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The second pinching theorem for hypersurfaces with constant mean curvature in a sphere

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Abstract We generalize the second pinching theorem for minimal hypersurfaces in a sphere due to Peng–Terng, Wei–Xu, Zhang, and Ding–Xin to the case of hypersurfaces with small constant mean curvature. Let M^n be a compact hypersurface with constant mean curvature H in S^{n+1}. Denote by S the squared norm of the second fundamental form of M. We prove that there exist two positive constants $\gamma(n)$ and $\delta(n)$ depending only on n such that if $|H| \leq \gamma(n)$ and $\beta(n, H) \leq S \leq \beta(n, H) + \delta(n)$, then $S \equiv \beta(n, H)$ and M is one of the following cases: (i) $S^k\left(\sqrt{\frac{k}{n}}\right) \times S^{n-k}\left(\sqrt{\frac{n-k}{n}}\right)$, $1 \leq k \leq n-1$; (ii) $S^1\left(\frac{1}{\sqrt{1+\mu^2}}\right) \times S^{n-1}\left(\frac{\mu}{\sqrt{1+\mu^2}}\right)$. Here $\beta(n, H) = n + \frac{n^3}{2(n-1)}H^2 + \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2}$ and $\mu = \frac{n|H|+\sqrt{n^2H^2+4(n-1)}}{2}$.

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1 Introduction

Let M^n be an n-dimensional compact hypersurface with constant mean curvature H in an $(n+1)$-dimensional unit sphere S^{n+1}. Denote by S the squared length of the second fundamental form of M and R its scalar curvature. Then $R = n(n-1) + n^2H^2 - S$.

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When $H = 0$, the famous pinching theorem due to Simons [12], Lawson [8], and Chern, do Carmo and Kobayashi ([2]) says that if $S \leq n$, then $S \equiv 0$ or $S \equiv n$, i.e., M must be the great sphere S^n or the Clifford torus $S^k(\sqrt{\frac{k}{n}}) \times S^{n-k}(\sqrt{\frac{n-k}{n}})$, $1 \leq k \leq n-1$. Further discussions have been carried out by many other authors (see [6,9,13,16,17,22,23], etc.). In 1970s, Chern proposed the following conjectures.

Chern Conjecture I. *Let M be a compact minimal hypersurface with constant scalar curvature in S^{n+1}. Then the possible values of S a discrete set. In particular, if $n \leq S \leq 2n$, then $S = n$, or $S = 2n$.*

Chern Conjecture II. *Let M be a compact minimal hypersurface in S^{n+1}. If $n \leq S \leq 2n$, then $S \equiv n$, or $S \equiv 2n$.*

In 1983, Peng and Terng made breakthrough on the Chern conjectures I and II. They [10] proved that if M is a compact minimal hypersurface with constant scalar curvature in the unit sphere S^{n+1}, and if $n \leq S \leq n + \frac{1}{12n}$, then $S = n$. Moreover, Peng and Terng [11] proved that if M is a compact minimal hypersurface in the unit sphere S^{n+1}, and if $n \leq 5$ and $n \leq S \leq n + \tau_1(n)$, where $\tau_1(n)$ is a positive constant depending only on n, then $S \equiv n$. During the past three decades, there have been some important progress on these aspects(see [1,4,5,7,14,15,24,25], etc.). In 1993, Chang [1] solved Chern Conjecture I for the case of dimension 3. In [4,24], Cheng, Ishikawa and Yang obtained some interesting results on the Chern conjectures.

In 2007, Suh–Yang and Wei–Xu made some progress on Chern Conjectures, respectively. Suh and Yang [14] proved that if M is a compact minimal hypersurface with constant scalar curvature in S^{n+1}, and if $n \leq S \leq n + \frac{3}{7}n$, then $S = n$ and M is a minimal Clifford torus. Meanwhile, Wei and Xu [15] proved that if M is a compact minimal hypersurface in S^{n+1}, $n = 6, 7$, and if $n \leq S \leq n + \tau_2(n)$, where $\tau_2(n)$ is a positive constant depending only on n, then $S \equiv n$ and M is a minimal Clifford torus. Later, Zhang [25] extended the second pinching theorem due to Peng–Terng [11] and Wei–Xu [15] to 8-dimensional compact minimal hypersurfaces in a unit sphere. Recently Ding and Xin [7] obtained the following pinching theorem for n-dimensional minimal hypersurfaces in a sphere.

