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Rigidity of Einstein manifolds with positive scalar curvature

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Abstract We prove that if $M^n (n \geq 4)$ is a compact Einstein manifold whose normalized scalar curvature and sectional curvature satisfy pinching condition $R_0 > \sigma_n K_{\max}$, where $\sigma_n \in (\frac{1}{4}, 1)$ is an explicit positive constant depending only on n, then M must be isometric to a spherical space form. Moreover, we prove that if an $n(\geq 4)$-dimensional compact Einstein manifold satisfies $K_{\min} \geq \eta_n R_0$, where $\eta_n \in (\frac{1}{4}, 1)$ is an explicit positive constant, then M is locally symmetric. It should be emphasized that the pinching constant η_n is optimal when n is even. We then obtain some rigidity theorems for Einstein manifolds under $(n-2)$-th Ricci curvature and normalized scalar curvature pinching conditions. Finally we extend the theorems above to Einstein submanifolds in a Riemannian manifold, and prove that if M is an $n(\geq 4)$-dimensional compact Einstein submanifold in the simply connected space form $F^N(c)$ with constant curvature $c \geq 0$, and the normalized scalar curvature R_0 of M satisfies $R_0 > \frac{A_n}{A_n+4n-8}(c+H^2)$, where $A_n = n^3 - 5n^2 + 8n$, and H is the mean curvature of M, then M is isometric to a standard n-sphere.

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1 Introduction

It plays an important role in Riemannian geometry to study the rigidity of Einstein manifolds. To classify the Einstein manifolds satisfying some curvature pinching condition is an important problem, which was initiated by Berger [1]. In 1974, Tachibana [23] proved that a compact Einstein manifold with positive curvature operator is isometric to a spherical space form. Later, Micallef and Wang [16] proved that a four-dimensional Einstein manifold with nonnegative isotropic curvature is locally symmetric. In 2000, Yang [28] obtained the following rigidity theorem on four-dimensional Einstein manifolds with positive sectional curvature.

Theorem A *Let M be a 4-dimensional compact Einstein manifold with $Ric_M = 3$. If the sectional curvature of M satisfies*

$$K_M \geq \epsilon_0 \equiv \frac{\sqrt{1249} - 23}{40} \approx 0.308529,$$

then M is isometric to either the unit 4-sphere S^4, the 4-dimensional real projective space $\mathbb{R}P^4$, or the complex projective space $\mathbb{C}P^2$.

In 2003, the pinching constant above was improved to 0.292893 by de Araujo Costa in [9]. More discussions about the Einstein manifolds can be seen in [2,13,19], etc. Recently, Brendle [8] generalized Micallef and Wang's theorem [16] for 4-dimensional Einstein manifolds to higher dimensional cases.

Theorem B *Let M be an $n(\geq 4)$-dimensional compact Einstein manifold. If M has positive isotropic curvature, then M is isometric to a spherical space form. Moreover, if M has nonnegative isotropic curvature, then M is locally symmetric.*

Let $K(\pi)$ be the sectional curvature of M for 2-plane $\pi \subset T_xM$. Set $K_{\max}(x) := \max_{\pi \subset T_xM} K(\pi)$, $K_{\min}(x) := \min_{\pi \subset T_xM} K(\pi)$. Denote by $Ric^{(k)}$ the k-th Ricci curvature of M (see Definition 2.2 below). The following rigidity theorem can be viewed as a consequence of Theorem B.

Theorem C *Let M be an $n(\geq 4)$-dimensional compact Einstein manifold. If $K_{\min} \geq \frac{1}{4}K_{\max}$, and the strict inequality holds for some point $x_0 \in M$, then M is isometric to a spherical space form.*

The purpose of this paper is to prove some new rigidity theorems for Einstein manifolds and submanifolds. In Sect. 3, we prove the following rigidity theorem for compact Einstein manifolds with positive scalar curvature.

Theorem 1.1 *Let M be an $n(\geq 4)$-dimensional compact Einstein manifold. Denote by $R_0 := c$ the normalized scalar curvature of M. We have*

(i) *If $R_0 > \sigma_n K_{\max}$, then M is isometric to a spherical space form of constant curvature c.*

(ii) *If* $K_{\min} \geq \eta_n R_0 > 0$, *then* M *is locally symmetric. In particular, if* M *is simply connected, then* M *is isometric to either the standard n-sphere* $S^n(\frac{1}{\sqrt{c}})$ *or the complex projective space* $\mathbb{C}P^m(\tilde{c})$ *with* $n = 2m$.

Here

$$\begin{aligned}\sigma_n &= 1 - \frac{6}{5(n-1)},\\ \eta_n &= 1 - \frac{3}{n+2},\\ \tilde{c} &= \frac{4(n-1)}{n+2}c.\end{aligned}$$

Furthermore, we obtain the following rigidity theorem.

Theorem 1.2 *Let* M *be an* $n(\geq 4)$*-dimensional compact Einstein manifold. Denote by* $R_0 := c$ *and* $Ric^{(n-2)}$ *the normalized scalar curvature and the* $(n-2)$*-th Ricci curvature of* M. *We have*

(i) *If* $Ric^{(n-2)}_{\min} > \tau_n(n-2)R_0$, *then* M *is isometric to a spherical space form of constant curvature c.*
(ii) *If* $(n-2)R_0 \geq \mu_n Ric^{(n-2)}_{\max} > 0$, *then* M *is locally symmetric. In particular, if* M *is simply connected, then* M *is isometric to either the standard n-sphere* $S^n(\frac{1}{\sqrt{c}})$ *or the complex projective space* $\mathbb{C}P^m(\tilde{c})$ *with* $n = 2m$.

Here

$$\begin{aligned}\tau_n &= 1 - \frac{6}{(n-2)(5n-11)},\\ \mu_n &= 1 - \frac{3}{(n-1)(n+1)},\\ \tilde{c} &= \frac{4(n-1)}{n+2}c.\end{aligned}$$

Remark 1.1 We see from Example 3.1 that the pinching constants η_n and μ_n are optimal when n is even.

Let M be an n-dimensional compact submanifold in an N-dimensional Riemannian manifold $\overline{M}^N$ with mean curvature H. In Sect. 4, we extend the theorems above to Einstein submanifolds in a Riemannian manifold with arbitrary codimension, and prove the following rigidity theorem.

Theorem 1.3 *Let* M *be an* $n(\geq 4)$*-dimensional compact Einstein submanifold in the Riemannian manifold* $\overline{M}^N$. *If* $S \leq \frac{16}{3}\left(\overline{K}_{\min} - \frac{1}{4}\overline{K}_{\max}\right) + \frac{n^2H^2}{n-2}$, *and the strict inequality holds for some point* $x_0 \in M$, *then* M *is isometric to a spherical space form.*

Remark 1.2 When M is a compact Einstein submanifold of codimension zero, Theorem 1.3 reduces to Theorem C.

In particular, we obtain the following rigidity theorem for Einstein submanifolds in a space form.

Theorem 1.4 *Let M be an $n(\geq 4)$-dimensional compact Einstein submanifold in the simply connected space form $F^N(c)$ with constant curvature c. If the normalized scalar curvature R_0 of M satisfies*

$$R_0 > \frac{A_n}{A_n + 4n - 8}(c + H^2),$$

where $A_n = n^3 - 5n^2 + 8n$, then M is isometric to a spherical space form. Moreover, if $c \geq 0$, then M is isometric to a standard n-sphere.

