Boundedness of the images of period maps and applications

Kefeng Liu and Yang Shen

Abstract

We prove a conjecture of Griffiths on simultaneous normalization of all periods which asserts that the image of the lifted period map on the universal cover lies in a bounded domain in a complex Euclidean space. As an application we prove that the Teichmüller spaces of a large class of projective manifolds have complex affine structures.

0 Introduction

Let $\Phi : S \to D/\Gamma$ be a period map which arises from geometry. This means that we have an algebraic family $f : \mathfrak{X} \to S$ of polarized algebraic manifolds over a quasi-projective manifold S, such that for any $q \in S$, the point $\Phi(q)$, modulo the action of the monodromy group Γ, represents the Hodge structure of the n-th primitive cohomology of the fiber $f^{-1}(q)$.

Since period map is locally liftable, we can lift the period map to $\widetilde{\Phi} : \mathcal{T} \to D$ by taking the universal cover $\mathcal{T}$ of S such that the diagram

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\widetilde{\Phi}} & D \\ \downarrow{\pi} & & \downarrow{\pi} \\ S & \xrightarrow{\Phi} & D/\Gamma \end{array} \tag{1}$$

is commutative.

In [12], Griffiths raised the following conjecture as Conjecture 10.1 in Section 10, which is now the main theorem of our paper.

Theorem 1. (Griffiths Conjecture) *Given $f : \mathfrak{X} \to S$, there exists a simultaneous normalization of all the periods $\Phi(X_s)$ $(s \in S)$. More precisely, the image $\widetilde{\Phi}(\mathcal{T})$ lies in a bounded domain in a complex Euclidean space.*

Our method of proving this conjecture is to first consider the restriction of the period map on $\check{\mathcal{T}}$, an open subset of $\mathcal{T}$, with $\mathcal{T} \setminus \check{\mathcal{T}} \subset \mathcal{T}$ an analytic subvariety. Then we prove that the image of the restricted period map is bounded in a complex Euclidean space, and apply the Riemann extension theorem to get the boundedness of the period map on $\mathcal{T}$. To explain the main ideas in more detail, we need to use the Lie theory description of period domain and period map.

First recall that the period domain D can be realized as quotient of real Lie groups $D = G_{\mathbb{R}}/V$, and its compact dual $\check{D}$ as quotient of complex Lie groups $\check{D} = G_{\mathbb{C}}/B$, where $V = B \cap G_{\mathbb{R}}$. The Hodge structure at a fixed point o in D induces a Hodge structure of weight zero on the Lie algebra $\mathfrak{g}$ of $G_{\mathbb{C}}$ as

$$\mathfrak{g} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}^{k,-k} \quad \text{with} \quad \mathfrak{g}^{k,-k} = \{X \in \mathfrak{g} |\ XH^{r,n-r} \subseteq H^{r+k,n-r-k},\ \forall\ r\}.$$

See Section 1 for the detail of the above notations. Then the Lie algebra of B is $\mathfrak{b} = \bigoplus_{k\geq 0} \mathfrak{g}^{k,-k}$ and the holomorphic tangent space $\mathrm{T}^{1,0}_o D$ of D at the base point o is naturally isomorphic to

$$\mathfrak{g}/\mathfrak{b} \simeq \oplus_{k\geq 1}\mathfrak{g}^{-k,k} \triangleq \mathfrak{n}_+.$$

Clearly we can identify the nilpotent Lie subalgebra $\mathfrak{n}_+$ to the complex Euclidean space $\mathrm{T}^{1,0}_o D$ with induced inner product from the homogenous metric on D at o. We denote the corresponding unipotent group by

$$N_+ = \exp(\mathfrak{n}_+)$$

which is considered as a complex Euclidean space with induced Euclidean metric from $\mathfrak{n}_+$. Since $N_+ \cap B = \{\mathrm{Id}\}$, we can identify the unipotent group $N_+ \subseteq G_{\mathbb{C}}$ to its orbit $N_+(o) \subseteq \check{D}$ so that the notation $N_+ \subseteq \check{D}$ is meaningful in this sense. With this we define

$$\check{\mathcal{T}} = \widetilde{\Phi}^{-1}(N_+ \cap D)$$

and first prove that $\mathcal{T}\backslash\check{\mathcal{T}}$ is an analytic subvariety of $\mathcal{T}$ with $\operatorname{codim}_{\mathbb{C}}(\mathcal{T}\backslash\check{\mathcal{T}}) \geq 1$.

Next we prove that the image of the restricted period map

$$\widetilde{\Phi}|_{\check{\mathcal{T}}} : \check{\mathcal{T}} \to N_+ \cap D$$

is bounded in the complex Euclidean space N_+. To prove this we consider the subspace of of $\mathfrak{n}_+$,

$$\mathfrak{p}_+ = \oplus_{k \text{ odd}, k\geq 1}\mathfrak{g}^{-k,k} \subseteq \mathfrak{n}_+.$$

Let $\exp(\mathfrak{p}_+) \subseteq N_+$ which is considered as a complex Euclidean subspace of N_+, and

$$P_+ : N_+ \cap D \to \exp(\mathfrak{p}_+) \cap D$$

be the induced projection map. The restricted period map $\widetilde{\Phi} : \check{\mathcal{T}} \to N_+ \cap D$, composed with the projection map P_+, $\widetilde{\Phi}_+ = P_+ \circ \widetilde{\Phi}$, gives a holomorphic map

$$\widetilde{\Phi}_+ : \check{\mathcal{T}} \to \exp(\mathfrak{p}_+) \cap D.$$

Then we prove that the image of $\widetilde{\Phi}_+$ is bounded with respect to the Euclidean metric on $\exp(\mathfrak{p}_+) \subseteq N_+$. The main idea of the argument is from Harish-Chandra's proof of his embedding theorem of Hermitian symmetric spaces as bounded symmetric domains in complex Euclidean spaces. See pages 94 – 97 in [29] for details about Harish-Chandra's proof.

Our next step is to prove the boundedness of the image of $\widetilde{\Phi}(\check{\mathcal{T}})$ in N_+ by showing the finiteness of the map

$$P_+|_{\widetilde{\Phi}(\check{\mathcal{T}})} : \widetilde{\Phi}(\check{\mathcal{T}}) \to \exp(\mathfrak{p}_+) \cap D$$

which is the restriction of the projection P_+ to $\widetilde{\Phi}(\check{\mathcal{T}})$. Here the geometry of the period map and period domain, especially the Griffiths transversality and the completion of the base manifold S, plays very crucial role in our argument.

Finally we can apply the Riemann extension theorem to conclude the first main theorem of our paper, Theorem 1′, which implies the Griffiths conjecture.

Theorem 1′. *The image of the period map* $\widetilde{\Phi} : \mathcal{T} \to D$ *lies in* $N_+ \cap D$ *as a bounded subset with* N_+ *as the complex Euclidean space.*

As an application of the above result and our method, we will prove that there exist affine structures on the Teichmüller spaces of a large class of projective manifolds.

More precisely we consider the moduli space of certain polarized manifolds with level m structure, and let $\mathcal{Z}_m$ be one of the irreducible components of the moduli space. Then the Teichmüller space $\mathcal{T}$ is defined as the universal cover of $\mathcal{Z}_m$. Let m_0 be a positive integer. As a technical assumption we will require that $\mathcal{Z}_m$ is smooth and carries an analytic family

$$f_m : \mathcal{U}_m \to \mathcal{Z}_m$$

of polarized manifolds with level m structure for all $m \geq m_0$. It is easy to show that $\mathcal{T}$ is independent of the levels as given in Lemma 5.2. Furthermore we introduce the notion of strong local Torelli in Definition 6.3. Our second main result is as follows.

Theorem 2. *Assume that* $\mathcal{Z}_m$ *is smooth with an analytic family for all* $m \geq m_0$*, and strong local Torelli holds on the Teichmüller space* $\mathcal{T}$*. Then there exists a complex affine structure on* $\mathcal{T}$*.*

Applications of affine structures on the Teichmüller and Torelli spaces can be found in [24] and [25].

The idea of our proof of Theorem 2 goes as follows. Let $\mathfrak{a} \subseteq \mathfrak{n}_+$ be the abelian subalgebra of $\mathfrak{n}_+$ defined by the tangent map of the period map,

$$\mathfrak{a} = d\widetilde{\Phi}_p(\mathrm{T}^{1,0}_p\mathcal{T})$$

where p is a base point in $\mathcal{T}$ with $\widetilde{\Phi}(p) = o$, and let

$$A = \exp(\mathfrak{a}) \subseteq N_+.$$

Then A can be considered as a complex Euclidean subspace of N_+. Let

$$P : \ N_+ \cap D \to A \cap D$$

be the projection map, which induces the holomorphic map

$$\Psi : \mathcal{T} \to A \cap D$$

with $\Psi = P \circ \widetilde{\Phi}$. We will prove that under the assumption in Theorem 2, the holomorphic map Ψ is an immersion into A which induces the affine structure on $\mathcal{T}$.

This paper is organized as follows. We review the basics of period domain from Lie group point of view and introduce the open subset $\check{\mathcal{T}}$ in $\mathcal{T}$ in Section 1. Also proved in Section 1 are several lemmas about the geometry of the Griffiths completion S' of S and its universal cover $\mathcal{T}'$, which are crucial to our arguments. In Section 2 we review some basic facts of period domain, in particular the existence and basic property of strongly orthogonal noncompact positive roots and the corresponding maximal abelian subspace as given in Lemma 2.3, from Lie algebra point of view.

In Section 3, we first prove in Lemma 3.1 the boundedness of the period map on $\check{\mathcal{T}}$ composed with the projection to the subspace $\exp(\mathfrak{p}_+) \cap D$, by using the method of Harish-Chandra. In fact we will prove that

$$\exp(\mathfrak{p}_+) \cap D \subset \exp(\mathfrak{p}_+)$$

is a bounded domain in the complex Euclidean space $\exp(\mathfrak{p}_+)$.

Then in Theorem 3.6, we prove the boundedness of the period map on $\check{\mathcal{T}}$ by using the geometric structures of the period domain D, among which the most important property is the Griffiths transversality of the period map on $\mathcal{T}'$, and the Griffiths completion of the base manifold S. In Section 4, our first main result is proved as Theorem 4.1 by using the Riemann extension theorem.

In Section 5 we apply our results to the moduli space $\mathcal{Z}_m$ with level m structure and its universal cover $\mathcal{T}$ which we call the Teichmüller space. Assuming $\mathcal{Z}_m$ is smooth for $m \geq m_0$ for some positive integer m_0, and strong local Torelli holds for $\mathcal{T}$, we prove in Section 6 that there exists an affine structure on the Teichmüller space $\mathcal{T}$ induced by a holomorphic immersion $\Psi : \mathcal{T} \to A \cap D$.

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1 Period domains and Lie groups

In this section we review the definitions and basic properties of period domains and period maps from Lie theory point of views. We consider the nilpotent Lie subalgebra $\mathfrak{n}_+$ and define the corresponding unipotent group to be $N_+ = \exp(\mathfrak{n}_+)$. Since $N_+ \cap B = \{\mathrm{Id}\}$, we can identify the unipotent group $N_+ \subseteq G_{\mathbb{C}}$ to its orbit $N_+(o) \subseteq \check{D}$ of a base point $o \in D$. Then we define

$$\check{\mathcal{T}} = \widetilde{\Phi}^{-1}(N_+ \cap D)$$

and show that $\mathcal{T} \backslash \check{\mathcal{T}}$ is an analytic subvariety of $\mathcal{T}$ with $\mathrm{codim}_{\mathbb{C}}(\mathcal{T} \backslash \check{\mathcal{T}}) \geq 1$.

Let $H_{\mathbb{Z}}$ be a fixed lattice and $H = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ the complexification. Let n be a positive integer, and Q a bilinear form on $H_{\mathbb{Z}}$ which is symmetric if n is even and skew-symmetric if n is odd. Let $h^{i,n-i}$, $0 \leq i \leq n$, be integers such that $\sum_{i=0}^{n} h^{i,n-i} = \dim_{\mathbb{C}} H$. The period domain D for the polarized Hodge structures of type

$$\{H_{\mathbb{Z}}, Q, h^{i,n-i}\}$$

is the set of all the collections of the subspaces $H^{i,n-i}$, $0 \leq i \leq n$, of H such that

$$H = \bigoplus_{0 \leq i \leq n} H^{i,n-i},\ H^{i,n-i} = \overline{H^{n-i,i}},\ \dim_{\mathbb{C}} H^{i,n-i} = h^{i,n-i} \text{ for } 0 \leq i \leq n,$$

and on which Q satisfies the Hodge-Riemann bilinear relations,

$$Q\left(H^{i,n-i}, H^{j,n-j}\right) = 0 \text{ unless } i + j = n; \tag{2}$$

$$\left(\sqrt{-1}\right)^{2k-n} Q\left(v, \overline{v}\right) > 0 \text{ for } v \in H^{k,n-k} \setminus \{0\}. \tag{3}$$

Alternatively, in terms of Hodge filtrations, the period domain D is the set of all the collections of the filtrations

$$H = F^0 \supset F^1 \supset \cdots \supset F^n,$$

such that

$$\dim_{\mathbb{C}} F^i = f^i, \tag{4}$$

$$H = F^i \oplus \overline{F^{n-i+1}},\ \text{ for } 0 \leq i \leq n,$$

where $f^i = h^{n,0} + \cdots + h^{i,n-i}$, and on which Q satisfies the Hodge-Riemann bilinear relations in the form of Hodge filtrations

$$Q\left(F^i, F^{n-i+1}\right) = 0; \tag{5}$$

$$Q\left(Cv, \overline{v}\right) > 0 \text{ if } v \neq 0, \tag{6}$$

where C is the Weil operator given by

$$Cv = \left(\sqrt{-1}\right)^{2k-n} v$$

for $v \in F^k \cap \overline{F^{n-k}}$.

Let (X, L) be a polarized manifold with $\dim_{\mathbb{C}} X = n$, which means that X is a projective manifold and L is an ample line bundle on X. For simplicity we use the same notation L to denote the first Chern class of L which acts on the cohomology groups by wedge product. Then the n-th primitive cohomology groups $H^n_{pr}(X, \mathbb{C})$ of X is defined by

$$H^n_{pr}(X, \mathbb{C}) = \ker\{L : \, H^n(X, \mathbb{C}) \to H^{n+2}(X, \mathbb{C})\}.$$

Let $\Phi : \, S \to D/\Gamma$ be the period map from geometry. More precisely we have an algebraic family

$$f : \, \mathfrak{X} \to S$$

of polarized algebraic manifolds over a quasi-projective manifold S, such that for any $q \in S$, the point $\Phi(q)$, modulo certain action of the monodromy group Γ, represents the Hodge structure of the n-th primitive cohomology group $H^n_{pr}(X_q, \mathbb{C})$ of the fiber $X_q = f^{-1}(q)$. Here $H \simeq H^n_{pr}(X_q, \mathbb{C})$ for any $q \in S$.

Recall that the monodromy group Γ, or global monodromy group, is the image of the representation of $\pi_1(S)$ in $\mathrm{Aut}(H_{\mathbb{Z}}, Q)$, the group of automorphisms of $H_{\mathbb{Z}}$ preserving Q.

By taking a finite index torsion-free subgroup of Γ, we can assume that Γ is torsion-free, therefore D/Γ is smooth. This way we can just proceed on a finite cover of S without loss of generality. We refer the reader to the proof of Lemma IV-A, pages 705 – 706 in [38] for such standard construction.

Since period map is locally liftable, we can lift the period map to $\widetilde{\Phi}$:

$\mathcal{T} \to D$ by taking the universal cover $\mathcal{T}$ of S such that the diagram

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\widetilde{\Phi}} & D \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ S & \xrightarrow{\Phi} & D/\Gamma \end{array} \tag{7}$$

is commutative.

Now we fix a point p in $\mathcal{T}$ and its image $o = \widetilde{\Phi}(p)$ as the reference points or base points, and denote the Hodge decomposition corresponding to the point $o = \widetilde{\Phi}(p)$ as

$$H^n_{pr}(X_p, \mathbb{C}) = H^{n,0}_p \oplus H^{n-1,1}_p \oplus \cdots \oplus H^{1,n-1}_p \oplus H^{0,n}_p,$$

where $H^{i,n-i}_p = H^{i,n-i}_{pr}(X_p)$ for $0 \leq i \leq n$, and the Hodge filtration by

$$H^n_{pr}(X_p, \mathbb{C}) = F^0_p \supset F^1_p \supset \cdots \supset F^n_p$$

with $F^i_p = H^{n,0}_p \oplus \cdots \oplus H^{i,n-i}_p$ for $0 \leq i \leq n$.

Let us introduce the notion of adapted basis for the given Hodge decomposition or Hodge filtration. We call a basis

$$\xi = \left\{\xi_0, \cdots, \xi_{f^n-1}, \xi_{f^n}, \cdots, \xi_{f^{n-1}-1} \cdots, \xi_{f^{k+1}}, \cdots, \xi_{f^k-1}, \cdots, \xi_{f^1}, \cdots, \xi_{f^0-1}\right\}$$

of $H^n_{pr}(X_p, \mathbb{C})$ an adapted basis for the given Hodge decomposition if it satisfies

$$H^{k,n-k}_p = \mathrm{Span}_{\mathbb{C}} \left\{\xi_{f^{k+1}}, \cdots, \xi_{f^k-1}\right\}.$$

We call a basis

$$\zeta = \left\{\zeta_0, \cdots, \zeta_{f^n-1}, \zeta_{f^n}, \cdots, \zeta_{f^{n-1}-1} \cdots, \zeta_{f^{k+1}}, \cdots, \zeta_{f^k-1}, \cdots, \zeta_{f^1}, \cdots, \zeta_{f^0-1}\right\}$$

of $H^n_{pr}(X_p, \mathbb{C})$ an adapted basis for the given filtration if it satisfies $F^k_p = \mathrm{Span}_{\mathbb{C}}\{\zeta_0, \cdots, \zeta_{f^k-1}\}$. For convenience, we set $f^{n+1} = 0$ and $m = f^0$.