Theorem A *Let M be an n-dimensional compact minimal hypersurface in a unit sphere S^{n+1}, and S the squared length of the second fundamental form of M. Then there exists a positive constant $\tau(n)$ depending only on n such that if $n \leq S \leq n+\tau(n)$, then $S \equiv n$, i.e., M is a Clifford torus.*

The pinching phenomenon for hypersurfaces of constant mean curvature in spheres is much more complicated than the minimal hypersurface case (see [16,18]). In [16], Xu proved the following pinching theorem for submanifolds with parallel mean curvature in a sphere.

Theorem B *Let M be an n-dimensional compact submanifold with parallel mean curvature vector $(H \neq 0)$ in an $(n+p)$-dimensional unit sphere S^{n+p}. If $S \leq \alpha(n, H)$, then either M is pseudo-umbilical, or $S \equiv \alpha(n, H)$ and M is the isoparametric hypersurface $S^{n-1}(\frac{1}{\sqrt{1+\lambda^2}}) \times S^1(\frac{\lambda}{\sqrt{1+\lambda^2}})$ in a great sphere S^{n+1}. In particular, if M is a compact hypersurface with constant mean curvature $H(\neq 0)$ in S^{n+1}, then M is either* *a totally umbilical sphere* $S^n(\frac{1}{\sqrt{1+H^2}})$*, or a Clifford hypersurface* $S^{n-1}(\frac{1}{\sqrt{1+\lambda^2}}) \times S^1(\frac{\lambda}{\sqrt{1+\lambda^2}})$*. Here* $\alpha(n,H) = n + \frac{n^3H^2}{2(n-1)} - \frac{n(n-2)|H|}{2(n-1)}\sqrt{n^2H^2+4(n-1)}$ *and* $\lambda = \frac{n|H|+\sqrt{n^2H^2+4(n-1)}}{2(n-1)}$.

In [19], Xu and Tian generalized Suh–Yang's pinching theorem [14] to the case where M is a compact hypersurface with constant scalar curvature and small constant mean curvature in S^{n+1}. The following second pinching theorem for hypersurfaces with small constant mean curvature was proved for $n \leq 7$ by Cheng et al. [3] and Xu–Zhao [20] respectively, and for $n = 8$ by Xu [21].

Theorem C *Let M be an n-dimensional compact hypersurface with constant mean curvature* $H(\neq 0)$ *in a unit sphere* S^{n+1}*,* $n \leq 8$*. Then there exist two positive constants* $\gamma_0(n)$ *and* $\delta_0(n)$ *depending only on n such that if* $|H| \leq \gamma_0(n)$*, and* $\beta(n,H) \leq S < \beta(n,H) + \delta_0(n)$*, then* $S \equiv \beta(n,H)$ *and* $M = S^1(\frac{1}{\sqrt{1+\mu^2}}) \times S^{n-1}(\frac{\mu}{\sqrt{1+\mu^2}})$*. Here* $\beta(n,H) = n + \frac{n^3}{2(n-1)}H^2 + \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4+4(n-1)H^2}$ *and* $\mu = \frac{n|H|+\sqrt{n^2H^2+4(n-1)}}{2}$.

In this paper, we prove the second pinching theorem for n-dimensional hypersurfaces with constant mean curvature, which is a generalization of Theorems A and C.

Main Theorem. *Let M be an n-dimensional compact hypersurface with constant mean curvature H in a unit sphere* S^{n+1}*. Then there exist two positive constants* $\gamma(n)$ *and* $\delta(n)$ *depending only on n such that if* $|H| \leq \gamma(n)$*, and* $\beta(n,H) \leq S \leq \beta(n,H) + \delta(n)$*, then* $S \equiv \beta(n,H)$ *and M is one of the following cases: (i)* $S^k(\sqrt{\frac{k}{n}}) \times S^{n-k}(\sqrt{\frac{n-k}{n}})$*,* $1 \leq k \leq n-1$*; (ii)* $S^1(\frac{1}{\sqrt{1+\mu^2}}) \times S^{n-1}(\frac{\mu}{\sqrt{1+\mu^2}})$*. Here* $\beta(n,H) = n + \frac{n^3}{2(n-1)}H^2 + \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4+4(n-1)H^2}$ *and* $\mu = \frac{n|H|+\sqrt{n^2H^2+4(n-1)}}{2}$.