2 Notation and lemmas

Let M^n be an $n(\geq 4)$-dimensional submanifold in an N-dimensional Riemannian manifold $\overline{M}^N$. We shall make use of the following convention on the range of indices.

$$1 \leq A, B, C, \ldots \leq N; \;\; 1 \leq i, j, k, \ldots \leq n;$$
$$\text{if } N \geq n + 1, \;\; n + 1 \leq \alpha, \beta, \gamma, \ldots \leq N.$$

For an arbitrary fixed point $x \in M \subset \overline{M}$, we choose an orthonormal local frame field $\{e_A\}$ in $\overline{M}^N$ such that e_i's are tangent to M. Denote by $\{\omega_A\}$ the dual frame field of $\{e_A\}$. Let

$$Rm = \sum_{i,j,k,l} R_{ijkl}\omega_i \otimes \omega_j \otimes \omega_k \otimes \omega_l,$$
$$\overline{Rm} = \sum_{A,B,C,D} \overline{R}_{ABCD}\omega_A \otimes \omega_B \otimes \omega_C \otimes \omega_D$$

be the Riemannian curvature tensors of M and $\overline{M}$, respectively. Denote by h the second fundamental form of M. When $N = n$, h is identically equal to zero. When $N \geq n+1$, we set

$$h = \sum_{\alpha,i,j} h^\alpha_{ij}\omega_i \otimes \omega_j \otimes e_\alpha.$$

Then we have the Gauss equation

$$R_{ijkl} = \overline{R}_{ijkl} + \langle h(e_i, e_k), h(e_j, e_l)\rangle - \langle h(e_i, e_l), h(e_j, e_k)\rangle. \tag{2.1}$$

The squared norm S of the second fundamental form of M is given by

$$S := \sum_{\alpha,i,j} (h^\alpha_{ij})^2.$$

We put

$$\xi := \frac{1}{n}\sum_{\alpha,i} h^{\alpha}_{ii} e_{\alpha}, \quad H := \|\xi\| = \frac{1}{n}\sqrt{\sum_{\alpha}\Big(\sum_{i} h^{\alpha}_{ii}\Big)^2}.$$

Definition 2.1 (See also [29], P.349) ξ and H are called the mean curvature vector and mean curvature of M, respectively.

Denote by $K(\cdot)$, $\overline{K}(\cdot)$, $Ric(\cdot)$, $\overline{Ric}(\cdot)$, R and $\overline{R}$ the sectional curvatures, the Ricci curvatures and the scalar curvatures of M and $\overline{M}$, respectively. Then we have

$$Ric(e_i) = \sum_{j} R_{ijij}, \quad \overline{Ric}(e_A) = \sum_{B} \overline{R}_{ABAB},$$
$$R = \sum_{i,j} R_{ijij}, \quad \overline{R} = \sum_{A,B} \overline{R}_{ABAB}.$$

Set

$$K_{\min}(x) = \min_{\pi\subset T_xM} K(\pi), \ K_{\max}(x) = \max_{\pi\subset T_xM} K(\pi),$$
$$\overline{K}_{\min}(x) = \min_{\pi\subset T_x\overline{M}} \overline{K}(\pi), \ \overline{K}_{\max}(x) = \max_{\pi\subset T_x\overline{M}} \overline{K}(\pi).$$

Then by Berger's inequality (See e.g. [5], Proposition 1.9), we have

$$|R_{ijkl}| \le \frac{2}{3}(K_{\max} - K_{\min}) \tag{2.2}$$

for all distinct indices i, j, k, l, and

$$|\overline{R}_{ABCD}| \le \frac{2}{3}(\overline{K}_{\max} - \overline{K}_{\min}) \tag{2.3}$$

for all distinct indices A, B, C, D. We set

$$Ric_{\min}(x) = \min_{u\in U_xM} Ric(u), \ \overline{Ric}_{\min}(x) = \min_{u\in U_x\overline{M}} \overline{Ric}(u),$$
$$Ric_{\max}(x) = \max_{u\in U_xM} Ric(u), \ \overline{Ric}_{\max}(x) = \max_{u\in U_x\overline{M}} \overline{Ric}(u).$$

For any unit tangent vector $u \in U_xM$ at point $x \in M$, let V^k_x be a k-dimensional subspace of T_xM satisfying $u \perp V^k_x$. Choose an orthonormal basis $\{e_i\}$ in T_xM such that $e_{j_0} = u$, $span\{e_{j_1}, \ldots, e_{j_k}\} = V^k_x$, where the indices $1 \le j_0, j_1, \ldots, j_k \le n$ are distinct with each other. We set

$$Ric^{(k)}(u; V^k_x) = Ric^{(k)}([e_{j_0}, \ldots, e_{j_k}]) = \sum_{q=1}^{k} R_{j_0 j_q j_0 j_q}. \tag{2.4}$$

We extend an orthonormal s-frame $\{e_{j_0}, \dots, e_{j_{s-1}}\}$ in $T_x M$ to $(k+1)$-frame $\{e_{j_0}, \dots, e_{j_k}\}$ for $1 \le s \le k+1 \le n$ and set

$$R^{(k,s)}([e_{j_0}, \dots, e_{j_k}]) = \sum_{p=0}^{s-1} \sum_{q=0}^{k} R_{j_p j_q j_p j_q}, \tag{2.5}$$

$$R^{(k)}([e_{j_0}, \dots, e_{j_k}]) = R^{(k,k+1)}([e_{j_0}, \dots, e_{j_k}]) = \sum_{p=0}^{k} \sum_{q=0}^{k} R_{j_p j_q j_p j_q}. \tag{2.6}$$

Definition 2.2 We call $Ric^{(k)}(u; V_x^k)$, $R^{(k,s)}([e_{j_0}, \dots, e_{j_k}])$, and $R^{(k)}([e_{j_0}, \dots, e_{j_k}])$ the k-th Ricci curvature, (k, s)-curvature and k-th scalar curvatrure of M, respectively.

Denote by $Ric_{\min}^{(k)}(x)$, $R_{\min}^{(k,s)}(x)$ and $Ric_{\max}^{(k)}(x)$, $R_{\max}^{(k,s)}(x)$ the minimum and maximum of the k-th Ricci curvature and (k, s)-curvature at point $x \in M$ for any orthonormal $(k+1)$-frame in $T_x M$. Set

$$R_{\min}^{(k)} = R_{\min}^{(k,k+1)}, \quad R_{\max}^{(k)} = R_{\max}^{(k,k+1)}.$$

The geometry and topology of k-th Ricci curvature was initiated by Hartman [12] in 1979, and developed by Wu [24] and Shen [20,21], etc.. By the definition above, it is seen that the Ricci curvature of M is equal to the $(n-1)$-th Ricci curvature and $(n-1, 1)$-curvature; the scalar curvature of M is equal to $(n-1, n)$-curvature and $(n-1)$-th scalar curvature. If M is Einstein, then

$$Ric_{\min} = Ric_{\max} = \frac{R}{n} = constant. \tag{2.7}$$

For any unit tangent vector $u \in U_x\overline{M}$ at point $x \in \overline{M}$, let V_x^k be a k-dimensional subspace of $T_x\overline{M}$ satisfying $u \perp V_x^k$. Choose an orthonormal basis $\{e_A\}$ in $T_x\overline{M}$ such that $e_{A_0} = u,\ span\{e_{A_1}, \dots, e_{A_k}\} = V_x^k$, where the indices $1 \le A_0, A_1, \dots, A_k \le N$ are distinct with each other. We define the k-th Ricci curvature as follows.

$$\overline{Ric}^{(k)}(u; V_x^k) = \sum_{q=1}^{k} \overline{R}_{A_0 A_q A_0 A_q}. \tag{2.8}$$

Moreover, we define the k-th scalar curvature of $\overline{M}$ as follows.

$$\overline{R}^{(k)}([e_{A_0}, \dots, e_{A_k}]) = \sum_{p=0}^{k} \sum_{q=0}^{k} \overline{R}_{A_p A_q A_p A_q}. \tag{2.9}$$

Denote by $\overline{Ric}_{\min}^{(k)}(x)$, $\overline{R}_{\min}^{(k)}(x)$ and $\overline{Ric}_{\max}^{(k)}(x)$, $\overline{R}_{\max}^{(k)}(x)$ the minimum and maximum of the curvatures defined above at point $x \in \overline{M}$.

The following nonexistence theorem for stable currents in a compact Riemannian manifold M isometrically immersed into the simply connected space form $F^N(c)$ is employed to eliminate the homology groups $H_q(M; \mathbb{Z})$ for $0 < q < n$, which was initiated by Lawson-Simons [14] and extended by Xin [25].

Theorem 2.1 *Let M^n be a compact submanifold in $F^N(c)$ with $c \geq 0$. Assume that*

$$\sum_{k=q+1}^{n} \sum_{i=1}^{q} [2|h(e_i, e_k)|^2 - \langle h(e_i, e_i), h(e_k, e_k)\rangle] < q(n-q)c$$

holds for any orthonormal basis $\{e_i\}$ of T_xM at any point $x \in M$, where q is an integer satisfying $0 < q < n$. Then there are no stable q-currents in M. Moreover,

$$H_q(M; \mathbb{Z}) = H_{n-q}(M; \mathbb{Z}) = 0,$$

where $H_i(M; \mathbb{Z})$ is the i-th homology group of M with integer coefficients, and $\pi_1(M) = 0$ when $q = 1$.