The blocks of an $m \times m$ matrix T are set as follows. For each $0 \leq \alpha, \beta \leq n$, the (α, β)-th block $T^{\alpha,\beta}$ denotes

$$T^{\alpha,\beta} = (T_{ij})_{f^{-\alpha+n+1} \leq i \leq f^{-\alpha+n}-1,\ f^{-\beta+n+1} \leq j \leq f^{-\beta+n}-1}, \tag{8}$$

where T_{ij} are the entries of the matrix T. In particular, $T = [T^{\alpha,\beta}]_{0\leq\alpha,\beta\leq n}$ is called a block lower triangular matrix if $T^{\alpha,\beta} = 0$ whenever $\alpha < \beta$.

Let $H_{\mathbb{F}} = H^n_{pr}(X, \mathbb{F})$, where $\mathbb{F}$ can be chosen as $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$. Then $H = H_{\mathbb{C}}$ under this notation. We define the complex Lie group

$$G_{\mathbb{C}} = \{g \in GL(H_{\mathbb{C}})|\ Q(gu, gv) = Q(u, v) \text{ for all } u, v \in H_{\mathbb{C}}\},$$

and the real one

$$G_{\mathbb{R}} = \{g \in GL(H_{\mathbb{R}})|\ Q(gu, gv) = Q(u, v) \text{ for all } u, v \in H_{\mathbb{R}}\}.$$

We also have

$$G_{\mathbb{Z}} = \mathrm{Aut}(H_{\mathbb{Z}}, Q) = \{g \in GL(H_{\mathbb{Z}})|\ Q(gu, gv) = Q(u, v) \text{ for all } u, v \in H_{\mathbb{Z}}\}.$$

Griffiths in [9] showed that $G_{\mathbb{C}}$ acts on $\check{D}$ transitively, so does $G_{\mathbb{R}}$ on D. The stabilizer of $G_{\mathbb{C}}$ on $\check{D}$ at the base point o is

$$B = \{g \in G_{\mathbb{C}}| gF^k_p = F^k_p,\ 0 \leq k \leq n\},$$

and the one of $G_{\mathbb{R}}$ on D is $V = B \cap G_{\mathbb{R}}$. Thus we can realize $\check{D}$, D as

$$\check{D} = G_{\mathbb{C}}/B, \text{ and } D = G_{\mathbb{R}}/V$$

so that $\check{D}$ is an algebraic manifold and $D \subseteq \check{D}$ is an open complex submanifold.

The Lie algebra $\mathfrak{g}$ of the complex Lie group $G_{\mathbb{C}}$ is

$$\mathfrak{g} = \{X \in \mathrm{End}(H_{\mathbb{C}})|\ Q(Xu, v) + Q(u, Xv) = 0, \text{ for all } u, v \in H_{\mathbb{C}}\},$$

and the real subalgebra

$$\mathfrak{g}_0 = \{X \in \mathfrak{g}|\ XH_{\mathbb{R}} \subseteq H_{\mathbb{R}}\}$$

is the Lie algebra of $G_{\mathbb{R}}$. Note that $\mathfrak{g}$ is a simple complex Lie algebra and contains $\mathfrak{g}_0$ as a real form, i.e. $\mathfrak{g} = \mathfrak{g}_0 \oplus \sqrt{-1}\mathfrak{g}_0$.

On the linear space $\mathrm{Hom}(H_{\mathbb{C}}, H_{\mathbb{C}})$ we can give a Hodge structure of weight zero by

$$\mathfrak{g} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}^{k,-k} \quad \text{with} \quad \mathfrak{g}^{k,-k} = \{X \in \mathfrak{g}|\ XH^{r,n-r}_p \subseteq H^{r+k,n-r-k}_p,\ \forall r\}.$$

By the definition of B, the Lie algebra $\mathfrak{b}$ of B has the form $\mathfrak{b} = \bigoplus_{k\geq 0} \mathfrak{g}^{k,-k}$. Then the Lie algebra $\mathfrak{v}_0$ of V is

$$\mathfrak{v}_0 = \mathfrak{g}_0 \cap \mathfrak{b} = \mathfrak{g}_0 \cap \mathfrak{b} \cap \overline{\mathfrak{b}} = \mathfrak{g}_0 \cap \mathfrak{g}^{0,0}.$$

With the above isomorphisms, the holomorphic tangent space of $\check{D}$ at the base point is naturally isomorphic to $\mathfrak{g}/\mathfrak{b}$.

Let us consider the nilpotent Lie subalgebra $\mathfrak{n}_+ := \oplus_{k\geq 1}\mathfrak{g}^{-k,k}$. Then one gets the isomorphism $\mathfrak{g}/\mathfrak{b} \cong \mathfrak{n}_+$. We denote the corresponding unipotent Lie group to be

$$N_+ = \exp(\mathfrak{n}_+).$$

As $\mathrm{Ad}(g)(\mathfrak{g}^{k,-k})$ is in $\bigoplus_{i\geq k} \mathfrak{g}^{i,-i}$ for each $g \in B$, the subspace $\mathfrak{b}\oplus\mathfrak{g}^{-1,1}/\mathfrak{b} \subseteq \mathfrak{g}/\mathfrak{b}$ defines an $\mathrm{Ad}(B)$-invariant subspace. By left translation via $G_{\mathbb{C}}$, $\mathfrak{b} \oplus \mathfrak{g}^{-1,1}/\mathfrak{b}$ gives rise to a $G_{\mathbb{C}}$-invariant holomorphic subbundle of the holomorphic tangent bundle. It will be denoted by $\mathrm{T}^{1,0}_h\check{D}$, and will be referred to as the horizontal tangent subbundle. One can check that this construction does not depend on the choice of the base point.

The horizontal tangent subbundle, restricted to D, determines a subbundle $\mathrm{T}^{1,0}_hD$ of the holomorphic tangent bundle $\mathrm{T}^{1,0}D$ of D. The $G_{\mathbb{C}}$-invariance of $\mathrm{T}^{1,0}_h\check{D}$ implies the $G_{\mathbb{R}}$-invariance of $\mathrm{T}^{1,0}_hD$. Note that the horizontal tangent subbundle $\mathrm{T}^{1,0}_hD$ can also be constructed as the associated bundle of the principle bundle $V \to G_{\mathbb{R}} \to D$ with the adjoint representation of V on the space $\mathfrak{b} \oplus \mathfrak{g}^{-1,1}/\mathfrak{b}$.

Let $\mathscr{F}^k$, $0 \leq k \leq n$ be the Hodge bundles on D with fibers $\mathscr{F}^k_s = F^k_s$ for any $s \in D$. As another interpretation of the horizontal bundle in terms of the Hodge bundles $\mathscr{F}^k \to \check{D}$, $0 \leq k \leq n$, one has

$$\mathrm{T}^{1,0}_h\check{D} \simeq \mathrm{T}^{1,0}\check{D} \cap \bigoplus_{k=1}^{n} \mathrm{Hom}(\mathscr{F}^k/\mathscr{F}^{k+1}, \mathscr{F}^{k-1}/\mathscr{F}^k). \tag{9}$$

As in [33], a holomorphic map $\Psi : M \to \check{D}$ of a complex manifold M into $\check{D}$ is called horizontal, if the tangent map $d\Psi : \mathrm{T}^{1,0}M \to \mathrm{T}^{1,0}\check{D}$ takes values in $\mathrm{T}^{1,0}_h\check{D}$. The period map $\widetilde{\Phi} : \mathcal{T} \to D$ is horizontal due to Griffiths transversality. See [10] for more details.

Remark 1.1. We remark that the elements in N_+ can be realized as nonsingular block lower triangular matrices with identity blocks in the diagonal; the elements in B can be realized as nonsingular block upper triangular matrices. If $c, c' \in N_+$ such that $cB = c'B$ in $\check{D}$, then

$$c'^{-1}c \in N_+ \cap B = \{I\},$$

i.e. $c = c'$. This means that the matrix representation in N_+ of the unipotent orbit $N_+(o)$ is unique. Therefore with the fixed base point $o \in \check{D}$, we can identify N_+ with its unipotent orbit $N_+(o)$ in $\check{D}$ by identifying an element $c \in N_+$ with $[c] = cB$ in $\check{D}$. Therefore our notation $N_+ \subseteq \check{D}$ is meaningful. In particular, when the base point o is in D, we have $N_+ \cap D \subseteq D$.

Now we define

$$\check{\mathcal{T}} = \widetilde{\Phi}^{-1}(N_+ \cap D).$$

We first prove that $\mathcal{T} \backslash \check{\mathcal{T}}$ is an analytic subvariety of $\mathcal{T}$ with $\operatorname{codim}_{\mathbb{C}}(\mathcal{T} \backslash \check{\mathcal{T}}) \geq 1$.

Lemma 1.2. *Let $p \in \mathcal{T}$ be the base point with $\widetilde{\Phi}(p) = \{F_p^n \subseteq F_p^{n-1} \subseteq \cdots \subseteq F_p^0\}$. Let $q \in \mathcal{T}$ be any point with $\widetilde{\Phi}(q) = \{F_q^n \subseteq F_q^{n-1} \subseteq \cdots \subseteq F_q^0\}$, then $\widetilde{\Phi}(q) \in N_+$ if and only if F_q^k is isomorphic to F_p^k for all $0 \leq k \leq n$.*

Proof. For any $q \in \mathcal{T}$, we choose an arbitrary adapted basis $\{\zeta_0, \cdots, \zeta_{m-1}\}$ for the given Hodge filtration $\{F_q^n \subseteq F_q^{n-1} \subseteq \cdots \subseteq F_q^0\}$. We fix $\{\eta_0, \cdots, \eta_{m-1}\}$ as the adapted basis for the Hodge filtration $\{F_p^n \subseteq F_p^{n-1} \subseteq \cdots \subseteq F_p^0\}$ at the base point p. Let $[A^{i,j}(q)]_{0 \leq i,j \leq n}$ be the transition matrix between the basis $\{\eta_0, \cdots, \eta_{m-1}\}$ and $\{\zeta_0, \cdots, \zeta_{m-1}\}$ for the same vector space $H_{\mathbb{C}}$, where $A^{i,j}(q)$ are the corresponding blocks. Then

$$\widetilde{\Phi}(q) \in N_+ = N_+B/B \subseteq \check{D}$$

if and only if its matrix representation $[A^{i,j}(q)]_{0 \leq i,j \leq n}$ can be decomposed as $L(q) \cdot U(q)$, where $L(q)$ is a nonsingular block lower triangular matrix with identities in the diagonal blocks, and $U(q)$ is a nonsingular block upper triangular matrix.

By basic linear algebra, we know that $[A^{i,j}(q)]$ has such decomposition if and only if $\det[A^{i,j}(q)]_{0 \leq i,j \leq k} \neq 0$ for any $0 \leq k \leq n$. In particular, we know that $[A(q)^{i,j}]_{0 \leq i,j \leq k}$ is the transition map between the bases of F_p^k and F_q^k. Therefore, $\det([A(q)^{i,j}]_{0 \leq i,j \leq k}) \neq 0$ if and only if F_q^k is isomorphic to F_p^k. □

Proposition 1.3. *The subset $\check{\mathcal{T}}$ is an open complex submanifold in $\mathcal{T}$, and $\mathcal{T}\backslash\check{\mathcal{T}}$ is an analytic subvariety of $\mathcal{T}$ with $\mathrm{codim}_{\mathbb{C}}(\mathcal{T}\backslash\check{\mathcal{T}}) \geq 1$.*

Proof. From Lemma 1.2, one can see that $\check{D} \setminus N_+ \subseteq \check{D}$ is defined as an analytic subvariety by the equations

$$\{q \in \check{D} : \det((A^{i,j}(q))_{0\leq i,j\leq k}) = 0 \text{ for some } 0 \leq k \leq n\}.$$

Therefore N_+ is dense in $\check{D}$, and that $\check{D} \setminus N_+$ is an analytic subvariety, which is closed in $\check{D}$ and with $\mathrm{codim}_{\mathbb{C}}(\check{D}\backslash N_+) \geq 1$.

We consider the period map $\widetilde{\Phi} : \mathcal{T} \to \check{D}$ as a holomorphic map to $\check{D}$, then

$$\mathcal{T} \setminus \check{\mathcal{T}} = \widetilde{\Phi}^{-1}(\check{D} \setminus N_+)$$

is the preimage of $\check{D} \setminus N_+$ of the holomorphic map $\widetilde{\Phi}$. Therefore $\mathcal{T} \setminus \check{\mathcal{T}}$ is also an analytic subvariety and a closed set in $\mathcal{T}$. Because $\mathcal{T}$ is smooth and connected, $\mathcal{T}$ is irreducible. If $\dim_{\mathbb{C}}(\mathcal{T} \setminus \check{\mathcal{T}}) = \dim_{\mathbb{C}} \mathcal{T}$, then $\mathcal{T} \setminus \check{\mathcal{T}} = \mathcal{T}$ and $\check{\mathcal{T}} = \emptyset$, but this contradicts to the fact that the reference point p is in $\check{\mathcal{T}}$. Thus we conclude that $\dim_{\mathbb{C}}(\mathcal{T} \setminus \check{\mathcal{T}}) < \dim_{\mathbb{C}} \mathcal{T}$, and consequently $\mathrm{codim}_{\mathbb{C}}(\mathcal{T}\backslash\check{\mathcal{T}}) \geq 1$. □

We come back to the setup in [12]. Since S is quasi-projective, by Hironaka's resolution of singularity theorem, S admits a compactification $\overline{S}$ such that $\overline{S}$ is a compact smooth algebraic variety, and $\overline{S} \setminus S$ is a divisor with simple normal crossings. Let $S' \supseteq S$ be the subset of $\overline{S}$ to which the period map $\Phi : S \to D/\Gamma$ extends continuously and let $\Phi' : S' \to D/\Gamma$ be the extended map. Then one has the commutative diagram

$$S \xrightarrow{\;i\;} S' \xrightarrow{\;\Phi'\;} D/\Gamma, \qquad \Phi = \Phi' \circ i : S \to D/\Gamma.$$

with $i : S \to S'$ the inclusion map.

Lemma 1.4. *S' is an open complex submanifold of $\overline{S}$ and the complex codimension of $\overline{S} \setminus S'$ is at least one.*

Proof. As a standard procedure, after going to a finite cover we may assume the monodromy group Γ is torsion-free, as given in Lemma IV-A in [38]. We will give two proofs of the lemma.

First we can use Theorem 9.6 of Griffiths in [11], see also Corollary 13.4.6 in [2], to get the Zariski open submanifold S'' of $\overline{S}$, where S'' contains all points of finite monodromy in $\overline{S}$, and hence of trivial monodromy in $\overline{S}$, since Γ is torsion-free. Then the extended period map

$$\Phi'' : S'' \to D/\Gamma$$

is a proper holomorphic map.

Indeed, as proved in Theorem 3.1 of [41], which follows directly from Propositions 9.10 and 9.11 of [11], $\overline{S}\backslash S''$ consists of those components of the simple normal crossing divisors in $\overline{S}\backslash S$ around which the orders of the monodromy are infinite, therefore S'' is a Zariski open submanifold in $\overline{S}$. For a more detailed discussion about this, see also pages 705 – 706 in [38]. Since S'' is Zariski open in $\overline{S}$, we only need to prove that $S' = S''$.

In fact, by the definition of S', we see that $S'' \subseteq S'$. Conversely, for any point $q \in S'$ with image $u = \Phi'(q) \in D/\Gamma$, we can choose the points $q_k \in S$, $k = 1, 2, \cdots$ such that $q_k \longrightarrow q$ with images $u_k = \Phi(q_k) \longrightarrow u$ as $k \longrightarrow \infty$. Since $\Phi'' : S'' \to D/\Gamma$ is proper, the sequence

$$\{q_k\}_{k=1}^{\infty} \subset (\Phi'')^{-1}(\{u_k\}_{k=1}^{\infty})$$

has a limit point q in S'', therefore $q \in S''$ and $S' \subseteq S''$. From this we see that $S' = S''$ is an open complex submanifold of $\overline{S}$ with $\mathrm{codim}_{\mathbb{C}}(\overline{S} \setminus S') \geq 1$.