2 Preliminaries

Let M^n be an n-dimensional compact hypersurface with constant mean curvature in a unit sphere S^{n+1}. We shall make use of the following convention on the range of indices.

$$1 \leq A, B, C, \ldots, \leq n+1, \quad 1 \leq i, j, k, \ldots, \leq n.$$

For an arbitrary fixed point $x \in M \subset S^{n+1}$, we choose an orthonormal local frame field $\{e_A\}$ in S^{n+1} such that e_i's are tangent to M. Let $\{\omega_A\}$ be the dual frame fields of $\{e_A\}$ and $\{\omega_{AB}\}$ the connection 1-forms of S^{n+1}. Restricting to M, we have

$$\omega_{n+1i} = \sum_j h_{ij}\omega_j, \quad h_{ij} = h_{ji}. \tag{1}$$

Let h be the second fundamental form of M. Denote by R, H and S the scalar curvature, mean curvature and squared length of the second fundamental form of M, respectively. Then we have

$$h = \sum_{i,j} h_{ij}\omega_i \otimes \omega_j, \tag{2}$$

$$S = \sum_{i,j} h_{ij}^2, \quad H = \frac{1}{n}\sum_i h_{ii}, \tag{3}$$

$$R = n(n-1) + n^2H^2 - S. \tag{4}$$

We choose e_{n+1} such that $H = \frac{1}{n}\sum_i h_{ii} \geq 0$. Denote by h_{ijk}, h_{ijkl} and h_{ijklm} the first, second and third covariant derivatives of the second fundamental tensor h_{ij}, respectively. Then we have

$$\nabla h = \sum_{i,j,k} h_{ijk}\omega_i \otimes \omega_j \otimes \omega_k, \quad h_{ijk} = h_{ikj}, \tag{5}$$

$$h_{ijkl} = h_{ijlk} + \sum_m h_{mj}R_{mikl} + \sum_m h_{im}R_{mjkl}, \tag{6}$$

$$h_{ijklm} = h_{ijkml} + \sum_r h_{rjk}R_{rilm} + \sum_r h_{irk}R_{rjlm} + \sum_r h_{ijr}R_{rklm}. \tag{7}$$

At each fixed point $x \in M$, we take orthonormal frames $\{e_i\}$ such that $h_{ij} = \lambda_i\delta_{ij}$ for all i, j. Then $\sum_i \lambda_i = nH$ and $\sum_i \lambda_i^2 = S$. By a direct computation, we have

$$\frac{1}{2}\Delta S = S(n-S) - n^2H^2 + nHf_3 + |\nabla h|^2, \tag{8}$$

$$\begin{aligned}\frac{1}{2}\Delta|\nabla h|^2 &= (2n+3-S)|\nabla h|^2 - \frac{3}{2}|\nabla S|^2 + |\nabla^2 h|^2 \\ &\quad + \sum_{i,j,k,l,m}(6h_{ijk}h_{ilm}h_{jl}h_{km} - 3h_{ijk}h_{ijl}h_{km}h_{ml}) \\ &\quad + 3nH\sum_{i,j,k,l} h_{ijk}h_{jlk}h_{li} \\ &= (2n+3-S)|\nabla h|^2 - \frac{3}{2}|\nabla S|^2 + |\nabla^2 h|^2 \\ &\quad + 3(2B-A) + 3nHC,\end{aligned} \tag{9}$$

where

$$f_k = \sum_i \lambda_i^k, \quad A = \sum_{i,j,k} h_{ijk}^2\lambda_i^2, \quad B = \sum_{i,j,k} h_{ijk}^2\lambda_i\lambda_j, \quad C = \sum_{i,j,k} h_{ijk}^2\lambda_i.$$

Using a similar method as in [10], we obtain

$$h_{ijij} = h_{jiji} + t_{ij}, \tag{10}$$

$$|\nabla^2 h|^2 \geq \frac{3}{4}\sum_{i\neq j} t_{ij}^2 = \frac{3}{4}\sum_{i,j} t_{ij}^2, \tag{11}$$

and

$$3(A-2B) \leq aS|\nabla h|^2, \tag{12}$$

where $t_{ij} = (\lambda_i - \lambda_j)(1+\lambda_i\lambda_j)$ and $a = \frac{\sqrt{17}+1}{2}$. From (11), we have

$$|\nabla^2 h|^2 \geq \frac{3}{2}[Sf_4 - f_3^2 - S^2 - S(S-n) - n^2H^2 + 2nHf_3]. \tag{13}$$