For submanifolds with positive Ricci curvature, we have the following lemma.

Lemma 2.1 [26] *Let M be an $n(\geq 4)$-dimensional compact submanifold in $F^N(c)$ with $c \geq 0$. If the Ricci curvature of M satisfies*

$$Ric_M > \frac{n(n-1)}{n+2}(c + H^2),$$

then M is simply connected.

Proof From Gauss equation, we have

$$Ric(e_i) = (n-1)c + \sum_{\alpha,k} [h_{ii}^{\alpha} h_{kk}^{\alpha} - (h_{ik}^{\alpha})^2]. \tag{2.10}$$

Moreover, we have

$$-S + n^2H^2 + n(n-1)c = R \geq nRic_{\min}.$$

This implies that

$$S - nH^2 \leq n(n-1)(c + H^2) - nRic_{\min}. \tag{2.11}$$

It follows from (2.10), (2.11) and the assumption that

$$\begin{aligned}
&\sum_{k=2}^{n} [2|h(e_1, e_k)|^2 - \langle h(e_1, e_1), h(e_k, e_k)\rangle] - (n-1)c \\
&= 2\sum_{\alpha} \sum_{k=2}^{n} (h_{1k}^{\alpha})^2 - \sum_{\alpha} \sum_{k=2}^{n} h_{11}^{\alpha} h_{kk}^{\alpha} - (n-1)c
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\alpha}\sum_{k=2}^{n}(h_{1k}^{\alpha})^2 - Ric(e_1) \\
&\le \frac{1}{2}(S - nH^2) - Ric(e_1) \\
&\le \frac{1}{2}[n(n-1)(c+H^2) - (n+2)Ric_{\min}] \\
&< 0. \qquad (2.12)
\end{aligned}
$$

Hence the assertion follows from Theorem 2.1. This proves Lemma 2.1. □

Lemma 2.2 [11] *Let M be a compact Riemannian manifold of dimension n. If M has nonnegative isotropic curvature and has positive isotropic curvature for some point in M, then M admits a metric with positive isotropic curvature.*

3 Rigidity theorems for Einstein manifolds

In this section, we will give the proof of Theorems 1.1 and 1.2.

Lemma 3.1 *Let M be an $n(\ge 4)$-dimensional compact Einstein manifold. Assume M satisfies one of the following conditions:*

(i) $Ric_{\min}^{(k)} > (k - \frac{6}{5})K_{\max}$ *for some integer* $k \in [2, n-1]$*;*
(ii) $Ric_{\min}^{(k)} > \frac{5k-6}{5k-1}Ric_{\max}^{(k+1)}$ *for some integer* $k \in [2, n-2]$*;*
(iii) $Ric_{\min}^{(k)} > \frac{5k-6}{5k^2+9k-8}R_{\max}^{(k+1)}$ *for some integer* $k \in [2, n-2]$*;*
(iv) $Ric_{\min}^{(k)} > \frac{(k+2)(5k-6)}{s(5k^2+9k-8)}R_{\max}^{(k+1,s)}$ *for some integers* $k \in [2, n-2]$ *and* $s \in [2, k+2]$.

Then M is isometric to a spherical space form.

Proof (i) It follows from (2.4) that

$$K_{\min} \ge Ric_{\min}^{(k)} - (k-1)K_{\max}. \qquad (3.1)$$

Suppose $\{e_1, e_2, e_3, e_4\}$ is an orthonormal four-frame. It follows from Berger's inequality (2.2) that

$$R_{1234} \le \frac{2}{3}(K_{\max} - K_{\min}).$$

This together with (3.1) implies that

$$
\begin{aligned}
&R_{1313} + R_{2323} + R_{1414} + R_{2424} - 2R_{1234} \\
&\ge 2Ric_{\min}^{(k)} - 2(k-2)K_{\max} - \frac{4}{3}(K_{\max} - K_{\min}) \\
&\ge 2Ric_{\min}^{(k)} - 2(k-2)K_{\max} - \frac{4}{3}[kK_{\max} - Ric_{\min}^{(k)}] \\
&\ge \frac{10}{3}\Big[Ric_{\min}^{(k)} - \Big(k - \frac{6}{5}\Big)K_{\max}\Big]. \qquad (3.2)
\end{aligned}
$$

This together with the assumption implies that M has positive isotropic curvature. It follows from Theorem B that M is isometric to a spherical space form.

(ii) It's easy to get from (2.4) that

$$K_{\max} \leq Ric^{(k+1)}_{\max} - Ric^{(k)}_{\min}. \tag{3.3}$$

This together with the assumption implies that

$$\begin{aligned} Ric^{(k)}_{\min} &> \frac{5k-6}{5k-1} Ric^{(k+1)}_{\max} \\ &\geq \frac{5k-6}{5k-1}(K_{\max} + Ric^{(k)}_{\min}). \end{aligned}$$

A direct calculation show that $Ric^{(k)}_{\min} > (k - \frac{6}{5})K_{\max}$, and the conclusion follows from (i).

(iii) By Definition 2.2, we obtain

$$K_{\max} \leq \frac{1}{2}[R^{(k+1)}_{\max} - (k+3)Ric^{(k)}_{\min}]. \tag{3.4}$$

It follows from (3.4) and the assumption that

$$\begin{aligned} Ric^{(k)}_{\min} &> \frac{5k-6}{5k^2+9k-8} R^{(k+1)}_{\max} \\ &\geq \frac{5k-6}{5k^2+9k-8}[2K_{\max} + (k+3)Ric^{(k)}_{\min}]. \end{aligned}$$

Then we obtain $Ric^{(k)}_{\min} > (k - \frac{6}{5})K_{\max}$. This together with (i) implies that M has constant sectional curvature.

(iv) From (2.5) and (2.6), we have

$$\frac{R^{(k+1,s)}_{\max}}{s(k+1)} \geq \frac{R^{(k+1)}_{\max}}{(k+1)(k+2)}, \tag{3.5}$$

which together with the assumption implies

$$\begin{aligned} Ric^{(k)}_{\min} &> \frac{(k+2)(5k-6)}{s(5k^2+9k-8)} R^{(k+1,s)}_{\max} \\ &\geq \frac{5k-6}{5k^2+9k-8} R^{(k+1)}_{\max}. \end{aligned} \tag{3.6}$$

Then the assertion follows from (iii).

This proves the lemma. □

Lemma 3.2 *Let M be an $n(\geq 4)$-dimensional compact Einstein manifold. Suppose one of the following conditions holds:*

(i) $K_{\min} > \frac{1}{k+3} Ric^{(k)}_{\max}$ *for some integer* $k \in [2, n-1]$*;*
(ii) $Ric^{(k+1)}_{\min} > \frac{k+4}{k+3} Ric^{(k)}_{\max}$ *for some integer* $k \in [2, n-2]$*;*
(iii) $R^{(k+1)}_{\min} > \frac{k^2+6k+11}{k+3} Ric^{(k)}_{\max}$ *for some integer* $k \in [2, n-2]$*;*
(iv) $R^{(k+1,s)}_{\min} > \frac{s(k^2+6k+11)}{(k+2)(k+3)} Ric^{(k)}_{\max}$ *for some integers* $k \in [2, n-2]$ *and* $s \in [2, k+2]$.

Then M is isometric to a spherical space form.

Proof (i) From (2.4), we obtain that

$$K_{\max} \leq Ric^{(k)}_{\max} - (k-1) K_{\min}. \tag{3.7}$$

Suppose $\{e_1, e_2, e_3, e_4\}$ is an orthonormal four-frame. Combing (2.2) and (3.7), we get

$$\begin{aligned} & R_{1313} + R_{2323} + R_{1414} + R_{2424} - 2R_{1234} \\ & \geq 4K_{\min} - \frac{4}{3}(K_{\max} - K_{\min}) \\ & \geq \frac{16}{3} K_{\min} - \frac{4}{3}[Ric^{(k)}_{\max} - (k-1) K_{\min}] \\ & \geq \frac{4}{3}[(k+3) K_{\min} - Ric^{(k)}_{\max}]. \end{aligned} \tag{3.8}$$

This together with the assumption implies M has positive isotropic curvature. From Theorem B, we see that M has constant sectional curvature.

