\mathbf{M}_0 < 1$ *(depending on* $\delta, Y_m, \mathbf{K}$*) such that if* $\mathbf{M} \leq \mathbf{M}_0$*, the solution* U_ϵ *of (2.14) and the* $\widehat{\mathcal{V}}_k$ *in (2.17) satisfy*

$$\|\nabla(U_\epsilon - \widehat{\mathcal{V}}_k)\|_{L^\infty(\mathbb{D}_f^\epsilon)} \leq c(\epsilon^k + \|F_\epsilon - \widehat{\mathcal{F}}\|_{L^{n+\delta}(\mathbb{D}_f^\epsilon)}),$$

where c *is a constant independent of* ϵ*. See (2.15) for* $\widehat{\mathcal{F}}$*.*

Let us also consider the following problem: Find $U_\epsilon \in H^1(\mathbb{D}_f^\epsilon)$ satisfying

$$\begin{cases} -\nabla\cdot(\mathbf{K}_\epsilon\nabla U_\epsilon) = F_\epsilon & \text{in } \mathbb{D}_f^\epsilon, \\ \mathbf{K}_\epsilon\nabla U_\epsilon\cdot\vec{\mathbf{n}}_\epsilon = 0 & \text{on } \partial\mathbb{D}_m^\epsilon, \\ U_\epsilon \text{ satisfies periodic boundary conditions on } \partial\mathbb{D}, \\ \displaystyle\int_{\mathbb{D}} \Pi_\epsilon U_\epsilon dx = 0. \end{cases} \tag{2.18}$$

Under A6–A7 and $F_\epsilon \in L^2(\mathbb{D}_f^\epsilon)$ with $\displaystyle\int_{\mathbb{D}_f^\epsilon} F_\epsilon(x)dx = 0$, (2.18) is solvable uniquely. By Lemma 2.1 and tracing the argument of Theorem 2.7 [3], there is a subsequence of $\{U_\epsilon, F_\epsilon\}$ (same notation for subsequence) satisfying

$$\begin{cases} \Pi_\epsilon U_\epsilon \to U & \text{in } L^2(\mathbb{D}) \text{ strongly} \\ \mathbf{K}_\epsilon\nabla U_\epsilon\mathcal{X}_{\mathbb{D}_f^\epsilon} \to \mathcal{K}\nabla U & \text{in } L^2(\mathbb{D}) \text{ weakly} \qquad \text{as } \epsilon\to 0, \\ F_\epsilon\mathcal{X}_{\mathbb{D}_f^\epsilon} \to |Y_f|\breve{\mathcal{F}} & \text{in } L^2(\mathbb{D}) \text{ weakly} \end{cases}$$

where $U \in H^1(\mathbb{D}), \breve{\mathcal{F}} \in L^2(\mathbb{D})$. See (2.4) for $\mathcal{K}$. The U above satisfies

$$\begin{cases} -\nabla\cdot(\mathcal{K}\nabla U) = |Y_f|\breve{\mathcal{F}} & \text{in } \mathbb{D}, \\ U \text{ satisfies periodic boundary conditions on } \partial\mathbb{D}, \\ \displaystyle\int_{\mathbb{D}} U dx = 0. \end{cases} \tag{2.19}$$

For any $k \geq 1$, let us define $\breve{\Phi}_k \equiv \sum_{j=0}^{k-1} \epsilon^j \breve{\varphi}_j$ in $\mathbb{D}$, where $\breve{\varphi}_0 = U$ (i.e., the solution of (2.19)) and $\breve{\varphi}_j \in H^1(\mathbb{D})$ for each $j = 1, \cdots, k-1$ is the solution of

$$\begin{cases} \displaystyle\sum_{|\vec{\mathbf{i}}|=2} \mathcal{N}_2^{\vec{\mathbf{i}}}\partial_{\vec{\mathbf{i}}}\breve{\varphi}_j = -\sum_{q=0}^{j-1}\sum_{|\vec{\mathbf{i}}|=j+2-q} \mathcal{N}_{j+2-q}^{\vec{\mathbf{i}}}\partial_{\vec{\mathbf{i}}}\breve{\varphi}_q \quad \text{in } \mathbb{D}, \\ \breve{\varphi}_j \text{ satisfies periodic boundary condition on } \partial\mathbb{D}, \\ \displaystyle\int_{\mathbb{D}} \breve{\varphi}_j(y)dy = 0, \end{cases} \tag{2.20}$$

and define

$$\breve{\mathcal{V}}_k(x) \equiv \breve{\Phi}_k(x) + \sum_{\ell=1}^{k+1}\sum_{|\vec{\mathbf{i}}|=\ell} \mathbb{X}_{\ell,\epsilon}^{\vec{\mathbf{i}}}(x)\partial_{\vec{\mathbf{i}}}\breve{\Phi}_k(x) \qquad \text{in } x\in\mathbb{D}_f^\epsilon. \tag{2.21}$$

Note (2.20) is uniquely solvable in $H^1(\mathbb{D})$ if $F_\epsilon \in H^{2k}(\mathbb{D})$ for $k \geq 2$.

Theorem 2.6. *Besides A6–A7 and A10, if* $\displaystyle\int_{\mathbb{D}_f^\epsilon} F_\epsilon dx = 0$*, the solution* U_ϵ *of (2.18) and the* $\breve{\mathcal{V}}_k$ *in (2.21) satisfy*

$$\|\nabla(U_\epsilon - \breve{\mathcal{V}}_k)\|_{L^\infty(\mathbb{D}_f^\epsilon)} \leq c(\epsilon^k + \|F_\epsilon - \breve{\mathcal{F}}\|_{L^{n+\delta}(\mathbb{D}_f^\epsilon)}),$$

where c *is a constant independent of* ϵ*. See (2.19) for* $\breve{\mathcal{F}}$*.*

Proofs of Theorem 2.5 and Theorem 2.6 are given in subsection 6.2.

3. A priori $W^{2,p}$ estimates

This section is to derive a priori $W^{2,p}$ estimates for the solutions of elliptic equations around the Neumann boundary. Let $\mathbb{R}^n_+ \equiv \{(x_1, \cdots, x_n)|x_n > 0\}$, $\partial\mathbb{R}^n_+ \equiv \{(x_1, \cdots, x_n)|x_n = 0\}$, and $\vec{\mathbf{n}}$ is a unit vector normal to $\partial\mathbb{R}^n_+$. Consider the following problem

$$\begin{cases} \nabla \cdot (\mathbb{K}\nabla\Psi + \mathbb{Q}) + \mathbb{P}\nabla\Psi + \mathbb{E}\Psi = \mathbb{G} & \text{in } B_1(0) \cap \mathbb{R}^n_+, \\ (\mathbb{K}\nabla\Psi + \mathbb{Q}) \cdot \vec{\mathbf{n}} = 0 & \text{on } B_1(0) \cap \partial\mathbb{R}^n_+. \end{cases} \tag{3.1}$$

Lemma 3.1. *If $\mathbb{K} \in W^{1,\infty}(B_1(0) \cap \mathbb{R}^n_+)$, $\mathbb{K} > \tau > 0$, $\mathbb{Q} \in W^{1,p}(B_1(0) \cap \mathbb{R}^n_+)$, $\mathbb{P}, \mathbb{E} \in L^\infty(B_1(0) \cap \mathbb{R}^n_+)$, and $\mathbb{G} \in L^p(B_1(0) \cap \mathbb{R}^n_+)$ for some $p \in (1, \infty)$, then any solution of (3.1) satisfies*

$$\|\Psi\|_{W^{2,p}(B_{1/2}(0)\cap\mathbb{R}^n_+)} \le c(\|\Psi, \mathbb{G}\|_{L^p(B_1(0)\cap\mathbb{R}^n_+)} + \|\mathbb{Q}\|_{W^{1,p}(B_1(0)\cap\mathbb{R}^n_+)}),$$

where c is a constant depending on $p, \tau, \|\mathbb{K}, \nabla\mathbb{K}, \mathbb{P}, \mathbb{E}\|_{L^\infty(B_1(0)\cap\mathbb{R}^n_+)}$.

Proof. By Theorem 7.53 [2], there exists a $\Phi \in W^{2,p}(\mathbb{R}^n_+)$ satisfying

$$\begin{cases} (\mathbb{K}\nabla\Phi + \mathbb{Q}) \cdot \vec{\mathbf{n}} = 0 \qquad \text{on } B_{3/4}(0) \cap \partial\mathbb{R}^n_+, \\ \|\Phi\|_{W^{2,p}(B_{3/4}(0)\cap\mathbb{R}^n_+)} \le c\|\mathbb{Q}\|_{W^{1-1/p,p}(B_1(0)\cap\partial\mathbb{R}^n_+)} \le c\|\mathbb{Q}\|_{W^{1,p}(B_1(0)\cap\mathbb{R}^n_+)}, \end{cases} \tag{3.2}$$

where c is a constant. If we define $\psi \equiv \Psi - \Phi$, then

$$\begin{cases} \nabla \cdot (\mathbb{K}\nabla\psi) + \mathbb{P}\nabla\psi + \mathbb{E}\psi \\ \qquad = \mathbb{G} - \nabla \cdot (\mathbb{K}\nabla\Phi + \mathbb{Q}) - \mathbb{P}\nabla\Phi - \mathbb{E}\Phi \equiv \mathcal{G} & \text{in } B_{3/4}(0) \cap \mathbb{R}^n_+, \\ \mathbb{K}\nabla\psi \cdot \vec{\mathbf{n}} = 0 & \text{on } B_{3/4}(0) \cap \partial\mathbb{R}^n_+. \end{cases}$$

We do odd extension for $\mathbb{P}$ and even extension for $\mathbb{K}, \psi, \mathbb{E}, \mathcal{G}$. Same notation is used for the extended function. Then ψ in the domain $B_{3/4}(0)$ satisfies

$$\nabla \cdot (\mathbb{K}\nabla\psi) + \mathbb{P}\nabla\psi + \mathbb{E}\psi = \mathcal{G} \qquad \text{in } B_{3/4}(0).$$

By Theorem 9.11 [12], we obtain $\|\psi\|_{W^{2,p}(B_{1/2}(0))} \le c\|\psi, \mathcal{G}\|_{L^p(B_{3/4}(0))}$, where c is a constant depending on $p, \tau, \|\mathbb{K}, \nabla\mathbb{K}, \mathbb{P}, \mathbb{E}\|_{L^\infty(B_1(0)\cap\mathbb{R}^n_+)}$. Together with (3.2), we prove the lemma. □

Assume $x_0 \in \partial Y_m$ and $B_{1/4}(x_0) \subset Y$.

Lemma 3.2. *Consider the following problem*

$$\begin{cases} \sum_{i=1}^n \partial_i(\sum_{j=1}^n \mathbb{K}^{(i,j)}\partial_j\Psi + \mathbb{Q}^{(i)}) + \mathbb{E}\Psi = \mathbb{G} & \textit{in } Y_f, \\ \sum_{i=1}^n(\sum_{j=1}^n \mathbb{K}^{(i,j)}\partial_j\Psi + \mathbb{Q}^{(i)})\mathbf{n}^{(i)} = 0 & \textit{on } \partial Y_m, \end{cases} \tag{3.3}$$

where $\vec{\mathbf{n}} = (\mathbf{n}^{(1)}, \mathbf{n}^{(2)}, \cdots, \mathbf{n}^{(n)})$ is a unit vector normal to ∂Y_m. If Y_m is a $C^{1,1}$ domain, $\mathbb{K}^{(i,j)} \in W^{1,\infty}(Y_f)$, $\mathbb{K} \equiv (\mathbb{K}^{(i,j)}) > \tau > 0$, $\mathbb{Q} \equiv (\mathbb{Q}^{(1)}, \cdots, \mathbb{Q}^{(n)}) \in W^{1,p}(Y_f)$, $\mathbb{E} \in L^\infty(Y_f)$, and $\mathbb{G} \in L^p(Y_f)$, then any solution of (3.3) satisfies

$$\|\Psi\|_{W^{2,p}(B_{1/8}(x_0)\cap Y_f)} \le c(\|\Psi, \mathbb{G}\|_{L^p(Y_f)} + \|\mathbb{Q}\|_{W^{1,p}(Y_f)}) \qquad \textit{for } p \in (1, \infty),$$

where c is a constant depending on $p, \tau, Y_m, \|\mathbb{K}, \nabla\mathbb{K}, \mathbb{E}\|_{L^\infty(Y_f)}$.

Proof. By assumption, at each point $x_0 \in \partial Y_m$ there exist a neighborhood $B_{1/8}(x_0)$ of x_0 and a $C^{1,1}$ diffeomorphism $\mathcal{L}$ that straightens the boundary ∂Y_m in $B_{1/8}(x_0)$. Set $D = B_{1/8}(x_0) \cap Y_f$, $\widehat{D} = \mathcal{L}(D)$, $S = B_{1/8}(x_0) \cap \partial Y_m \subset \partial D$, and $\widehat{S} = \mathcal{L}(S) \subset \partial\widehat{D}$. Under the mapping $y = \mathcal{L}(x) = (\mathcal{L}^{(1)}(x), \cdots, \mathcal{L}^{(n)}(x))$, if we define $\widehat{\Psi}(y) \equiv \Psi(x)$, then problem (3.3) is transformed to

$$\begin{cases} \sum_{i=1}^n \partial_i(\sum_{j=1}^n \widehat{\mathbb{K}}^{(i,j)}\partial_j\widehat{\Psi} + \widehat{\mathbb{Q}}^{(i)}) + \sum_{i=1}^n \widehat{\mathbb{P}}^{(i)}\partial_i\widehat{\Psi} + \widehat{\mathbb{E}}\widehat{\Psi} = \widehat{\mathbb{G}} & \text{in } \widehat{D}, \\ \sum_{i=1}^n (\sum_{j=1}^n \widehat{\mathbb{K}}^{(i,j)}\partial_j\widehat{\Psi} + \widehat{\mathbb{Q}}^{(i)})\mathbf{n}^{(y_i)} = 0 & \text{on } \widehat{S}, \end{cases}$$

where $\vec{\mathbf{n}}_y = (\mathbf{n}^{(y_1)}, \mathbf{n}^{(y_2)}, \cdots, \mathbf{n}^{(y_n)})$ is a unit vector normal to $\widehat{S}$ and

$$\begin{cases} \widehat{\mathbb{K}}^{(i,j)}(y) \equiv \sum_{k,m=1}^n \mathbb{K}^{(k,m)}\partial_k\mathcal{L}^{(i)}\partial_m\mathcal{L}^{(j)}(x), \\ \widehat{\mathbb{Q}}^{(i)}(y) \equiv \sum_{k=1}^n \mathbb{Q}^{(k)}\partial_k\mathcal{L}^{(i)}(x), \\ \widehat{\mathbb{P}}^{(i)}(y) \equiv -\sum_{j,k,m=1}^n \mathbb{K}^{(m,j)}\partial_{k,m}\mathcal{L}^{(k)}\partial_j\mathcal{L}^{(i)}(x), \\ \widehat{\mathbb{E}}(y) \equiv \mathbb{E}(x), \\ \widehat{\mathbb{G}}(y) \equiv \mathbb{G}(x) + \sum_{k,m=1}^n \mathbb{Q}^{(m)}\partial_{k,m}\mathcal{L}^{(k)}(x). \end{cases}$$

Clearly, $\widehat{\mathbb{K}}^{(i,j)}, \widehat{\mathbb{Q}}^{(i)}, \widehat{\mathbb{P}}^{(i)}, \widehat{\mathbb{E}}, \widehat{\mathbb{G}}$ satisfy the requirements of Lemma 3.1. This lemma follows from Lemma 3.1. □

If $\mathbf{K}, \mathbf{T} \in C^{0,1}_{per}(\mathbb{R}^n)$, $\mathbf{K} > \tau > 0$, $\mathbf{P}, \mathbf{E} \in L^\infty_{per}(\mathbb{R}^n)$, and Y_m is a $C^{1,1}$ domain, the $\mathbb{S}^{(\lambda)}, \mathbb{Y}^{(i)}$ in (2.1)–(2.2) and the $\mathbb{X}_k^{(i_0,i_1,\cdots,i_s)}$ in (2.3) and (2.5)–(2.9) satisfy, by Theorem 9.11 [12] and Lemma 3.2,

$$\|\mathbb{S}^{(\lambda)}, \mathbb{Y}^{(i)}, \mathbb{X}_k^{(i_0,i_1,\cdots,i_s)}\|_{W^{2,p}(Y_f)} \leq c(\|\mathbf{K}, \mathbf{T}\|_{C^{0,1}(Y_f)} + \|\mathbf{P}, \mathbf{E}\|_{L^\infty(Y_f)}), \tag{3.4}$$

where $p \in (1, \infty)$, $\lambda \in [0, 1]$, $s \in \{0, 1, \cdot\cdot, k\}$, $k \geq 1$, $i_0 = 0$, $i, i_1, \cdots, i_s \in \{1, \cdots, n\}$, and c is a constant independent of λ, k. (3.4) implies the functions in (2.4) satisfy

$$|\mathcal{K}|, |\mathcal{P}|, |\mathcal{T}_\lambda|, |\mathcal{E}_{\lambda,\omega}| \leq c(\|\mathbf{K}, \mathbf{T}\|_{C^{0,1}(Y_f)} + \|\mathbf{P}, \mathbf{E}\|_{L^\infty(Y_f)}), \tag{3.5}$$

where c is a constant independent of $\lambda, \omega \in [0, 1]$.