By a computation, we obtain

$$\begin{aligned}
\frac{1}{3}\sum_{i,j} h_{ij}(f_3)_{ij} &= \frac{1}{3}\sum_k \lambda_k (f_3)_{kk} \\
&= \sum_k \lambda_k \left(\sum_i h_{iikk}\lambda_i^2 + 2\sum_{i,j} h_{ijk}^2\lambda_i\right) \\
&= \sum_{i,k} h_{iikk}\lambda_k\lambda_i^2 + 2\sum_{i,j,k} h_{ijk}^2\lambda_i\lambda_k \\
&= \sum_{i,k}[h_{kkii} + (\lambda_i-\lambda_k)(1+\lambda_i\lambda_k)]\lambda_k\lambda_i^2 + 2B \\
&= \sum_i \left(\frac{S_{ii}}{2} - \sum_{j,k} h_{ijk}^2\right)\lambda_i^2 + \sum_{i,k}\lambda_i^2\lambda_k(\lambda_i-\lambda_k)(1+\lambda_i\lambda_k) + 2B \\
&= \sum_{i,j,k}\frac{h_{ik}h_{kj}}{2}S_{ij} + nHf_3 - S^2 - f_3^2 + Sf_4 - (A-2B).
\end{aligned} \tag{14}$$

Since $\int_M \sum_{i,j} h_{ij}(f_3)_{ij} dM = 0$, we drive the following integral formula.

$$\begin{aligned}
\int_M (A-2B)dM &= \int_M \left(nHf_3 - S^2 - f_3^2 + Sf_4 + \sum_{i,j,k}\frac{h_{ik}h_{kj}}{2}S_{ij}\right)dM \\
&= \int_M \left(nHf_3 - S^2 - f_3^2 + Sf_4 - \sum_{i,j,k}(h_{ik}h_{kj})_j\frac{S_i}{2}\right)dM \\
&= \int_M \left(nHf_3 - S^2 - f_3^2 + Sf_4 - \sum_{i,j,k} h_{ikj}h_{kj}\frac{S_i}{2}\right.
\end{aligned}$$

$$-\sum_{i,j,k} h_{ik}h_{kjj}\frac{S_i}{2}\Bigg) dM$$

$$= \int_M \left(nHf_3 - S^2 - f_3^2 + Sf_4 - \sum_{i,j,k} h_{ikj}h_{kj}\frac{S_i}{2}\right) dM$$

$$= \int_M \left(nHf_3 - S^2 - f_3^2 + Sf_4 - \frac{|\nabla S|^2}{4}\right) dM. \tag{15}$$

3 Proof of Main Theorem

The key to the proof of Main Theorem is to establish some integral equalities and inequalities on the second fundamental form of M and its covariant derivatives by the parameter method.

To simplify the computation, we introduce the tracefree second fundamental form $\phi = \sum_{i,j} \phi_{ij}\omega_i \otimes \omega_j$, where $\phi_{ij} = h_{ij} - H\delta_{ij}$. If $h_{ij} = \lambda_i\delta_{ij}$, then $\phi_{ij} = \mu_i\delta_{ij}$, where $\mu_i = \lambda_i - H$. Putting $\Phi = |\phi|^2$ and $\bar{f}_k = \sum_i \mu_i^k$, we get $\Phi = S - nH^2$, $f_3 = \bar{f}_3 + 3H\Phi + nH^3$ and $f_4 = \bar{f}_4 + 4H\bar{f}_3 + 6H^2\Phi + nH^4$. From (8), we obtain

$$\begin{aligned}\frac{1}{2}\Delta\Phi &= S(n - S) - n^2H^2 + nHf_3 + |\nabla h|^2\\ &= -\Phi^2 + n\Phi + nH\bar{f}_3 + nH^2\Phi + |\nabla\phi|^2\\ &= -F(\Phi) + |\nabla\phi|^2,\end{aligned} \tag{16}$$

where $F(\Phi) = \Phi^2 - n\Phi - nH^2\Phi - nH\bar{f}_3$. Therefore, we have

$$|\nabla\Phi|^2 = \frac{1}{2}\Delta\Phi^2 - \Phi\Delta\Phi = \frac{1}{2}\Delta\Phi^2 + 2\Phi F(\Phi) - 2\Phi|\nabla\phi|^2, \tag{17}$$

and

$$\int_M F(\Phi)dM = \int_M |\nabla\phi|^2 dM. \tag{18}$$

Lemma 1 (See [16]) *Let $a_1, a_2, \ldots, a_n$ be real numbers satisfying $\sum_i a_i = 0$ and $\sum_i a_i^2 = a$. Then*

$$\left|\sum_i a_i^3\right| \le \frac{n-2}{\sqrt{n(n-1)}} a^{\frac{3}{2}},$$

and the equality holds if and only if at least $n - 1$ numbers of a_i's are same with each other.