(ii) It's seen from (2.4) that

$$K_{\min} \geq Ric^{(k+1)}_{\min} - Ric^{(k)}_{\max}. \tag{3.9}$$

Then we get from the assumption that

$$\begin{aligned} K_{\min} & \geq Ric^{(k+1)}_{\min} - Ric^{(k)}_{\max} \\ & > \frac{k+4}{k+3} Ric^{(k)}_{\max} - Ric^{(k)}_{\max} \\ & = \frac{1}{k+3} Ric^{(k)}_{\max}. \end{aligned} \tag{3.10}$$

Therefore, the assertion follows from (i).

(iii) It follows from (2.4), (2.6) and the assumption that

$$K_{\min} \geq \frac{1}{2}[R^{(k+1)}_{\min} - (k+3)Ric^{(k)}_{\max}]$$
$$> \frac{1}{2}\Big[\frac{k^2+6k+11}{k+3}Ric^{(k)}_{\max} - (k+3)Ric^{(k)}_{\max}\Big]$$
$$= \frac{1}{k+3}Ric^{(k)}_{\max}. \tag{3.11}$$

This together with (i) implies that M has constant sectional curvature.

(iv) By (2.5) and (2.6), we get

$$\frac{R^{(k+1,s)}_{\min}}{s(k+1)} \leq \frac{R^{(k+1)}_{\min}}{(k+1)(k+2)}, \tag{3.12}$$

which together with the assumption implies

$$Ric^{(k)}_{\max} < \frac{(k+2)(k+3)}{s(k^2+6k+11)}R^{(k+1,s)}_{\min}$$
$$\leq \frac{k+3}{k^2+6k+11}R^{(k+1)}_{\min}. \tag{3.13}$$

It follows from (iii) that M is isometric to a spherical space form.

This proves the lemma. □

Lemma 3.3 *Let M be an $n(\geq 4)$-dimensional compact Einstein manifold. If one of the following conditions holds:*

(i) $R^{(k)}_{\min} > \left(k^2+k-\frac{24}{7}\right)K_{\max}$ *for some integer* $k \in [3, n-1]$;

(ii) $R^{(k,s)}_{\min} > \frac{s(7k^2+7k-24)}{7(k+1)}K_{\max}$ *for some integers* $k \in [3, n-1]$ *and* $s \in [2, k+1]$;

(iii) $K_{\min} > \frac{1}{ks+6}R^{(k,s)}_{\max}$ *for some integers* $k \in [1, n-1]$ *and* $s \in [2, k+1]$;

(iv) $K_{\min} > \frac{1}{k^2+k+6}R^{(k)}_{\max}$ *for some integer* $k \in [1, n-1]$,

then M is isometric to a spherical space form.

Proof (i) It follows from (2.6) that

$$R^{(k)}_{\min} \leq 2K_{\min} + [k(k+1)-2]K_{\max}.$$

Then we have

$$K_{\min} \geq \frac{1}{2}[R^{(k)}_{\min} - (k^2+k-2)K_{\max}]. \tag{3.14}$$

Suppose $\{e_1, e_2, e_3, e_4\}$ is an orthonormal four-frame. From (2.2), (3.14) and the assumption we get

$$\begin{aligned}
&R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} \\
&\geq \frac{1}{2}\{R^{(k)}_{\min} - [k(k+1) - 8]K_{\max}\} - \frac{4}{3}(K_{\max} - K_{\min}) \\
&\geq \frac{1}{2}\{R^{(k)}_{\min} - [k(k+1) - 8]K_{\max}\} - \frac{2}{3}[k(k+1)K_{\max} - R^{(k)}_{\min}] \\
&\geq \frac{7}{6}\left[R^{(k)}_{\min} - \left(k^2 + k - \frac{24}{7}\right)K_{\max}\right] \\
&> 0.
\end{aligned} \tag{3.15}$$

Therefore, M has positive isotropic curvature. By Theorem B, we see that M has constant sectional curvature.

(ii) By Definition 2.2, we have

$$\frac{R^{(k)}_{\min}}{k(k+1)} \geq \frac{R^{(k,s)}_{\min}}{ks}. \tag{3.16}$$

This together with the assumption implies

$$R^{(k)}_{\min} \geq \frac{k+1}{s}R^{(k,s)}_{\min} > \left(k^2 + k - \frac{24}{7}\right)K_{\max}. \tag{3.17}$$

Then the assertion follows from (i).

(iii) It follows from (2.5) that

$$K_{\max} \leq \frac{1}{2}\left[R^{(k,s)}_{\max} - (ks - 2)K_{\min}\right]. \tag{3.18}$$

Suppose $\{e_1, e_2, e_3, e_4\}$ is an orthonormal four-frame. Then we get from (2.2) and (3.18) that

$$\begin{aligned}
&R_{1313} + R_{2323} + R_{1414} + R_{2424} - 2R_{1234} \\
&\geq 4K_{\min} - \frac{4}{3}(K_{\max} - K_{\min}) \\
&\geq \frac{16}{3}K_{\min} - \frac{2}{3}[Ric^{(k,s)}_{\max} - (ks - 2)K_{\min}] \\
&\geq \frac{2}{3}[(ks + 6)K_{\min} - R^{(k)}_{\max}].
\end{aligned} \tag{3.19}$$

This together with the assumption implies M has positive isotropic curvature. Therefore, M is isometric to a spherical space form.

(iv) By taking $s = k + 1$ in (iii), we get the conclusion.

This completes the proof of Lemma 3.3. □

Proof of Theorem 1.1 Since M is Einstein, we know from (2.7) that the normalized scalar curvature and the Ricci curvature satisfy $R_0 = \frac{Ric_{\min}}{n-1} = \frac{Ric_{\max}}{n-1}$.

(i) By taking $k = n - 1$ in conditions (i) of Lemmas 3.1, we get that M has constant sectional curvature.

(ii) By taking $k = n - 1$ in (3.8), we see that if $K_{\min} \geq \frac{n-1}{n+2} R_0$, then M has nonnegative isotropic curvature. This together with Theorem B implies that M is locally symmetric. When M is simply connected, we assume that M is not isometric to the standard n-sphere. Then we claim that for every point $x \in M$ there exists an orthonormal four-frame $\{e_1, e_2, e_3, e_4\}$ such that

$$R_{1313} + R_{2323} + R_{1414} + R_{2424} - 2R_{1234} = 0. \tag{3.20}$$

Otherwise, M has positive isotropic curvature at some point in M. This together with Lemma 2.2 implies that M admits a metric with positive isotropic curvature. A result due to Harish [11] says that a compact locally symmetric space which admits a metric of positive isotropic curvature has constant sectional curvature. So M is isometric to the standard n-sphere. This is a contradiction. From (3.8), (3.20) and the assumption, we have

$$K_{\min} - \frac{1}{4} K_{\max} \equiv 0,$$

and

$$K_{\min} = \frac{n-1}{n+2} R_0 = \frac{n-1}{n+2} c = constant. \tag{3.21}$$

Therefore, M admits a metric with weakly 1/4-pinched sectional curvature in the global sense. Following Berger's classification theorem we obtain that M either is homeomorphic to S^n or isometric to a compact rank one symmetric space(CROSS). Since M is locally symmetric, a topological sphere would have to be of constant positive sectional curvature. This contradicts the assumption that M is not isometric to the standard n-sphere. By (3.21) and a simple computation, we know that M is isometric the complex projective space $\mathbb{C}\mathrm{P}^m(\tilde{c})$ with $n = 2m$, where $\tilde{c} = \frac{4(n-1)}{n+2} c$.

This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2 (i) By taking $k = n - 2$ in condition (ii) of Lemma 3.1, we know that M has constant sectional curvature.

(ii) From

$$K_{\min} \geq Ric_{\min} - Ric_{\max}^{(n-2)},$$

and the assumption

$$(n-2)R_0 \geq \frac{n^2-4}{n^2-1} Ric_{\max}^{(n-2)},$$

we see that $K_{\min} \geq \frac{n-1}{n+2} R_0$ and the complex projective space satisfies the equality $K_{\min} = Ric_{\min} - Ric_{\max}^{(n-2)}$. Hence the assertion follows from (ii) of Theorem 1.1.

This proves Theorem 1.2. □

From the proof of Theorems 1.1 and 1.2, we have the following corollary.

Corollary 3.1 *Let M be an $n(\geq 4)$-dimensional compact Einstein Riemannian manifold. Denote by $R_0 := c$ the normalized scalar curvature of M. Assume one of the following conditions holds:*

(i) $K_{\min} > \eta_n R_0$;
(ii) $(n-2)R_0 > \mu_n Ric_{\max}^{(n-2)}$.