For the second proof, note that $\overline{S}$ is smooth and $S \subseteq \overline{S}$ is Zariski open, we only need to show that S' is open in $\overline{S}$. To prove this we use the compactification space $\overline{D/\Gamma}$. There are several natural notions of the compactification space $\overline{D/\Gamma}$, see [20], or the exposition by Griffiths in [7], or discussions in [6, Page 2], [6, Page 29, 30], in which it is proved that the period map has continuous, even holomorphic extension. We can choose any one of them together with the continuous extension of the period map as

$$\overline{\Phi} : \overline{S} \to \overline{D/\Gamma}.$$

By the definition of S', $S' = \overline{\Phi}^{-1}(D/\Gamma)$. Since D/Γ is open and dense in the compacification $\overline{D/\Gamma}$, S' is therefore an open and dense submanifold of $\overline{S}$. □

Note that, since Γ is torsion-free, the extended period map

$$\Phi' : S' \to D/\Gamma$$

is still locally liftable. Let $\mathcal{T}'$ be the universal cover of S' with the covering map $\pi' : \mathcal{T}' \to S'$. We then have the following commutative diagram

$$\begin{array}{ccccc} \mathcal{T} & \xrightarrow{i_{\mathcal{T}}} & \mathcal{T}' & \xrightarrow{\widetilde{\Phi}'} & D \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi_D} \\ S & \xrightarrow{i} & S' & \xrightarrow{\Phi'} & D/\Gamma, \end{array} \tag{10}$$

where $i_{\mathcal{T}}$ is the lifting of $i \circ \pi$ with respect to the covering map $\pi' : \mathcal{T}' \to S'$ and $\widetilde{\Phi}'$ is the lifting of $\Phi' \circ \pi'$ with respect to the covering map $\pi_D : D \to D/\Gamma$. Then $\widetilde{\Phi}'$ is continuous. There are different choices of $i_{\mathcal{T}}$ and $\widetilde{\Phi}'$, but Lemma A.1 in the Appendix shows that we can choose $i_{\mathcal{T}}$ and $\widetilde{\Phi}'$ such that

$$\widetilde{\Phi} = \widetilde{\Phi}' \circ i_{\mathcal{T}}.$$

Let $\mathcal{T}_0 \subseteq \mathcal{T}'$ be defined by $\mathcal{T}_0 = i_{\mathcal{T}}(\mathcal{T})$, then we have the following lemma.

Lemma 1.5. *$\mathcal{T}_0 = \pi'^{-1}(S)$, and $i_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}_0$ is a covering map.*

Proof. The proof is an elementary argument in basic topology. First, from diagram (10), we see that

$$\pi'(\mathcal{T}_0) = \pi'(i_{\mathcal{T}}(\mathcal{T})) = i(\pi(\mathcal{T})) = S,$$

hence $\mathcal{T}_0 \subseteq \pi'^{-1}(S)$.

Conversely, for any $q \in \pi'^{-1}(S)$, we need to prove that $q \in \mathcal{T}_0$. Let $p = \pi'(q) \in S$. If there exists $r \in \pi^{-1}(p)$ such that $i_{\mathcal{T}}(r) = q$, then we are done. Otherwise, we can draw a curve γ from $i_{\mathcal{T}}(r)$ to q for some $r \in \pi^{-1}(p)$, as $\mathcal{T}'$ is connected and thus path connected. Then we get a circle $\beta = \pi'(\gamma)$ in S'. But Lemma A.2 in the Appendix implies that we can choose the circle β contained in S. Note that $p \in \beta$.

Since $\pi : \mathcal{T} \to S$ is covering map, we can lift β to a unique curve $\widetilde{\gamma}$ from r to some $r' \in \pi^{-1}(p)$. Notice that both γ and $i_{\mathcal{T}}(\widetilde{\gamma})$ map to β via the the covering map $\pi' : \mathcal{T}' \to S'$, that is, both γ and $i_{\mathcal{T}}(\widetilde{\gamma})$ are the lifts of β

starting from the same point $i_{\mathcal{T}}(r)$. By the uniqueness of homotopy lifting, $i_{\mathcal{T}}(r') = q$, i.e. $q \in i_{\mathcal{T}}(\mathcal{T}) = \mathcal{T}_0$.

To show that $i_{\mathcal{T}}$ is a covering map, note that for any small enough open neighborhood U in $\mathcal{T}_0$, the restricted map

$$\pi'|_U : U \to V = \pi'(U) \subset S$$

is biholomorphic, and there exists a disjoint union $\cup_i V_i$ of open subsets in $\mathcal{T}$ such that $\cup_i V_i = (\pi)^{-1}(V)$ and $\pi|_{V_i} : V_i \to V$ is biholomorphic. Then from the commutativity of diagram (10), we have that

$$\cup_i V_i = (i_{\mathcal{T}})^{-1}(U)$$

and $i_{\mathcal{T}}|_{V_i} : V_i \to U$ is biholomorphic. Therefore $i_{\mathcal{T}} : \mathcal{T} \to \mathcal{T}_0$ is a covering map.

□

We remark that as the lifts of holomorphic maps i and Φ' to the corresponding universal covers, both $i_{\mathcal{T}}$ and $\widetilde{\Phi}'$ are holomorphic maps. This fact can also be directly proved by using Theorem 9.6 in [11] and the Riemann extension theorem.

Actually we have proved that $i_{\mathcal{T}}$ is a covering map of complex manifolds, so it is holomorphic. Lemma 1.5 implies that $\mathcal{T}_0$ is an open complex submanifold of $\mathcal{T}'$ and $\mathrm{codim}_{\mathbb{C}}(\mathcal{T}' \setminus \mathcal{T}_0) \geq 1$. Since $\widetilde{\Phi} = \widetilde{\Phi}' \circ i_{\mathcal{T}}$ is holomorphic,

$$\widetilde{\Phi}'_0 = \widetilde{\Phi}'|_{\mathcal{T}_0} : \mathcal{T}_0 \to D$$

is also holomorphic. Since $\widetilde{\Phi}'_0$ has a continuous extension $\widetilde{\Phi}'$, we can apply the Riemann extension theorem for which we only need to show that $\mathcal{T}' \setminus \mathcal{T}_0$ is an analytic subvariety of $\mathcal{T}'$. In fact, since $\overline{S} \setminus S$ is a union of simple normal crossing divisors, from Lemma 1.4, we see that the subset $S' \setminus S$ consists of those divisors in S' around which the monodromy group is trivial. Therefore $S' \setminus S$ is an analytic subvariety of S'.

On the other hand, from Lemma 1.5, we know that $\mathcal{T}' \setminus \mathcal{T}_0$ is the inverse image of $S' \setminus S$ under the covering map $\pi' : \mathcal{T}' \to S'$, and this implies that $\mathcal{T}' \setminus \mathcal{T}_0$ is an analytic subvariety of $\mathcal{T}'$.

Note that the fact that $S' \setminus S$, therefore $\mathcal{T}' \setminus \mathcal{T}_0$, is analytic subvariety is contained in Theorem 9.6 in [11]. We refer the readers to page 156 of [11] for other related discussions.

Lemma 1.6. *The extended holomorphic map* $\widetilde{\Phi}' : \mathcal{T}' \to D$ *satisfies the Griffiths transversality.*

Proof. Let $\mathrm{T}^{1,0}_h D$ be the horizontal subbundle. Since $\widetilde{\Phi}' : \mathcal{T}' \to D$ is a holomorphic map, the tangent map

$$d\widetilde{\Phi}' : \mathrm{T}^{1,0}\mathcal{T}' \to \mathrm{T}^{1,0}D$$

is continuous. We only need to show that the image of $d\widetilde{\Phi}'$ is contained in the horizontal tangent bundle $\mathrm{T}^{1,0}_h D$.

Since $\mathrm{T}^{1,0}_h D$ is closed in $\mathrm{T}^{1,0}D$, the set $(d\widetilde{\Phi}')^{-1}(\mathrm{T}^{1,0}_h D)$ is closed in $\mathrm{T}^{1,0}\mathcal{T}'$. But $\widetilde{\Phi}'|_{\mathcal{T}_0}$ satisfies the Griffiths transversality, i.e. $(d\widetilde{\Phi}')^{-1}(\mathrm{T}^{1,0}_h D)$ contains $\mathrm{T}^{1,0}\mathcal{T}_0$, which is open in $\mathrm{T}^{1,0}\mathcal{T}'$. Hence $(d\widetilde{\Phi}')^{-1}(\mathrm{T}^{1,0}_h D)$ contains the closure of $\mathrm{T}^{1,0}\mathcal{T}_0$, which is $\mathrm{T}^{1,0}\mathcal{T}'$. □

Finally we remark that from the result in page 171 of the book of Grauert-Remmert [5], we know that as analytic varieties, S, S', $\mathcal{T}$, $\mathcal{T}'$ and $\check{\mathcal{T}}$ are all irreducible.

2 Period domains and Lie algebras

In this section, we describe the structure of the Lie algebra $\mathfrak{g}$ by using root system. Lemma 2.3 about existence of strongly orthogonal noncompact positive roots is particularly important to our proof of Theorem 3.1. Most results in this section are from [14] and [33] to which the reader can refer for detailed proofs.

Now we define the Weil operator $\theta : \mathfrak{g} \to \mathfrak{g}$ by

$$\theta(X) = (-1)^k X, \text{ for } X \in \mathfrak{g}^{k,-k}.$$

Then θ is an involutive automorphism of $\mathfrak{g}$, and defined over $\mathbb{R}$. Let $\mathfrak{k}$ and $\mathfrak{p}$ be the $(+1)$ and (-1) eigenspaces of θ respectively. Considering the types, we have

$$\mathfrak{k} = \oplus_{k \text{ even}} \mathfrak{g}^{k,-k}, \ \mathfrak{p} = \oplus_{k \text{ odd}} \mathfrak{g}^{k,-k}.$$

Set

$$\mathfrak{k}_0 = \mathfrak{k} \cap \mathfrak{g}_0, \quad \mathfrak{p}_0 = \mathfrak{p} \cap \mathfrak{g}_0.$$

Then we have the decompositions

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad \mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$$

with the property that

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}.$$

Let $\mathfrak{g}_c = \mathfrak{k}_0 \oplus \sqrt{-1}\mathfrak{p}_0$. Then $\mathfrak{g}_c$ is also a real form of $\mathfrak{g}$. Let us denote the complex conjugation of $\mathfrak{g}$ with respect to the real form $\mathfrak{g}_c$ by τ_c, and the complex conjugation of $\mathfrak{g}$ with respect to the real form $\mathfrak{g}_0$ by τ_0.

Recall that on the complex linear space $H_{\mathbb{C}}$ we can define an Hermitian inner product $(\cdot, \cdot)$ induced by the Poincaré bilinear form Q as

$$(u, v) = Q(Cu, \overline{v}) \quad u, v \in H_{\mathbb{C}}, \tag{11}$$

where C is the Weil operator on $H_{\mathbb{C}}$ defined over $\mathbb{R}$. Thus C can be considered as an element in $G_{\mathbb{R}}$, whose adjoint action on $\mathfrak{g}$ is just θ. For any $Z = X + \sqrt{-1}Y \in \mathfrak{g}_c$, where $X \in \mathfrak{k}_0$ and $Y \in \mathfrak{p}_0$, one can easily check that

$$(Z \cdot u, v) = -(u, Z \cdot v), \forall\, u, v \in H_{\mathbb{C}}. \tag{12}$$

Hence $\mathfrak{g}_c$ is the intersection of $\mathfrak{g}$ with the algebra of all skew Hermitian transforms with respect to the Hermitian inner product $(\cdot, \cdot)$. As described by Schmid in [33], one has the following result,

- $\mathfrak{g}_c$ is a compact real form of $\mathfrak{g}$, and the Killing form B

  $$B(X, Y) = \mathrm{Trace}(\mathrm{ad}X \circ \mathrm{ad}Y), \quad X, Y \in \mathfrak{g}$$

  restricts to a negative definite bilinear form $B|_{\mathfrak{g}_c}$ on $\mathfrak{g}_c$. Moreover, one has an Hermitian inner product $-B(\theta\, \cdot, \bar{\cdot})$ on $\mathfrak{g}$, making $\mathfrak{g}$ an Hermitian complex linear space.

Following this result, we have that $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ is the Cartan decomposition.

Now we define the Lie subgroup $G_c \subseteq G_{\mathbb{C}}$ as a connected subgroup corresponding to $\mathfrak{g}_c \subseteq \mathfrak{g}$. Then (12) implies that the elements in G_c preserve the Hermitian inner product $(\cdot, \cdot)$, i.e. G_c is contained in the unitary subgroup of $G_{\mathbb{C}}$. As noted by Schmid, a compact real form in a connected complex semisimple Lie group is always connected and is its own normalizer, which implies that G_c is the unitary subgroup of $G_{\mathbb{C}}$, and hence compact.

The intersection

$$K = G_c \cap G_{\mathbb{R}}$$

is a compact subgroup of $G_{\mathbb{R}}$ with Lie algebra $\mathfrak{g}_c \cap \mathfrak{g}_0 = \mathfrak{k}_0$. The following statements are in pages 278 – 279 of [33],

- K is a maximal compact subgroup of $G_{\mathbb{R}}$ and it meets every connected component of $G_{\mathbb{R}}$.
- $G_c \cap B = V$, which implies $V \subseteq K$ and their Lie algebras $\mathfrak{v}_0 \subseteq \mathfrak{k}_0$.

In [14], Griffiths and Schmid observed the following result,

- There exists a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$ such that $\mathfrak{h}_0 \subseteq \mathfrak{v}_0 \subseteq \mathfrak{k}_0$ and $\mathfrak{h}_0$ is also a Cartan subalgebra of $\mathfrak{k}_0$;

Denote $\mathfrak{h}$ to be the complexification of $\mathfrak{h}_0$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ such that $\mathfrak{h} \subseteq \mathfrak{v} \subseteq \mathfrak{k}$.

Now we review the root systems we need. Write $\mathfrak{h}_0^* = \mathrm{Hom}(\mathfrak{h}_0, \mathbb{R})$ and $\mathfrak{h}_{\mathbb{R}}^* = \sqrt{-1}\mathfrak{h}_0^*$. Then $\mathfrak{h}_{\mathbb{R}}^*$ can be identified with $\mathfrak{h}_{\mathbb{R}} := \sqrt{-1}\mathfrak{h}_0$ by the restricted Killing form $B|_{\mathfrak{h}_{\mathbb{R}}}$ on $\mathfrak{h}_{\mathbb{R}}$. Let $\rho \in \mathfrak{h}_{\mathbb{R}}^* \simeq \mathfrak{h}_{\mathbb{R}}$, one can define the following subspace of $\mathfrak{g}$,

$$\mathfrak{g}^{\rho} = \{x \in \mathfrak{g} : [h, x] = \rho(h)x \text{ for all } h \in \mathfrak{h}\}.$$

An element $\varphi \in \mathfrak{h}_{\mathbb{R}}^* \simeq \mathfrak{h}_{\mathbb{R}}$ is called a root of $\mathfrak{g}$ with respect to $\mathfrak{h}$, or $\mathfrak{h}$-root, if $\mathfrak{g}^{\varphi} \neq \{0\}$.

Let $\Delta \subseteq \mathfrak{h}_{\mathbb{R}}^* \simeq \mathfrak{h}_{\mathbb{R}}$ denote the space of nonzero $\mathfrak{h}$-roots. Then each root space $\mathfrak{g}^{\varphi}$ with respect to some $\varphi \in \Delta$ is one-dimensional over $\mathbb{C}$, generated by a root vector e_{φ}.

Since the involution θ is a Lie algebra automorphism fixing $\mathfrak{k}$, we have that for any $h \in \mathfrak{h}$ and $\varphi \in \Delta$,

$$\theta([h, \theta(e_\varphi)]) = [h, e_\varphi]$$

and hence

$$[h, \theta(e_\varphi)] = \theta([h, e_\varphi]) = \varphi(h)\theta(e_\varphi).$$

Thus $\theta(e_\varphi)$ is also a root vector belonging to the root φ, so e_φ must be an eigenvector of θ. It follows that there is a decomposition of the roots Δ into the union $\Delta_\mathfrak{k} \cup \Delta_\mathfrak{p}$ of compact roots and non-compact roots with root spaces $\mathbb{C}e_\varphi \subseteq \mathfrak{k}$ and $\mathfrak{p}$ respectively.

The adjoint representation of $\mathfrak{h}$ on $\mathfrak{g}$ determins a decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\varphi \in \Delta} \mathfrak{g}^\varphi.$$

Although stated in a general version, the results of Chapter III of [16], Theorem 6.3, Theorem 7.1, Theorem 7.2 and Corollary 7.3, can be applied to get the following properties.

Proposition 2.1. *There exist a basis* $\{h_i : 1 \leq i \leq l\}$ *of* $\mathfrak{h}_\mathbb{R}$ *and a Weyl basis* $\{e_\varphi : \varphi \in \Delta\}$ *such that*

$$\begin{aligned} &\tau_c(h_i) = \tau_0(h_i) = -h_i, &&\text{for any } 1 \leq i \leq l; \\ &\tau_c(e_\varphi) = \tau_0(e_\varphi) = -e_{-\varphi}, &&\text{for any } \varphi \in \Delta_\mathfrak{k}; \\ &\tau_0(e_\varphi) = -\tau_c(e_\varphi) = e_{-\varphi}, &&\text{for any } \varphi \in \Delta_\mathfrak{p}, \end{aligned} \tag{13}$$

and

$$\mathfrak{k}_0 = \mathfrak{h}_0 + \sum_{\varphi \in \Delta_\mathfrak{k}} \mathbb{R}(e_\varphi - e_{-\varphi}) + \sum_{\varphi \in \Delta_\mathfrak{k}} \mathbb{R}\sqrt{-1}(e_\varphi + e_{-\varphi}); \tag{14}$$

$$\mathfrak{p}_0 = \sum_{\varphi \in \Delta_\mathfrak{p}} \mathbb{R}(e_\varphi + e_{-\varphi}) + \sum_{\varphi \in \Delta_\mathfrak{p}} \mathbb{R}\sqrt{-1}(e_\varphi - e_{-\varphi}). \tag{15}$$

We also have the following simple observation.