From Lemma 1, we get

$$\begin{aligned} F(\Phi) &\geq \Phi^2 - n\Phi - nH^2\Phi - \frac{n(n-2)H\Phi^{\frac{3}{2}}}{\sqrt{n(n-1)}} \\ &= \Phi\left[\Phi - \frac{n(n-2)H\Phi^{\frac{1}{2}}}{\sqrt{n(n-1)}} - n(1+H^2)\right] \\ &\geq 0, \end{aligned} \tag{19}$$

provided

$$\Phi \geq \beta_0(n,H) := n + \frac{n^3}{2(n-1)}H^2 + \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2} - nH^2.$$

Moreover, $F(\Phi) = 0$ if and only if $\Phi = \beta_0(n,H)$.

Set

$$G = \sum_{i,j}(\lambda_i - \lambda_j)^2(1 + \lambda_i\lambda_j)^2.$$

Then we have

$$G = 2[Sf_4 - f_3^2 - S^2 - S(S-n) + 2nHf_3 - n^2H^2]. \tag{20}$$

This together with (8) and (15) implies

$$\frac{1}{2}\int_M G\,dM = \int_M \left[(A - 2B) - |\nabla h|^2 + \frac{1}{4}|\nabla S|^2\right] dM. \tag{21}$$

Lemma 2 *Let M be an $n(\geq 4)$-dimensional compact hypersurface with constant mean curvature in S^{n+1}. If $S \geq \beta(n,H)$, then we have*

$$3(A - 2B) \leq 2S|\nabla h|^2 + C_1(n)|\nabla h|^2 G^{\frac{1}{3}},$$

where $C_1(n) = (\sqrt{17} - 3)[6(\sqrt{17} + 1)]^{-\frac{1}{3}}(\frac{2}{\sqrt{17}} - \frac{\sqrt{2}}{17} - \frac{1}{n})^{-\frac{2}{3}}$.

Proof We derive the estimate above at each fixed point $x \in M$. If $\lambda_j^2 - 4\lambda_i\lambda_j \leq 2S$ for all $i \neq j$, then we get the desired estimate immediately. Otherwise, we assume that there exist $i \neq j$, such that $\lambda_j^2 - 4\lambda_i\lambda_j = tS > 2S$.

We get

$$S \geq \lambda_i^2 + \lambda_j^2 = \left(\frac{tS - \lambda_j^2}{4\lambda_j}\right)^2 + \lambda_j^2. \tag{22}$$

Then

$$\lambda_j^2 \leq \frac{1}{17}\left(t+8+4\sqrt{4+t-t^2}\right)S, \quad 2<t\leq\frac{\sqrt{17}+1}{2}, \tag{23}$$

which implies

$$-\lambda_i\lambda_j \geq \frac{1}{17}\left(4t-2-\sqrt{4+t-t^2}\right)S \geq 0.26S > \frac{S}{n} \geq 1. \tag{24}$$

On the other hand, we have

$$(\lambda_i-\lambda_j)^2 = \left(\frac{\lambda_j}{2}+\lambda_i\right)^2 + \frac{3}{4}\left(\lambda_j^2-4\lambda_i\lambda_j\right) \geq \frac{3t}{4}S. \tag{25}$$

By the definition of G, we get

$$\begin{aligned} G &\geq 2(\lambda_i-\lambda_j)^2(1+\lambda_i\lambda_j)^2 \\ &\geq \frac{3t}{2}S(1+\lambda_i\lambda_j)^2 \\ &\geq \frac{3t}{2}S\left(-\lambda_i\lambda_j-\frac{S}{n}\right)^2 \\ &\geq \frac{3t}{2}\left[\frac{1}{17}(4t-2-\sqrt{4+t-t^2})-\frac{1}{n}\right]^2 S^3. \end{aligned} \tag{26}$$