Remark 3.1. The $C^{1,\alpha}$ norm and the $W^{2,p}$ norm of the solution of (1.1) in general are not bounded uniformly in ϵ even if $\|Q_\epsilon\|_{W^{1,p}(\Omega_f^\epsilon)}, \|F_\epsilon\|_{L^p(\Omega_f^\epsilon)}$ are bounded uniformly in ϵ. For example, suppose $B_1(0) \subset \mathbb{D}$ and η is a bell-shaped smooth function satisfying $\eta \in C_0^\infty(B_1(0))$, $\eta \in [0, 1]$, and $\eta(x) = 1$ in $B_{1/2}(0)$. By (2.3) for $i = 1$ and (2.10) for $\mathbb{X}_{1,\epsilon}^{(0,1)}$ and $\epsilon \in (0, 1)$, we know

$$\begin{cases} -\nabla \cdot \big(\mathbf{K}_\epsilon(\nabla(\eta\mathbb{X}_{1,\epsilon}^{(0,1)}) - \mathbb{X}_{1,\epsilon}^{(0,1)}\nabla\eta + \eta\vec{e}_1)\big) & \\ \quad = -\mathbf{K}_\epsilon(\nabla\mathbb{X}_{1,\epsilon}^{(0,1)} + \vec{e}_1)\nabla\eta & \text{in } \mathbb{D}_f^\epsilon, \\ \mathbf{K}_\epsilon\big(\nabla(\eta\mathbb{X}_{1,\epsilon}^{(0,1)}) - \mathbb{X}_{1,\epsilon}^{(0,1)}\nabla\eta + \eta\vec{e}_1\big) \cdot \vec{\mathbf{n}}_\epsilon = 0 & \text{on } \partial\mathbb{D}_m^\epsilon, \\ |\eta\mathbb{X}_{1,\epsilon}^{(0,1)}| = 0 & \text{on } \partial\mathbb{D}. \end{cases}$$

If $\mathbf{K} \in C^{0,1}_{per}(\mathbb{R}^n)$ and Y_m is a $C^{1,1}$ domain, by (3.4), we see that

$$\|\mathbb{X}^{(0,1)}_{1,\epsilon}\nabla\eta - \eta\vec{e}_1\|_{W^{1,\infty}(\mathbb{D}^\epsilon_f)} + \|\mathbf{K}_\epsilon(\nabla\mathbb{X}^{(0,1)}_{1,\epsilon} + \vec{e}_1)\nabla\eta\|_{L^\infty(\mathbb{D}^\epsilon_f)}$$

is bounded uniformly in ϵ. But $\|\eta\mathbb{X}^{(0,1)}_{1,\epsilon}\|_{C^{1,\alpha}(\mathbb{D}^\epsilon_f)}$ for $\alpha \in (0,1)$ and $\|\eta\mathbb{X}^{(0,1)}_{1,\epsilon}\|_{W^{2,p}(\mathbb{D}^\epsilon_f)}$ for $p \in (1,\infty)$ are not bounded uniformly in ϵ.

4. Uniform Hölder estimate

We shall prove a uniform Hölder estimate in ϵ for problem (1.1) (that is, Theorem 2.1). The Hölder estimate in the interior region of Ω is considered in subsection 4.1 and the estimate around the boundary $\partial\Omega$ is in subsection 4.2. A1–A3 are assumed in this section.

4.1. *Interior Hölder estimate*

For convenience, $\overline{B_1(0)} \subset \Omega$.

Lemma 4.1. *For any $\delta > 0$, there are $\theta_1, \theta_2 \in (0,1)$ (depending on $\delta, Y_f, \mathbf{K}, \mathbf{M}$) with $\theta_1 < \theta_2^2$ and there is a $\epsilon_0 \in (0,1)$ (depending on $\theta_2, \delta, Y_f, \mathbf{K}, \mathbf{M}$) so that if*

$$\begin{cases} -\nabla\cdot(\mathbf{K}_\nu\nabla\psi_\nu + \lambda_\nu\mathbf{T}_\nu\psi_\nu + \mathbb{Q}_\nu) \\ \qquad +\lambda_\nu\mathbf{P}_\nu\nabla\psi_\nu + \omega_\nu\mathbf{E}_\nu\psi_\nu = \mathbb{G}_\nu \quad \text{in } B_1(0)\cap\Omega^\nu_f, \\ (\mathbf{K}_\nu\nabla\psi_\nu + \lambda_\nu\mathbf{T}_\nu\psi_\nu + \mathbb{Q}_\nu)\cdot\vec{\mathbf{n}}_\nu = 0 \quad \text{on } B_1(0)\cap\partial\Omega^\nu_m, \end{cases} \tag{4.1}$$

and if

$$\begin{cases} \nu \in (0,\epsilon_0), \quad \theta \in [\theta_1,\theta_2], \quad \lambda_\nu, \omega_\nu \in [0,1], \\ \|\psi_\nu\|_{L^2(B_1(0)\cap\Omega^\nu_f)}, \epsilon_0^{-1}\|\mathbb{Q}_\nu, \mathbb{G}_\nu\|_{L^{n+\delta}(B_1(0)\cap\Omega^\nu_f)} \le 1, \end{cases} \tag{4.2}$$

then

$$⨍_{B_\theta(0)} |\Pi_\nu\psi_\nu - (\Pi_\nu\psi_\nu)_{0,\theta}|^2 dx \le \theta^{2\mu}, \tag{4.3}$$

where $\mu \equiv \frac{\delta}{n+\delta}$. See section 2 for the definitions of $\Pi_\nu, (\Pi_\nu\psi_\nu)_{0,\theta}$.

Proof. Consider

$$-\nabla\cdot(\mathcal{K}\nabla\psi + \mathcal{T}_\lambda\psi) + \lambda\mathcal{P}\nabla\psi + \mathcal{E}_{\lambda,\omega}\psi = 0 \qquad \text{in } B_{3/4}(0), \tag{4.4}$$

where $\mathcal{K}, \mathcal{T}_\lambda, \mathcal{P}, \mathcal{E}_{\lambda,\omega}$ are defined in (2.4) and $\lambda, \omega \in [0,1]$. Note that $\mathcal{K}$ is a symmetric positive definite matrix. By energy method, (3.5), and Theorem 9.11 [12],

$$\|\psi\|_{C^{1,0}(\overline{B_{1/2}(0)})} \le c\|\psi\|_{L^2(B_{3/4}(0))},$$

where c is a constant depending on $Y_f, \mathbf{K}, \mathbf{M}$. If $\mu' \in (\mu, 1)$, then, by Theorem 1.2 in page 70 [11],

$$⨍_{B_\theta(0)} |\psi - (\psi)_{0,\theta}|^2 dx \le \theta^{2\mu'} ⨍_{B_{3/4}(0)} \psi^2 dx \tag{4.5}$$

for θ sufficiently small (depending on $Y_f, \mathbf{K}, \mathbf{M}$). Fix $\theta_1, \theta_2 \in (0, 1/2)$ such that (1) $\theta_1 < \theta_2^2$ and (2) equation (4.5) holds for any $\theta \in [\theta_1, \theta_2]$.

Now we claim (4.3). If not, there is a sequence $\{\theta_\nu, \lambda_\nu, \omega_\nu, \psi_\nu, \mathbb{Q}_\nu, \mathbb{G}_\nu\}$ satisfying (4.1) and

$$\begin{cases} \theta_\nu \in [\theta_1, \theta_2], \quad \lambda_\nu, \omega_\nu \in [0,1], \\ \|\psi_\nu\|_{L^2(B_1(0)\cap\Omega_f^\nu)} \le 1, \\ \lim_{\nu\to 0} \|\mathbb{Q}_\nu, \mathbb{G}_\nu\|_{L^{n+\delta}(B_1(0)\cap\Omega_f^\nu)} = 0, \\ \fint_{B_{\theta_\nu}(0)} |\Pi_\nu\psi_\nu - (\Pi_\nu\psi_\nu)_{0,\theta_\nu}|^2 dx > \theta_\nu^{2\mu}. \end{cases} \tag{4.6}$$

By tracing the proof of Theorem 2.7 [3] and employing Lemma 2.1, we can extract a subsequence (same notation for subsequence) such that, as $\nu \to 0$,

$$\begin{cases} \theta_\nu \to \theta \in [\theta_1, \theta_2], \\ \lambda_\nu, \omega_\nu \to \lambda, \omega \in [0,1], \\ \Pi_\nu\psi_\nu \to \psi & \text{in } L^2(B_{3/4}(0)) \text{ strongly,} \\ (\mathbf{K}_\nu\nabla\psi_\nu + \lambda_\nu\mathbf{T}_\nu\psi_\nu)\mathcal{X}_{\Omega_f^\nu} \to \mathcal{K}\nabla\psi + \mathcal{T}_\lambda\psi & \text{in } L^2(B_{3/4}(0)) \text{ weakly,} \\ (\lambda_\nu\mathbf{P}_\nu\nabla\psi_\nu + \omega_\nu\mathbf{E}_\nu\psi_\nu)\mathcal{X}_{\Omega_f^\nu} \to \lambda\mathcal{P}\nabla\psi + \mathcal{E}_{\lambda,\omega}\psi & \text{in } L^2(B_{3/4}(0)) \text{ weakly,} \end{cases} \tag{4.7}$$

where $\mathcal{K}, \mathcal{T}_\lambda, \mathcal{P}, \mathcal{E}_{\lambda,\omega}$ are defined in (2.4). The ψ in (4.7) satisfies (4.4). By (4.5)–(4.7),

$$\begin{aligned} \theta^{2\mu} &= \lim_{\nu\to 0} \theta_\nu^{2\mu} \le \lim_{\nu\to 0} \fint_{B_{\theta_\nu}(0)} |\Pi_\nu\psi_\nu - (\Pi_\nu\psi_\nu)_{0,\theta_\nu}|^2 dx \\ &= \fint_{B_\theta(0)} |\psi - (\psi)_{0,\theta}|^2 dx \le \theta^{2\mu'} \fint_{B_{3/4}(0)} \psi^2 dx. \end{aligned} \tag{4.8}$$

If θ_2 is small enough, the right hand side of (4.8) is less than $\theta^{2\mu''}$ for $\mu'' \in (\mu, \mu')$. So we have $\theta^{2\mu} \le \theta^{2\mu''}$ for $\mu'' \in (\mu, \mu')$, which is impossible. So we prove (4.3). □

Lemma 4.2. *For any $\delta \in (0,3)$, there are $\theta_1, \theta_2 \in (0,1)$ (depending on $\delta, Y_f, \mathbf{K}, \mathbf{M}$) with $\theta_1 < \theta_2^2$ and there is a $\epsilon_0 \in (0,1)$ (depending on $\theta_2, \delta, Y_f, \mathbf{K}, \mathbf{M}$) so that if*

$$\begin{cases} -\nabla\cdot(\mathbf{K}_\epsilon\nabla\Psi_\epsilon + \beta_\epsilon\mathbf{T}_\epsilon\Psi_\epsilon + Q_\epsilon) \\ \qquad +\beta_\epsilon\mathbf{P}_\epsilon\nabla\Psi_\epsilon + \ell_\epsilon\mathbf{E}_\epsilon\Psi_\epsilon = G_\epsilon & \text{in } B_1(0)\cap\Omega_f^\epsilon, \\ (\mathbf{K}_\epsilon\nabla\Psi_\epsilon + \beta_\epsilon\mathbf{T}_\epsilon\Psi_\epsilon + Q_\epsilon)\cdot\vec{\mathbf{n}}_\epsilon = 0 & \text{on } B_1(0)\cap\partial\Omega_m^\epsilon, \end{cases} \tag{4.9}$$

and if $\epsilon \in (0, \epsilon_0)$, $\theta \in [\theta_1, \theta_2]$, and $\beta_\epsilon, \ell_\epsilon \in [0,1]$, then, for any k satisfying $\epsilon/\theta^k \le \epsilon_0$,

$$\fint_{B_{\theta^k}(0)} |\Pi_\epsilon\Psi_\epsilon - (\Pi_\epsilon\Psi_\epsilon)_{0,\theta^k}|^2 dx \le \theta^{2k\mu} J_\epsilon^2, \tag{4.10}$$

where $\mu \equiv \frac{\delta}{n+\delta}$ and $J_\epsilon \equiv \frac{4}{\epsilon_0}(\|\Psi_\epsilon\|_{L^2(B_1(0)\cap\Omega_f^\epsilon)} + \|Q_\epsilon, G_\epsilon\|_{L^{n+\delta}(B_1(0)\cap\Omega_f^\epsilon)})$.