Lemma 2.2. *Let* Δ *be the set of* $\mathfrak{h}$*-roots as above. Then for each root* $\varphi \in \Delta$, *there is an integer* $-n \leq k \leq n$ *such that* $e_\varphi \in \mathfrak{g}^{k,-k}$. *In particular, if* $e_\varphi \in \mathfrak{g}^{k,-k}$, *then* $e_{-\varphi} = \pm\tau_0(e_\varphi) \in \mathfrak{g}^{-k,k}$ *for any* $-n \leq k \leq n$.

Proof. Let φ be a root, and e_φ be the generator of the root space $\mathfrak{g}^\varphi$, then $e_\varphi = \sum_{k=-n}^{n} e^{-k,k}$, where $e^{-k,k} \in \mathfrak{g}^{-k,k}$. Because $\mathfrak{h} \subseteq \mathfrak{v} \subseteq \mathfrak{g}^{0,0}$, the Lie bracket $[h, e^{-k,k}] \in \mathfrak{g}^{-k,k}$ for each k. Then the condition $[h, e_\varphi] = \varphi(h)e_\varphi$ implies that

$$\sum_{k=-n}^{n} [h, e^{-k,k}] = \sum_{k=-n}^{n} \varphi(h)e^{-k,k} \quad \text{for each } h \in \mathfrak{h}.$$

By comparing the type, we get

$$[h, e^{-k,k}] = \varphi(h)e^{-k,k} \quad \text{for each } h \in \mathfrak{h}.$$

Therefore $e^{-k,k} \in g^\varphi$ for each k. As $\{e^{-k,k}\}_{k=-n}^{n}$ forms a linear independent set, andg^φ is one dimensional, thus there is only one k with $-n \le k \le n$ and $e^{-k,k} \ne 0$. □

After introducing the lexicographic order (cf. page 41 in [42] or page 416 in [39]) in the real vector space $\mathfrak{h}_\mathbb{R}$, we have the notion of positive roots, and can define

$$\Delta^+ = \{\varphi > 0 : \varphi \in \Delta\}; \quad \Delta^+_\mathfrak{p} = \Delta^+ \cap \Delta_\mathfrak{p}; \quad \Delta^+_\mathfrak{k} = \Delta^+ \cap \Delta_\mathfrak{k}.$$

Recall that two different roots $\varphi, \psi \in \Delta$ are said to be strongly orthogonal if and only if

$$\varphi \pm \psi \notin \Delta \cup \{0\},$$

which is denoted by $\varphi \perp\!\!\!\perp \psi$. With the lexicographic order one can prove the following well-known result. See page 91 of [29], or pages 141 – 143 of [42] for the proof of (1), and page 247 of [16] for the proof of (2).

Lemma 2.3. *(1) There exists a set of strongly orthogonal noncompact positive roots $\Lambda = \{\varphi_1, \cdots, \varphi_r\} \subseteq \Delta^+_\mathfrak{p}$ such that*

$$\mathfrak{A}_0 = \sum_{i=1}^{r} \mathbb{R}\left(e_{\varphi_i} + e_{-\varphi_i}\right)$$

is a maximal abelian subspace in $\mathfrak{p}_0$.

(2) Let $\mathfrak{A}'_0$ be an arbitrary maximal abelian subspaces of $\mathfrak{p}_0$, then there exists an element $k \in K$ such that $\mathrm{Ad}(k) \cdot \mathfrak{A}_0 = \mathfrak{A}'_0$. Moreover, we have

$$\mathfrak{p}_0 = \bigcup_{k \in K} \mathrm{Ad}(k) \cdot \mathfrak{A}_0,$$

where Ad *denotes the adjoint action of K on $\mathfrak{A}_0$.*

3 Boundedness of the period map

In previous two sections, we have discussed the basic properties of the period domains and period maps from Lie group and Lie algebra point of views. In particular, we have introduced $\check{\mathcal{T}}$ which is Zariski open in $\mathcal{T}$. In this section, we will prove the boundedness of the image of the restricted period map

$$\widetilde{\Phi}|_{\check{\mathcal{T}}} : \check{\mathcal{T}} \to N_+ \cap D$$

in N_+ by using the geometric structures of the period domain and period map as discussed in the previous sections.

Recall that we have fixed the base points $p \in \mathcal{T}$ and

$$o = \widetilde{\Phi}(p) \in D.$$

Then the unipotent Lie group N_+ can be viewed as a subset in $\check{D}$ by identifying it to its orbit $N_+(o)$ in $\check{D}$. At the base point $\widetilde{\Phi}(p) = o \in N_+ \cap D$, the tangent spaces satisfy

$$\mathrm{T}^{1,0}_o N_+ = \mathrm{T}^{1,0}_o D \simeq \mathfrak{n}_+,$$

and the exponential map

$$\exp : \mathfrak{n}_+ \to N_+$$

is an isomorphism. There is a natural homogeneous metric on D induced from the Killing form as studied in [14], which we will call the Hodge metric. The Hodge metric on $\mathrm{T}^{1,0}_o D$ induces an Euclidean metric on $\mathfrak{n}_+$, as well as on N_+ so that $\exp : \mathfrak{n}_+ \to N_+$ is an isometry.

Also recall that we have defined

$$\check{\mathcal{T}} = \widetilde{\Phi}^{-1}(N_+ \cap D)$$

and have shown that $\mathcal{T}\backslash\check{\mathcal{T}}$ is an analytic subvariety of $\mathcal{T}$ with $\mathrm{codim}_{\mathbb{C}}(\mathcal{T}\backslash\check{\mathcal{T}}) \geq 1$. We will prove that the image of

$$\widetilde{\Phi} : \check{\mathcal{T}} \to N_+ \cap D$$

is bounded in N_+ with respect to the Euclidean metric on N_+.

Let

$$\mathfrak{p}_+ = \mathfrak{p}/(\mathfrak{p} \cap \mathfrak{b}) = \mathfrak{p} \cap \mathfrak{n}_+ \subseteq \mathfrak{n}_+$$

denote a subspace of $\mathrm{T}^{1,0}_o D \simeq \mathfrak{n}_+$. Clearly $\mathfrak{p}_+$ can be viewed as an Euclidean subspace of $\mathfrak{n}_+$. Similarly $\exp(\mathfrak{p}_+)$ can be viewed as an Euclidean subspace of N_+ with the induced metric from N_+. Define the projection map

$$P_+ : N_+ \cap D \to \exp(\mathfrak{p}_+) \cap D$$

by

$$P_+ = \exp \circ p_+ \circ \exp^{-1} \tag{16}$$

where $\exp^{-1} : N_+ \to \mathfrak{n}_+$ is the inverse of the isometry $\exp : \mathfrak{n}_+ \to N_+$, and $p_+ : \mathfrak{n}_+ \to \mathfrak{p}_+$ is the projection map from the complex Euclidean space $\mathfrak{n}_+$ to its Euclidean subspace $\mathfrak{p}_+$.

The restricted period map $\widetilde{\Phi} : \check{\mathcal{T}} \to N_+ \cap D$, composed with the projection map P_+, gives a holomorphic map, $\widetilde{\Phi}_+ = P_+ \circ \widetilde{\Phi}|_{\check{\mathcal{T}}}$,

$$\widetilde{\Phi}_+ : \check{\mathcal{T}} \to \exp(\mathfrak{p}_+) \cap D. \tag{17}$$

The following lemma actually proves that $\exp(\mathfrak{p}_+) \cap D$ is a bounded set in the complex Euclidean space $\exp(\mathfrak{p}_+)$.

Lemma 3.1. *The image of the holomorphic map* $\widetilde{\Phi}_+ : \check{\mathcal{T}} \to \exp(\mathfrak{p}_+) \cap D$ *is bounded in* $\exp(\mathfrak{p}_+)$ *with respect to the Euclidean metric on* $\exp(\mathfrak{p}_+) \subseteq N_+$.

Proof. We need to show that there exists $0 \leq C < \infty$ such that for any $q \in \check{\mathcal{T}}$, $d_E(\widetilde{\Phi}_+(p), \widetilde{\Phi}_+(q)) \leq C$, where d_E is the Euclidean distance on $\exp(\mathfrak{p}_+)$.

By the definition of $\exp(\mathfrak{p}_+)$, for any $t \in \exp(\mathfrak{p}_+)$ there is a unique $Y \in \mathfrak{p}_+$ such that $\exp(Y) \cdot \overline{o} = t$, where $\overline{o} = P_+(o)$ is the base point in $\exp(\mathfrak{p}_+) \cap D$, and $\exp(Y) \cdot \overline{o}$ denotes the left translation of the base point $\overline{o}$ by $\exp(Y)$. Similarly, for any $s \in \exp(\mathfrak{p}_+) \cap D$, there also exists an $X \in \mathfrak{p}_0$ such that $\exp(X) \cdot \overline{o} = s$.

Next we analyze the point $\exp(X) \cdot \overline{o}$ considered as a point in $\exp(\mathfrak{p}_+)$ by using the method of Harish-Chandra's proof of his famous embedding theorem for Hermitian symmetric spaces. See pages 94 – 97 in [29].

Since $X \in \mathfrak{p}_0$, by Lemma 2.3, there exists $k \in K$ such that $X \in \mathrm{Ad}(k) \cdot \mathfrak{A}_0$. As the adjoint action of K on $\mathfrak{p}_0$ is unitary action and we are considering the length in this proof, we may simply assume that $X \in \mathfrak{A}_0$ up to a unitary transformation.

Let $\Lambda = \{\varphi_1, \cdots, \varphi_r\} \subseteq \Delta_{\mathfrak{p}}^+$ be a set of strongly orthogonal roots given in Lemma 2.3. We denote $x_{\varphi_i} = e_{\varphi_i} + e_{-\varphi_i}$ and $y_{\varphi_i} = \sqrt{-1}(e_{\varphi_i} - e_{-\varphi_i})$ for any $\varphi_i \in \Lambda$. Then

$$\mathfrak{A}_0 = \mathbb{R}x_{\varphi_1} \oplus \cdots \oplus \mathbb{R}x_{\varphi_r}, \quad \text{and} \quad \mathfrak{A}_c = \mathbb{R}y_{\varphi_1} \oplus \cdots \oplus \mathbb{R}y_{\varphi_r},$$

are maximal abelian spaces in $\mathfrak{p}_0$ and $\sqrt{-1}\mathfrak{p}_0$ respectively. For any $X \in \mathfrak{A}_0$ there exists $\lambda_i \in \mathbb{R}$ for $1 \le i \le r$ such that

$$X = \lambda_1 x_{\varphi_1} + \lambda_2 x_{\varphi_2} + \cdots + \lambda_r x_{\varphi_r}.$$

Since $\mathfrak{A}_0$ is commutative, we have

$$\exp(tX) = \prod_{i=1}^{r} \exp(t\lambda_i x_{\varphi_i}).$$

Now for each $\varphi_i \in \Lambda$, we have $\mathrm{Span}_{\mathbb{C}}\{e_{\varphi_i}, e_{-\varphi_i}, h_{\varphi_i}\} \simeq \mathfrak{sl}_2(\mathbb{C})$ with

$$h_{\varphi_i} \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e_{\varphi_i} \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad e_{-\varphi_i} \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix};$$

and $\mathrm{Span}_{\mathbb{R}}\{x_{\varphi_i}, y_{\varphi_i}, \sqrt{-1}h_{\varphi_i}\} \simeq \mathfrak{sl}_2(\mathbb{R})$ with

$$\sqrt{-1}h_{\varphi_i} \mapsto \begin{bmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{bmatrix}, \, x_{\varphi_i} \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \, y_{\varphi_i} \mapsto \begin{bmatrix} 0 & \sqrt{-1} \\ -\sqrt{-1} & 0 \end{bmatrix}.$$

Since $\Lambda = \{\varphi_1, \cdots, \varphi_r\}$ is a set of strongly orthogonal roots, we have that

$$\begin{aligned} \mathfrak{g}_{\mathbb{C}}(\Lambda) &= \mathrm{Span}_{\mathbb{C}}\{e_{\varphi_i}, e_{-\varphi_i}, h_{\varphi_i}\}_{i=1}^r \simeq (\mathfrak{sl}_2(\mathbb{C}))^r, \\ \text{and} \quad \mathfrak{g}_{\mathbb{R}}(\Lambda) &= \mathrm{Span}_{\mathbb{R}}\{x_{\varphi_i}, y_{\varphi_i}, \sqrt{-1}h_{\varphi_i}\}_{i=1}^r \simeq (\mathfrak{sl}_2(\mathbb{R}))^r. \end{aligned}$$

In fact, we know that for any $\varphi, \psi \in \Lambda$ with $\varphi \neq \psi$, $[e_{\pm\varphi}, e_{\pm\psi}] = 0$ since φ is strongly orthogonal to ψ; $[h_\varphi, h_\psi] = 0$, since $\mathfrak{h}$ is abelian; and

$$[h_\varphi, e_{\pm\psi}] = [[e_\varphi, e_{-\varphi}], e_{\pm\psi}] = -[[e_{-\varphi}, e_{\pm\psi}], e_\varphi] - [[e_{\pm\psi}, e_\varphi], e_{-\varphi}] = 0.$$

Then we denote

$$G_{\mathbb{C}}(\Lambda) = \exp(\mathfrak{g}_{\mathbb{C}}(\Lambda)) \simeq (SL_2(\mathbb{C}))^r$$

and

$$G_{\mathbb{R}}(\Lambda) = \exp(\mathfrak{g}_{\mathbb{R}}(\Lambda)) = (SL_2(\mathbb{R}))^r,$$

which are subgroups of $G_{\mathbb{C}}$ and $G_{\mathbb{R}}$ respectively. With the fixed reference point $o = \widetilde{\Phi}(p)$, we denote $D(\Lambda) = G_{\mathbb{R}}(\Lambda)(o)$ and $S(\Lambda) = G_{\mathbb{C}}(\Lambda)(o)$ to be the corresponding orbits of these two subgroups, respectively. Then we have the following isomorphisms,

$$D(\Lambda) = G_{\mathbb{R}}(\Lambda) \cdot B/B \simeq G_{\mathbb{R}}(\Lambda)/G_{\mathbb{R}}(\Lambda) \cap V, \tag{18}$$

$$S(\Lambda) \cap (N_+B/B) = (G_{\mathbb{C}}(\Lambda) \cap N_+) \cdot B/B \simeq G_{\mathbb{C}}(\Lambda) \cap N_+. \tag{19}$$

With the above notations, we will show that

(i) $D(\Lambda) \subseteq S(\Lambda) \cap (N_+B/B) \subseteq \check{D}$;

(ii) $D(\Lambda)$ is bounded inside $S(\Lambda) \cap (N_+B/B)$.

By Lemma 2.2, we know that for each pair of roots $\{e_{\varphi_i}, e_{-\varphi_i}\}$, there exists a positive integer k such that either $e_{\varphi_i} \in \mathfrak{g}^{-k,k} \subseteq \mathfrak{n}_+$ and $e_{-\varphi_i} \in \mathfrak{g}^{k,-k}$, or $e_{\varphi_i} \in \mathfrak{g}^{k,-k}$ and $e_{-\varphi_i} \in \mathfrak{g}^{-k,k} \subseteq \mathfrak{n}_+$. For simplicity of notations, given each pair of root vectors $\{e_{\varphi_i}, e_{-\varphi_i}\}$, we may assume the one in $\mathfrak{g}^{-k,k} \subseteq \mathfrak{n}_+$ to be e_{φ_i} and denote the one in $\mathfrak{g}^{k,-k}$ by $e_{-\varphi_i}$. In this way, one can check that $\{\varphi_1, \cdots, \varphi_r\}$ may not be a set in $\Delta^+_{\mathfrak{p}}$, but it is a set of strongly orthogonal roots in $\Delta_{\mathfrak{p}}$. In this case, for any two different vectors $e_{\varphi_i}, e_{\varphi_j}$ in $\{e_{\varphi_1}, e_{\varphi_2}, \cdots, e_{\varphi_r}\}$, the Hermitian inner product

$$\begin{aligned} -B(\theta(e_{\varphi_i}), \overline{e_{\varphi_j}}) &= -B(-e_{\varphi_i}, e_{-\varphi_j}) \\ &= B(1/2[h_{\varphi_i}, e_{\varphi_i}], e_{-\varphi_j}) \\ &= -B(1/2e_{\varphi_i}, [h_{\varphi_i}, e_{-\varphi_j}]) = 0. \end{aligned}$$

Hence the basis $\{e_{\varphi_1}, e_{\varphi_2}, \cdots, e_{\varphi_r}\}$ can be chosen as an orthonormal basis.