Then M is isometric to a spherical space form of constant curvature c. Here η_n is defined as in Theorem 1.1 and μ_n is defined as in Theorem 1.2.

By (3.8), (3.9) and a direct calculation, we know that if M is an $n(\geq 4)$-dimensional compact manifold satisfies condition (i) or (ii) in Lemma 3.2, then

$$R_{1313} + \lambda^2 R_{1414} + R_{2323} + \lambda^2 R_{2424} - 2\lambda R_{1234} > 0$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$ and all $\lambda \in \mathbb{R}$. Using Brendle's convergence result for Ricci flow [4] and taking $k = n-1$ in condition (i) and $k = n-2$ in condition (ii), we get the following differentiable sphere theorem.

Theorem 3.1 *Let M be an $n(\geq 4)$-dimensional compact Riemannian manifold. Assume one of the following conditions holds:*

(i) $(n-1)K_{\min} > \eta_n Ric_{\max}$;
(ii) $(n-2)Ric_{\min} > \mu_n (n-1) Ric_{\max}^{(n-2)}$.

Then M is diffeomorphic to a spherical space form. In particular, if M is simply connected, then M is diffeomorphic to S^n. Here η_n is defined as in Theorem 1.1 and μ_n is defined as in Theorem 1.2.

For further discussions about the Ricci flow and sphere theorem, we refer to see [3,5–7,10,15,17,18,22,27].

Example 3.1 Let R_0 be the normalized scalar curvature of a Riemannian manifold. By a direct computation, we have the normalized scalar curvatures of the compact rank one symmetric spaces (CROSS) with standard metrics.

$$R_0(\mathbb{C}\mathrm{P}^m) = \frac{m+1}{4m-2}, \ \dim_{\mathbb{R}}(\mathbb{C}\mathrm{P}^m) = 2m, \ m \geq 2;$$
$$R_0(\mathbb{H}\mathrm{P}^m) = \frac{m+2}{4m-1}, \ \dim_{\mathbb{R}}(\mathbb{H}\mathrm{P}^m) = 4m, \ m \geq 2;$$
$$R_0(\mathbb{O}\mathrm{P}^2) = \frac{3}{5}, \ \dim_{\mathbb{R}}(\mathbb{O}\mathrm{P}^2) = 16.$$

On the other hand, $K_{\min}(\mathbb{C}\mathrm{P}^m) = K_{\min}(\mathbb{H}\mathrm{P}^m) = K_{\min}(\mathbb{O}\mathrm{P}^2) = \frac{1}{4}$. Then we know that the curvatures of $\mathbb{C}\mathrm{P}^m$ satisfy

$$K_{\min} = \eta_{2m} \frac{Ric_{\max}}{(2m-1)} = \eta_{2m} R_0,$$

and

$$(2m-2)R_0 = (2m-2)\frac{Ric_{\min}}{2m-1} = \mu_{2m} Ric_{\max}^{(2m-2)}.$$

These mean the pinching constants η_n and μ_n are optimal when n is even.

Motivated by Theorem 1.1 and Example 3.1, we would like to propose the following conjectures.

Conjecture A *Let $M^n (n \geq 4)$ be a compact Einstein manifold. If $R_0 > \frac{3}{5}K_{\max}$, then M is isometric to a spherical space form.*

Conjecture B *Let $M^n (n \geq 4)$ be an even dimensional compact simply connected Einstein manifold. If $K_M \leq 1$ and $R_0 \geq c_n$, where*

$$c_n = \begin{cases} \dfrac{n+2}{4(n-1)} & \text{for } n = 4 \text{ or } 4k+2,\ k \in \mathbb{Z}^+, \\ \dfrac{n+8}{4(n-1)} & \text{for } n = 4k,\ k \in \mathbb{Z}^+ \bigcap [2, \infty) \text{ and } k \neq 4, \\ \dfrac{3}{5} & \text{for } n = 16, \end{cases}$$

then M is either isometric to the standard n-sphere, or a compact rank one symmetric space.

4 Einstein submanifolds with arbitrary codimension

In this section, we extend the theorems in Sect. 3 to Einstein submanifolds in a general Riemannian manifold. For compact submanifolds, we first prove Theorem 1.3.

Proof of Theorem 1.3 Setting $S_\alpha = \sum_{i,j=1}^n (h_{ij}^\alpha)^2$, we obtain

$$\left(\sum_{i=1}^n h_{ii}^\alpha\right)^2 = (n-2)\left[\sum_{i=1}^n (h_{ii}^\alpha)^2 + \sum_{i\neq j}(h_{ij}^\alpha)^2 + \frac{(\sum_{i=1}^n h_{ii}^\alpha)^2}{n-2} - S_\alpha\right]. \quad (4.1)$$

Note that for all distinct p, q, m, l

$$\begin{aligned}\left(\sum_{i=1}^n h_{ii}^\alpha\right)^2 &\leq (n-2)\left[(h_{pp}^\alpha + h_{qq}^\alpha)^2 + (h_{mm}^\alpha + h_{ll}^\alpha)^2 + \sum_{i\neq p,q,m,l}(h_{ii}^\alpha)^2\right] \\ &= (n-2)\left[\sum_{i=1}^n (h_{ii}^\alpha)^2 + 2h_{pp}^\alpha h_{qq}^\alpha + 2h_{mm}^\alpha h_{ll}^\alpha\right].\end{aligned}$$

This together with (4.1) implies

$$2h_{pp}^\alpha h_{qq}^\alpha + 2h_{mm}^\alpha h_{ll}^\alpha \geq \sum_{i\neq j}(h_{ij}^\alpha)^2 + \frac{(\sum_{i=1}^n h_{ii}^\alpha)^2}{n-2} - S_\alpha, \quad (4.2)$$

for all distinct p, q, m, l. Suppose $\{e_1, e_2, e_3, e_4\}$ is an orthonormal four-frame. From (2.1), we get

$$\begin{aligned}
&R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} \\
&\quad = \overline{R}_{1313} + \overline{R}_{1414} + \overline{R}_{2323} + \overline{R}_{2424} - 2\overline{R}_{1234} \\
&\qquad + \sum_\alpha \Big[h_{11}^\alpha h_{33}^\alpha + h_{22}^\alpha h_{44}^\alpha + h_{22}^\alpha h_{33}^\alpha + h_{11}^\alpha h_{44}^\alpha \\
&\qquad - (h_{13}^\alpha)^2 - (h_{23}^\alpha)^2 - (h_{24}^\alpha)^2 - (h_{14}^\alpha)^2 - 2(h_{13}^\alpha h_{24}^\alpha - h_{14}^\alpha h_{23}^\alpha)\Big] \\
&\quad \geq \overline{R}_{1313} + \overline{R}_{1414} + \overline{R}_{2323} + \overline{R}_{2424} - 2\overline{R}_{1234} \\
&\qquad + \sum_\alpha \Big[h_{11}^\alpha h_{33}^\alpha + h_{22}^\alpha h_{44}^\alpha + h_{22}^\alpha h_{33}^\alpha + h_{11}^\alpha h_{44}^\alpha \\
&\qquad - 2(h_{13}^\alpha)^2 - 2(h_{23}^\alpha)^2 - 2(h_{24}^\alpha)^2 - 2(h_{14}^\alpha)^2\Big].
\end{aligned} \tag{4.3}$$

It follows from Berger's inequality (2.3) and (4.2) that

$$\overline{R}_{1234} \leq \frac{2}{3}(\overline{K}_{\max} - \overline{K}_{\min}),$$

$$h_{11}^\alpha h_{33}^\alpha + h_{22}^\alpha h_{44}^\alpha + h_{22}^\alpha h_{33}^\alpha + h_{11}^\alpha h_{44}^\alpha \geq \sum_{i\neq j}(h_{ij}^\alpha)^2 + \frac{(\sum_{i=1}^n h_{ii}^\alpha)^2}{n-2} - S_\alpha.$$