Proof. Let c denote a constant independent of $\epsilon, \beta_\epsilon, \ell_\epsilon$. This is done by induction on k. For $k = 1$, we define $\psi_\epsilon \equiv \frac{\Psi_\epsilon}{J_\epsilon}$, $\mathbb{Q}_\epsilon \equiv \frac{Q_\epsilon}{J_\epsilon}$, $\mathbb{G}_\epsilon \equiv \frac{G_\epsilon}{J_\epsilon}$, $\lambda_\epsilon \equiv \beta_\epsilon$, $\omega_\epsilon \equiv \ell_\epsilon$. Then these functions satisfy (4.1) and (4.2) with $\nu = \epsilon$. By Lemma 4.1,

$$\fint_{B_\theta(0)} |\Pi_\epsilon \psi_\epsilon - (\Pi_\epsilon \psi_\epsilon)_{0,\theta}|^2 dx \le \theta^{2\mu}.$$

This implies (4.10) for $k = 1$. By A2–A3, energy method, Theorem 7.26 [12], and Lemma 2.1, any solution of (4.9) satisfies

$$\big\|\Pi_\epsilon \Psi_\epsilon\big\|_{L^{\frac{2n}{n-2}}(B_{4/5}(0))} + \|\Psi_\epsilon\|_{H^1(B_{4/5}(0)\cap\Omega_f^\epsilon)} \le c\|\Psi_\epsilon, Q_\epsilon, G_\epsilon\|_{L^2(B_1(0)\cap\Omega_f^\epsilon)}. \tag{4.11}$$

If (4.10) holds for some k satisfying $\epsilon/\theta^k \le \epsilon_0$, we define

$$\begin{cases} \psi_{\epsilon/\theta^k}(x) \equiv J_\epsilon^{-1}\theta^{-k\mu}\big(\Psi_\epsilon(\theta^k x) - (\Pi_\epsilon \Psi_\epsilon)_{0,\theta^k}\big) \\ \mathbb{Q}_{\epsilon/\theta^k}(x) \equiv J_\epsilon^{-1}\theta^{k(1-\mu)}(Q_\epsilon(\theta^k x) + \beta_\epsilon \mathbf{T}_\epsilon(\theta^k x)(\Pi_\epsilon\Psi_\epsilon)_{0,\theta^k}) \\ \mathbb{G}_{\epsilon/\theta^k}(x) \equiv J_\epsilon^{-1}\theta^{k(2-\mu)}\big(G_\epsilon(\theta^k x) - \ell_\epsilon \mathbf{E}_\epsilon(\theta^k x)(\Pi_\epsilon\Psi_\epsilon)_{0,\theta^k}\big) \qquad \text{in } B_1(0)\cap\Omega_f^\epsilon/\theta^k. \\ \lambda_{\epsilon/\theta^k} \equiv \theta^k \beta_\epsilon \\ \omega_{\epsilon/\theta^k} \equiv \theta^{2k}\ell_\epsilon \end{cases}$$

See section 2 for the definition of $\Omega_f^\epsilon/\theta^k$. Then these functions satisfy

$$\begin{cases} -\nabla\cdot(\mathbf{K}_{\epsilon/\theta^k}\nabla\psi_{\epsilon/\theta^k} + \lambda_{\epsilon/\theta^k}\mathbf{T}_{\epsilon/\theta^k}\psi_{\epsilon/\theta^k} + \mathbb{Q}_{\epsilon/\theta^k}) \\ \qquad +\lambda_{\epsilon/\theta^k}\mathbf{P}_{\epsilon/\theta^k}\nabla\psi_{\epsilon/\theta^k} + \omega_{\epsilon/\theta^k}\mathbf{E}_{\epsilon/\theta^k}\psi_{\epsilon/\theta^k} = \mathbb{G}_{\epsilon/\theta^k} \quad \text{in } B_1(0)\cap\Omega_f^\epsilon/\theta^k, \\ (\mathbf{K}_{\epsilon/\theta^k}\nabla\psi_{\epsilon/\theta^k} + \lambda_{\epsilon/\theta^k}\mathbf{T}_{\epsilon/\theta^k}\psi_{\epsilon/\theta^k} + \mathbb{Q}_{\epsilon/\theta^k})\cdot\vec{\mathbf{n}}_{\epsilon/\theta^k} = 0 \quad \text{on } B_1(0)\cap\partial\Omega_m^\epsilon/\theta^k, \end{cases}$$

where $\vec{\mathbf{n}}_{\epsilon/\theta^k}$ is a unit vector normal to $\partial\Omega_m^\epsilon/\theta^k$. By A3,

$$\begin{aligned} \|\mathbb{Q}_{\epsilon/\theta^k}\|_{L^{n+\delta}(B_1(0)\cap\Omega_f^\epsilon/\theta^k)} &\le J_\epsilon^{-1}\|Q_\epsilon\|_{L^{n+\delta}(B_{\theta^k}(0)\cap\Omega_f^\epsilon)} + J_\epsilon^{-1}\theta^{k(1-\mu)}\mathbf{M}|(\Pi_\epsilon\Psi_\epsilon)_{0,\theta^k}| \\ &\le J_\epsilon^{-1}\|Q_\epsilon\|_{L^{n+\delta}(B_{\theta^k}(0)\cap\Omega_f^\epsilon)} + cJ_\epsilon^{-1}\theta^{k(2-\mu-n/2)}\mathbf{M}\|\Pi_\epsilon\Psi_\epsilon\|_{L^{\frac{2n}{n-2}}(B_{\theta^k}(0))}. \end{aligned}$$

By induction, (4.11), $\delta \in (0,3)$, and small θ_2, we see

$$\begin{cases} \lambda_{\epsilon/\theta^k}, \omega_{\epsilon/\theta^k} \in [0,1], \\ \|\psi_{\epsilon/\theta^k}\|_{L^2(B_1(0)\cap\Omega_f^\epsilon/\theta^k)}, \epsilon_0^{-1}\|\mathbb{Q}_{\epsilon/\theta^k}, \mathbb{G}_{\epsilon/\theta^k}\|_{L^{n+\delta}(B_1(0)\cap\Omega_f^\epsilon/\theta^k)} \le 1. \end{cases}$$

By Lemma 4.1 (take $\nu = \epsilon/\theta^k$), we obtain

$$\fint_{B_\theta(0)} \big|\Pi_{\epsilon/\theta^k}\psi_{\epsilon/\theta^k} - (\Pi_{\epsilon/\theta^k}\psi_{\epsilon/\theta^k})_{0,\theta}\big|^2 dx \le \theta^{2\mu}. \tag{4.12}$$

Note, by Lemma 2.1,

$$\fint_{B_\theta(0)} \big|\Pi_{\epsilon/\theta^k}\psi_{\epsilon/\theta^k} - (\Pi_{\epsilon/\theta^k}\psi_{\epsilon/\theta^k})_{0,\theta}\big|^2 dx = \fint_{B_{\theta^{k+1}}(0)} \frac{\big|\Pi_\epsilon\Psi_\epsilon - (\Pi_\epsilon\Psi_\epsilon)_{0,\theta^{k+1}}\big|^2}{J_\epsilon^2\ \theta^{2k\mu}} dx. \tag{4.13}$$

Equations (4.12)–(4.13) imply (4.10) for $k+1$ case. □

Lemma 4.3. *For any $\delta \in (0,3)$, there is a $\mu_* \in (0, \frac{\delta}{n+\delta})$ (depending on $\delta, Y_f, \mathbf{K}, \mathbf{M}$) such that if $\epsilon \in (0,1)$ and $\beta_\epsilon, \ell_\epsilon \in [0,1]$, then any solution of (4.9) satisfies*

$$[\Psi_\epsilon]_{C^{0,\mu_*}(\overline{B_{1/2}(0)\cap\Omega_f^\epsilon})} \le c(\|\Psi_\epsilon\|_{L^2(B_1(0)\cap\Omega_f^\epsilon)} + \|Q_\epsilon, G_\epsilon\|_{L^{n+\delta}(B_1(0)\cap\Omega_f^\epsilon)}), \tag{4.14}$$

where c is a constant independent of $\epsilon, \beta_\epsilon, \ell_\epsilon$.

Proof. Let $\theta_1, \theta_2, \epsilon_0, \mu, J_\epsilon$ be same as those in Lemma 4.2, define $\epsilon_* \equiv \epsilon_0\theta_2/2$, and let $\epsilon \le \epsilon_*$. Denote by c a constant independent of $\epsilon, \beta_\epsilon, \ell_\epsilon$. Because of $\theta_1 < \theta_2^2$, for any $r \in [\epsilon/\epsilon_0, \theta_2]$, there are $\theta \in [\theta_1, \theta_2]$ and $k \in \mathbb{N}$ satisfying $r = \theta^k$. Lemma 4.2 implies, for any $r \in [\epsilon/\epsilon_0, \theta_2]$,

$$\fint_{B_r(0)} |\Pi_\epsilon \Psi_\epsilon - (\Pi_\epsilon \Psi_\epsilon)_{0,r}|^2 \, dx \le r^{2\mu} J_\epsilon^2. \tag{4.15}$$

Since $2\epsilon/\epsilon_0 \in [\epsilon/\epsilon_0, \theta_2]$, we define, for any $\tilde{\mu} \in (0, \mu)$,

$$\begin{cases} \psi_1(x) \equiv J_\epsilon^{-1} \epsilon^{-\tilde{\mu}} \left(\Psi_\epsilon(\epsilon x) - (\Pi_\epsilon \Psi_\epsilon)_{0,2\epsilon/\epsilon_0} \right) \\ \mathbb{Q}_1(x) \equiv J_\epsilon^{-1} \epsilon^{1-\tilde{\mu}} (Q_\epsilon(\epsilon x) + \beta_\epsilon \mathbf{T}_\epsilon(\epsilon x)(\Pi_\epsilon \Psi_\epsilon)_{0,2\epsilon/\epsilon_0}) \\ \mathbb{G}_1(x) \equiv J_\epsilon^{-1} \epsilon^{2-\tilde{\mu}} \big(G_\epsilon(\epsilon x) - \ell_\epsilon \mathbf{E}_\epsilon(\epsilon x)(\Pi_\epsilon \Psi_\epsilon)_{0,2\epsilon/\epsilon_0} \big) \quad \text{in } B_{\frac{2}{\epsilon_0}}(0) \cap \Omega_f^\epsilon/\epsilon. \\ \lambda_1 \equiv \epsilon \beta_\epsilon \\ \omega_1 \equiv \epsilon^2 \ell_\epsilon \end{cases} \tag{4.16}$$

Then those functions satisfy

$$\begin{cases} -\nabla \cdot (\mathbf{K} \nabla \psi_1 + \lambda_1 \mathbf{T} \psi_1 + \mathbb{Q}_1) + \lambda_1 \mathbf{P} \nabla \psi_1 + \omega_1 \mathbf{E} \psi_1 = \mathbb{G}_1 & \text{in } B_{\frac{2}{\epsilon_0}}(0) \cap \Omega_f^\epsilon/\epsilon, \\ (\mathbf{K} \nabla \psi_1 + \lambda_1 \mathbf{T} \psi_1 + \mathbb{Q}_1) \cdot \vec{\mathbf{n}}_{\epsilon/\epsilon} = 0 & \text{on } B_{\frac{2}{\epsilon_0}}(0) \cap \partial\Omega_m^\epsilon/\epsilon, \end{cases}$$

where $\vec{\mathbf{n}}_{\epsilon/\epsilon}$ is a unit vector normal to $\partial\Omega_m^\epsilon/\epsilon$. Take $r = \frac{2\epsilon}{\epsilon_0}$ in (4.15) to get

$$\begin{cases} \lambda_1, \omega_1 \in [0, 1], \\ \|\psi_1\|_{L^2(B_{2/\epsilon_0}(0) \cap \Omega_f^\epsilon/\epsilon)} + \|\mathbb{Q}_1, \mathbb{G}_1\|_{L^{n+\delta}(B_{2/\epsilon_0}(0) \cap \Omega_f^\epsilon/\epsilon)} \le \tilde{c}, \end{cases}$$

where $\tilde{c}$ is independent of $\tilde{\mu}, \epsilon, \beta_\epsilon, \ell_\epsilon$. By tracing the proof of Theorem 8.24 [12], there is a $\mu_* < \mu$ (Note μ_* is independent of $\tilde{\mu}$ in (4.16)) such that

$$[\psi_1]_{C^{0,\mu_*}(\overline{B_{1/\epsilon_0}(0) \cap \Omega_f^\epsilon/\epsilon})} \le c. \tag{4.17}$$

If the $\tilde{\mu}$ in (4.16) is taken to be the μ_* in (4.17), by Lemma 2.1, we see that (4.15) with μ replaced by μ_* also holds for $r \le \epsilon/\epsilon_0$. Which implies that (4.15) with μ replaced by μ_* holds $r \le \theta_2$. Then we shift the origin to any point $z \in B_{1/2}(0)$, repeat above argument, and see that (4.15) with 0 (resp. μ) replaced by $z \in B_{1/2}(0)$ (resp. μ_*) also holds for $r \le \theta_2$. By Theorem 1.2 in page 70 [11], we conclude

For any $\delta \in (0, 3)$*, there are* $\mu_* \in (0, \frac{\delta}{n+\delta})$ *and* $\epsilon_* \in (0, 1)$ *(depending on* $\delta, Y_f, \mathbf{K}, \mathbf{M}$*) such that if* $\epsilon \in (0, \epsilon_*)$ *and* $\beta_\epsilon, \ell_\epsilon \in [0, 1]$*, then any solution of (4.9) satisfies (4.14).*

By tracing the proof of Theorem 8.24 [12], we also see

Let $\delta, \epsilon_*, \mu_*$ *be same as above. If* $\epsilon \in [\epsilon_*, 1)$ *and* $\beta_\epsilon, \ell_\epsilon \in [0, 1]$*, then any solution of (4.9) satisfies (4.14).*

Combining the above two results, we prove Lemma 4.3. □

4.2. *Boundary Hölder estimate*

Assume $0 \in \partial\Omega$. By A1, there is a function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ satisfying

$$\begin{cases} \|\phi\|_{C^{1,1}(\mathbb{R}^{n-1})} \le c, \quad \phi(0) = |\nabla\phi(0)| = 0, \\ B_1(0) \cap \Omega/t = B_1(0) \cap \{(x', x_n) \in \mathbb{R}^n |\ tx_n > \phi(tx')\} \quad \text{if } t \in (0, 1]. \end{cases}$$

Define $B_1(0) \cap \Omega/t \equiv B_1(0) \cap \{(x', x_n) \in \mathbb{R}^n |\ x_n > 0\}$ if $t = 0$.