Therefore, we have the following description of the above groups,

$$G_{\mathbb{R}}(\Lambda) = \exp(\mathfrak{g}_{\mathbb{R}}(\Lambda)) = \exp(\mathrm{Span}_{\mathbb{R}}\{x_{\varphi_1}, y_{\varphi_1}, \sqrt{-1}h_{\varphi_1}, \cdots, x_{\varphi_r}, y_{\varphi_r}, \sqrt{-1}h_{\varphi_r}\})$$
$$G_{\mathbb{R}}(\Lambda) \cap V = \exp(\mathfrak{g}_{\mathbb{R}}(\Lambda) \cap \mathfrak{v}_0) = \exp(\mathrm{Span}_{\mathbb{R}}\{\sqrt{-1}h_{\varphi_1}, \cdot, \sqrt{-1}h_{\varphi_r}\})$$
$$G_{\mathbb{C}}(\Lambda) \cap N_+ = \exp(\mathfrak{g}_{\mathbb{C}}(\Lambda) \cap \mathfrak{n}_+) = \exp(\mathrm{Span}_{\mathbb{C}}\{e_{\varphi_1}, e_{\varphi_2}, \cdots, e_{\varphi_r}\}).$$

Thus by the isomorphisms in (18) and (19), we have

$$
\begin{aligned}
D(\Lambda) &\simeq \prod_{i=1}^{r} \exp(\mathrm{Span}_{\mathbb{R}}\{x_{\varphi_i}, y_{\varphi_i}, \sqrt{-1}h_{\varphi_i}\})/\exp(\mathrm{Span}_{\mathbb{R}}\{\sqrt{-1}h_{\varphi_i}\}, \\
S(\Lambda) \cap (N_+B/B) &\simeq \prod_{i=1}^{r} \exp(\mathrm{Span}_{\mathbb{C}}\{e_{\varphi_i}\}).
\end{aligned}
$$

Let us denote

$$G_{\mathbb{C}}(\varphi_i) = \exp(\mathrm{Span}_{\mathbb{C}}\{e_{\varphi_i}, e_{-\varphi_i}, h_{\varphi_i}) \simeq SL_2(\mathbb{C}),$$

$S(\varphi_i) = G_{\mathbb{C}}(\varphi_i)(o)$, and

$$G_{\mathbb{R}}(\varphi_i) = \exp(\mathrm{Span}_{\mathbb{R}}\{x_{\varphi_i}, y_{\varphi_i}, \sqrt{-1}h_{\varphi_i}\}) \simeq SL_2(\mathbb{R}),$$

$D(\varphi_i) = G_{\mathbb{R}}(\varphi_i)(o)$.

Now each point in $S(\varphi_i) \cap (N_+B/B)$ can be represented by

$$\exp(ze_{\varphi_i}) = \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} \quad \text{for some } z \in \mathbb{C}.$$

Thus $S(\varphi_i) \cap (N_+B/B) \simeq \mathbb{C}$. In order to see $D(\varphi_i)$ in $G_{\mathbb{C}}/B$, we decompose each point in $D(\varphi_i)$ as follows. Let $z = a + bi$ for some $a, b \in \mathbb{R}$, then

$$
\begin{aligned}
\exp(ax_{\varphi_i} + by_{\varphi_i}) &= \begin{bmatrix} \cosh|z| & \frac{z}{|z|}\sinh|z| \\ \frac{\overline{z}}{|z|}\sinh|z| & \cosh|z| \end{bmatrix} \qquad (20) \\
&= \begin{bmatrix} 1 & \frac{z}{|z|}\tanh|z| \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (\cosh|z|)^{-1} & 0 \\ 0 & \cosh|z| \end{bmatrix} \\
&\quad \begin{bmatrix} 1 & 0 \\ \frac{\overline{z}}{|z|}\tanh|z| & 1 \end{bmatrix} \\
&= \exp\left[(\frac{z}{|z|}\tanh|z|)e_{\varphi_i}\right] \exp\left[-\log(\cosh|z|)h_{\varphi_i}\right] \\
&\quad \exp\left[(\frac{\overline{z}}{|z|}\tanh|z|)e_{-\varphi_i}\right] \\
&\equiv \exp\left[(\frac{z}{|z|}\tanh|z|)e_{\varphi_i}\right] \pmod{B}.
\end{aligned}
$$

So elements of $D(\varphi_i)$ in $G_{\mathbb{C}}/B$ can be represented by $\exp[(z/|z|)(\tanh|z|)e_{\varphi_i}]$, i.e.

$$\begin{bmatrix} 1 & \frac{z}{|z|}\tanh|z| \\ 0 & 1 \end{bmatrix},$$

in which $\frac{z}{|z|}\tanh|z|$ is a point in the unit disc $\mathfrak{D}$ of the complex plane.

So we have proved that $D(\varphi_i)$ is a unit disc $\mathfrak{D}$ in the complex plane $S(\varphi_i)\cap(N_+B/B)\simeq\mathbb{C}$. Therefore

$$D(\Lambda)\simeq\mathfrak{D}^r \quad \text{and} \quad S(\Lambda)\cap N_+\simeq\mathbb{C}^r.$$

So we have obtained both (i) and (ii). As a consequence, we get that for any $q\in\check{\mathcal{T}}$, $\widetilde{\Phi}_+(q)\in D(\Lambda)$. This implies

$$d_E(\widetilde{\Phi}_+(p),\widetilde{\Phi}_+(q))\leq\sqrt{r}$$

where d_E is the Eulidean distance on $S(\Lambda)\cap(N_+B/B)$.

To complete the proof, we only need to show that $S(\Lambda)\cap(N_+B/B)$ is totally geodesic in N_+B/B. In fact, the tangent space of N_+ at the base point is $\mathfrak{n}_+$ and the tangent space of $S(\Lambda)\cap N_+B/B$ at the base point is $\mathrm{Span}_{\mathbb{C}}\{e_{\varphi_1},e_{\varphi_2},\cdots,e_{\varphi_r}\}$. Since $\mathrm{Span}_{\mathbb{C}}\{e_{\varphi_1},e_{\varphi_2},\cdots,e_{\varphi_r}\}$ is a Lie subalgebra of $\mathfrak{n}_+$, the corresponding orbit $S(\Lambda)\cap N_+B/B$ is totally geodesic in N_+B/B. Here recall that the basis $\{e_{\varphi_1},e_{\varphi_2},\cdots,e_{\varphi_r}\}$ is an orthonormal basis. □

Now we briefly explain the geometry related to the two projection maps

$$P_+:\ N_+\cap D\to\exp(\mathfrak{p}_+),\ \pi:\ D\to G_{\mathbb{R}}/K$$

which follows from the proof of Lemma 3.1.

If $X\in\mathfrak{A}_0$, i.e. X has the expression

$$X=\sum_i t_i(e_{\varphi_i}+e_{-\varphi_i})=\sum_i t_i x_{\varphi_i},$$

then there exists a unique $Y\in\mathfrak{p}_+$ defined by

$$Y=\sum_i\tanh(t_i)e_{\varphi_i}$$

such that $\exp(X)\bar{o} = \exp(Y)\bar{o}$, which follows from equation (20) in the proof of Lemma 3.1. Recall that $e_{\varphi_i} \in \mathfrak{n}_+$ for each i.

In general, if $X \in \mathfrak{p}_0$, then from Lemma 2.3, there exists $k \in K$ such that

$$\widetilde{X} = \mathrm{Ad}(k)(X) \in \mathfrak{A}_0$$

with $\widetilde{X} = \sum_i t_i(e_{\varphi_i} + e_{-\varphi_i})$ and

$$X = \sum_i t_i[\mathrm{Ad}(k^{-1})(e_{\varphi_i}) + \mathrm{Ad}(k^{-1})(e_{-\varphi_i})].$$

Now define

$$\widetilde{Y} = \sum_i \tanh(t_i) e_{\varphi_i}.$$

Then by equation (20), we have $\exp(\widetilde{X})(k\bar{o}) = \exp(\widetilde{Y})(k\bar{o})$, of which both sides are considered as left translations of the base point $k\bar{o}$. By translating back to $\bar{o}$, we have

$$\exp(X)\bar{o} = \exp(Y)\bar{o},$$

where Y is given by

$$Y = \sum_i \tanh(t_i)\mathrm{Ad}(k^{-1})(e_{\varphi_i}).$$

Now we define a map ι from $G_{\mathbb{R}}/K \simeq \exp(\mathfrak{p}_0)$ to $\exp(\mathfrak{p}_+)$, by mapping $\exp(X)\bar{o}$ to $\exp(Y)\bar{o}$ with $Y \in \mathfrak{p}_+$ which is defined for each $X \in \mathfrak{p}_0$ uniquely as above. Hence

$$\iota : G_{\mathbb{R}}/K \to \exp(\mathfrak{p}_+), \; \iota(\exp(X)\bar{o}) = \exp(Y)\bar{o}$$

is a smooth embedding. Note that in general ι is not holomorphic.

From Lemma 3.1, the image of ι is precisely $\exp(\mathfrak{p}_+) \cap D$, which is a bounded domain in the complex Euclidean space $\exp(\mathfrak{p}_+)$.

For a more detailed description of the above explicit correspondence from $\mathfrak{p}_0$ to $\mathfrak{p}_+$ in proving the Harish-Chandra embedding theorem, please see Lemma 7.11 in pages 390 – 391 in [16], pages 94 – 97 of [29], or the discussion in pages 463 – 466 in [42].

Next, from the definition of Hodge metric on D in [14], we know that $\pi : D \to G_{\mathbb{R}}/K$ is a Riemannian submersion with the natural homogeneous metrics on D and $G_{\mathbb{R}}/K$. For more detail about this, see [19].

Corollary 3.2. *The natural projection* $\pi : D \to G_{\mathbb{R}}/K$*, when restricted to the underlying real manifold of* $\exp(\mathfrak{p}_+) \cap D$*, is given by the diffeomorphism*

$$\pi_+ : \exp(\mathfrak{p}_+) \cap D \longrightarrow \exp(\mathfrak{p}_0) \xrightarrow{\simeq} G_{\mathbb{R}}/K,$$

and the diagram

$$\begin{array}{ccc} N_+ \cap D & \xrightarrow{\pi} & G_{\mathbb{R}}/K \\ \downarrow{\scriptstyle P_+} & \nearrow{\scriptstyle \pi_+} & \\ \exp(\mathfrak{p}_+) \cap D & & \end{array} , \qquad (21)$$

is commutative.

Proof. From the discussion above, one sees that the diffeomorphism from $\exp(\mathfrak{p}_+) \cap D$ to $G_{\mathbb{R}}/K \simeq \exp(\mathfrak{p}_0)$ is explicitly given by identifying the point

$$\exp(Y)\bar{o} \in \exp(\mathfrak{p}_+) \cap D$$

to the point

$$\exp(X)\bar{o} \in \exp(\mathfrak{p}_0),$$

with the correspondence between X and Y given as above.

On the other hand, since $\pi : D \to G_{\mathbb{R}}/K$ is a Riemannian submersion, the real geodesic

$$c(t) = \exp(tX)$$

in $\exp(\mathfrak{p}_+) \cap D$ with $X \in \mathfrak{p}_0$ connecting the based point o and any point $z \in \exp(\mathfrak{p}_+) \cap D$ is the horizontal lift of the geodesic $\pi(c(t))$ in $G_{\mathbb{R}}/K$. This is a basic fact in Riemmanian submersion as given in, for example, Corollary 26.12 in page 339 of [27].

Hence the natural projection $\pi : D \to G_{\mathbb{R}}/K$ maps $c(t)$ isometrically to $\pi(c(t))$. From this one sees that the projection π, when restricted to the underlying real manifold of $\exp(\mathfrak{p}_+) \cap D$, is given by the diffeomorphism

$$\pi_+ : \exp(\mathfrak{p}_+) \cap D \longrightarrow \exp(\mathfrak{p}_0) \xrightarrow{\simeq} G_{\mathbb{R}}/K. \qquad (22)$$

We know that differential map $d(P_+)_o$ of the projection map

$$P_+ : N_+ \cap D \to \exp(\mathfrak{p}_+) \cap D$$

at the base point o kills the vertical tangent space at o of the fiber bundle $\pi : D \to G_{\mathbb{R}}/K$. Moreover, the differential map $d(P_+)_q$ at any point $q = go \in N_+ \cap D$, for $g \in G_{\mathbb{R}}$, is given by

$$d(P_+)_q = dg \circ d(P_+)_o,$$

where go denotes the left translation of the base point o and dg denotes its tangent map.

On the other hand, by [14] we know that the vertical and horizontal bundles in D of the fiber bundle $\pi : D \to G_{\mathbb{R}}/K$ are both invariant under left translations. Hence $d(P_+)_q$ kills the vertical fiber at q. This, together with the diffeomorphism in (22), implies that diagram (21) is commutative. □

Let the projection map

$$\pi : N_+ \cap D \to G_{\mathbb{R}}/K$$

be the restriction of $\pi : D \to G_{\mathbb{R}}/K$ to $N_+ \cap D \subset D$. From the discussion above, one sees that the projection map

$$P_+ : N_+ \cap D \to \exp(\mathfrak{p}_+) \cap D$$

defined before Lemma 3.1 can be considered as the composition of the projection map $\pi : N_+ \cap D \to G_{\mathbb{R}}/K$ and the embedding ι, such that the following diagram is commutative

$$\begin{array}{ccc} N_+ \cap D & & \\ \downarrow{\scriptstyle \pi} & \overset{P_+}{\searrow} & \\ G_{\mathbb{R}}/K & \xrightarrow{\iota} & \exp(\mathfrak{p}_+) \cap D \subset \exp(\mathfrak{p}_+). \end{array} \tag{23}$$

Note that the tangent map of P_+, which is the composite of the following maps

$$\mathfrak{n}_+ \to \mathfrak{p}_0 \to \mathfrak{p}_+,$$

maps the holomorphic tangent space $\mathfrak{n}_+$ of $N_+ \cap D$ to the holomorphic tangent space $\mathfrak{p}_+$ of $\exp(\mathfrak{p}_+)$.

From this point of view, the commutativity of diagram (21) is equivalent to the commutative diagram (23) with

$$\iota = \pi_+^{-1}.$$

Next, for convenience of discussion, we introduce the notion of horizontal slice which is a reformulation of integral submanifolds of the horizontal distribution induced by the period map on the period domain D.

For any $q \in \mathcal{T}$ and $s = \widetilde{\Phi}(q)$. By Griffiths transversality we know that there exists a small open neighborhood U of $q \in \mathcal{T}$ such that the image of the tangent map of $\widetilde{\Phi}$ at any point in U lies in the horizontal subbundle $\mathrm{T}^{1,0}_h D$. If needed, here we can always take U arbitrarily small.

Definition 3.3. *We call $\widetilde{\Phi}(U)$ a horizontal slice of the period map $\widetilde{\Phi}$ passing through s.*

As proved in Proposition 3 of Chapter 2 in [34], for the period map $\widetilde{\Phi} : \mathcal{T} \to D$, there is a narrow Whitney stratification $\mathcal{T} = \cup_i \mathcal{T}_i$ such that if

$$\widetilde{\Phi}_i = \widetilde{\Phi}|_{\mathcal{T}_i},$$

the rank of the tangent map $d\widetilde{\Phi}_i$ is constant on $\mathcal{T}_i$. Here narrow means that for any open neighborhood U of any point in $\mathcal{T}$, $U = \cup_i(\mathcal{T}_i \cap U)$ induces a Whitney stratification of U. See Definition 1 in Chapter 2 of [34]. Therefore when the open neighborhood U of $q \in \mathcal{T}$ is small, we have

$$\widetilde{\Phi}(U) = \cup_i L_i$$

where each $L_i = \widetilde{\Phi}(\mathcal{T}_i \cap U)$ is a smooth complex manifold. We remark that there are only finitely many $\mathcal{T}_i$'s around q, as $\mathcal{T}$ is of finite dimension, and each L_i is an integral submanifold of the horizontal distribution induced by the period map.

Since the tangent map of the period map $\widetilde{\Phi}$ is continuous on tangent spaces, the Whitney conditions are still satisfied for the stratification $\widetilde{\Phi}(U) = \cup_i L_i$, i.e. $\widetilde{\Phi}(U) = \cup_i L_i$ is also a Whitney stratification. See for example, page 36 in [31] about the details related to the Whitney stratifications. In fact here we can also directly use the Whitney stratification of $\widetilde{\Phi}(U)$ for the proof of the following Lemma 3.4, while using the Whitney stratification for $\mathcal{T}$ makes the geometric picture of the period map more transparent.

Note that, for the extended period map $\widetilde{\Phi}' : \mathcal{T}' \to D$, we still have the notion of horizontal slices, since the extended period map $\widetilde{\Phi}'$ still satisfies Griffiths transversality due to Lemma 1.6.

Let $f : D \to M$ be a smooth map from the period domain D to a smooth manifold M. The map f is said to be locally injective on the horizontal slices of $\widetilde{\Phi}$, provided that for any horizontal slice $\widetilde{\Phi}(U)$ through $s = \widetilde{\Phi}(q)$ with $\widetilde{\Phi}(U) = \cup_i L_i$, we can take the open neighborhood U of $q \in \mathcal{T}$ small enough such that f is injective on each L_i.

Note that, by the Griffiths transversality, we know that at any point $t \in L_i$ the corresponding real tangent spaces satisfy

$$\mathrm{T}_t L_i \subset \mathrm{T}_{h,t} D \subset \mathrm{T}_{\bar{t}} G_{\mathbb{R}}/K \simeq \mathfrak{p}_0$$

where $\bar{t} = \pi(t)$. Here the inclusion $\mathrm{T}_{h,t}D \subset \mathrm{T}_{\bar{t}} G_{\mathbb{R}}/K$ is induced by the tangent map of π at t. Therefore the tangent map of $\pi|_{L_i} : L_i \to G_{\mathbb{R}}/K$ at $t \in L_i$ is injective, and π is injective in a small neighborhood of t in L_i.

In the proof of the following lemma, we will only discuss the horizontal slices for $\widetilde{\Phi}$. The proof is the same for $\widetilde{\Phi}'$.