This together with (4.3) implies that

$$\begin{aligned}
&R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} \\
&\quad \geq 4\overline{K}_{\min} - \frac{4}{3}(\overline{K}_{\max} - \overline{K}_{\min}) \\
&\qquad + \sum_\alpha \Big[\sum_{i\neq j}(h_{ij}^\alpha)^2 + \frac{(\sum_{i=1}^n h_{ii}^\alpha)^2}{n-2} - S_\alpha \\
&\qquad - 2(h_{13}^\alpha)^2 - 2(h_{23}^\alpha)^2 - 2(h_{24}^\alpha)^2 - 2(h_{14}^\alpha)^2\Big] \\
&\quad \geq \frac{16}{3}\Big(\overline{K}_{\min} - \frac{1}{4}\overline{K}_{\max}\Big) + \frac{n^2H^2}{n-2} - S.
\end{aligned} \tag{4.4}$$

Then it follows from the assumption that M has nonnegative isotropic curvature and has positive isotropic curvature for some point $x_0 \in M$. This together with Lemma 2.2 implies that M admits a metric of positive isotropic curvature. Since M is Einstein, it follows from Theorem B that M is locally symmetric. A result due to Harish [11] says that a compact locally symmetric space which admits a metric of positive isotropic curvature has constant sectional curvature. This completes the proof of Theorem 1.3. □

Theorem 4.1 *Let M be an $n(\geq 4)$-dimensional compact Einstein submanifold in the Riemannian manifold $\overline{M}^N$ with codimension $N - n \geq 0$. If M satisfies one of the following conditions:*

(i) $S < \frac{10}{3}\Big(\overline{Ric}_{\min} - \frac{5N-11}{5}\overline{K}_{\max}\Big) + \frac{n^2H^2}{n-2}$;

(ii) $S < \frac{4(N+1)}{3}\Big(\overline{Ric}_{\min} - \frac{N+2}{N+1}\overline{Ric}^{(N-2)}_{\max}\Big) + \frac{n^2H^2}{n-2}$;

(iii) $S < \frac{4(N+2)}{3}\Big(\overline{K}_{\min} - \frac{1}{N+2}\overline{Ric}_{\max}\Big) + \frac{n^2H^2}{n-2}$;

(iv) $S < \frac{2(5N-11)}{3}\Big(\overline{Ric}^{(N-2)}_{\min} - \frac{5N-16}{5N-11}\overline{Ric}_{\max}\Big) + \frac{n^2H^2}{n-2}$,

then M is isometric to a spherical space form.

Proof If $N = n$, i.e., the codimension is zero, then S and H are equal to zero. Hence M is an Einstein manifold, and the assertion follows from (2.7), Theorems 1.1 and 1.2.

If $N > n$, we consider the following cases.

(i) From (2.8), we get that

$$\overline{K}_{\min} \geq \overline{Ric}^{(k)}_{\min} - (k-1)\overline{K}_{\max}. \tag{4.5}$$

Suppose $\{e_1, e_2, e_3, e_4\}$ is an orthonormal four-frame. From (2.3), (4.2) and (4.5), we have

$$\begin{aligned}
&R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} \\
&\quad = \overline{R}_{1313} + \overline{R}_{1414} + \overline{R}_{2323} + \overline{R}_{2424} - 2\overline{R}_{1234} \\
&\qquad + \sum_\alpha \Big[h^\alpha_{11}h^\alpha_{33} + h^\alpha_{22}h^\alpha_{44} + h^\alpha_{22}h^\alpha_{33} + h^\alpha_{11}h^\alpha_{44} \\
&\qquad -(h^\alpha_{13})^2 - (h^\alpha_{23})^2 - (h^\alpha_{24})^2 - (h^\alpha_{14})^2 - 2(h^\alpha_{13}h^\alpha_{24} - h^\alpha_{14}h^\alpha_{23})\Big] \\
&\quad \geq 2\overline{Ric}^{(k)}_{\min} - 2(k-2)\overline{K}_{\max} - \frac{4}{3}(\overline{K}_{\max} - \overline{K}_{\min}) \\
&\qquad + \sum_\alpha \left[\sum_{i\neq j}(h^\alpha_{ij})^2 + \frac{(\sum_{i=1}^n h^\alpha_{ii})^2}{n-2} - S_\alpha \right.\\
&\qquad \left. -2(h^\alpha_{13})^2 - 2(h^\alpha_{23})^2 - 2(h^\alpha_{24})^2 - 2(h^\alpha_{14})^2 \right] \\
&\quad \geq 2\overline{Ric}^{(k)}_{\min} - 2(k-2)\overline{K}_{\max} - \frac{4}{3}(k\overline{K}_{\max} - \overline{Ric}^{(k)}_{\min}) + \frac{n^2H^2}{n-2} - S \\
&\quad \geq \frac{10}{3}\Big[\overline{Ric}^{(k)}_{\min} - \Big(k - \frac{6}{5}\Big)\overline{K}_{\max}\Big] + \frac{n^2H^2}{n-2} - S.
\end{aligned} \tag{4.6}$$

Taking $k = N - 1$ in (4.6), we get from the assumption that M has positive isotropic curvature. This together with Theorem B implies that M is isometric to a spherical space form.

(ii) It follows from (2.8) that

$$\overline{K}_{\max} \leq \overline{Ric}_{\max}^{(N-2)} - (N-3)\overline{K}_{\min}, \tag{4.7}$$

$$\overline{K}_{\min} \geq \overline{Ric}_{\min} - \overline{Ric}_{\max}^{(N-2)}. \tag{4.8}$$

This together with the assumption implies that

$$\begin{aligned} S &< \frac{4(N+1)}{3}\left(\overline{Ric}_{\min} - \frac{N+2}{N+1}\overline{Ric}_{\max}^{(N-2)}\right) + \frac{n^2H^2}{n-2} \\ &\leq \frac{4(N+1)}{3}\left(\overline{K}_{\min} + \overline{Ric}_{\max}^{(N-2)} - \frac{N+2}{N+1}\overline{Ric}_{\max}^{(N-2)}\right) + \frac{n^2H^2}{n-2} \\ &\leq \frac{4(N+1)}{3}\left[\overline{K}_{\min} - \frac{1}{N+1}(\overline{K}_{\max} + (N-3)\overline{K}_{\min})\right] + \frac{n^2H^2}{n-2} \\ &= \frac{16}{3}\left(\overline{K}_{\min} - \frac{1}{4}\overline{K}_{\max}\right) + \frac{n^2H^2}{n-2}. \end{aligned}$$

Then the assertion follows from Theorem 1.3.

(iii) We know that

$$\overline{K}_{\max} \leq \overline{Ric}_{\max} - (N-2)\overline{K}_{\min}. \tag{4.9}$$

It follows from (4.9) and the assumption that

$$\begin{aligned} S &< \frac{4(N+2)}{3}\left(\overline{K}_{\min} - \frac{1}{N+2}\overline{Ric}_{\max}\right) + \frac{n^2H^2}{n-2} \\ &\leq \frac{4(N+2)}{3}\left[\overline{K}_{\min} - \frac{1}{N+2}(\overline{K}_{\max} + (N-2)\overline{K}_{\min})\right] + \frac{n^2H^2}{n-2} \\ &= \frac{16}{3}\left(\overline{K}_{\min} - \frac{1}{4}\overline{K}_{\max}\right) + \frac{n^2H^2}{n-2}. \end{aligned}$$

This together with Theorem 1.3 implies that M has constant sectional curvature.

(iv) It's seen from (2.8) that

$$\overline{K}_{\max} \leq \overline{Ric}_{\max} - \overline{Ric}_{\min}^{(N-2)}. \tag{4.10}$$

Taking $k = N-2$, for any orthonormal four-frame $\{e_1, e_2, e_3, e_4\}$, we get from (4.6), (4.10) that

$$\begin{aligned} &R_{1313} + R_{2323} + R_{1414} + R_{2424} - 2R_{1234} \\ &\geq \frac{10}{3}\left[\overline{Ric}_{\min}^{(N-2)} - \left(N - \frac{16}{5}\right)\overline{K}_{\max}\right] + \frac{n^2H^2}{n-2} - S \end{aligned}$$

$$\geq \frac{10}{3}\Big[\overline{Ric}^{(N-2)}_{\min} - \Big(N - \frac{16}{5}\Big)(\overline{Ric}_{\max} - \overline{Ric}^{(N-2)}_{\min})\Big] + \frac{n^2H^2}{n-2} - S$$
$$= \frac{2(5N-11)}{3}\Big(\overline{Ric}^{(N-2)}_{\min} - \frac{5N-16}{5N-11}\overline{Ric}_{\max}\Big) + \frac{n^2H^2}{n-2} - S.$$

This together with the assumption implies that M has positive isotropic curvature. Therefore, we see from Theorem B that M is isometric to a spherical space form.