Lemma 4.4. *For any $\delta > 0$, there are $\tilde{\theta}_1, \tilde{\theta}_2 \in (0, 1)$ (depending on $\delta, Y_f, \mathbf{K}, \mathbf{M}, \Omega$) satisfying $\tilde{\theta}_1 < \tilde{\theta}_2^2$ and there is a $\tilde{\epsilon}_0 \in (0, 1)$ (depending on $\tilde{\theta}_2, \delta, Y_f, \mathbf{K}, \mathbf{M}, \Omega$) satisfying $\tilde{\epsilon}_0 < \epsilon_0$ (ϵ_0 is that in Lemma 4.1) such that if*

$$\begin{cases} -\nabla \cdot (\mathbf{K}_{\epsilon/\nu}\nabla\psi_{\epsilon,\nu} + \lambda_{\epsilon,\nu}\mathbf{T}_{\epsilon/\nu}\psi_{\epsilon,\nu} + \mathbb{Q}_{\epsilon,\nu}) & \\ \qquad +\lambda_{\epsilon,\nu}\mathbf{P}_{\epsilon/\nu}\nabla\psi_{\epsilon,\nu} + \omega_{\epsilon,\nu}\mathbf{E}_{\epsilon/\nu}\psi_{\epsilon,\nu} = \mathbb{G}_{\epsilon,\nu} & \text{in } B_1(0) \cap \Omega_f^\epsilon/\nu, \\ (\mathbf{K}_{\epsilon/\nu}\nabla\psi_{\epsilon,\nu} + \lambda_{\epsilon,\nu}\mathbf{T}_{\epsilon/\nu}\psi_{\epsilon,\nu} + \mathbb{Q}_{\epsilon,\nu}) \cdot \vec{\mathbf{n}}_{\epsilon/\nu} = 0 & \text{on } B_1(0) \cap \partial\Omega_m^\epsilon/\nu, \\ \psi_{\epsilon,\nu} = 0 & \text{on } B_1(0) \cap \partial\Omega/\nu, \end{cases} \tag{4.18}$$

and if

$$\begin{cases} \epsilon, \epsilon/\nu \in (0, \tilde{\epsilon}_0), \quad \nu \in (0, 1], \quad \tilde{\theta} \in [\tilde{\theta}_1, \tilde{\theta}_2], \quad \lambda_{\epsilon,\nu}, \omega_{\epsilon,\nu} \in [0, 1], \\ \|\psi_{\epsilon,\nu}\|_{L^2(B_1(0)\cap\Omega_f^\epsilon/\nu)}, \tilde{\epsilon}_0^{-1}\|\mathbb{Q}_{\epsilon,\nu}, \mathbb{G}_{\epsilon,\nu}\|_{L^{n+\delta}(B_1(0)\cap\Omega_f^\epsilon/\nu)} \le 1, \end{cases}$$

then

$$⨍_{B_{\tilde{\theta}}(0)\cap\Omega/\nu} |\Pi_{\epsilon/\nu}\psi_{\epsilon,\nu}|^2 dx \le \tilde{\theta}^{2\mu}, \tag{4.19}$$

where $\mu \equiv \frac{\delta}{n+\delta}$.

Proof. Consider the following problem

$$\begin{cases} -\nabla \cdot (\mathcal{K}\nabla\psi + \mathcal{T}_\lambda\psi) + \lambda\mathcal{P}\nabla\psi + \mathcal{E}_{\lambda,\omega}\psi = 0 & \text{in } B_{3/4}(0) \cap \Omega/t, \\ \psi = 0 & \text{on } B_{3/4}(0) \cap \partial\Omega/t, \end{cases} \tag{4.20}$$

where $\mathcal{K}, \mathcal{T}_\lambda, \mathcal{P}, \mathcal{E}_{\lambda,\omega}$ are defined in (2.4), $t \in [0, 1]$, and $\lambda, \omega \in [0, 1]$. By (3.5) and Theorem 9.13 [12],

$$\|\psi\|_{C^{1,0}(\overline{B_{1/2}(0)\cap\Omega/t})} \le c\|\psi\|_{L^2(B_{3/4}(0)\cap\Omega/t)}, \tag{4.21}$$

where c is a constant depending on $Y_f, \mathbf{K}, \mathbf{M}, \Omega$ but independent of t. If $\mu' \in (\mu, 1)$, then, by (4.21),

$$⨍_{B_{\tilde{\theta}}(0)\cap\Omega/t} \psi^2 dx \le \tilde{\theta}^{2\mu'} ⨍_{B_{3/4}(0)\cap\Omega/t} \psi^2 dx, \tag{4.22}$$

for sufficiently small $\tilde{\theta}$ (depending on $Y_f, \mathbf{K}, \mathbf{M}, \Omega$ but independent of t). Fix $\tilde{\theta}_1, \tilde{\theta}_2 \in (0, 1/2)$ such that (1) $\tilde{\theta}_1 < \tilde{\theta}_2^2$ and (2) equation (4.22) holds for any $\tilde{\theta} \in [\tilde{\theta}_1, \tilde{\theta}_2]$.

We claim (4.19). If not, there is a sequence $\{\nu_\epsilon, \tilde\theta_{\epsilon,\nu_\epsilon}, \lambda_{\epsilon,\nu_\epsilon}, \omega_{\epsilon,\nu_\epsilon}, \psi_{\epsilon,\nu_\epsilon}, \mathbb{Q}_{\epsilon,\nu_\epsilon}, \mathbb{G}_{\epsilon,\nu_\epsilon}\}$ satisfying (4.18) and

$$\begin{cases} \epsilon, \epsilon/\nu_\epsilon \to 0, \quad \nu_\epsilon \in (0,1], \quad \tilde\theta_{\epsilon,\nu_\epsilon} \in [\tilde\theta_1, \tilde\theta_2], \quad \lambda_{\epsilon,\nu_\epsilon}, \omega_{\epsilon,\nu_\epsilon} \in [0,1], \\ \|\psi_{\epsilon,\nu_\epsilon}\|_{L^2(B_1(0)\cap\Omega_f^\epsilon/\nu_\epsilon)} \le 1, \\ \lim\limits_{\epsilon/\nu_\epsilon\to 0} \|\mathbb{Q}_{\epsilon,\nu_\epsilon}, \mathbb{G}_{\epsilon,\nu_\epsilon}\|_{L^{n+\delta}(B_1(0)\cap\Omega_f^\epsilon/\nu_\epsilon)} = 0, \\ \fint_{B_{\tilde\theta_{\epsilon,\nu_\epsilon}}(0)\cap\Omega/\nu_\epsilon} \left|\Pi_{\epsilon/\nu_\epsilon}\psi_{\epsilon,\nu_\epsilon}\right|^2 dx > |\tilde\theta_{\epsilon,\nu_\epsilon}|^{2\mu}. \end{cases} \tag{4.23}$$

By compactness principle and by tracing the proof of Theorem 2.7 [3], we can extract a subsequence (same notation for subsequence) such that, as $\epsilon/\nu_\epsilon \to 0$,

$$\begin{cases} \tilde\theta_{\epsilon,\nu_\epsilon} \to \tilde\theta \in [\tilde\theta_1, \tilde\theta_2], \\ \nu_\epsilon, \lambda_{\epsilon,\nu_\epsilon}, \omega_{\epsilon,\nu_\epsilon} \to \nu, \lambda, \omega \in [0,1], \\ \Pi_{\epsilon/\nu_\epsilon}\psi_{\epsilon,\nu_\epsilon} \to \psi & \text{in } L^2(B_{3/4}(0)\cap\Omega/\nu) \text{ strongly}, \\ (\mathbf{K}_{\epsilon/\nu_\epsilon}\nabla\psi_{\epsilon,\nu_\epsilon} + \lambda_{\epsilon,\nu_\epsilon}\mathbf{T}_{\epsilon/\nu_\epsilon}\psi_{\epsilon,\nu_\epsilon})\mathcal{X}_{\Omega_f^\epsilon/\nu_\epsilon} \\ \qquad \to \mathcal{K}\nabla\psi + \mathcal{T}_\lambda\psi & \text{in } L^2(B_{3/4}(0)\cap\Omega/\nu) \text{ weakly}, \\ (\lambda_{\epsilon,\nu_\epsilon}\mathbf{P}_{\epsilon/\nu_\epsilon}\nabla\psi_{\epsilon,\nu_\epsilon} + \omega_{\epsilon,\nu_\epsilon}\mathbf{E}_{\epsilon/\nu_\epsilon}\psi_{\epsilon,\nu_\epsilon})\mathcal{X}_{\Omega_f^\epsilon/\nu_\epsilon} \\ \qquad \to \lambda\mathcal{P}\nabla\psi + \mathcal{E}_{\lambda,\omega}\psi & \text{in } L^2(B_{3/4}(0)\cap\Omega/\nu) \text{ weakly}, \end{cases} \tag{4.24}$$

where $\mathcal{K}, \mathcal{T}_\lambda, \mathcal{P}, \mathcal{E}_{\lambda,\omega}$ are defined in (2.4). The ψ in (4.24) is a solution of (4.20) with $t = \nu$. By (4.22)–(4.24),

$$\begin{aligned} \tilde\theta^{2\mu} &= \lim_{\epsilon/\nu_\epsilon\to 0} |\tilde\theta_{\epsilon,\nu_\epsilon}|^{2\mu} \le \lim_{\epsilon/\nu_\epsilon\to 0} \fint_{B_{\tilde\theta_{\epsilon,\nu_\epsilon}}(0)\cap\Omega/\nu_\epsilon} \left|\Pi_{\epsilon/\nu_\epsilon}\psi_{\epsilon,\nu_\epsilon}\right|^2 dx \\ &= \fint_{B_{\tilde\theta}(0)\cap\Omega/\nu} \psi^2 dx \le \tilde\theta^{2\mu'} \fint_{B_{3/4}(0)\cap\Omega/\nu} \psi^2 dx. \end{aligned} \tag{4.25}$$

If $\tilde\theta_2$ is small enough, the right hand side of (4.25) is less than $\tilde\theta^{2\mu''}$ for $\mu'' \in (\mu, \mu')$. So we have $\tilde\theta^{2\mu} \le \tilde\theta^{2\mu''}$ for $\mu'' \in (\mu, \mu')$, which is impossible. Therefore, there is a $\tilde\epsilon_0$ such that (4.19) holds for $\epsilon/\nu < \tilde\epsilon_0$. Clearly, $\tilde\epsilon_0$ can be chosen so that $\tilde\epsilon_0 < \epsilon_0$. □

Lemma 4.5. *For any $\delta > 0$, there are $\tilde\theta_1, \tilde\theta_2 \in (0,1)$ (depending on $\delta, Y_f, \mathbf{K}, \mathbf{M}, \Omega$) satisfying $\tilde\theta_1 < \tilde\theta_2^2$ and there is a $\tilde\epsilon_0 \in (0,1)$ (depending on $\tilde\theta_2, \delta, Y_f, \mathbf{K}, \mathbf{M}, \Omega$) satisfying $\tilde\epsilon_0 < \epsilon_0$ (ϵ_0 is that in Lemma 4.2) such that if*

$$\begin{cases} -\nabla\cdot(\mathbf{K}_\epsilon\nabla\Psi_\epsilon + \mathbf{T}_\epsilon\Psi_\epsilon + Q_\epsilon) + \mathbf{P}_\epsilon\nabla\Psi_\epsilon + \mathbf{E}_\epsilon\Psi_\epsilon = G_\epsilon & \text{in } B_1(0)\cap\Omega_f^\epsilon, \\ (\mathbf{K}_\epsilon\nabla\Psi_\epsilon + \mathbf{T}_\epsilon\Psi_\epsilon + Q_\epsilon)\cdot\vec{\mathbf{n}}_\epsilon = 0 & \text{on } B_1(0)\cap\partial\Omega_m^\epsilon, \\ \Psi_\epsilon = 0 & \text{on } B_1(0)\cap\partial\Omega, \end{cases} \tag{4.26}$$

and if $\epsilon \in (0, \tilde\epsilon_0)$ and $\tilde\theta \in [\tilde\theta_1, \tilde\theta_2]$, then, for any k satisfying $\epsilon/\tilde\theta^k \le \tilde\epsilon_0$,

$$\fint_{B_{\tilde\theta^k}(0)\cap\Omega} \left|\Pi_\epsilon\Psi_\epsilon\right|^2 dx \le \tilde\theta^{2k\mu}\tilde J_\epsilon^2, \tag{4.27}$$

where $\mu \equiv \frac{\delta}{n+\delta}$ *and* $\tilde{J}_\epsilon \equiv \frac{1}{\tilde{\epsilon}_0}(\|\Psi_\epsilon\|_{L^2(B_1(0)\cap\Omega_f^\epsilon)} + \|Q_\epsilon, G_\epsilon\|_{L^{n+\delta}(B_1(0)\cap\Omega_f^\epsilon)})$.

Proof. The proof is similar to that of Lemma 4.2 and is done by induction on k. Define $\psi_{\epsilon,1} \equiv \frac{\Psi_\epsilon}{\tilde{J}_\epsilon}$, $\mathbb{Q}_{\epsilon,1} \equiv \frac{Q_\epsilon}{\tilde{J}_\epsilon}$, $\mathbb{G}_{\epsilon,1} \equiv \frac{G_\epsilon}{\tilde{J}_\epsilon}$, $\lambda_{\epsilon,1} = \omega_{\epsilon,1} = 1$. Then these functions satisfy (4.18) with $\nu = 1$. (4.27) for $k = 1$ is deduced from Lemma 4.4. Suppose (4.27) holds for some k satisfying $\epsilon/\tilde{\theta}^k \le \tilde{\epsilon}_0$, then we define

$$\begin{cases} \psi_{\epsilon,\tilde{\theta}^k}(x) \equiv \tilde{J}_\epsilon^{-1}\tilde{\theta}^{-k\mu}\Psi_\epsilon(\tilde{\theta}^k x) \\ \mathbb{Q}_{\epsilon,\tilde{\theta}^k}(x) \equiv \tilde{J}_\epsilon^{-1}\tilde{\theta}^{k(1-\mu)}Q_\epsilon(\tilde{\theta}^k x) \\ \mathbb{G}_{\epsilon,\tilde{\theta}^k}(x) \equiv \tilde{J}_\epsilon^{-1}\tilde{\theta}^{k(2-\mu)}G_\epsilon(\tilde{\theta}^k x) & \text{in } B_1(0)\cap\Omega_f^\epsilon/\tilde{\theta}^k. \\ \lambda_{\epsilon,\tilde{\theta}^k} \equiv \tilde{\theta}^k \\ \omega_{\epsilon,\tilde{\theta}^k} \equiv \tilde{\theta}^{2k} \end{cases}$$

Then these functions satisfy

$$\begin{cases} -\nabla\cdot(\mathbf{K}_{\epsilon/\tilde{\theta}^k}\nabla\psi_{\epsilon,\tilde{\theta}^k} + \lambda_{\epsilon,\tilde{\theta}^k}\mathbf{T}_{\epsilon/\tilde{\theta}^k}\psi_{\epsilon,\tilde{\theta}^k} + \mathbb{Q}_{\epsilon,\tilde{\theta}^k}) \\ \qquad +\lambda_{\epsilon,\tilde{\theta}^k}\mathbf{P}_{\epsilon/\tilde{\theta}^k}\nabla\psi_{\epsilon,\tilde{\theta}^k} + \omega_{\epsilon,\tilde{\theta}^k}\mathbf{E}_{\epsilon/\tilde{\theta}^k}\psi_{\epsilon,\tilde{\theta}^k} = \mathbb{G}_{\epsilon,\tilde{\theta}^k} & \text{in } B_1(0)\cap\Omega_f^\epsilon/\tilde{\theta}^k, \\ (\mathbf{K}_{\epsilon/\tilde{\theta}^k}\nabla\psi_{\epsilon,\tilde{\theta}^k} + \lambda_{\epsilon,\tilde{\theta}^k}\mathbf{T}_{\epsilon/\tilde{\theta}^k}\psi_{\epsilon,\tilde{\theta}^k} + \mathbb{Q}_{\epsilon,\tilde{\theta}^k})\cdot\vec{\mathbf{n}}_{\epsilon/\tilde{\theta}^k} = 0 & \text{on } B_1(0)\cap\partial\Omega_m^\epsilon/\tilde{\theta}^k, \\ \psi_{\epsilon,\tilde{\theta}^k} = 0 & \text{on } B_1(0)\cap\partial\Omega/\tilde{\theta}^k, \end{cases}$$

where $\vec{\mathbf{n}}_{\epsilon/\tilde{\theta}^k}$ is a unit vector normal to $\partial\Omega_m^\epsilon/\tilde{\theta}^k$. By induction, we see

$$\begin{cases} \lambda_{\epsilon,\tilde{\theta}^k}, \omega_{\epsilon,\tilde{\theta}^k} \in [0,1], \\ \|\psi_{\epsilon,\tilde{\theta}^k}\|_{L^2(B_1(0)\cap\Omega_f^\epsilon/\tilde{\theta}^k)}, \tilde{\epsilon}_0^{-1}\|\mathbb{Q}_{\epsilon,\tilde{\theta}^k}, \mathbb{G}_{\epsilon,\tilde{\theta}^k}\|_{L^{n+\delta}(B_1(0)\cap\Omega_f^\epsilon/\tilde{\theta}^k)} \le 1. \end{cases}$$

Then we follow the argument of Lemma 4.2 and employ Lemma 4.4 with $\nu = \tilde{\theta}^k$ to obtain (4.27) with $k+1$ in place of k. □

Lemma 4.6. *For any* $\delta \in (0,3)$*, there is a* $\tilde{\mu}_* \in (0, \frac{\delta}{n+\delta})$ *(depending on* $\delta, Y_f, \mathbf{K}, \mathbf{M}, \Omega$*) such that, if* $\epsilon \in (0,1)$*, any solution of (4.26) satisfies*

$$[\Psi_\epsilon]_{C^{0,\tilde{\mu}_*}(\overline{B_{1/2}(0)\cap\Omega_f^\epsilon})} \le c(\|\Psi_\epsilon\|_{L^2(B_1(0)\cap\Omega_f^\epsilon)} + \|Q_\epsilon, G_\epsilon\|_{L^{n+\delta}(B_1(0)\cap\Omega_f^\epsilon)}), \tag{4.28}$$

where c *is a constant independent of* ϵ.