Lemma 3.4. *The projection map $\pi : D \to G_{\mathbb{R}}/K$ is locally injective on horizontal slices.*

Proof. From the above discussion, we see that the lemma is an obvious corollary from the Griffiths transversality, if $\widetilde{\Phi}(U)$ is smooth. The proof for general case is essentially the same, except that we need to use the Whitney stratification of $\widetilde{\Phi}(U)$ and apply the Griffiths transversality on each stratum. This should be standard in stratified spaces as discussed, for example, in Section 3.8 of Chapter 1 in [31]. We give the detailed argument for reader's convenience.

Let $s = \widetilde{\Phi}(q) \in D$ and U be a small open neighborhood of q. As described above, we have the Whitney stratification $\widetilde{\Phi}(U) = \cup_i L_i$ and each L_i can be identified to the image $\widetilde{\Phi}(\mathcal{T}_i \cap U)$ of the stratum $\mathcal{T}_i$.

From Theorem 2.1.2 of [31], we know that the tangent bundle $\mathrm{T}\widetilde{\Phi}(U)$ is a stratified space with a smooth structure, such that the projection $\mathrm{T}\widetilde{\Phi}(U) \to \widetilde{\Phi}(U)$ is smooth and a morphism of stratified spaces. For any sequence of points $\{s_k\}$ in a stratum L_i converging to s, the limit of the tangent spaces,

$$\lim_{k\to\infty} \mathrm{T}_{s_k} L_i = \mathrm{T}_s L_i$$

exists by the Whitney conditions, and is defined as the generalized tangent space at s in page 44 of [4]. Also see the discussion in page 64 of [31]. Denote

$\overline{s} = \pi(s)$. With these notations understood, and by the Griffiths transversality, we get the following relations for the corresponding real tangent spaces,

$$\mathrm{T}_s\widetilde{\Phi}(U) = \cup_i \mathrm{T}_s L_i \simeq (d\widetilde{\Phi})_q(\mathrm{T}_q\mathcal{T}) \subset (\mathfrak{g}^{-1,1} \oplus \mathfrak{g}^{1,-1}) \cap \mathfrak{g}_0 \subset \mathrm{T}_{\overline{s}} G_{\mathbb{R}}/K \simeq \mathfrak{p}_0.$$

This implies that the tangent map of

$$\pi|_{\widetilde{\Phi}(U)} : \widetilde{\Phi}(U) \to G_{\mathbb{R}}/K$$

at s is injective in the sense of stratified space, or equivalently it is injective on each $\mathrm{T}_s L_i$ considered as generalized tangent space.

Hence we can find a small open neighborhood V of s in D, such that the restriction of π to $\widetilde{\Phi}(U) \cap V$,

$$\pi|_{\widetilde{\Phi}(U)\cap V} : \widetilde{\Phi}(U) \cap V \to G_{\mathbb{R}}/K,$$

is an immersion in the sense of stratified spaces, or equivalently injective on each stratum $L_i \cap U$. Now we take U in $\mathcal{T}$ containing q small enough such that $\widetilde{\Phi}(U) \subset V$. With such a choice of U, π is injective on the horizontal slice $\widetilde{\Phi}(U)$ in the sense of stratified spaces, and hence injective on each stratum L_i. □

Lemma 3.5. *For any $z \in \widetilde{\Phi}_+(\check{\mathcal{T}}) \subset \exp(\mathfrak{p}_+) \cap D$, we have*

$$P_+^{-1}(z) \cap \widetilde{\Phi}(\check{\mathcal{T}}) = \pi^{-1}(z') \cap \widetilde{\Phi}(\check{\mathcal{T}}),$$

where $z' = \pi(z) \in G_{\mathbb{R}}/K$.

Proof. Using the commutative diagram (21) in Corollary 3.2,

$$\begin{array}{ccc} N_+ \cap D & \xrightarrow{\ \pi\ } & G_{\mathbb{R}}/K \\ \downarrow{\scriptstyle P_+} & \nearrow{\scriptstyle \pi_+} & \\ \exp(\mathfrak{p}_+) \cap D & & \end{array},$$

we see that any two points in $N_+ \cap D$ are mapped to the same point in $\exp(\mathfrak{p}_+) \cap D$ via P_+, if and only if they are are mapped to the same point in $G_{\mathbb{R}}/K$ via π. Hence for any

$$z \in \widetilde{\Phi}_+(\check{\mathcal{T}}) \subset \exp(\mathfrak{p}_+) \cap D,$$

the projection map P_+ maps the fiber $\pi^{-1}(z') \cap \widetilde{\Phi}(\check{\mathcal{T}})$ onto the point

$$z \in \exp(\mathfrak{p}_+) \cap D,$$

where $z' = \pi(z) \in G_{\mathbb{R}}/K$. □

Theorem 3.6. *The image of the restriction of the period map* $\widetilde{\Phi} : \check{\mathcal{T}} \to N_+ \cap D$ *is bounded in* N_+ *with respect to the Euclidean metric on* N_+.

Proof. In Lemma 3.1, we already proved that the image of

$$\widetilde{\Phi}_+ = P_+ \circ \widetilde{\Phi} : \check{\mathcal{T}} \to \exp(\mathfrak{p}_+) \cap D \subset \exp(\mathfrak{p}_+)$$

is bounded with respect to the Euclidean metric on $\exp(\mathfrak{p}_+) \subseteq N_+$. Now together with the Griffiths transversality on S' and $\mathcal{T}'$, we will deduce the boundedness of the image of $\widetilde{\Phi} : \check{\mathcal{T}} \to N_+ \cap D$ from the boundedness of the image of $\widetilde{\Phi}_+$.

Our proof can be divided into two steps. It is an elementary argument to apply the Griffiths transversality on $\mathcal{T}'$.

(i) We claim that there are only finite points in $(P_+|_{\widetilde{\Phi}(\check{\mathcal{T}})})^{-1}(z)$, for any $z \in \widetilde{\Phi}_+(\check{\mathcal{T}})$.

Otherwise, by Lemma 3.5, we have $\{q_i\}_{i=1}^{\infty} \subseteq \check{\mathcal{T}}$ and

$$\{y_i = \widetilde{\Phi}(q_i)\}_{i=1}^{\infty} \subseteq (P_+|_{\widetilde{\Phi}(\check{\mathcal{T}})})^{-1}(z)$$

with limiting point $y_\infty \in \pi^{-1}(z') \simeq K/V$, since K/V is compact. We project the points q_i to $q_i' \in S$ via the universal covering map $\pi : \mathcal{T} \to S$. There must be infinitely many q_i''s. Otherwise, we have a subsequence $\{q_{j_k}\}$ of $\{q_j\}$ such that $\pi(q_{j_k}) = q_{i_0}'$ for some i_0 and

$$y_{j_k} = \widetilde{\Phi}(q_{j_k}) = \gamma_k \widetilde{\Phi}(q_{j_0}) = \gamma_k y_{j_0},$$

where $\gamma_k \in \Gamma$ is the monodromy action. Since Γ is discrete, the subsequence $\{y_{j_k}\}$ is not convergent, which is a contradiction.

Now we project the points q_i on S via the universal covering map $\pi : \mathcal{T} \to S$ and still denote them by q_i without confusion. Then after going to a subsequence if needed, we may assume the sequence $\{q_i\}_{i=1}^{\infty} \subseteq S$ has a limiting point q_∞ in $\overline{S}$, where $\overline{S}$ is the compactification of S with simple normal crossing compactification divisors. By continuity the period map $\Phi : S \to D/\Gamma$ can be extended over q_∞ with

$$\Phi'(q_\infty) = \pi_D(y_\infty) \in D/\Gamma,$$

where $\pi_D : D \to D/\Gamma$ is the projection map. Thus q_∞ lies S'.

Now we can regard the sequence $\{q_i\}_{i=1}^\infty$ as a convergent sequence in S' with limiting point $q_\infty \in S'$. Let V be a small open neighborhood of q_∞ in S', and U be an open neighborhood in $\mathcal{T}'$ such that $\pi' : U \to V$ is a diffeomorphism under the universal covering map $\pi' : \mathcal{T}' \to S'$. We can choose a sequence $\{\widetilde{q}_i\}_{i=1}^\infty \subseteq \mathcal{T}'$ with limiting point $\widetilde{q}_\infty \in U$ such that

$$\pi'(\widetilde{q}_i) = q_i,$$

$\widetilde{q}_i$ lies in U for i large, and

$$\widetilde{\Phi}'(\widetilde{q}_i) = y_i \in D,$$

for $i \geq 1$ and $i = \infty$.

Since the extended period map $\widetilde{\Phi}' : \mathcal{T}' \to D$ still satisfies the Griffiths transversality by Lemma 1.6, we can choose a small neighborhood U of $\widetilde{q}_\infty$ such that

$$\widetilde{\Phi}'(U) = \cup_i L_i$$

is a disjoint union of the Whitney stratifications in Lemma 3.4 and π is injective on each L_i. By passing to a subsequence, we may assume that the points $\widetilde{q}_i$ for i sufficiently large are mapped into some stratum L_i, which is a contradiction, since the points $y_i = \widetilde{\Phi}'(\widetilde{q}_i)$ all lie in the same fiber $\pi^{-1}(z') \simeq K/V$.

Denote $P_+|_{\widetilde{\Phi}'(\mathcal{T}')}$ to be the restricted map

$$P_+|_{\widetilde{\Phi}'(\mathcal{T}')} : N_+ \cap \widetilde{\Phi}'(\mathcal{T}') \to \exp(\mathfrak{p}_+) \cap D.$$

In fact, the same argument also proves that there are finite points in

$$(P_+|_{\widetilde{\Phi}'(\mathcal{T}')})^{-1}(z),$$

for any $z \in \widetilde{\Phi}_+(\check{\mathcal{T}})$. Furthermore, we have the following conclusion.

(ii) The restricted map

$$P_+|_{\widetilde{\Phi}'(\mathcal{T}')} : N_+ \cap \widetilde{\Phi}'(\mathcal{T}') \to P_+(N_+ \cap \widetilde{\Phi}'(\mathcal{T}'))$$

is a finite holomorphic map, or equivalently, a finite ramified covering map.

From (i) and the definition of finite map in complex analytic geometry as given in page 47 of [5], we only need to show that $P_+|_{\widetilde{\Phi}'(\mathcal{T}')}$ is closed. In fact, we know that $\Phi' : S' \to D/\Gamma$ is a proper map, and hence $\Phi'(S')$ is closed in D/Γ. So

$$\widetilde{\Phi}'(\mathcal{T}') = \pi_D^{-1}(\Phi'(S'))$$

is also closed in D, where $\pi_D : D \to D/\Gamma$ is the projection map.

Then any closed subset E of $\widetilde{\Phi}'(\mathcal{T}')$ is also a closed subset of D. Since $\pi : D \to G_{\mathbb{R}}/K$ is a proper map, one sees that π is a closed map, which implies that $\pi(E)$ is closed in $G_{\mathbb{R}}/K$. Moreover, from the proof of Lemma 3.5, one sees that $P_+(E)$ is diffeomorphic to $\pi(E)$ through the diffeomorphism

$$\exp(\mathfrak{p}_+) \cap D \simeq \mathfrak{p}_0 \simeq G_{\mathbb{R}}/K,$$

which implies that $P_+(E)$ is closed in $\exp(\mathfrak{p}_+) \cap D$. Therefore we have proved that $P_+|_{\widetilde{\Phi}'(\mathcal{T}')}$ is a closed map.

As proved in page 171 of [5], the image of an irreducible complex variety under holomorphic map is still irreducible. Since $\mathcal{T}'$ is irreducible, we know that $\widetilde{\Phi}'(\mathcal{T}')$ is an irreducible analytic subvariety of $\check{D}$. The intersection $N_+ \cap \widetilde{\Phi}'(\mathcal{T}')$ is equal to $\widetilde{\Phi}'(\mathcal{T}')$ minus the proper analytic subvariety

$$\widetilde{\Phi}'(\mathcal{T}') \cap (\check{D} \setminus N_+).$$

Hence, from the results in page 171 of the book of Grauert-Remmert [5], we get that $N_+ \cap \widetilde{\Phi}'(\mathcal{T}')$ and $P_+(N_+ \cap \widetilde{\Phi}'(\mathcal{T}'))$ are both irreducible.

By the result in page 179 of [5], we deduce that the projection map

$$P_+|_{\widetilde{\Phi}'(\mathcal{T}')} : N_+ \cap \widetilde{\Phi}'(\mathcal{T}') \to P_+(N_+ \cap \widetilde{\Phi}'(\mathcal{T}'))$$

is a finite holomorphic map, or equivalently, a finite ramified covering map.

Let $r(z)$ be the cardinality of the fiber $(P_+|_{\widetilde{\Phi}'(\mathcal{T}')})^{-1}(z)$ for any $z \in P_+(N_+ \cap \widetilde{\Phi}'(\mathcal{T}'))$. From the result in page 135 in [5], we know that $r(z) = r$ is constant on $P_+(N_+ \cap \widetilde{\Phi}'(\mathcal{T}'))$ outside the ramified locus which is an analytic subset. From the above proof of (i), we also know that r is finite. More precisely we have proved that $P_+|_{\widetilde{\Phi}'(\mathcal{T}')}$ is an r-sheeted ramified covering, which together with Lemma 3.1, implies that the image $\widetilde{\Phi}(\check{\mathcal{T}}) \subseteq N_+ \cap D$ is bounded. $\square$

In early versions of this paper we wrote a more elementary proof of (ii), which only used the Griffiths transversality and a simple limiting argument similar to the proof of (i). The new proof we give here is more illuminating, since it has the advantage of involving more geometric structures of the image of period map and period domain.

4 Proof of the Griffiths conjecture

In this section, we prove the Griffiths conjecture by using the boundedness of the restricted period map $\widetilde{\Phi}|_{\check{\mathcal{T}}} : \check{\mathcal{T}} \to N_+ \cap D$ and the Riemann extension theorem. This is our first main result.

Theorem 4.1. *The image of $\widetilde{\Phi} : \mathcal{T} \to D$ lies in $N_+ \cap D$ and is bounded with respect to the Euclidean metric on N_+.*

Proof. According to Proposition 1.3, $\mathcal{T} \setminus \check{\mathcal{T}}$ is an analytic subvariety of $\mathcal{T}$, the complex codimension of $\mathcal{T} \setminus \check{\mathcal{T}}$ in $\mathcal{T}$ is at least one. By Theorem 3.6, the image of the holomorphic map $\widetilde{\Phi} : \check{\mathcal{T}} \to N_+ \cap D$ is bounded in N_+ with respect to the Euclidean metric. Thus by the Riemann extension theorem, there exists a holomorphic map $\widetilde{\Phi}_{\mathcal{T}} : \mathcal{T} \to N_+$, such that

$$\widetilde{\Phi}_{\mathcal{T}}|_{\check{\mathcal{T}}} = \widetilde{\Phi}|_{\check{\mathcal{T}}}.$$

Since as holomorphic maps, $\widetilde{\Phi}_{\mathcal{T}}$ and $\widetilde{\Phi}$ agree on the open subset $\check{\mathcal{T}}$, they must be the same on the entire $\mathcal{T}$. Therefore, the image of $\widetilde{\Phi}$ is in $N_+ \cap D$, and the image is bounded with respect to the Euclidean metric on N_+. As a consequence, we also get

$$\mathcal{T} = \check{\mathcal{T}} = \widetilde{\Phi}^{-1}(N_+ \cap D).$$

$\square$

Recall that in the discussion after Lemma 1.5, we have proved $\mathcal{T}_0 = i_{\mathcal{T}}(\mathcal{T})$ is an analytic subvariety of $\mathcal{T}'$ with $\operatorname{codim}_{\mathbb{C}}(\mathcal{T}' \backslash \mathcal{T}_0) \geq 1$. Since $\widetilde{\Phi}(\mathcal{T}) = \widetilde{\Phi}_0(\mathcal{T}_0)$ where $\widetilde{\Phi}'_0 = \widetilde{\Phi}'|_{\mathcal{T}_0}$, one can also get the boundedness of the restricted map $\widetilde{\Phi}'_0 : \mathcal{T}_0 \to N_+ \cap D$ from Theorem 4.1. By applying Riemann extension theorem, we immediately get the boundedness of the extended period map $\widetilde{\Phi}' : \mathcal{T}' \to D$.

Corollary 4.2. *The image of the extended period map $\widetilde{\Phi}' : \mathcal{T}' \to D$ also lies in $N_+ \cap D$ and is bounded with respect to the Euclidean metric on N_+.*

Proof. We give another more direct and elementary proof. In fact, let $q' \in \mathcal{T}'$ be any point and $\{q_k\}$ be a sequence of points in $\mathcal{T}_0$ with limit q', then we have

$$\widetilde{\Phi}'(q') = \lim_{k \to \infty} \widetilde{\Phi}_0(q_k).$$

Note that $\widetilde{\Phi}(\mathcal{T}) = \widetilde{\Phi}_0(\mathcal{T}_0)$ implies that the image of $\widetilde{\Phi}_0$ is bounded. Therefore the boundedness of $\widetilde{\Phi}'$ follows from the boundedness of $\widetilde{\Phi}_0$ immediately. □

Since the image of the projection map

$$P_+ : N_+ \cap D \to \exp(\mathfrak{p}_+) \cap D$$

is bounded in the complex Euclidean space $\exp(\mathfrak{p}_+)$, and $D \setminus (N_+ \cap D)$ is a proper analytic subvariety, from the Hartogs extension theorem, we get that P_+ extends to a holomorphic map

$$P_+ : D \to \exp(\mathfrak{p}_+) \cap D.$$

Moreover, since $\widetilde{\Phi}'(\mathcal{T}') \subset N_+ \cap D$, as a direct consequence of Corollary 4.2 and Theorem 3.6, we have the following corollary.