We define an auxiliary function

$$\zeta(t) = \frac{t}{(t-2)^3}\left[\frac{1}{17}(4t-2-\sqrt{4+t-t^2})-\frac{1}{n}\right]^2, \quad 2<t\leq\frac{\sqrt{17}+1}{2}.$$

Then we have

$$\begin{aligned} \zeta(t) &\geq \frac{t}{(t-2)^3}\left[\frac{1}{17}\left(4t-2-\sqrt{2}\right)-\frac{1}{n}\right]^2 \\ &\geq \inf_{2<t\leq\frac{\sqrt{17}+1}{2}} \frac{t}{(t-2)^3}\left[\frac{1}{17}\left(4t-2-\sqrt{2}\right)-\frac{1}{n}\right]^2 \\ &= \frac{4(\sqrt{17}+1)}{(\sqrt{17}-3)^3}\left(\frac{2}{\sqrt{17}}-\frac{\sqrt{2}}{17}-\frac{1}{n}\right)^2. \end{aligned} \tag{27}$$

Hence

$$\begin{aligned} (\lambda_j^2-4\lambda_i\lambda_j-2S)^3 &= (t-2)^3S^3 \\ &\leq \frac{2G}{3\zeta(t)} \end{aligned}$$

$$\leq \frac{(\sqrt{17}-3)^3}{6(\sqrt{17}+1)}\left(\frac{2}{\sqrt{17}}-\frac{\sqrt{2}}{17}-\frac{1}{n}\right)^{-2} G$$
$$= \left(C_1(n)G^{\frac{1}{3}}\right)^3. \tag{28}$$

This implies

$$\begin{aligned} 3(A-2B) \leq & \sum_{i,j,k\, distinct} \left[2\left(\lambda_i^2+\lambda_j^2+\lambda_k^2\right)-\left(\lambda_i+\lambda_j+\lambda_k\right)^2\right] h_{ijk}^2 \\ & + 3\sum_{i\neq j}(\lambda_j^2-4\lambda_i\lambda_j)h_{iij}^2 \\ \leq & \, 2S \sum_{i,j,k\ distinct} h_{ijk}^2 + 3\sum_{i\neq j} h_{iij}^2\left(2S+C_1(n)G^{\frac{1}{3}}\right) \\ \leq & \, 2S|\nabla h|^2 + C_1(n)|\nabla h|^2 G^{\frac{1}{3}}. \end{aligned} \tag{29}$$

□

Proof of Main Theorem (i) When $H=0$, the assertion follows from Theorem A.

(ii) When $H \neq 0$, the assertion for lower dimensional cases ($n \leq 8$) was verified in [3,20,21]. We consider the case for $n \geq 4$. From (10) and (11), we see that $G=\sum_{i,j} t_{ij}^2$ and $|\nabla^2 h|^2 \geq \frac{3}{4}G$. Letting $0<\theta<1$, we have

$$\int_M |\nabla^2 h|^2 dM \geq \left[\frac{3(1-\theta)}{4}+\frac{3\theta}{4}\right]\int_M G dM. \tag{30}$$

From (9), (21), Lemma 2 and Young's inequality, we drive the following inequality.

$$\begin{aligned} & \frac{3(1-\theta)}{4}\int_M G dM \\ \leq & \int_M \left[(S-2n-3)|\nabla h|^2+\frac{3}{2}|\nabla S|^2+3(A-2B)-3nHC-\frac{3\theta}{4}G\right]dM \\ = & \int_M \left(S-2n-3+\frac{3\theta}{2}\right)|\nabla h|^2 dM + \left(3-\frac{3\theta}{2}\right)\int_M (A-2B)dM \\ & + \left(\frac{3}{2}-\frac{3\theta}{8}\right)\int_M |\nabla S|^2 dM - 3nH\int_M C dM \\ \leq & \int_M \left(S-2n-3+\frac{3\theta}{2}\right)|\nabla h|^2 dM + \left(1-\frac{\theta}{2}\right)\int_M \Big(2S|\nabla h|^2 \\ & + C_1(n)|\nabla h|^2 G^{\frac{1}{3}}\Big) dM + \left(\frac{3}{2}-\frac{3\theta}{8}\right)\int_M |\nabla S|^2 dM - 3nH\int_M C dM \end{aligned}$$

$$\leq \int_M \left[(3-\theta)S - 2n - 3 + \frac{3\theta}{2}\right] |\nabla h|^2 dM + \frac{3(1-\theta)}{4} \int_M G dM$$
$$+ C_2(n,\theta) \int_M |\nabla h|^3 dM + \left(\frac{3}{2} - \frac{3\theta}{8}\right) \int_M |\nabla S|^2 dM$$
$$- 3nH \int_M C dM, \tag{31}$$

where $C_2(n,\theta) = \frac{4}{9} C_1(n)^{\frac{3}{2}} \left(1 - \frac{\theta}{2}\right)^{\frac{3}{2}} (1-\theta)^{-\frac{1}{2}}$.