Proof. Let $\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\epsilon}_0, \mu, \tilde{J}_\epsilon$ be those in Lemma 4.5, define $\tilde{\epsilon}_* \equiv \min\{\tilde{\epsilon}_0\tilde{\theta}_2/3, \epsilon_*\}$ where ϵ_* is the one in the proof of Lemma 4.3, and let $\epsilon \le \tilde{\epsilon}_*$. Denote by c a constant independent of ϵ. By energy method, A2–A3, Theorem 7.26 [12], and Lemma 2.1, any solution of (4.26) satisfies

$$\left\|\Pi_\epsilon\Psi_\epsilon\right\|_{L^{\frac{2n}{n-2}}(B_{3/4}(0)\cap\Omega)} + \|\Psi_\epsilon\|_{H^1(B_{3/4}(0)\cap\Omega_f^\epsilon)} \le c\|\Psi_\epsilon, Q_\epsilon, G_\epsilon\|_{L^2(B_1(0)\cap\Omega_f^\epsilon)}. \tag{4.29}$$

For any $x \in B_{\tilde{\theta}_2/3}(0)\cap\Omega_f^\epsilon$, define $\rho(x) \equiv |x - x_0|$ where $x_0 \in \partial\Omega$ satisfying $|x - x_0| = \min_{y\in\partial\Omega}|x - y|$. Then we have either case (1) $\rho(x) > \frac{2\epsilon}{3\tilde{\epsilon}_0}$ or case (2) $\rho(x) \le \frac{2\epsilon}{3\tilde{\epsilon}_0}$.

We now consider case (1). Because of $\tilde{\theta}_1 < \tilde{\theta}_2^2$, for any $r \in [\epsilon/\tilde{\epsilon}_0, \tilde{\theta}_2]$, there are $\tilde{\theta} \in [\tilde{\theta}_1, \tilde{\theta}_2]$ and $k \in \mathbb{N}$ satisfying $r = \tilde{\theta}^k$. Since $\rho(x) \in [\frac{2\epsilon}{3\tilde{\epsilon}_0}, \frac{\tilde{\theta}_2}{3}]$, by Lemma 4.5,

$$\fint_{B_r(x_0)\cap\Omega} |\Pi_\epsilon \Psi_\epsilon|^2 \, dy \le r^{2\mu} \tilde{J}_\epsilon^2 \quad \text{for } r \in [\tfrac{3}{2}\rho(x), \tilde{\theta}_2].$$

So

$$\fint_{B_s(x)\cap\Omega} \left|\Pi_\epsilon \Psi_\epsilon - (\Pi_\epsilon \Psi_\epsilon)_{x,s}\right|^2 dy \le c s^{2\mu} \tilde{J}_\epsilon^2 \quad \text{for } s \in [\tfrac{\rho(x)}{2}, \tfrac{\tilde{\theta}_2}{3}]. \tag{4.30}$$

Shift the coordinate system so that the x is located at the origin. For any $\tilde{\mu} < \mu$, we define, in $B_1(x) \cap \Omega_f^\epsilon/\rho(x)$,

$$\begin{cases} \psi_{\epsilon,\rho(x)}(y) \equiv \tilde{J}_\epsilon^{-1} \rho^{-\tilde{\mu}}(x) \big(\Psi_\epsilon(\rho(x)y) - (\Pi_\epsilon \Psi_\epsilon)_{x,\rho(x)}\big), \\ \mathbb{Q}_{\epsilon,\rho(x)}(y) \equiv \tilde{J}_\epsilon^{-1} \rho^{1-\tilde{\mu}}(x) \big(Q_\epsilon(\rho(x)y) + \mathbf{T}_\epsilon(\rho(x)y)(\Pi_\epsilon \Psi_\epsilon)_{x,\rho(x)}\big), \\ \mathbb{G}_{\epsilon,\rho(x)}(y) \equiv \tilde{J}_\epsilon^{-1} \rho^{2-\tilde{\mu}}(x) \big(G_\epsilon(\rho(x)y) - \mathbf{E}_\epsilon(\rho(x)y)(\Pi_\epsilon \Psi_\epsilon)_{x,\rho(x)}\big), \\ \lambda_{\epsilon,\rho(x)} \equiv \rho(x), \\ \omega_{\epsilon,\rho(x)} \equiv \rho^2(x). \end{cases} \tag{4.31}$$

Then these functions satisfy

$$\begin{cases} -\nabla \cdot (\mathbf{K}_{\epsilon/\rho(x)} \nabla \psi_{\epsilon,\rho(x)} + \lambda_{\epsilon,\rho(x)} \mathbf{T}_{\epsilon/\rho(x)} \psi_{\epsilon,\rho(x)} + \mathbb{Q}_{\epsilon,\rho(x)}) \\ \qquad + \lambda_{\epsilon,\rho(x)} \mathbf{P}_{\epsilon/\rho(x)} \nabla \psi_{\epsilon,\rho(x)} + \omega_{\epsilon,\rho(x)} \mathbf{E}_{\epsilon/\rho(x)} \psi_{\epsilon,\rho(x)} \\ \qquad = \mathbb{G}_{\epsilon,\rho(x)} \qquad\qquad \text{in } B_1(x) \cap \Omega_f^\epsilon/\rho(x), \\ (\mathbf{K}_{\epsilon/\rho(x)} \nabla \psi_{\epsilon,\rho(x)} + \lambda_{\epsilon,\rho(x)} \mathbf{T}_{\epsilon/\rho(x)} \psi_{\epsilon,\rho(x)} + \mathbb{Q}_{\epsilon,\rho(x)}) \cdot \vec{\mathbf{n}}_{\epsilon/\rho(x)} \\ \qquad = 0 \qquad\qquad \text{on } B_1(x) \cap \partial\Omega_m^\epsilon/\rho(x), \end{cases} \tag{4.32}$$

where $\vec{\mathbf{n}}_{\epsilon/\rho(x)}$ is a unit vector normal to $\partial\Omega_m^\epsilon/\rho(x)$. Take $s = \rho(x)$ in (4.30) to see, by (4.29),

$$\begin{cases} \lambda_{\epsilon,\rho(x)}, \omega_{\epsilon,\rho(x)} \in [0,1], \\ \|\psi_{\epsilon,\rho(x)}\|_{L^2(B_1(x)\cap\Omega_f^\epsilon/\rho(x))} + \frac{1}{\tilde{\epsilon}_0} \|\mathbb{Q}_{\epsilon,\rho(x)}, \mathbb{G}_{\epsilon,\rho(x)}\|_{L^{n+\delta}(B_1(x)\cap\Omega_f^\epsilon/\rho(x))} \le \tilde{c}, \end{cases}$$

where $\tilde{c}$ is independent of $\tilde{\mu}, \epsilon$. Apply Lemma 4.3 to (4.32) to obtain

$$[\psi_{\epsilon,\rho(x)}]_{C^{0,\tilde{\mu}_\dagger}(\overline{B_{1/2}(x)\cap\Omega_f^\epsilon/\rho(x)})} \le c \quad \text{for some } \tilde{\mu}_\dagger < \mu. \tag{4.33}$$

Note the $\tilde{\mu}_\dagger$ in (4.33) is independent of the $\tilde{\mu}$ in (4.31). If the $\tilde{\mu}$ in (4.31) is taken to be the $\tilde{\mu}_\dagger$ in (4.33), by Theorem 1.2 in page 70 [11] and Lemma 2.1, we obtain

$$\fint_{B_s(x)\cap\Omega} \left|\Pi_\epsilon \Psi_\epsilon - (\Pi_\epsilon \Psi_\epsilon)_{x,s}\right|^2 dy \le c s^{2\tilde{\mu}_\dagger} \tilde{J}_\epsilon^2 \quad \text{for } s < \tfrac{\rho(x)}{2}. \tag{4.34}$$

Now we consider case (2). Again because of $\tilde{\theta}_1 < \tilde{\theta}_2^2$, for any $r \in [\epsilon/\tilde{\epsilon}_0, \tilde{\theta}_2]$, there are $\tilde{\theta} \in [\tilde{\theta}_1, \tilde{\theta}_2]$ and $k \in \mathbb{N}$ satisfying $r = \tilde{\theta}^k$. By Lemma 4.5,

$$\fint_{B_r(x_0)\cap\Omega} |\Pi_\epsilon \Psi_\epsilon|^2 \, dy \le r^{2\mu} \tilde{J}_\epsilon^2 \quad \text{for } r \in [\epsilon/\tilde{\epsilon}_0, \tilde{\theta}_2]. \tag{4.35}$$

This implies

$$\fint_{B_s(x)\cap\Omega} |\Pi_\epsilon\Psi_\epsilon - (\Pi_\epsilon\Psi_\epsilon)_{x,s}|^2\, dy \le cs^{2\mu}\tilde{J}_\epsilon^2 \qquad \text{for } s\in[\tfrac{\epsilon}{3\tilde{\epsilon}_0}, \tfrac{\tilde{\theta}_2}{3}]. \tag{4.36}$$

Shift the coordinate system so that the x is located at the origin. For any $\tilde{\mu} < \mu$, we define

$$\begin{cases} \psi_1(y) \equiv \tilde{J}_\epsilon^{-1}\epsilon^{-\tilde{\mu}}\left(\Psi_\epsilon(\epsilon y) - (\Pi_\epsilon\Psi_\epsilon)_{x,\epsilon/\tilde{\epsilon}_0}\right) \\ \mathbb{Q}_1(y) \equiv \tilde{J}_\epsilon^{-1}\epsilon^{1-\tilde{\mu}}\big(Q_\epsilon(\epsilon y) + \mathbf{T}_\epsilon(\epsilon y)(\Pi_\epsilon\Psi_\epsilon)_{x,\epsilon/\tilde{\epsilon}_0}\big) \\ \mathbb{G}_1(y) \equiv \tilde{J}_\epsilon^{-1}\epsilon^{2-\tilde{\mu}}\big(G_\epsilon(\epsilon y) - \mathbf{E}_\epsilon(\epsilon y)(\Pi_\epsilon\Psi_\epsilon)_{x,\epsilon/\tilde{\epsilon}_0}\big) \quad \text{in } B_{\frac{1}{\tilde{\epsilon}_0}}(x)\cap\Omega_f^\epsilon/\epsilon, \\ \lambda_1 \equiv \epsilon \\ \omega_1 \equiv \epsilon^2 \end{cases} \tag{4.37}$$

and

$$\psi_{b_\epsilon}(y) \equiv -\tilde{J}_\epsilon^{-1}\epsilon^{-\tilde{\mu}}(\Pi_\epsilon\Psi_\epsilon)_{x,\epsilon/\tilde{\epsilon}_0} \qquad \text{in } B_{\frac{1}{\tilde{\epsilon}_0}}(x)\cap\Omega/\epsilon.$$

By (4.35), $|\psi_{b_\epsilon}|$ is a constant independent of $\epsilon, \tilde{\mu}$. Functions in (4.37) satisfy

$$\begin{cases} -\nabla\cdot(\mathbf{K}\nabla\psi_1 + \lambda_1\mathbf{T}\psi_1 + \mathbb{Q}_1) + \lambda_1\mathbf{P}\nabla\psi_1 + \omega_1\mathbf{E}\psi_1 = \mathbb{G}_1 & \text{in } B_{\frac{1}{\tilde{\epsilon}_0}}(x)\cap\Omega_f^\epsilon/\epsilon, \\ (\mathbf{K}\nabla\psi_1 + \lambda_1\mathbf{T}\psi_1 + \mathbb{Q}_1)\cdot\vec{\mathbf{n}}_{\epsilon/\epsilon} = 0 & \text{on } B_{\frac{1}{\tilde{\epsilon}_0}}(x)\cap\partial\Omega_m^\epsilon/\epsilon, \\ \psi_1 = \psi_{b_\epsilon} & \text{on } B_{\frac{1}{\tilde{\epsilon}_0}}(x)\cap\partial\Omega/\epsilon, \end{cases}$$

where $\vec{\mathbf{n}}_{\epsilon/\epsilon}$ is a unit vector normal to $\partial\Omega_m^\epsilon/\epsilon$. Since $\frac{3\epsilon}{\tilde{\epsilon}_0}\in[\epsilon/\tilde{\epsilon}_0, \tilde{\theta}_2]$, we take $s = \frac{\epsilon}{\tilde{\epsilon}_0}$ in (4.36) to see, by (4.29),

$$\begin{cases} \lambda_1, \omega_1 \in [0,1], \\ \|\psi_1\|_{L^2(B_{\frac{1}{\tilde{\epsilon}_0}}(x)\cap\Omega_f^\epsilon/\epsilon)} + \|\mathbb{Q}_1, \mathbb{G}_1\|_{L^{n+\delta}(B_{\frac{1}{\tilde{\epsilon}_0}}(x)\cap\Omega_f^\epsilon/\epsilon)} + \|\psi_{b_\epsilon}\|_{C^{0,1}(B_{\frac{1}{\tilde{\epsilon}_0}}(x)\cap\Omega/\epsilon)} \le \tilde{c}, \end{cases}$$

where $\tilde{c}$ is independent of $\tilde{\mu}, \epsilon$. By Theorem 8.29 [12] and tracing the proof of Theorem 8.24 [12],

$$[\psi_1]_{C^{0,\tilde{\mu}^\dagger}(\overline{B_{1/2\tilde{\epsilon}_0}(x)\cap\Omega_f^\epsilon/\epsilon})} \le c \quad \text{for some } \tilde{\mu}^\dagger < \mu. \tag{4.38}$$

Note the $\tilde{\mu}^\dagger$ in (4.38) is independent of the $\tilde{\mu}$ in (4.37). If the $\tilde{\mu}$ in (4.37) is taken to be the $\tilde{\mu}^\dagger$ in (4.38), Theorem 1.2 in page 70 [11] and Lemma 2.1 imply that

$$\fint_{B_s(x)\cap\Omega} |\Pi_\epsilon\Psi_\epsilon - (\Pi_\epsilon\Psi_\epsilon)_{x,s}|^2\, dy \le cs^{2\tilde{\mu}^\dagger}\tilde{J}_\epsilon^2 \qquad \text{for } s \le \tfrac{\epsilon}{2\tilde{\epsilon}_0}. \tag{4.39}$$

Now define $\tilde{\mu}_* \equiv \min\{\tilde{\mu}_\dagger, \tilde{\mu}^\dagger\}$. Then (4.30), (4.34), (4.36), (4.39), and Theorem 1.2 in page 70 [11] imply

For any $\delta\in(0,3)$, there are $\tilde{\mu}_*\in(0, \frac{\delta}{n+\delta})$ and $\tilde{\epsilon}_*\in(0,1)$ (depending on $\delta, Y_f, \mathbf{K}, \mathbf{M}, \Omega$) such that if $\epsilon\in(0,\tilde{\epsilon}_*)$, then any solution of (4.26) satisfies (4.28).