Corollary 4.3. *The restriction of the projection map*

$$P_+ : \widetilde{\Phi}'(\mathcal{T}') \to P_+(\widetilde{\Phi}'(\mathcal{T}'))$$

is a finite ramified covering map.

5 Moduli spaces and their extensions

In this section, we introduce the definitions of moduli space with level m structure and Teichmüller space of polarized manifolds. Then we study three extensions of the moduli space of polarized manifolds and prove the equivalence of them.

Most results in this section are standard and well-known. For example, one can refer to [32] for the knowledge of moduli space, and [21], [37] for the knowledge of deformation theory.

Let (X, L) be a polarized manifold. The moduli space $\mathcal{M}$ of polarized manifolds is the complex analytic space parameterizing the isomorphism classes of polarized manifolds with the isomorphism defined by

$$(X, L) \sim (X', L') \Longleftrightarrow \exists \text{ biholomorphic map } f : X \to X' \text{ s.t. } f^*L' = L.$$

We fix a lattice Λ with a pairing Q_0, where Λ is isomorphic to $H^n(X_0, \mathbb{Z})/\mathrm{Tor}$ for some $(X_0, L_0) \in \mathcal{M}$ and Q_0 is defined by the cup-product. For a polarized manifold (X, L), we define a marking γ as an isometry of the lattices

$$\gamma : (\Lambda, Q_0) \to (H^n(X, \mathbb{Z})/\mathrm{Tor}, Q).$$

For any integer $m \geq 3$, we define an m-equivalent relation of two markings on (X, L) by

$$\gamma \sim_m \gamma' \text{ if and only if } \gamma' \circ \gamma^{-1} - \mathrm{Id} \in m \cdot \mathrm{End}(H^n(X, \mathbb{Z})/\mathrm{Tor}),$$

and denote by $[\gamma]_m$ the set of all the m-equivalent classes of γ. Then we call $[\gamma]_m$ a level m structure on the polarized manifold (X, L). Two polarized manifolds with level m structure $(X, L, [\gamma]_m)$ and $(X', L', [\gamma']_m)$ are said to be isomorphic if there exists a biholomorphic map

$$f : X \to X'$$

such that $f^*L' = L$ and

$$f^*\gamma' \sim_m \gamma,$$

where $f^*\gamma'$ is given by $\gamma' : (\Lambda, Q_0) \to (H^n(X', \mathbb{Z})/\mathrm{Tor}, Q)$ composed with the induced map

$$f^* : (H^n(X', \mathbb{Z})/\mathrm{Tor}, Q) \to (H^n(X, \mathbb{Z})/\mathrm{Tor}, Q).$$

Let $\mathcal{Z}_m$ be one of the irreducible components of the moduli space of polarized manifolds with level m structure, which parameterizes the isomorphism classes of polarized manifolds with level m structure.

From now on, we will assume that the irreducible component $\mathcal{Z}_m$ defined as above is a smooth complex manifold with an analytic family $f_m : \mathcal{U}_m \to \mathcal{Z}_m$ of polarized manifolds with level m structure for all $m \geq m_0$, where $m_0 \geq 3$ is some fixed integer. For simplicity, we can also assume that $m_0 = 3$.

Let $\mathcal{T}_m$ be the universal cover of $\mathcal{Z}_m$ with covering map $\pi_m : \mathcal{T}_m \to \mathcal{Z}_m$. Then we have an analytic family $g_m : \mathcal{V}_m \to \mathcal{T}_m$ such that the following diagram is cartesian

$$\begin{array}{ccc} \mathcal{V}_m & \longrightarrow & \mathcal{U}_m \\ \Big\downarrow{\scriptstyle g_m} & & \Big\downarrow{\scriptstyle f_m} \\ \mathcal{T}_m & \longrightarrow & \mathcal{Z}_m \end{array}$$

i.e. $\mathcal{V}_m = \mathcal{U}_m \times_{\mathcal{Z}_m} \mathcal{T}_m$. Such a family is called the pull-back family. We call $\mathcal{T}_m$ the Teichmüller space of polarized manifolds with level m structure.

The proof of the following lemma is obvious.

Lemma 5.1. *Assume that $\mathcal{Z}_m$ is smooth for $m \geq 3$, then the Teichmüller space $\mathcal{T}_m$ is smooth and the pull-back family $g_m : \mathcal{V}_m \to \mathcal{T}_m$ is an analytic family.*

We will give two proofs of the following lemma, which allows us to simply denote $\mathcal{T}_m$ by $\mathcal{T}$, the analytic family by $g : \mathcal{V} \to \mathcal{T}$ and the covering map by $\pi_m : \mathcal{T} \to \mathcal{Z}_m$.

Lemma 5.2. *The Teichmüller space $\mathcal{T}_m$ does not depend on the choice of m.*

Proof. The first proof uses the construction of moduli space with level m structure, see Lecture 10 of [32], or pages 692 – 693 of [41]. Let m_1 and m_2 be two different integers, and

$$\mathcal{U}_{m_1} \to \mathcal{Z}_{m_1},\ \mathcal{U}_{m_2} \to \mathcal{Z}_{m_2}$$

be two analytic families with level m_1 structure and level m_2 structure respectively. Let $\mathcal{T}_{m_1}$ and $\mathcal{T}_{m_2}$ be the universal covering space of $\mathcal{Z}_{m_1}$ and $\mathcal{Z}_{m_2}$ respectively. Let $m = m_1 m_2$ and consider the analytic family $\mathcal{U}_m \to \mathcal{Z}_m$.

From the discussion in page 130 of [32] or pages 692 – 693 of [41], we know that $\mathcal{Z}_m$ is a covering space of both $\mathcal{Z}_{m_1}$ and $\mathcal{Z}_{m_2}$. Let $\mathcal{T}$ be the universal covering space of $\mathcal{Z}_m$. Since $\mathcal{Z}_m$ is a covering space of both $\mathcal{Z}_{m_1}$ and $\mathcal{Z}_{m_2}$, we conclude that $\mathcal{T}$ is the universal cover of both $\mathcal{Z}_{m_1}$ and $\mathcal{Z}_{m_2}$, i.e.

$$\mathcal{T}_{m_1} \simeq \mathcal{T}_{m_2} \simeq \mathcal{T}.$$

If the analytic family $f_m : \mathcal{U}_m \to \mathcal{Z}_m$ is universal as defined in page 9 of [37], then we have a second proof. Let m_1, m_2 be two different integers ≥ 3, and let $\mathcal{T}_{m_1}$ and $\mathcal{T}_{m_2}$ be the corresponding Teichmüller space with the universal families

$$g_{m_1} : \mathcal{V}_{m_1} \to \mathcal{T}_{m_1},\ g_{m_2} : \mathcal{V}_{m_2} \to \mathcal{T}_{m_2}$$

respectively. Then for any point $p \in \mathcal{T}_{m_1}$ and the fiber $X_p = g_{m_1}^{-1}(p)$ over p, there exists $q \in \mathcal{T}_{m_2}$ such that $Y_q = g_{m_2}^{-1}(q)$ is biholomorphic to X_p. By the definition of universal family, we can find a local neighborhood U_p of p and a holomorphic map

$$h_p : U_p \to \mathcal{T}_{m_2},\ p \mapsto q$$

such that the map h_p is uniquely determined. Since $\mathcal{T}_{m_1}$ is simply-connected, all the local holomorphic maps

$$\{h_p : U_p \to \mathcal{T}_{m_2},\ p \in \mathcal{T}_{m_1}\}$$

patches together to give a global holomorphic map $h : \mathcal{T}_{m_1} \to \mathcal{T}_{m_2}$ which is well-defined. Moreover h is unique since it is unique on each local neighborhood of $\mathcal{T}_{m_1}$. Similarly we have a holomorphic map $h' : \mathcal{T}_{m_2} \to \mathcal{T}_{m_1}$ which is also unique. Then h and h' are inverse to each other by the uniqueness of h and h'. Therefore $\mathcal{T}_{m_1}$ and $\mathcal{T}_{m_2}$ are biholomorphic. $\square$

From now on we will denote $\mathcal{T} = \mathcal{T}_m$ for any $m \geq 3$ and call $\mathcal{T}$ the Teichmüller space of polarized manifolds.

As before, we can define the period map $\Phi_{\mathcal{Z}_m} : \mathcal{Z}_m \to D/\Gamma$ and the lifted period map $\widetilde{\Phi} : \mathcal{T} \to D$ such that the diagram

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\widetilde{\Phi}} & D \\ \Big\downarrow{\scriptstyle \pi_m} & & \Big\downarrow{\scriptstyle \pi} \\ \mathcal{Z}_m & \xrightarrow{\Phi_{\mathcal{Z}_m}} & D/\Gamma \end{array}$$

is commutative.

From now on, we will also assume that the local Torelli theorem holds, i.e. the tangent map of the lifted period map $\widetilde{\Phi}$ is injective at every point of the Teichmüller space $\mathcal{T}$.

Now we prove a lemma concerning the monodromy group Γ on $\mathcal{Z}_m$ with level $m \geq 3$, which will be used in the following discussion.

Lemma 5.3. *Let γ be the image of some element of $\pi_1(\mathcal{Z}_m)$ in Γ under the monodromy representation. Suppose that γ is of finite order, then γ is trivial. Therefore we can assume that Γ is torsion-free and D/Γ is smooth.*

Proof. Look at the period map locally as $\Phi_{\mathcal{Z}_m} : \Delta^* \to D/\Gamma$. Assume that γ is the monodromy action corresponding to the generator of the fundamental group of Δ^*. We lift the period map to $\widetilde{\Phi} : \mathbb{H} \to D$, where $\mathbb{H}$ is the upper half plane and the covering map from $\mathbb{H}$ to Δ^* is

$$z \mapsto \exp(2\pi\sqrt{-1}z).$$

Then $\widetilde{\Phi}(z+1) = \gamma\widetilde{\Phi}(z)$ for any $z \in \mathbb{H}$. But $\widetilde{\Phi}(z+1)$ and $\widetilde{\Phi}(z)$ correspond to the same point when descending onto $\mathcal{Z}_m$, therefore by the definition of $\mathcal{Z}_m$ we have

$$\gamma \equiv \mathrm{I} \bmod (m).$$

But γ is also in $\mathrm{Aut}(H_{\mathbb{Z}})$, applying Serre's lemma [35] or Lemma 2.4 in [41], we have $\gamma = \mathrm{I}$. □

As in the discussion before Lemma 1.4, we now assume that $\mathcal{Z}_m$ is quasi-projective such that $\mathcal{Z}_m$ admits a compactification $\overline{\mathcal{Z}}_m$ such that $\overline{\mathcal{Z}}_m$ is a smooth projective variety, and $\overline{\mathcal{Z}}_m \setminus \mathcal{Z}_m$ consists of divisors with simple normal crossings.

Let $\mathcal{Z}_m' \supseteq \mathcal{Z}_m$ be the maximal subset of $\overline{\mathcal{Z}}_m$ to which the period map $\Phi_{\mathcal{Z}_m} : \mathcal{Z}_m \to D/\Gamma$ extends continuously and let $\Phi_{\mathcal{Z}_m'} : \mathcal{Z}_m' \to D/\Gamma$ be the extended map. Then one has the commutative diagram

$$\mathcal{Z}_m \xrightarrow{i} \mathcal{Z}_m' \xrightarrow{\Phi_{\mathcal{Z}_m'}} D/\Gamma, \qquad \Phi_{\mathcal{Z}_m} = \Phi_{\mathcal{Z}_m'} \circ i.$$

with $i : \mathcal{Z}_m \to \mathcal{Z}'_m$ is the inclusion map.

Since $\overline{\mathcal{Z}}_m \setminus \mathcal{Z}_m$ is a divisor with simple normal crossings, for any point in $\overline{\mathcal{Z}}_m \setminus \mathcal{Z}_m$ we can find a neighborhood U of that point, which is isomorphic to a polycylinder Δ^n, such that

$$U \cap \mathcal{Z}_m \simeq (\Delta^*)^k \times \Delta^{N-k}.$$

Let T_i, $1 \le i \le k$ be the image of the generator of the fundamental group of the i-th copy Δ^* in $(\Delta^*)^k$ under the representation of monodromy action, then the T_i's are called the Picard-Lefschetz transformations. Let us define the subspace $\mathcal{Z}''_m \subset \overline{\mathcal{Z}}_m$ which contains $\mathcal{Z}_m$ and the points in $\overline{\mathcal{Z}}_m \setminus \mathcal{Z}_m$ around which the Picard-Lefschetz transformations are of finite order, hence trivial by Lemma 5.3.

In the proof of Lemma 1.4, we have proved that $\mathcal{Z}'_m = \mathcal{Z}''_m$ is Zariski open in $\overline{\mathcal{Z}}_m$ and the extended period map $\Phi_{\mathcal{Z}'_m} : \mathcal{Z}'_m \to D/\Gamma$ is a proper holomorphic map.

In summary we have the following proposition which is just Theorem 9.6 in [11].

Proposition 5.4. *$\mathcal{Z}'_m$ is an open complex submanifold of $\overline{\mathcal{Z}}_m$ with $\mathrm{codim}_{\mathbb{C}}(\overline{\mathcal{Z}}_m \setminus \mathcal{Z}'_m) \ge 1$ and $\mathcal{Z}'_m \setminus \mathcal{Z}_m$ consists of the points around which the Picard-Lefschetz transformations are trivial. Moreover the extended period map $\Phi_{\mathcal{Z}'_m} : \mathcal{Z}'_m \to D/\Gamma$ is a proper holomorphic map.*

Recall that we have assumed that the local Torelli theorem holds at each point in the Teichmüller space, we will use it to give a geometric interpretation of $\mathcal{Z}'_m$, which we call the Hodge metric completion of $\mathcal{Z}_m$.

In [14], Griffiths and Schmid studied the natural homogeneous metric, which we call the Hodge metric, on the period domain D. In particular, the Hodge metric h is a complete homogeneous metric on D. Assuming local Torelli, then the tangent maps of the period map $\Phi_{\mathcal{Z}_m}$ and the lifted period map $\widetilde{\Phi}$ are injective. It follows from [14] that the pull-back metrics of h on $\mathcal{Z}_m$ and $\mathcal{T}$, via $\Phi_{\mathcal{Z}_m}$ and $\widetilde{\Phi}$ respectively, are both well-defined Kähler metrics. For convenience we still call the pull-back metrics the Hodge metrics.

Let us take $\mathcal{Z}^H_m$ to be the completion of $\mathcal{Z}_m$ in $\overline{\mathcal{Z}}_m$ with respect to the Hodge metric. Then $\mathcal{Z}^H_m$ is the smallest complete space with respect to the

Hodge metric that contains $\mathcal{Z}_m$. Here we recall some basic properties about metric completion space we are using in this paper. First note that the metric completion space of a connected space is still connected. Therefore, $\mathcal{Z}_m^H$ is connected.

Suppose (X, d) is a metric space with the metric d. Then the metric completion space of (X, d) is unique in the following sense: if $\overline{X}_1$ and $\overline{X}_2$ are complete metric spaces that both contain X as a dense subset, then there exists an isometry

$$f : \overline{X}_1 \to \overline{X}_2$$

such that $f|_X$ is the identity map on X. Moreover, the metric completion space $\overline{X}$ of X is the smallest complete metric space containing X in the sense that any other complete space that contains X as a subspace must also contain $\overline{X}$ as a subspace. Hence the Hodge metric completion space $\mathcal{Z}_m^H$ is unique up to isometry, although the compact space $\overline{\mathcal{Z}}_m$ may not be unique. This means that our definition of $\mathcal{Z}_m^H$ is intrinsic.

Moreover, suppose $\overline{X}$ is the metric completion space of the metric space (X, d). If there is a continuous map $f : X \to Y$ which is an isometry with Y a complete space, then there exists a continuous extension $\overline{f} : \overline{X} \to Y$ such that $\overline{f}|_X = f$. Since D/Γ together with the Hodge metric h is complete, we can extend the period map to a continuous map

$$\Phi_{\mathcal{Z}_m^H} : \mathcal{Z}_m^H \to D/\Gamma.$$

Thus $\mathcal{Z}_m^H \subseteq \mathcal{Z}'_m$ by the definition of $\mathcal{Z}'_m$. Conversely the points in $\mathcal{Z}'_m \setminus \mathcal{Z}_m$ are mapped into D/Γ, and hence lie in the Hodge metric completion $\mathcal{Z}_m^H$ of $\mathcal{Z}_m$. Then $\mathcal{Z}'_m \subseteq \mathcal{Z}_m^H$. Therefore together with Proposition 5.4 we have proved the following proposition.