This completes the proof of Theorem 4.1. □

Theorem 4.2 *Let M be an $n(\geq 4)$-dimensional compact submanifold in an N-dimensional Riemannian manifold $\overline{M}^N$. Denote by $\overline{R}_0$ the normalized scalar curvature of $\overline{M}$. Assume that M satisfies one of the following conditions:*

(i) $S < \frac{7N(N-1)}{6}(\overline{R}_0 - \sigma'_N \overline{K}_{\max}) + \frac{n^2H^2}{n-2}$;
(ii) $S < \frac{2N(N^2-1)}{3(N-2)}[(N-2)\overline{R}_0 - \mu'_N \overline{Ric}^{(N-2)}_{\max}] + \frac{n^2H^2}{n-2}$;
(iii) $S < \frac{2(N^2-N+6)}{3}(\overline{K}_{\min} - \eta'_N \overline{R}_0) + \frac{n^2H^2}{n-2}$;
(iv) $S < \frac{5N^2-11N-6}{3}[\overline{Ric}^{(N-2)}_{\min} - \tau'_N(N-2)\overline{R}_0] + \frac{n^2H^2}{n-2}$.

Then M is isometric to a spherical space form. Here

$$\sigma'_N = 1 - \frac{24}{7N(N-1)},$$
$$\mu'_N = 1 - \frac{6}{N(N-1)(N+1)},$$
$$\eta'_N = 1 - \frac{6}{N^2-N+6},$$
$$\tau'_N = 1 - \frac{12}{(N-2)(5N^2-11N-6)}.$$

Proof (i) Since

$$\overline{R}^{(k)}_{\min} \leq 2\overline{K}_{\min} + [k(k+1)-2]\overline{K}_{\max},$$

we have

$$\overline{K}_{\min} \geq \frac{1}{2}[\overline{R}^{(k)}_{\min} - (k^2+k-2)\overline{K}_{\max}]. \tag{4.11}$$

Suppose $\{e_1, e_2, e_3, e_4\}$ is an orthonormal four-frame. From (2.1), (2.3), (4.2) and (4.11), we get

$$\begin{aligned} &R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} \\ &\quad = \overline{R}_{1313} + \overline{R}_{1414} + \overline{R}_{2323} + \overline{R}_{2424} - 2\overline{R}_{1234} \\ &\qquad + \sum_\alpha \Big[h^\alpha_{11}h^\alpha_{33} + h^\alpha_{22}h^\alpha_{44} + h^\alpha_{22}h^\alpha_{33} + h^\alpha_{11}h^\alpha_{44} \\ &\qquad - (h^\alpha_{13})^2 - (h^\alpha_{23})^2 - (h^\alpha_{24})^2 - (h^\alpha_{14})^2 - 2(h^\alpha_{13}h^\alpha_{24} - h^\alpha_{14}h^\alpha_{23})\Big] \end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{2}\{\overline{R}^{(k)}_{\min} - [k(k+1)-8]\overline{K}_{\max}\} - \frac{4}{3}(\overline{K}_{\max} - \overline{K}_{\min}) \\
&\quad + \sum_{\alpha}\left[\sum_{i\neq j}(h^{\alpha}_{ij})^2 + \frac{(\sum_{i=1}^{n} h^{\alpha}_{ii})^2}{n-2} - S_{\alpha}\right. \\
&\quad \left. -2(h^{\alpha}_{13})^2 - 2(h^{\alpha}_{23})^2 - 2(h^{\alpha}_{24})^2 - 2(h^{\alpha}_{14})^2\right] \\
&\geq \frac{1}{2}[\overline{R}^{(k)}_{\min} - (k^2+k-8)\overline{K}_{\max}] \\
&\quad - \frac{2}{3}[k(k+1)\overline{K}_{\max} - \overline{R}^{(k)}_{\min}] + \frac{n^2H^2}{n-2} - S \\
&\geq \frac{7}{6}\left[\overline{R}^{(k)}_{\min} - \left(k^2+k-\frac{24}{7}\right)\overline{K}_{\max}\right] + \frac{n^2H^2}{n-2} - S.
\end{aligned} \tag{4.12}
$$

Taking $k = N-1$, we see from (4.12) and the assumption that M has constant sectional curvature.

(ii) It follows from (2.8) that

$$\overline{K}_{\min} \geq \frac{1}{2}[\overline{R} - (N+1)\overline{Ric}^{(N-2)}_{\max}]. \tag{4.13}$$

This together with (4.7) and the assumption implies that

$$
\begin{aligned}
S &< \frac{2N(N^2-1)}{3(N-2)}\left[\frac{N-2}{N(N-1)}\overline{R} - \left(1 - \frac{6}{N(N^2-1)}\right)\overline{Ric}^{(N-2)}_{\max}\right] + \frac{n^2H^2}{n-2} \\
&\leq \frac{2N(N^2-1)}{3(N-2)}\left[\frac{N-2}{N(N-1)}\left(2\overline{K}_{\min} + (N+1)\overline{Ric}^{(N-2)}_{\max}\right)\right. \\
&\quad \left. - \left(1 - \frac{6}{N(N^2-1)}\right)\overline{Ric}^{(N-2)}_{\max}\right] + \frac{n^2H^2}{n-2} \\
&= \frac{4}{3}[(N+1)\overline{K}_{\min} - \overline{Ric}^{(N-2)}_{\max}] + \frac{n^2H^2}{n-2} \\
&\leq \frac{4}{3}\{(N+1)\overline{K}_{\min} - [\overline{K}_{\max} + (N-3)\overline{K}_{\min}]\} + \frac{n^2H^2}{n-2} \\
&= \frac{16}{3}\left(\overline{K}_{\min} - \frac{1}{4}\overline{K}_{\max}\right) + \frac{n^2H^2}{n-2}.
\end{aligned}
$$

Applying Theorem 1.3, we conclude that M is isometric to a spherical space form.

(iii) We know that

$$\overline{K}_{\max} \leq \frac{1}{2}[\overline{R} - (N^2-N-2)\overline{K}_{\min}]. \tag{4.14}$$

Then we get from the assumption that

$$\begin{aligned} S &< \frac{2(N^2-N+6)}{3}\Big(\overline{K}_{\min} - \frac{1}{N^2-N+6}\overline{R}\Big) + \frac{n^2H^2}{n-2} \\ &\le \frac{2(N^2-N+6)}{3}\Big[\overline{K}_{\min} - \frac{1}{N^2-N+6}(2\overline{K}_{\max} \\ &\quad +(N^2-N-2)\overline{K}_{\min})\Big] + \frac{n^2H^2}{n-2} \\ &= \frac{16}{3}\Big(\overline{K}_{\min} - \frac{1}{4}\overline{K}_{\max}\Big) + \frac{n^2H^2}{n-2}. \end{aligned}$$

Hence, we get the conclusion from Theorem 1.3.

(iv) It follows from (2.8) and (2.9) that

$$\overline{K}_{\max} \le \frac{1}{2}[\overline{R} - (N+1)\overline{Ric}^{(N-2)}_{\min}]. \tag{4.15}$$

Taking $k = N-2$, we get from (4.6), (4.15) and the assumption that

$$\begin{aligned} &\overline{R}_{1313} + \overline{R}_{2323} + \overline{R}_{1414} + \overline{R}_{2424} - 2\overline{R}_{1234} \\ &\quad \ge \frac{10}{3}\Big[\overline{Ric}^{(N-2)}_{\min} - \Big(N - \frac{16}{5}\Big)\overline{K}_{\max}\Big] + \frac{n^2H^2}{n-2} - S \\ &\quad \ge \frac{10}{3}\Big[\overline{Ric}^{(N-2)}_{\min} - \frac{1}{2}\Big(N - \frac{16}{5}\Big)\Big(\overline{R} - (N+1)\overline{Ric}^{(N-2)}_{\min}\Big)\Big] + \frac{n^2H^2}{n-2} - S \\ &\quad = \frac{1}{3}\Big[(5N^2 - 11N - 6)\overline{Ric}^{(N-2)}_{\min} - (5N-16)\overline{R}\Big] + \frac{n^2H^2}{n-2} - S \\ &\quad > 0, \end{aligned}$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$. This together with Theorem B implies that M has constant sectional curvature.