By Theorem 8.29 [12] and tracing the proof of Theorem 8.24 [12], we also see

Let $\delta, \tilde{\epsilon}_*, \tilde{\mu}_*$ be same as above. If $\epsilon\in[\tilde{\epsilon}_*, 1)$, then any solution of (4.26) satisfies (4.28).

Combining the above two results, we prove Lemma 4.6. □

By A1–A4, energy method, partition of unity, Lemma 4.3, and Lemma 4.6, we obtain Theorem 2.1.

5. L^∞ convergence estimate

We now prove Theorem 2.2. By A5, Theorem 2.1, (2.4), (3.5), and Theorem 9.19 [12], the solution U of (1.4) satisfies

$$\|U\|_{W^{3,n+\delta}(\Omega)} \le c\|\nabla\cdot\mathcal{Q},\mathcal{F}\|_{W^{1,n+\delta}(\Omega)}, \tag{5.1}$$

where c is a constant depending on $\mathcal{K},\mathbf{M},\Omega$. If U_ϵ is the solution of (1.1) and U is the solution of (1.4), we define

$$\Psi_\epsilon(x) \equiv U_\epsilon(x) - U(x) - \sum_{\ell=1}^{2}\sum_{s=0}^{\ell}\sum_{|\vec{\mathbf{i}}|=s} \mathbb{X}^{\vec{\mathbf{i}}}_{\ell,\epsilon}(x)\partial_{\vec{\mathbf{i}}}U(x) \qquad \text{for } x\in\Omega_f^\epsilon,$$

where $\vec{\mathbf{i}} \equiv (i_0, i_1 \cdot\cdot, i_s), |\vec{\mathbf{i}}| \equiv s, i_0 = 0, i_1, \cdot\cdot, i_s \in \{1,\cdots,n\}, \mathbb{X}^{\vec{\mathbf{i}}}_{\ell,\epsilon} \equiv \mathbb{X}^{(i_0,i_1,\cdot\cdot,i_s)}_{\ell,\epsilon}, \partial_{\vec{\mathbf{i}}}U \equiv \partial_{i_1,\cdots,i_s}U$. See (2.3), (2.5)–(2.7), (2.10) for $\mathbb{X}^{(0,i_1,\cdots,i_s)}_{\ell,\epsilon}$. By (3.4) and (5.1), we obtain

$$\begin{cases} -\nabla\cdot\big(\mathbf{K}_\epsilon\nabla\Psi_\epsilon + \mathbf{T}_\epsilon\Psi_\epsilon + Q_\epsilon + \mathcal{O}_1(\epsilon^2)\big) \\ \quad +\mathbf{P}_\epsilon\nabla\Psi_\epsilon + \mathbf{E}_\epsilon\Psi_\epsilon = F_\epsilon - \nabla\cdot\mathcal{Q} - \mathcal{F} + \mathcal{O}_1(\epsilon) & \text{in } \partial\Omega_f^\epsilon, \\ (\mathbf{K}_\epsilon\nabla\Psi_\epsilon + \mathbf{T}_\epsilon\Psi_\epsilon + Q_\epsilon + \mathcal{O}_1(\epsilon^2))\cdot\vec{\mathbf{n}}_\epsilon = 0 & \text{on } \partial\Omega_m^\epsilon, \\ \Psi_\epsilon = \mathcal{O}_2(\epsilon) & \text{on } \partial\Omega, \end{cases}$$

where $\mathcal{O}_1(\nu)$ denotes a function satisfying $\|\mathcal{O}_1(\nu)\|_{L^{n+\delta}(\Omega_f^\epsilon)} \le c\nu$, $\mathcal{O}_2(\nu)$ is a function satisfying $\|\mathcal{O}_2(\nu)\|_{L^\infty(\Omega_f^\epsilon)} \le c\nu$, and c is a constant independent of ν. Decompose Ψ_ϵ as $\Psi_\epsilon = \widehat{\Psi}_\epsilon + \breve{\Psi}_\epsilon$, where $\widehat{\Psi}_\epsilon$ satisfies

$$\begin{cases} -\nabla\cdot\big(\mathbf{K}_\epsilon\nabla\widehat{\Psi}_\epsilon + \mathbf{T}_\epsilon\widehat{\Psi}_\epsilon + Q_\epsilon + \mathcal{O}_1(\epsilon^2)\big) \\ \quad +\mathbf{P}_\epsilon\nabla\widehat{\Psi}_\epsilon + \mathbf{E}_\epsilon\widehat{\Psi}_\epsilon = F_\epsilon - \nabla\cdot\mathcal{Q} - \mathcal{F} + \mathcal{O}_1(\epsilon) & \text{in } \partial\Omega_f^\epsilon, \\ \big(\mathbf{K}_\epsilon\nabla\widehat{\Psi}_\epsilon + \mathbf{T}_\epsilon\widehat{\Psi}_\epsilon + Q_\epsilon + \mathcal{O}_1(\epsilon^2)\big)\cdot\vec{\mathbf{n}}_\epsilon = 0 & \text{on } \partial\Omega_m^\epsilon, \\ \widehat{\Psi}_\epsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.2}$$

and $\breve{\Psi}_\epsilon$ satisfies

$$\begin{cases} -\nabla\cdot(\mathbf{K}_\epsilon\nabla\breve{\Psi}_\epsilon + \mathbf{T}_\epsilon\breve{\Psi}_\epsilon) + \mathbf{P}_\epsilon\nabla\breve{\Psi}_\epsilon + \mathbf{E}_\epsilon\breve{\Psi}_\epsilon = 0 & \text{in } \Omega_f^\epsilon, \\ (\mathbf{K}_\epsilon\nabla\breve{\Psi}_\epsilon + \mathbf{T}_\epsilon\breve{\Psi}_\epsilon)\cdot\vec{\mathbf{n}}_\epsilon = 0 & \text{on } \partial\Omega_m^\epsilon, \\ \breve{\Psi}_\epsilon = \mathcal{O}_2(\epsilon) & \text{on } \partial\Omega. \end{cases} \tag{5.3}$$

By Theorem 2.1, the solution of (5.2) satisfies

$$\|\widehat{\Psi}_\epsilon\|_{L^\infty(\Omega_f^\epsilon)} \le c(\epsilon + \|Q_\epsilon, F_\epsilon - \nabla\cdot\mathcal{Q} - \mathcal{F}\|_{L^{n+\delta}(\Omega_f^\epsilon)}), \tag{5.4}$$

where c is independent of ϵ. Next we claim that the solution of (5.3) satisfies

$$\|\breve{\Psi}_\epsilon\|_{L^\infty(\Omega_f^\epsilon)} = \|\breve{\Psi}_\epsilon\|_{L^\infty(\partial\Omega)} \le c\epsilon, \tag{5.5}$$

where c is a constant independent of ϵ.

Proof of the claim: Assume $\breve{\Psi}_\epsilon$ is not a constant in Ω_f^ϵ (otherwise (5.5) is clear), so $\|\breve{\Psi}_\epsilon\|_{L^\infty(\Omega_f^\epsilon)} > 0$. Theorem 3.5 [12] imply that the maximum value of $|\breve{\Psi}_\epsilon|$ in $\overline{\Omega_f^\epsilon}$ can not appear on any interior point of Ω_f^ϵ. So it is on the boundary $\partial\Omega_f^\epsilon$ of Ω_f^ϵ. If there is a $x_0 \in \partial\Omega_m^\epsilon$ satisfying $|\breve{\Psi}_\epsilon(x_0)| = \|\breve{\Psi}_\epsilon\|_{L^\infty(\Omega_f^\epsilon)} > 0$, then either $\breve{\Psi}_\epsilon(x_0) > 0$ or $\breve{\Psi}_\epsilon(x_0) < 0$. If $\breve{\Psi}_\epsilon(x_0) > 0$, Lemma 3.4 [12] and A5 imply $\mathbf{K}_\epsilon\nabla\breve{\Psi}_\epsilon \cdot \vec{\mathbf{n}}_\epsilon(x_0) > 0$, which is inconsistent with $(5.3)_2$. If $\breve{\Psi}_\epsilon(x_0) < 0$, we define $\zeta = -\breve{\Psi}_\epsilon$. Then ζ satisfies (5.3) and $\zeta(x_0) = \|\zeta\|_{L^\infty(\Omega_f^\epsilon)} > 0$. Lemma 3.4 [12] and A5 imply $\mathbf{K}_\epsilon\nabla\zeta\cdot\vec{\mathbf{n}}_\epsilon(x_0) > 0$, which is again inconsistent with $(5.3)_2$. So we know $\|\breve{\Psi}_\epsilon\|_{L^\infty(\Omega_f^\epsilon)} = \|\breve{\Psi}_\epsilon\|_{L^\infty(\partial\Omega)}$. Clearly $\|\breve{\Psi}_\epsilon\|_{L^\infty(\partial\Omega)} \le c\epsilon$ by $(5.3)_3$. So we prove the claim.

Theorem 2.2 follows from (3.4), (5.1), (5.4), and (5.5).

6. Uniform Lipschitz estimate

This section consists of two subsections. The first subsection is to prove a uniform Lipschitz bound in ϵ for the solutions of elliptic equations (that is, Theorem 2.3 and Theorem 2.4). The second subsection is to derive Lipschitz convergence estimates for the elliptic solutions (that is, Theorem 2.5 and Theorem 2.6).

6.1. *Uniform Lipschitz bound in ϵ*

Let us assume A6–A8 and $\overline{B_1(0)} \subset \mathbb{D}$ in this subsection.

Lemma 6.1. *For any $\delta \in (0,3)$, there are $\breve{\theta} \in (0,1)$ (depending on $\delta, Y_f, \mathbf{K}$) and $\breve{\epsilon}_0 \in (0,1)$ (depending on $\breve{\theta}, \delta, Y_f, \mathbf{K}$) such that if*

$$\begin{cases} -\nabla\cdot(\mathbf{K}_\nu\nabla\psi_\nu + \mathbb{Q}_\nu) + \omega_\nu\mathbf{E}_\nu\psi_\nu = \mathbb{G}_\nu & \text{in } B_1(0)\cap\mathbb{D}_f^\nu, \\ (\mathbf{K}_\nu\nabla\psi_\nu + \mathbb{Q}_\nu)\cdot\vec{\mathbf{n}}_\nu = 0 & \text{on } B_1(0)\cap\partial\mathbb{D}_m^\nu, \end{cases} \tag{6.1}$$

and if

$$\begin{cases} \nu \in (0,\breve{\epsilon}_0), \quad \omega_\nu \in [0,1], \quad \mathbf{M} \le 1, \\ \|\psi_\nu\|_{L^\infty(B_1(0)\cap\mathbb{D}_f^\nu)}, \breve{\epsilon}_0^{-1}\|\mathbb{Q}_\nu, \mathbb{G}_\nu\|_{L^{n+\delta}(B_1(0)\cap\mathbb{D}_f^\nu)} \le 1, \end{cases} \tag{6.2}$$

then

$$\sup_{|x|<\breve{\theta}} |\Pi_\nu\psi_\nu(x) - \Pi_\nu\psi_\nu(0) - (x + \Pi_\nu\mathbb{X}_\nu(x))\mathbf{b}_\nu| \le \breve{\theta}^{1+\frac{\mu}{2}}, \tag{6.3}$$

where $\mu \equiv \frac{\delta}{n+\delta}$, $\mathbf{b}_\nu \equiv \dfrac{\mathcal{K}^{-1}}{|B_{\breve{\theta}}(0)|}\displaystyle\int_{B_{\breve{\theta}}(0)\cap\mathbb{D}_f^\nu} \mathbf{K}_\nu\nabla\psi_\nu dx$, and $\mathcal{K}^{-1}$ is the inverse matrix of the symmetric positive definite matrix $\mathcal{K}$ in (2.4). See A8 for $\mathbf{M}$, (2.10) for $\mathbb{X}_\nu$, and Lemma 2.1 for Π_ν.

Proof. Let $\omega \in [0,1]$, $\mathcal{E}_{0,1} \le 1$ (see (2.4)), and ψ satisfy

$$-\nabla\cdot(\mathcal{K}\nabla\psi) + \omega\mathcal{E}_{0,1}\psi = 0 \qquad \text{in } B_{3/4}(0), \tag{6.4}$$

where $\mathcal{K}$ is defined in (2.4). By (3.5) and Theorem 6.2 [12],

$$\|\psi\|_{C^{2,0}(\overline{B_{1/2}(0)})} \le c\|\psi\|_{L^\infty(B_{3/4}(0))},$$

where c is a constant depending on $\mathcal{K}$. If $\mu' \in (\mu, 1)$, by Taylor's expansion,

$$\sup_{|x|<\theta} \left|\psi(x) - \psi(0) - x(\nabla\psi)_{0,\theta}\right| \le \theta^{1+\frac{\mu'}{2}} \|\psi\|_{L^\infty(B_{3/4}(0))} \tag{6.5}$$

for θ (depending on $\mathcal{K}$) sufficiently small.