Letting $\epsilon > 0$, from (16), we get

$$\begin{aligned}\int_M |\nabla h|^3 dM &= \int_M |\nabla \phi|^3 dM \\ &= \int_M |\nabla \phi| \left(F(\Phi) + \frac{1}{2}\Delta \Phi\right) dM \\ &= \int_M F(\Phi)|\nabla \phi| dM - \frac{1}{2} \int_M \nabla |\nabla \phi| \cdot \nabla \Phi dM \\ &\leq \int_M F(\Phi)|\nabla \phi| dM + \epsilon \int_M |\nabla^2 \phi|^2 dM + \frac{1}{16\epsilon} \int_M |\nabla \Phi|^2 dM.\end{aligned} \tag{32}$$

Since

$$|C| \leq \sqrt{S} |\nabla h|^2, \tag{33}$$

we have

$$0 \leq \int_M \left[(3 + 3\sqrt{n}H - \theta)(\Phi + nH^2) - 2n - 3 + \frac{3\theta}{2}\right] |\nabla \phi|^2 dM$$
$$+ C_2(n,\theta) \left[\int_M F(\Phi)|\nabla \phi| dM + \epsilon \int_M |\nabla^2 \phi|^2 dM + \frac{1}{16\epsilon} \int_M |\nabla \Phi|^2 dM\right]$$
$$+ \left(\frac{3}{2} - \frac{3\theta}{8}\right) \int_M |\nabla \Phi|^2 dM. \tag{34}$$

Substituting (12) and (33) into (9), we have

$$\begin{aligned}\int_M |\nabla^2\phi|^2 dM &= \int_M |\nabla^2 h|^2 dM\\ &\le \int_M \left[(S-2n-3)|\nabla h|^2 + \frac{3}{2}|\nabla S|^2 + aS|\nabla h|^2 - 3nHC\right] dM\\ &\le \int_M [(a+1+3\sqrt{n}H)S-2n-3]|\nabla\phi|^2 dM + \frac{3}{2}\int_M |\nabla S|^2 dM.\end{aligned} \tag{35}$$

Combining (16) and (17), we have

$$\begin{aligned}\int_M \frac{1}{2}|\nabla\Phi|^2 dM &= \int_M \Phi F(\Phi) dM - \int_M \Phi|\nabla\phi|^2 dM + \beta_0(n,H)\int_M |\nabla\phi|^2 dM\\ &\quad -\beta_0(n,H)\int_M F(\Phi) dM\\ &= \int_M (\Phi-\beta_0(n,H))F(\Phi) dM + \int_M (\beta_0(n,H)-\Phi)|\nabla\phi|^2 dM.\end{aligned} \tag{36}$$

Hence

$$\begin{aligned}0 &\le \int_M \Big\{ \left[3+3\sqrt{n}H-\theta+\epsilon C_2(n,\theta)(a+1+3\sqrt{n}H)\right](\Phi-\beta_0(n,H))\\ &\quad +\beta(n,H)\left[3+3\sqrt{n}H-\theta+\epsilon C_2(n,\theta)(a+1+3\sqrt{n}H)\right]\\ &\quad -2\left(\frac{3}{2}-\frac{3\theta}{8}+\frac{C_2(n,\theta)}{16\epsilon}+\frac{3\epsilon C_2(n,\theta)}{2}\right)(\Phi-\beta_0(n,H))\\ &\quad -2n-3+\frac{3\theta}{2}-\epsilon C_2(n,\theta)(2n+3)\Big\}|\nabla\phi|^2 dM\\ &\quad +2\left(\frac{3}{2}-\frac{3\theta}{8}+\frac{C_2(n,\theta)}{16\epsilon}+\frac{3\epsilon C_2(n,\theta)}{2}\right)\int_M (\Phi-\beta_0(n,H))F(\Phi) dM\\ &\quad +C_2(n,\theta)\int_M F(\Phi)|\nabla\phi| dM\\ &= \int_M \Big\{ D(n,H)\left[3+3\sqrt{n}H-\theta+\epsilon C_2(n,\theta)(a+1+3\sqrt{n}H)\right]\\ &\quad +(1-\theta)n-3+\frac{3\theta}{2}+3n^{\frac{3}{2}}H+\epsilon C_2(n,\theta)(an+3n^{\frac{3}{2}}H-n-3)\Big\}|\nabla\phi|^2 dM\end{aligned}$$