This completes the proof of Theorem 4.2. □

Proof of Theorem 1.4 Setting

$$T_\alpha := \frac{1}{n}\sum_i h^\alpha_{ii},$$

we have $\sum_\alpha T_\alpha^2 = H^2$, and

$$\begin{aligned} h^\alpha_{ii}h^\alpha_{jj} &= \frac{1}{2}[(h^\alpha_{ii} + h^\alpha_{jj} - 2T_\alpha)^2 - (h^\alpha_{ii} - T_\alpha)^2 - (h^\alpha_{jj} - T_\alpha)^2] \\ &\quad + T_\alpha(h^\alpha_{ii} - T_\alpha) + T_\alpha(h^\alpha_{jj} - T_\alpha) + T_\alpha^2 \end{aligned} \tag{4.16}$$

for $i, j = 1, \ldots, n$. Suppose $\{e_1, e_2, e_3, e_4\}$ is an orthonormal four-frame. From (2.1) and (4.16), we get

$$
\begin{aligned}
&R_{1313}+R_{2323}+R_{1414}+R_{2424}-2R_{1234}\\
&=4c+\sum_\alpha[h^\alpha_{11}h^\alpha_{33}-(h^\alpha_{13})^2+h^\alpha_{22}h^\alpha_{33}-(h^\alpha_{23})^2\\
&\quad+h^\alpha_{11}h^\alpha_{44}-(h^\alpha_{14})^2+h^\alpha_{22}h^\alpha_{44}-(h^\alpha_{24})^2-2(h^\alpha_{13}h^\alpha_{24}-h^\alpha_{14}h^\alpha_{23})]\\
&\geq 4c-2\sum_\alpha[(h^\alpha_{13})^2+(h^\alpha_{23})^2+(h^\alpha_{14})^2+(h^\alpha_{14})^2]\\
&\quad+\frac{1}{2}\sum_\alpha[(h^\alpha_{11}+h^\alpha_{33}-2T_\alpha)^2-(h^\alpha_{11}-T_\alpha)^2-(h^\alpha_{33}-T_\alpha)^2\\
&\quad+2T_\alpha(h^\alpha_{11}-T_\alpha)+2T_\alpha(h^\alpha_{33}-2T_\alpha)+2T^2_\alpha]\\
&\quad+\frac{1}{2}\sum_\alpha[(h^\alpha_{22}+h^\alpha_{33}-2T_\alpha)^2-(h^\alpha_{22}-T_\alpha)^2-(h^\alpha_{33}-T_\alpha)^2\\
&\quad+2T_\alpha(h^\alpha_{22}-T_\alpha)+2T_\alpha(h^\alpha_{33}-2T_\alpha)+2T^2_\alpha]\\
&\quad+\frac{1}{2}\sum_\alpha[(h^\alpha_{11}+h^\alpha_{44}-2T_\alpha)^2-(h^\alpha_{11}-T_\alpha)^2-(h^\alpha_{44}-T_\alpha)^2\\
&\quad+2T_\alpha(h^\alpha_{11}-T_\alpha)+2T_\alpha(h^\alpha_{44}-2T_\alpha)+2T^2_\alpha]\\
&\quad+\frac{1}{2}\sum_\alpha[(h^\alpha_{22}+h^\alpha_{44}-2T_\alpha)^2-(h^\alpha_{22}-T_\alpha)^2-(h^\alpha_{44}-T_\alpha)^2\\
&\quad+2T_\alpha(h^\alpha_{22}-T_\alpha)+2T_\alpha(h^\alpha_{44}-2T_\alpha)+2T^2_\alpha]\\
&\geq 4(c+H^2)-2\sum_\alpha[(h^\alpha_{13})^2+(h^\alpha_{23})^2+(h^\alpha_{14})^2+(h^\alpha_{14})^2]\\
&\quad+\sum_\alpha\sum_{i=1}^4[-(h^\alpha_{ii}-T_\alpha)^2+2T_\alpha(h^\alpha_{ii}-T_\alpha)]. \qquad (4.17)
\end{aligned}
$$

On the other hand, from Gauss equation (2.1), we obtain

$$
\begin{aligned}
Ric(e_i)=(n-1)(c+H^2)+(n-2)\sum_\alpha T_\alpha(h^\alpha_{ii}-T_\alpha)\\
-\sum_\alpha(h^\alpha_{ii}-T_\alpha)^2-\sum_{\alpha,j\neq i}(h^\alpha_{ij})^2
\end{aligned}
$$

for $i=1,\ldots,n$. So, we have

$$
\begin{aligned}
-\sum_\alpha(h^\alpha_{ii}-T_\alpha)^2\geq Ric_{\min}-(n-1)(c+H^2)\\
-(n-2)\sum_\alpha T_\alpha(h^\alpha_{ii}-T_\alpha)+\sum_{\alpha,j\neq i}(h^\alpha_{ij})^2, \qquad (4.18)
\end{aligned}
$$

and

$$
\sum_\alpha T_\alpha(h^\alpha_{ii}-T_\alpha)\geq\frac{1}{n-2}[Ric_{\min}-(n-1)(c+H^2)], \qquad (4.19)
$$

for $i = 1, \ldots, n$. This together with (4.17) and (4.18) implies that

$$\begin{aligned} &R_{1313} + R_{2323} + R_{1414} + R_{2424} - 2R_{1234} \\ &\quad \geq 4(c + H^2) - 2\sum_{\alpha}[(h_{13}^{\alpha})^2 + (h_{23}^{\alpha})^2 + (h_{14}^{\alpha})^2 + (h_{14}^{\alpha})^2] \\ &\qquad +4[Ric_{\min} - (n-1)(c + H^2)] - (n-4)\sum_{\alpha}\sum_{i=1}^{4} T_{\alpha}(h_{ii}^{\alpha} - T_{\alpha}) \\ &\qquad + \sum_{\alpha, j\neq 1}(h_{1j}^{\alpha})^2 + \sum_{\alpha, j\neq 2}(h_{2j}^{\alpha})^2 + \sum_{\alpha, j\neq 3}(h_{3j}^{\alpha})^2 + \sum_{\alpha, j\neq 4}(h_{4j}^{\alpha})^2 \\ &\quad \geq 4(c + H^2) + 4[Ric_{\min} - (n-1)(c + H^2)] \\ &\qquad -(n-4)\sum_{\alpha}\sum_{i=1}^{4} T_{\alpha}(h_{ii}^{\alpha} - T_{\alpha}). \end{aligned} \tag{4.20}$$

Since

$$nT_{\alpha} = \sum_{i=1}^{4} h_{ii}^{\alpha} + \sum_{j=5}^{n} h_{jj}^{\alpha},$$

we have

$$\begin{aligned} \sum_{i=1}^{4}(h_{ii}^{\alpha} - T_{\alpha}) &= nT_{\alpha} - \sum_{j=5}^{n} h_{jj}^{\alpha} - 4T_{\alpha} \\ &= -\sum_{i=5}^{n}(h_{ii}^{\alpha} - T_{\alpha}). \end{aligned} \tag{4.21}$$

Substituting (4.19) and (4.21) into (4.20), we obtain

$$\begin{aligned} &R_{1313} + R_{2323} + R_{1414} + R_{2424} - 2R_{1234} \\ &\quad \geq 4(c + H^2) + 4[Ric_{\min} - (n-1)(c + H^2)] \\ &\qquad +(n-4)\sum_{\alpha}\sum_{i=5}^{n} T_{\alpha}(h_{ii}^{\alpha} - T_{\alpha}) \\ &\quad \geq 4(c + H^2) + \left(4 + \frac{n-4}{n-2}\right)[Ric_{\min} - (n-1)(c + H^2)] \\ &\quad = 4(c + H^2) + \frac{n^2 - 4n + 8}{n-2}[(n-1)R_0 - (n-1)(c + H^2)]. \end{aligned} \tag{4.22}$$

The last equality above holds because M is Einstein, i.e.,

$$(n-1)R_0 = Ric_{\min}. \tag{4.23}$$

This together with the assumption implies that M has positive isotropic curvature. This shows that M is isometric to a spherical space form.

For the case $c \geq 0$, by a direct computation, we have

$$\frac{n(n^2 - 5n + 8)}{n^2 - 4n + 8}(c + H^2) \geq \frac{n(n-1)}{n+2}(c + H^2). \tag{4.24}$$

This together with the assumption, (4.23) and Lemma 2.1 implies that M is simply connected. Therefore, M is isometric to a standard n-sphere.

This completes the proof of Theorem 1.4. □

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