Now we fix a $\breve{\theta} < \frac{1}{2}$ such that (6.5) holds. We claim (6.3). If not, there is a sequence $\{\omega_\nu, \psi_\nu, \mathbb{Q}_\nu, \mathbb{G}_\nu\}$ satisfying (6.1) and

$$\begin{cases} \omega_\nu \in [0,1], \\ \|\psi_\nu\|_{L^\infty(B_1(0)\cap\mathbb{D}_f^\nu)} \le 1, \\ \lim\limits_{\nu\to 0} \|\mathbb{Q}_\nu, \mathbb{G}_\nu\|_{L^{n+\delta}(B_1(0)\cap\mathbb{D}_f^\nu)} = 0, \\ \sup\limits_{|x|<\breve{\theta}} |\Pi_\nu\psi_\nu(x) - \Pi_\nu\psi_\nu(0) - (x + \Pi_\nu\mathbb{X}_\nu(x))\mathbf{b}_\nu| > \breve{\theta}^{1+\frac{\mu}{2}}. \end{cases} \tag{6.6}$$

After extraction of a subsequence (same notation for subsequence), we have, by Theorem 2.7 [3] and Lemma 4.3,

$$\begin{cases} \omega_\nu \to \omega \in [0,1] & \\ \Pi_\nu\psi_\nu \to \psi & \text{in } L^\infty(B_{3/4}(0)) \text{ strongly} \\ \mathbf{K}_\nu\nabla\psi_\nu\mathcal{X}_{\mathbb{D}_f^\nu} \to \mathcal{K}\nabla\psi & \text{in } L^2(B_{3/4}(0)) \text{ weakly} \\ \omega_\nu\mathbf{E}_\nu\psi_\nu\mathcal{X}_{\mathbb{D}_f^\nu} \to \omega\mathcal{E}_{0,1}\psi & \text{in } L^2(B_{3/4}(0)) \text{ weakly} \end{cases} \quad \text{as } \nu \to 0, \tag{6.7}$$

where $\mathcal{K}, \mathcal{E}_{0,1}$ are defined in (2.4) and $\mathcal{E}_{0,1} \le 1$ by $(6.2)_1$. The ψ in (6.7) is a solution of (6.4). (3.4) and (6.5)–(6.7) imply

$$\begin{aligned} \breve{\theta}^{1+\frac{\mu}{2}} &\le \lim_{\nu\to 0} \sup_{|x|<\breve{\theta}} |\Pi_\nu\psi_\nu(x) - \Pi_\nu\psi_\nu(0) - (x + \Pi_\nu\mathbb{X}_\nu(x))\mathbf{b}_\nu| \\ &= \sup_{|x|<\breve{\theta}} \left|\psi(x) - \psi(0) - x(\nabla\psi)_{0,\breve{\theta}}\right| \le \breve{\theta}^{1+\frac{\mu'}{2}} \|\psi\|_{L^\infty(B_{3/4}(0))}, \end{aligned}$$

which is impossible if $\breve{\theta}$ is small. So (6.3) holds. □

Lemma 6.2. *Let $\delta, \breve{\theta}, \breve{\epsilon}_0, \mu$ be same as those in Lemma 6.1 and $\mathbf{M}\breve{\epsilon}_0^{-1} < 1$. For any $\epsilon \in (0, \breve{\epsilon}_0)$ and k satisfying $\epsilon/\breve{\theta}^k \le \breve{\epsilon}_0$, there are constants $\mathbf{a}_k^\epsilon$, $\mathbf{b}_k^\epsilon$ such that any solution of*

$$\begin{cases} -\nabla\cdot(\mathbf{K}_\epsilon\nabla\Psi_\epsilon + Q_\epsilon) + \mathbf{E}_\epsilon\Psi_\epsilon = G_\epsilon & \text{in } B_1(0)\cap\mathbb{D}_f^\epsilon, \\ (\mathbf{K}_\epsilon\nabla\Psi_\epsilon + Q_\epsilon)\cdot\vec{\mathbf{n}}_\epsilon = 0 & \text{on } B_1(0)\cap\partial\mathbb{D}_m^\epsilon, \end{cases} \tag{6.8}$$

satisfies

$$\begin{cases} \sup\limits_{|x|<\breve{\theta}^k} |\Pi_\epsilon\Psi_\epsilon(x) - \Pi_\epsilon\Psi_\epsilon(0) - \epsilon\mathbf{a}_k^\epsilon - (x + \Pi_\epsilon\mathbb{X}_\epsilon(x))\mathbf{b}_k^\epsilon| \le \breve{\theta}^{k(1+\frac{\mu}{2})}\breve{J}_\epsilon, \\ |\mathbf{a}_k^\epsilon| + |\mathbf{b}_k^\epsilon| \le c\breve{J}_\epsilon, \end{cases} \tag{6.9}$$

where c is a constant independent of ϵ *and*

$$\breve{J}_\epsilon \equiv 4\breve{\epsilon}_0^{-1}(\|\Psi_\epsilon\|_{L^\infty(B_1(0)\cap\mathbb{D}_f^\epsilon)} + \|\epsilon^{\frac{\mu}{2}-1}Q_\epsilon, G_\epsilon\|_{L^{n+\delta}(B_1(0)\cap\mathbb{D}_f^\epsilon)}).$$

Proof. This is proved by induction on k. Let c denote a constant independent of ϵ. Set $\psi_\epsilon \equiv \frac{\Psi_\epsilon}{\breve{J}_\epsilon}$, $\mathbb{Q}_\epsilon \equiv \frac{Q_\epsilon}{\breve{J}_\epsilon}$, $\mathbb{G}_\epsilon \equiv \frac{G_\epsilon}{\breve{J}_\epsilon}$, $\omega_\epsilon = 1$. Then they satisfy (6.1) and (6.2) with $\nu = \epsilon$. By Lemma 6.1, we obtain $(6.9)_1$ for $k = 1$ case, where $\mathbf{a}_1^\epsilon = 0$ and $\mathbf{b}_1^\epsilon = \dfrac{\mathcal{K}^{-1}}{|B_{\breve{\theta}}(0)|}\displaystyle\int_{B_{\breve{\theta}}(0)\cap\mathbb{D}_f^\epsilon} \mathbf{K}_\epsilon\nabla\Psi_\epsilon dx$. By energy method and A7–A8, any solution of (6.8) satisfies

$$\|\Psi_\epsilon\|_{H^1(B_{3/4}(0)\cap\mathbb{D}_f^\epsilon)} \le c\|\Psi_\epsilon, Q_\epsilon, G_\epsilon\|_{L^2(B_1(0)\cap\mathbb{D}_f^\epsilon)}. \tag{6.10}$$

(6.10) implies that $(6.9)_2$ for $k = 1$ also holds. If (6.9) holds for some k satisfying $\epsilon/\breve{\theta}^k \le \breve{\epsilon}_0$, we define, in $B_1(0)\cap\mathbb{D}_f^\epsilon/\breve{\theta}^k$,

$$\begin{cases} \psi_{\epsilon/\breve{\theta}^k}(x) \equiv \dfrac{\Psi_\epsilon(\breve{\theta}^k x) - \Pi_\epsilon\Psi_\epsilon(0) - \epsilon\mathbf{a}_k^\epsilon - \big(\breve{\theta}^k x + \mathbb{X}_\epsilon(\breve{\theta}^k x)\big)\mathbf{b}_k^\epsilon}{\breve{J}_\epsilon\ \breve{\theta}^{k(1+\mu/2)}}, \\ \mathbb{Q}_{\epsilon/\breve{\theta}^k}(x) \equiv \dfrac{Q_\epsilon(\breve{\theta}^k x)}{\breve{J}_\epsilon\ \breve{\theta}^{k\mu/2}}, \\ \mathbb{G}_{\epsilon/\breve{\theta}^k}(x) \equiv \dfrac{G_\epsilon(\breve{\theta}^k x) - \mathbf{E}_\epsilon(\breve{\theta}^k x)\big(\Pi_\epsilon\Psi_\epsilon(0) + \epsilon\mathbf{a}_k^\epsilon + \big(\breve{\theta}^k x + \mathbb{X}_\epsilon(\breve{\theta}^k x)\big)\mathbf{b}_k^\epsilon\big)}{\breve{J}_\epsilon\ \breve{\theta}^{k(\mu/2-1)}}, \\ \omega_{\epsilon/\breve{\theta}^k} = \breve{\theta}^{2k}. \end{cases}$$

Then they satisfy, by (2.3) and (2.10),

$$\begin{cases} -\nabla\cdot(\mathbf{K}_{\epsilon/\breve{\theta}^k}\nabla\psi_{\epsilon/\breve{\theta}^k} + \mathbb{Q}_{\epsilon/\breve{\theta}^k}) \\ \qquad +\omega_{\epsilon/\breve{\theta}^k}\mathbf{E}_{\epsilon/\breve{\theta}^k}\psi_{\epsilon/\breve{\theta}^k} = \mathbb{G}_{\epsilon/\breve{\theta}^k} & \text{in } B_1(0)\cap\mathbb{D}_f^\epsilon/\breve{\theta}^k, \\ (\mathbf{K}_{\epsilon/\breve{\theta}^k}\nabla\psi_{\epsilon/\breve{\theta}^k} + \mathbb{Q}_{\epsilon/\breve{\theta}^k})\cdot\vec{\mathbf{n}}_{\epsilon/\breve{\theta}^k} = 0 & \text{on } B_1(0)\cap\partial\mathbb{D}_m^\epsilon/\breve{\theta}^k, \end{cases} \tag{6.11}$$

where $\vec{\mathbf{n}}_{\epsilon/\breve{\theta}^k}$ is a unit vector normal to $\partial\mathbb{D}_m^\epsilon/\breve{\theta}^k$. By $(6.9)_1$ and A8,

$$\|\mathbb{G}_{\epsilon/\breve{\theta}^k}\|_{L^{n+\delta}(B_1(0)\cap\mathbb{D}_f^\epsilon/\breve{\theta}^k)} \le \breve{\theta}^{k\mu/2}\breve{J}_\epsilon^{-1}\|G_\epsilon\|_{L^{n+\delta}(B_1(0)\cap\mathbb{D}_f^\epsilon)}$$
$$+\breve{\theta}^{k(1-\mu/2)}\breve{J}_\epsilon^{-1}\mathbf{M}\big(\|\Pi_\epsilon\Psi_\epsilon\|_{L^\infty(B_{\breve{\theta}^k}(0))} + \breve{\theta}^{k(1+\frac{\mu}{2})}\breve{J}_\epsilon\big).$$

By induction, $\mathbf{M}\breve{\epsilon}_0^{-1} < 1$, (6.10), Lemma 4.3, and small $\breve{\theta}$, we have

$$\begin{cases} \omega_{\epsilon/\breve{\theta}^k} \in [0,1], \\ \|\psi_{\epsilon/\breve{\theta}^k}\|_{L^\infty(B_1(0)\cap\mathbb{D}_f^\epsilon/\breve{\theta}^k)}, \breve{\epsilon}_0^{-1}\|\mathbb{Q}_{\epsilon/\breve{\theta}^k}, \mathbb{G}_{\epsilon/\breve{\theta}^k}\|_{L^{n+\delta}(B_1(0)\cap\mathbb{D}_f^\epsilon/\breve{\theta}^k)} \le 1. \end{cases} \tag{6.12}$$

Apply Lemma 6.1 (take $\nu = \epsilon/\breve{\theta}^k$) to obtain

$$\sup_{|x|<\breve{\theta}} \big|\Pi_{\epsilon/\breve{\theta}^k}\psi_{\epsilon/\breve{\theta}^k}(x) - \Pi_{\epsilon/\breve{\theta}^k}\psi_{\epsilon/\breve{\theta}^k}(0) - \big(x + \Pi_{\epsilon/\breve{\theta}^k}\mathbb{X}_{\epsilon/\breve{\theta}^k}(x)\big)\mathbf{b}_{\epsilon/\breve{\theta}^k}\big| \le \breve{\theta}^{1+\frac{\mu}{2}}, \tag{6.13}$$

where $\mathbf{b}_{\epsilon/\breve{\theta}^k} \equiv \dfrac{\mathcal{K}^{-1}}{|B_{\breve{\theta}}(0)|}\displaystyle\int_{B_{\breve{\theta}}(0)\cap\mathbb{D}_f^\epsilon/\breve{\theta}^k} \mathbf{K}_{\epsilon/\breve{\theta}^k}\nabla\psi_{\epsilon/\breve{\theta}^k}dx$. By Lemma 2.1, (6.13) can be written as

$$\sup_{|x|<\breve{\theta}}\big|\Pi_\epsilon\Psi_\epsilon(\breve{\theta}^k x) - \Pi_\epsilon\Psi_\epsilon(0) + \epsilon\Pi_1\mathbb{X}(0)\mathbf{b}_k^\epsilon - \big(\breve{\theta}^k x + \Pi_\epsilon\mathbb{X}_\epsilon(\breve{\theta}^k x)\big)\mathbf{b}_k^\epsilon$$
$$-\breve{J}_\epsilon\ \breve{\theta}^{k(1+\mu/2)}\big(x + \breve{\theta}^{-k}\Pi_\epsilon\mathbb{X}_\epsilon(\breve{\theta}^k x)\big)\mathbf{b}_{\epsilon/\breve{\theta}^k}\big| \le \breve{J}_\epsilon\ \breve{\theta}^{(k+1)(1+\mu/2)}. \tag{6.14}$$

See (2.10) for $\mathbb{X}$. Define

$$\mathbf{a}_{k+1}^\epsilon \equiv -\Pi_1\mathbb{X}(0)\mathbf{b}_k^\epsilon \qquad \text{and} \qquad \mathbf{b}_{k+1}^\epsilon \equiv \mathbf{b}_k^\epsilon + \breve{J}_\epsilon\breve{\theta}^{k\mu/2}\mathbf{b}_{\epsilon/\breve{\theta}^k}. \tag{6.15}$$

Substituting (6.15) into (6.14) and making the change of variables $\breve{\theta}^k x$ to x, we obtain $(6.9)_1$. By (6.11) and (6.12), $|\mathbf{b}_{\epsilon/\breve{\theta}^k}|$ is bounded uniformly in ϵ, k. So $(6.9)_2$ holds by (6.15). □

Lemma 6.3. *For any $\delta \in (0,3)$, there is a $\breve{\epsilon}_0 \in (0,1)$ (depending on $\delta, Y_f, \mathbf{K}$) such that if $\epsilon \in (0,\breve{\epsilon}_0)$ and $\mathbf{M}\breve{\epsilon}_0^{-1} < 1$, then any solution of (6.8) satisfies*

$$\|\nabla\Psi_\epsilon\|_{L^\infty(B_{1/2}(0)\cap\mathbb{D}_f^\epsilon)} \le c(\breve{J}_\epsilon + \epsilon^{-\mu/2}|\!|\!|Q_\epsilon|\!|\!|_{W^{1,n+\delta}_{loc}(\mathbb{D}_f^\epsilon)}) \equiv cJ_\epsilon^*, \tag{6.16}$$

where c is a constant independent of ϵ. See Lemma 6.2 for $\mu, \breve{J}_\epsilon$ and see section 2 for $|\!|\!|\cdot|\!|\!|_{W^{1,n+\delta}_{loc}(\mathbb{D}_f^\epsilon)}$.