Proposition 5.5. *$\mathcal{Z}'_m = \mathcal{Z}_m^H$ is an open complex submanifold of $\overline{\mathcal{Z}}_m$ with $codim_{\mathbb{C}}(\overline{\mathcal{Z}}_m \setminus \mathcal{Z}_m^H) \geq 1$. In particular the definition of $\mathcal{Z}'_m$ is intrinsic, i.e. independent of the embedding $\mathcal{Z}_m \subseteq \overline{\mathcal{Z}}_m$. Moreover the extended period map $\Phi_{\mathcal{Z}_m^H} : \mathcal{Z}_m^H \to D/\Gamma$ is proper and holomorphic.*

From now on we will use notation $\mathcal{Z}_m^H$ to denote the extension of $\mathcal{Z}_m$. We define $\mathcal{T}_m^H$ to be the universal covering space of $\mathcal{Z}_m^H$ with the covering map

$\pi_m^H : \mathcal{T}_m^H \to \mathcal{Z}_m^H$. Since $\mathcal{Z}_m' = \mathcal{Z}_m^H$, diagram (10) can be rewritten as

$$\begin{array}{ccccc} \mathcal{T} & \xrightarrow{i_m} & \mathcal{T}_m^H & \xrightarrow{\Phi_m^H} & D \\ \downarrow{\scriptstyle \pi_m} & & \downarrow{\scriptstyle \pi_m^H} & & \downarrow{\scriptstyle \pi_D} \\ \mathcal{Z}_m & \xrightarrow{i} & \mathcal{Z}_m^H & \xrightarrow{\Phi_{\mathcal{Z}_m^H}} & D/\Gamma, \end{array} \tag{24}$$

where i_m is the lifting of $i \circ \pi_m$ with respect to the covering map $\pi_m^H : \mathcal{T}_m^H \to \mathcal{Z}_m^H$ and Φ_m^H is the lifting of $\Phi_{\mathcal{Z}_m^H} \circ \pi_m^H$ with respect to the covering map $\pi_D : D \to D/\Gamma$. Similar to (10), we also have that Φ_m^H is holomorphic, and we can choose the lifts i_m and Φ_m^H such that $\widetilde{\Phi} = \Phi_m^H \circ i_m$.

Similarly, if we define $\mathcal{T}_m \subseteq \mathcal{T}_m^H$ by $\mathcal{T}_m = i_m(\mathcal{T})$, Lemma 1.5 and Lemma 1.6 becomes the following two lemmas.

Lemma 5.6. *$\mathcal{T}_m = (\pi_m^H)^{-1}(\mathcal{Z}_m)$, and $i_m : \mathcal{T} \to \mathcal{T}_m$ is a covering map.*

Lemma 5.7. *The extended holomorphic map $\Phi_m^H : \mathcal{T}_m^H \to D$ satisfies the Griffiths transversality.*

6 Affine structures on Teichmüller spaces

In this section, we first apply Theorem 4.1 and Corollary 4.2 to the Teichmüller spaces defined in last section. Then we construct a holomorphic map $\Psi : \mathcal{T} \to A \cap D$ where A is an Euclidean subspace of N_+, and prove that this holomorphic map is an immersion, therefore defines a global complex affine structure on $\mathcal{T}$.

By Theorem 4.1, we have that the image of the period map $\widetilde{\Phi} : \mathcal{T} \to N_+ \cap D$ from the Teichmüller space $\mathcal{T}$ is bounded in N_+. Moreover we rephrase Corollary 4.2 in the setting of Techmüller space.

Corollary 6.1. *The image of the extended period map $\Phi_m^H : \mathcal{T}_m^H \to D$ lies in $N_+ \cap D$ and is bounded with respect to the Euclidean metric on N_+.*

Theorem 4.1 also implies that there exist bounded holomorphic functions $\Phi_{ij} : \mathcal{T} \to \mathbb{C}$ on $\mathcal{T}$, which is defined by the (i,j)-th entry of $\widetilde{\Phi}(q)$ for any $q \in \mathcal{T}$, where $\widetilde{\Phi}(q)$ is considered as a matrix in N_+.

Let us consider

$$\mathfrak{a} = d\widetilde{\Phi}_p(\mathrm{T}^{1,0}_p\mathcal{T}) \subseteq \mathrm{T}^{1,0}_o D \simeq \mathfrak{n}_+$$

where p is the base point in $\mathcal{T}$ with $\widetilde{\Phi}(p) = o$. By Griffiths transversality, $\mathfrak{a} \subseteq \mathfrak{g}^{-1,1}$ is an abelian subspace, therefore $\mathfrak{a} \subseteq \mathfrak{n}_+$ is an abelian subalgebra of $\mathfrak{n}_+$ determined by the tangent map of the period map

$$d\widetilde{\Phi} : \mathrm{T}^{1,0}\mathcal{T} \to \mathrm{T}^{1,0}D.$$

Consider the corresponding Lie group

$$A \triangleq \exp(\mathfrak{a}) \subseteq N_+.$$

Then A can be considered as a complex Euclidean subspace of N_+ with the induced Euclidean metric from N_+.

Define the projection map $P : N_+ \cap D \to A \cap D$ by

$$P = \exp \circ p \circ \exp^{-1}$$

where $\exp^{-1} : N_+ \to \mathfrak{n}_+$ is the inverse of the isometry $\exp : \mathfrak{n}_+ \to N_+$, and

$$p : \mathfrak{n}_+ \to \mathfrak{a}$$

is the projection map from the complex Euclidean space $\mathfrak{n}_+$ to its Euclidean subspace $\mathfrak{a}$.

The period map $\widetilde{\Phi} : \mathcal{T} \to N_+ \cap D$ composed with the projection P gives a holomorphic map, $\Psi = P \circ \widetilde{\Phi}$,

$$\Psi : \mathcal{T} \to A \cap D. \tag{25}$$

Let us recall the definition of complex affine structure on a complex manifold.

Definition 6.2. *Let M be a complex manifold of complex dimension n. If there is a coordinate cover $\{(U_i, \varphi_i);\, i \in I\}$ of M such that $\varphi_{ik} = \varphi_i \circ \varphi_k^{-1}$ is a holomorphic affine transformation on $\mathbb{C}^n$ whenever $U_i \cap U_k$ is not empty, then $\{(U_i, \varphi_i);\, i \in I\}$ is called a complex affine coordinate cover on M and it defines a holomorphic affine structure on M.*

We will prove that the map in (25) defines a global affine structure on the Teichmüller space $\mathcal{T}$, under the conditions in the following definition.

Definition 6.3. *The period map $\widetilde{\Phi}: \mathcal{T} \to D$ is said to satisfy strong local Torelli, if the local Torelli holds, i.e. the tangent map of the period map $\widetilde{\Phi}$ is injective at each point of $\mathcal{T}$, and furthermore, there exists a holomorphic subbundle $\mathcal{H}$ of the Hodge bundle*

$$\bigoplus_{k=1}^{n} \mathrm{Hom}(\mathscr{F}^k/\mathscr{F}^{k+1}, \mathscr{F}^{k-1}/\mathscr{F}^k),$$

such that the tangent map of the period map induces an isomorphism on $\mathcal{T}$ of the tangent bundle $\mathrm{T}^{1,0}\mathcal{T}$ of $\mathcal{T}$ to the Hodge subbundle $\mathcal{H}$,

$$d\widetilde{\Phi}: \mathrm{T}^{1,0}\mathcal{T} \xrightarrow{\sim} \mathcal{H}. \tag{26}$$

Recall that $\mathscr{F}^k$, $0 \leq k \leq n$ are Hodge bundles with fiber given by the corresponding linear subspaces in the Hodge filtration F_q^k for any $q \in \mathcal{T}$. Note that for variation of Hodge structure from geometry, the Hodge bundles naturally exist on $\mathcal{T}$ which are the same as pull-backs of the corresponding Hodge bundles on D. Therefore in the above definition, for convenience we have identified the Hodge bundles on $\mathcal{T}$ to the corresponding Hodge bundles on D.

The most famous example for which strong local Torelli holds are Calabi–Yau manifolds with the Hodge subbundle $\mathcal{H}$ given by $\mathrm{Hom}(\mathscr{F}^n, \mathscr{F}^{n-1}/\mathscr{F}^n)$. For more examples for which strong local Torelli holds, please see Section 2 of [25].

Theorem 6.4. *Assume that the irreducible component $\mathcal{Z}_m$ of the moduli space with level m structure is smooth with an analytic family for $m \geq m_0$ and the strong local Torelli holds for $\mathcal{T}$. Then there exists a complex affine structure on the Teimüller space $\mathcal{T}$.*

Proof. Let $\mathcal{H}$ be the Hodge subbundle in Definition 6.3 with the isomorphism

$$d\widetilde{\Phi}: \mathrm{T}^{1,0}\mathcal{T} \xrightarrow{\sim} \mathcal{H}. \tag{27}$$

Let $\mathcal{H}_A$ be the restriction of $\mathcal{H}$ to $A \cap D$. Since $\mathcal{H}_A$ is a Hodge subbundle, we have the natural commutative diagram

$$\begin{array}{ccc} \mathcal{H}_A|_o & \xrightarrow[\simeq]{d\exp(X)} & \mathcal{H}_A|_s \\ \downarrow \simeq & & \downarrow \\ \mathrm{T}^{1,0}_o(A \cap D) & \xrightarrow[\simeq]{d\exp(X)} & \mathrm{T}^{1,0}_s(A \cap D) \end{array}$$

at any point $s = \exp(X)$ in $A \cap D$ with $X \in \mathfrak{a}$. Here $d\exp(X)$ is the tangent map of the left translation by $\exp(X)$. Here note that the action $d\exp(X)$ on $\mathcal{H}_A$ is induced from the action on the homogeneous tangent bundle $\mathrm{T}^{1,0}D$, since $\mathcal{H}$ is a homogeneous subbundle of $\mathrm{T}^{1,0}D$. See [8] for the basic properties of homogeneous vector bundles.

Hence we have the isomorphism of tangent spaces $\mathrm{T}^{1,0}_s(A \cap D) \simeq \mathcal{H}_A|_s$ for any $s \in A \cap D$, which implies that

$$\mathrm{T}^{1,0}(A \cap D) \simeq \mathcal{H}_A$$

as holomorphic bundles on $A \cap D$. Note that the tangent map of the projection map $P : N_+ \cap D \to A \cap D$ maps the tangent bundle of D,

$$\mathrm{T}^{1,0}D \subset \bigoplus_{k=1}^{n} \bigoplus_{l=1}^{k} \mathrm{Hom}(\mathscr{F}^k/\mathscr{F}^{k+1}, \mathscr{F}^{k-l}/\mathscr{F}^{k-l+1})$$

onto its subbundle $\mathrm{T}^{1,0}(A \cap D) \simeq \mathcal{H}_A$. Therefore by using the translation invariance of the Hodge subbundle and the same argument as in the proof of Corollary 3.2, we deduce that the tangent map

$$d\Psi = dP \circ d\widetilde{\Phi} : \mathrm{T}^{1,0}\mathcal{T} \to \mathrm{T}^{1,0}(A \cap D)$$

at any point $q \in \mathcal{T}$ is explicitly given by

$$d\Psi_q : \mathrm{T}^{1,0}_q\mathcal{T} \to \oplus_{k=1}^{n}\mathrm{Hom}(\mathscr{F}^k_q/\mathscr{F}^{k+1}_q, \mathscr{F}^{k-1}_q/\mathscr{F}^k_q) \to \mathcal{H}_A|_{\Psi(q)} \simeq \mathrm{T}^{1,0}_{\Psi(q)}(A \cap D)$$

which is an isomorphism by (27). So we have proved that the holomorphic map $\Psi : \mathcal{T} \to A \cap D$ is nondegenerate which induces an affine structure on $\mathcal{T}$ from A. $\square$

It is well-known that the Teichmüller spaces of Riemann surfaces and hyperkähler manifolds have complex affine structures. There are many more examples of projective manifolds that satisfy the conditions in our theorem. In Section 2 of [25], we have verified that the moduli and Teichmüller spaces from the following examples satisfy the conditions of Theorem 6.4: K3 surfaces; Calabi-Yau manifolds; hyperkähler manifolds; smooth hypersurface of degree d in $\mathbb{P}^{n+1}$ satisfying $d|(n+2)$ and $d \geq 3$; arrangement of m hyperplanes in $\mathbb{P}^n$ with $m \geq n$; smooth cubic surface and cubic threefold. Hence their Teichmüller spaces all have complex affine structures.

A Two topological lemmas

In this appendix we give the proofs of two elementary topological lemmas that are used in the paper. They may be well-known. We include them here for reader's convenience.

Lemma A.1. *There exists a suitable choice of $i_{\mathcal{T}}$ and $\widetilde{\Phi}'$ such that $\widetilde{\Phi}' \circ i_{\mathcal{T}} = \widetilde{\Phi}$.*

Proof. Recall the following commutative diagram as in (10)

$$\begin{array}{ccccc} \mathcal{T} & \xrightarrow{i_{\mathcal{T}}} & \mathcal{T}' & \xrightarrow{\widetilde{\Phi}'} & D \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi_D} \\ S & \xrightarrow{i} & S' & \xrightarrow{\Phi'} & D/\Gamma. \end{array} \tag{28}$$

Fix a reference point $p \in \mathcal{T}$. The relations $i \circ \pi = \pi' \circ i_{\mathcal{T}}$ and $\Phi' \circ \pi' = \pi_D \circ \widetilde{\Phi}'$ imply that

$$\pi_D \circ \widetilde{\Phi}' \circ i_{\mathcal{T}} = \Phi' \circ i \circ \pi = \Phi \circ \pi.$$

Therefore $\widetilde{\Phi}' \circ i_{\mathcal{T}}$ is a lifting map of Φ. On the other hand $\widetilde{\Phi} : \mathcal{T} \to D$ is also a lifting of Φ. In order to make $\widetilde{\Phi}' \circ i_{\mathcal{T}} = \widetilde{\Phi}$, one only needs to choose the suitable $i_{\mathcal{T}}$ and $\widetilde{\Phi}'$ such that these two maps agree on the reference point, i.e. $\widetilde{\Phi}' \circ i_{\mathcal{T}}(p) = \widetilde{\Phi}(p)$.

For an arbitrary choice of $i_{\mathcal{T}}$, we have $i_{\mathcal{T}}(p) \in \mathcal{T}'$ and $\pi'(i_{\mathcal{T}}(p)) = i(\pi(p))$. Considering the point $i_{\mathcal{T}}(p)$ as a reference point in $\mathcal{T}'$, we can choose $\widetilde{\Phi}'(i_{\mathcal{T}}(p))$

to be any point in

$$\pi_D^{-1}(\Phi'(i(\pi(p)))) = \pi_D^{-1}(\Phi(\pi(p))).$$

Moreover the relation $\pi_D(\widetilde{\Phi}(p)) = \Phi(\pi(p))$ implies that

$$\widetilde{\Phi}(p) \in \pi_D^{-1}(\Phi(\pi(p))).$$

Therefore we can choose $\widetilde{\Phi}'$ such that $\widetilde{\Phi}'(i_{\mathcal{T}}(p)) = \widetilde{\Phi}(p)$. With this choice, we have $\widetilde{\Phi}' \circ i_{\mathcal{T}} = \widetilde{\Phi}$. $\square$

Lemma A.2. *Let $\pi_1(S)$ and $\pi_1(S')$ be the fundamental groups of S and S' respectively, and suppose the group morphism*

$$i_* : \pi_1(S) \to \pi_1(S')$$

is induced by the inclusion $i : S \to S'$. Then i_* is surjective.

Proof. First notice that S and S' are both smooth manifolds, and $S \subseteq S'$ is open. Thus for each point $q \in S' \setminus S$ there is a disc $D_q \subseteq S'$ with $q \in D_q$. Then the union of these discs

$$\bigcup_{q \in S' \setminus S} D_q$$

forms a manifold with open cover $\{D_q : q \in \cup_q D_q\}$. Because both S and S' are second countable spaces, there is a countable subcover $\{D_i\}_{i=1}^{\infty}$ such that $S' = S \cup \bigcup_{i=1}^{\infty} D_i$, where the D_i are open discs in S' for each i. Therefore, we have $\pi_1(D_i) = 0$ for all $i \geq 1$. Letting $S_k = S \cup \bigcup_{i=1}^{k} D_i$, we get

$$\pi_1(S_k) * \pi_1(D_{k+1}) = \pi_1(S_k) = \pi_1(S_{k-1} \cup D_k), \quad \text{for any } k.$$

We know that $\operatorname{codim}_{\mathbb{C}}(S' \backslash S) \geq 1$. Therefore since $D_{k+1} \backslash S_k \subseteq D_{k+1} \backslash S$, we have

$$\operatorname{codim}_{\mathbb{C}}(D_{k+1} \backslash S_k) \geq 1$$

for any k. As a consequence we can conclude that $D_{k+1} \cap S_k$ is path-connected. Hence we can apply the Van Kampen Theorem on $S_{k+1} = D_{k+1} \cup S_k$ to conclude that for every k, the following group homomorphism is surjective:

$$\pi_1(S_k) = \pi_1(S_k) * \pi_1(D_{k+1}) \twoheadrightarrow \pi_1(S_k \cup D_{k+1}) = \pi_1(S_{k+1}).$$

Thus we get the directed system:

$$\pi_1(S) \twoheadrightarrow \pi_1(S_1) \twoheadrightarrow \cdots\cdots \twoheadrightarrow \pi_1(S_k) \twoheadrightarrow \cdots\cdots$$

By taking the direct limit of this directed system, we get the surjectivity of the group homomorphism $\pi_1(S) \to \pi_1(S')$.

□

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Center of Mathematical Sciences, Zhejiang University, Hangzhou, Zhejiang 310027, China;
Department of Mathematics, University of California at Los Angeles, Los Angeles, CA 90095-1555, USA
e-mail: liu@math.ucla.edu, kefeng@cms.zju.edu.cn

Center of Mathematical Sciences, Zhejiang University, Hangzhou, Zhejiang 310027, China
e-mail: syliuguang2007@163.com