Proof. Denote by c a constant independent of ϵ and let $\breve{\theta}, \breve{\epsilon}_0$ be those in Lemma 6.2. Let $k \in \mathbb{N}$ such that $\epsilon/\breve{\theta}^k \le \breve{\epsilon}_0 < \epsilon/\breve{\theta}^{k+1}$. By Lemma 6.2,

$$\sup_{|x|<\frac{\epsilon}{\breve{\epsilon}_0}}|\Pi_\epsilon\Psi_\epsilon(x) - \Pi_\epsilon\Psi_\epsilon(0) - \epsilon\mathbf{a}_k^\epsilon - (x + \Pi_\epsilon\mathbb{X}_\epsilon(x))\,\mathbf{b}_k^\epsilon| \le c\Big|\frac{\epsilon}{\breve{\epsilon}_0}\Big|^{1+\frac{\mu}{2}}\breve{J}_\epsilon. \tag{6.17}$$

Define, in $B_{1/\breve{\epsilon}_0}(0)\cap\mathbb{D}_f^\epsilon/\epsilon$,

$$\begin{cases} \psi_1(x) \equiv \dfrac{\Psi_\epsilon(\epsilon x) - \Pi_\epsilon\Psi_\epsilon(0) - \epsilon\mathbf{a}_k^\epsilon - (\epsilon x + \mathbb{X}_\epsilon(\epsilon x))\mathbf{b}_k^\epsilon}{J_\epsilon^*\ \epsilon^{1+\mu/2}}, \\ \mathbb{Q}_1(x) \equiv \dfrac{Q_\epsilon(\epsilon x)}{J_\epsilon^*\ \epsilon^{\mu/2}}, \\ \mathbb{G}_1(x) \equiv \dfrac{G_\epsilon(\epsilon x) - \mathbf{E}(x)\big(\Pi_\epsilon\Psi_\epsilon(0) + \epsilon\mathbf{a}_k^\epsilon + (\epsilon x + \mathbb{X}_\epsilon(\epsilon x))\mathbf{b}_k^\epsilon\big)}{J_\epsilon^*\ \epsilon^{\mu/2-1}}, \\ \omega_1 \equiv \epsilon^2. \end{cases}$$

Then these functions satisfy, by (2.3) and (2.10),

$$\begin{cases} -\nabla\cdot(\mathbf{K}\nabla\psi_1 + \mathbb{Q}_1) + \omega_1\mathbf{E}\psi_1 = \mathbb{G}_1 & \text{in } B_{\frac{1}{\breve{\epsilon}_0}}(0)\cap\mathbb{D}_f^\epsilon/\epsilon, \\ (\mathbf{K}\nabla\psi_1 + \mathbb{Q}_1)\cdot\vec{\mathbf{n}}_{\epsilon/\epsilon} = 0 & \text{on } B_{\frac{1}{\breve{\epsilon}_0}}(0)\cap\partial\mathbb{D}_m^\epsilon/\epsilon. \end{cases}$$

By (6.17), A8, and Lemma 4.3,

$$\begin{cases} \omega_1 \in [0,1], \\ \|\psi_1\|_{L^\infty(B_{1/\breve{\epsilon}_0}(0)\cap\mathbb{D}_f^\epsilon/\epsilon)} \le c, \\ \|\mathbb{Q}_1\|_{W^{1,n+\delta}(B_{1/\breve{\epsilon}_0}(0)\cap\mathbb{D}_f^\epsilon/\epsilon)} + \|\mathbb{G}_1\|_{L^{n+\delta}(B_{1/\breve{\epsilon}_0}(0)\cap\mathbb{D}_f^\epsilon/\epsilon)} \le c. \end{cases}$$

By Lemma 3.2, we see

$$\|\psi_1\|_{W^{2,n+\delta}(B_{\frac{1}{8\breve{\epsilon}_0}}(0)\cap\mathbb{D}_f^\epsilon/\epsilon)} \leq c. \tag{6.18}$$

Since $\nabla\psi_1(x) = \frac{\nabla\Psi_\epsilon(\epsilon x)-(I+\nabla\mathbb{X}(x))\mathbf{b}_k^\epsilon}{\epsilon^{\mu/2}J_\epsilon^*}$, $|\nabla\Psi_\epsilon(\epsilon x)| \leq cJ_\epsilon^*$ for $x \in B_{\frac{1}{8\breve{\epsilon}_0}}(0) \cap \mathbb{D}_f^\epsilon/\epsilon$ by (3.4), (6.18), and Lemma 6.2. By shifting the coordinate system and repeating above argument, we see that this lemma holds. □

Let $\breve{\epsilon}_0$ be same as that in Lemma 6.3. From Lemma 3.2, we also see

For any $\delta \in (0,3)$ and $\epsilon \in [\breve{\epsilon}_0, 1)$, any solution of (6.8) satisfies (6.16).

Together with Lemma 6.3, we obtain

Lemma 6.4. *For $\delta \in (0,3)$, $\epsilon \in (0,1)$, and small $\mathbf{M}$ (depending on $\delta, Y_f, \mathbf{K}$), any solution of (6.8) satisfies (6.16).*

Theorem 2.3 and Theorem 2.4 follow from energy method, Poincaré inequality, Lemma 4.3, and Lemma 6.4.

6.2. *Proofs of Theorem 2.5 and Theorem 2.6*

In this subsection, we first prove Theorem 2.5 and then Theorem 2.6.

6.2.1. *Proof of Theorem 2.5*

Let $\widehat{\mathcal{V}}_k, \widehat{\Phi}_k (= \sum_{j=0}^{k-1}\epsilon^j\widehat{\varphi}_j)$ for $k \geq 1$ be those in (2.16)–(2.17). By assumptions of Theorem 2.5, (3.5), and Theorem 9.19 [12], the solution U of (2.15) satisfies

$$\|U\|_{W^{2k+2,n+\delta}(\mathbb{D})} \leq c\|\widehat{\mathcal{F}}\|_{W^{2k,n+\delta}(\mathbb{D})} \qquad \text{for } k \geq 1, \delta \in (0,3), \tag{6.19}$$

where c is a constant depending on $\mathcal{K}, \mathcal{E}_{0,1}$. (6.19) implies

$$\|\widehat{\mathcal{V}}_k\|_{W^{2,n+\delta}(\mathbb{D})} + \|\widehat{\Phi}_k\|_{W^{k+3,n+\delta}(\mathbb{D})} \leq c\|\widehat{\mathcal{F}}\|_{W^{2k,n+\delta}(\mathbb{D})} \qquad \text{for } k \geq 1, \delta \in (0,3), \tag{6.20}$$

where c is a constant depending on $\mathbf{K}, \mathbf{E}$. By (2.17), (3.4), and (6.19)–(6.20),

$$\begin{cases} \nabla\cdot\Big(\mathbf{K}_\epsilon\nabla\widehat{\mathcal{V}}_k + \mathcal{O}_3(\epsilon^{k+1})\Big) - \mathbf{E}_\epsilon\widehat{\mathcal{V}}_k = \displaystyle\sum_{\ell=2}^{k+1}\sum_{s=0}^{\ell}\sum_{|\vec{\mathbf{i}}|=s}\epsilon^{\ell-2}\mathcal{N}_\ell^{\vec{\mathbf{i}}}\partial_{\vec{\mathbf{i}}}\widehat{\Phi}_k + \mathcal{O}_4(\epsilon^k) & \text{in } \mathbb{D}_f^\epsilon, \\ \Big(\mathbf{K}_\epsilon\nabla\widehat{\mathcal{V}}_k + \mathcal{O}_3(\epsilon^{k+1})\Big)\cdot\vec{\mathbf{n}}_\epsilon = 0 & \text{on } \partial\mathbb{D}_m^\epsilon, \\ \widehat{\mathcal{V}}_k \text{ satisfies periodic boundary conditions on } \partial\mathbb{D}, \end{cases}$$

where $\mathcal{O}_3(\epsilon^{k+1}), \mathcal{O}_4(\epsilon^k)$ are periodic functions with period $\mathbb{D}$ and satisfy

$$\begin{cases} \|\mathcal{O}_3(\epsilon^{k+1})\|_{L^{n+\delta}(\mathbb{D}_f^\epsilon)} + \|\!|\mathcal{O}_3(\epsilon^{k+1})|\!\|_{W^{1,n+\delta}_{loc}(\mathbb{D}_f^\epsilon)} \leq c\epsilon^{k+1}, \\ \|\mathcal{O}_4(\epsilon^k)\|_{L^{n+\delta}(\mathbb{D}_f^\epsilon)} \leq c\epsilon^k, \end{cases}$$

for some constant c independent of ϵ. Since the $\widehat{\Phi}_k$ in (2.17) satisfies $\widehat{\Phi}_k \equiv \sum_{j=0}^{k-1} \epsilon^j \widehat{\varphi}_j$ in $\mathbb{D}$, we have, by (2.15)–(2.16),

$$\begin{aligned}
&\sum_{\ell=2}^{k+1}\sum_{s=0}^{\ell}\sum_{|\vec{\mathbf{i}}|=s} \epsilon^{\ell-2}\mathcal{N}_\ell^{\vec{\mathbf{i}}}\partial_{\vec{\mathbf{i}}}\widehat{\Phi}_k = \sum_{\ell=2}^{k+1}\sum_{j=0}^{k-1}\epsilon^{\ell-2+j}\left(\sum_{s=0}^{\ell}\sum_{|\vec{\mathbf{i}}|=s}\mathcal{N}_\ell^{\vec{\mathbf{i}}}\partial_{\vec{\mathbf{i}}}\widehat{\varphi}_j\right)\\
&= \sum_{s=0}^{2}\sum_{|\vec{\mathbf{i}}|=s}\mathcal{N}_2^{\vec{\mathbf{i}}}\partial_{\vec{\mathbf{i}}}\widehat{\varphi}_0 + \sum_{r=1}^{k-1}\epsilon^r\left(\sum_{s=0}^{2}\sum_{|\vec{\mathbf{i}}|=s}\mathcal{N}_2^{\vec{\mathbf{i}}}\partial_{\vec{\mathbf{i}}}\widehat{\varphi}_r + \sum_{q=0}^{r-1}\sum_{s=0}^{r+2-q}\sum_{|\vec{\mathbf{i}}|=s}\mathcal{N}_{r+2-q}^{\vec{\mathbf{i}}}\partial_{\vec{\mathbf{i}}}\widehat{\varphi}_q\right) + \mathcal{O}_4(\epsilon^k)\\
&= -\widehat{\mathcal{F}} + \mathcal{O}_4(\epsilon^k).
\end{aligned}$$

If U_ϵ solves (2.14) and if we define $\Psi_\epsilon \equiv U_\epsilon - \widehat{\mathcal{V}}_k$ in $\mathbb{D}_f^\epsilon$, then

$$\begin{cases}
-\nabla\cdot\left(\mathbf{K}_\epsilon\nabla\Psi_\epsilon + \mathcal{O}_3(\epsilon^{k+1})\right) + \mathbf{E}_\epsilon\Psi_\epsilon = F_\epsilon - \widehat{\mathcal{F}} + \mathcal{O}_4(\epsilon^k) & \text{in } \mathbb{D}_f^\epsilon,\\
\left(\mathbf{K}_\epsilon\nabla\Psi_\epsilon + \mathcal{O}_3(\epsilon^{k+1})\right)\cdot\vec{\mathbf{n}}_\epsilon = 0 & \text{on } \partial\mathbb{D}_m^\epsilon,\\
\Psi_\epsilon \text{ satisfies periodic boundary conditions on } \partial\mathbb{D}.
\end{cases}$$

By Theorem 2.3,

$$\|\nabla\Psi_\epsilon\|_{L^\infty(\mathbb{D}_f^\epsilon)} \le c(\epsilon^k + \|F_\epsilon - \widehat{\mathcal{F}}\|_{L^{n+\delta}(\mathbb{D}_f^\epsilon)}), \tag{6.21}$$

where c is a constant independent of ϵ. Which implies Theorem 2.5.

6.2.2. *Proof of Theorem 2.6*

The proof is similar to that of Theorem 2.5. Let $\check{\mathcal{V}}_k, \check{\Phi}_k (= \sum_{j=0}^{k-1}\epsilon^j\check{\varphi}_j)$ for $k \ge 1$ be those in (2.20)–(2.21). By assumptions of Theorem 2.6, and Theorem 9.19 [12], the solution U of (2.19) satisfies

$$\|U\|_{W^{2k+2,n+\delta}(\mathbb{D})} \le c\|\check{\mathcal{F}}\|_{W^{2k,n+\delta}(\mathbb{D})} \qquad \text{for } k \ge 1, \delta \in (0,3),$$

where c is a constant depending on $\mathcal{K}$. Which implies

$$\|\check{\mathcal{V}}_k\|_{W^{2,n+\delta}(\mathbb{D})} + \|\check{\Phi}_k\|_{W^{k+3,n+\delta}(\mathbb{D})} \le c\|\check{\mathcal{F}}\|_{W^{2k,n+\delta}(\mathbb{D})} \qquad \text{for } k \ge 1, \delta \in (0,3),$$

where c is a constant depending on $\mathbf{K}$. By (2.21) and (3.4),

$$\begin{cases}
\nabla\cdot\left(\mathbf{K}_\epsilon\nabla\check{\mathcal{V}}_k + \mathcal{O}_3(\epsilon^{k+1})\right) = \displaystyle\sum_{\ell=2}^{k+1}\sum_{|\vec{\mathbf{i}}|=\ell}\epsilon^{\ell-2}\mathcal{N}_\ell^{\vec{\mathbf{i}}}\partial_{\vec{\mathbf{i}}}\check{\Phi}_k + \mathcal{O}_4(\epsilon^k) & \text{in } \mathbb{D}_f^\epsilon,\\
\left(\mathbf{K}_\epsilon\nabla\check{\mathcal{V}}_k + \mathcal{O}_3(\epsilon^{k+1})\right)\cdot\vec{\mathbf{n}}_\epsilon = 0 & \text{on } \partial\mathbb{D}_m^\epsilon,\\
\check{\mathcal{V}}_k \text{ satisfies periodic boundary conditions on } \partial\mathbb{D},
\end{cases}$$

where $\mathcal{O}_3(\epsilon^{k+1}), \mathcal{O}_4(\epsilon^k)$ are defined same as §6.2.1. Since $\breve{\Phi}_k \equiv \sum_{j=0}^{k-1} \epsilon^j \breve{\varphi}_j$ in $\mathbb{D}$, by (2.19)–(2.20),

$$\begin{aligned}\sum_{\ell=2}^{k+1} \sum_{|\vec{\mathbf{i}}|=\ell} \epsilon^{\ell-2} \mathcal{N}_\ell^{\vec{\mathbf{i}}} \partial_{\vec{\mathbf{i}}} \breve{\Phi}_k &= \sum_{\ell=2}^{k+1} \sum_{j=0}^{k-1} \epsilon^{\ell-2+j} \sum_{|\vec{\mathbf{i}}|=\ell} \mathcal{N}_\ell^{\vec{\mathbf{i}}} \partial_{\vec{\mathbf{i}}} \breve{\varphi}_j \\ &= \sum_{|\vec{\mathbf{i}}|=2} \mathcal{N}_2^{\vec{\mathbf{i}}} \partial_{\vec{\mathbf{i}}} \breve{\varphi}_0 + \sum_{r=1}^{k-1} \epsilon^r \left(\sum_{|\vec{\mathbf{i}}|=2} \mathcal{N}_2^{\vec{\mathbf{i}}} \partial_{\vec{\mathbf{i}}} \breve{\varphi}_r + \sum_{q=0}^{r-1} \sum_{|\vec{\mathbf{i}}|=r+2-q} \mathcal{N}_{r+2-q}^{\vec{\mathbf{i}}} \partial_{\vec{\mathbf{i}}} \breve{\varphi}_q \right) + \mathcal{O}_4(\epsilon^k) \\ &= -\breve{\mathcal{F}} + \mathcal{O}_4(\epsilon^k).\end{aligned}$$

If U_ϵ solves (2.18), we define $\Psi_\epsilon \equiv U_\epsilon - \breve{\mathcal{V}}_k$ in $\mathbb{D}_f^\epsilon$. Theorem 2.4 implies Theorem 2.6.

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