$$
\begin{aligned}
&-\Bigg(\frac{\theta}{4}+\frac{C_2(n,\theta)}{8\epsilon}-3\sqrt{n}H\\
&+\epsilon C_2(n,\theta)\left(2-a-3\sqrt{n}H\right)\Bigg)\int_M(\Phi-\beta_0(n,H))|\nabla\phi|^2dM\\
&+\left(3-\frac{3\theta}{4}+\frac{C_2(n,\theta)}{8\epsilon}+3\epsilon C_2(n,\theta)\right)\int_M(\Phi-\beta_0(n,H))F(\Phi)dM\\
&+C_2(n,\theta)\int_M F(\Phi)|\nabla\phi|dM,
\end{aligned}
\tag{37}
$$

where $\beta(n,H)=\beta_0(n,H)+nH^2$ and $D(n,H)=\beta(n,H)-n$.

Note that

$$
\frac{\theta}{4}+\frac{C_2(n,\theta)}{8\epsilon}-3\sqrt{n}H+\epsilon C_2(n,\theta)(2-a-3\sqrt{n}H)\geq 0,
\tag{38}
$$

for all $\epsilon\in(0,\epsilon_1]$, where ϵ_1 is some positive constant. When $\beta(n,H)\leq S\leq \beta(n,H)+\epsilon^2$, we obtain

$$
\begin{aligned}
0\leq\int_M\Bigg[&(1-\theta)n-3+\frac{3\theta}{2}+3n^{\frac{3}{2}}H+D(n,H)(3+3\sqrt{n}H-\theta)\\
&+O(\epsilon,\theta,H)\Bigg]|\nabla\phi|^2dM+C_2(n,\theta)\int_M F(\Phi)|\nabla\phi|dM,
\end{aligned}
\tag{39}
$$

where

$$
\begin{aligned}
O(\epsilon,\theta,H)&=\epsilon D(n,H)C_2(n,\theta)\left(a+1+3\sqrt{n}H\right)\\
&\quad+\epsilon C_2(n,\theta)(an+3n^{\frac{3}{2}}H-n-3)\\
&\quad+\epsilon^2\left(3-\frac{3\theta}{4}+\frac{C_2(n,\theta)}{8\epsilon}+3\epsilon C_2(n,\theta)\right).
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&C_2(n,\theta)\int_M F(\Phi)|\nabla\phi|dM\\
&\leq\frac{3}{8}\int_M F(\Phi)dM+\frac{2C_2(n,\theta)^2}{3}\int_M F(\Phi)|\nabla\phi|^2dM.
\end{aligned}
\tag{40}
$$

Using Lemma 1, we drive an upper bound for $F(\Phi)$.

$$\begin{aligned} F(\Phi) &\le \Phi^2 - n\Phi - nH^2\Phi + \frac{n(n-2)H\Phi^{\frac{3}{2}}}{\sqrt{n(n-1)}} \\ &= \Phi\left[\Phi + \frac{n(n-2)H\Phi^{\frac{1}{2}}}{\sqrt{n(n-1)}} - n(1+H^2)\right] \\ &= \frac{\Phi\left(\Phi^{\frac{1}{2}} + \beta_0(n,H)^{\frac{1}{2}}\right)(\Phi - \alpha_0(n,H))}{\Phi^{\frac{1}{2}} + \alpha_0(n,H)^{\frac{1}{2}}}, \end{aligned} \tag{41}$$

where $\alpha_0(n,H) = \left[\frac{-n(n-2)H + n\sqrt{n^2H^2+4n-4}}{2\sqrt{n(n-1)}}\right]^2$.

When $\delta(n) \le \epsilon^2$ and $\epsilon \le 1$, we choose positive constant $\gamma_1(n)$ such that $n \le \Phi \le 2n$ and $\beta_0(n,H) \le 2n-1$ for all $H \le \gamma_1(n)$. We obtain

$$F(\Phi) \le 8n(\Phi - \alpha_0(n,H)) \le 8n\left(\epsilon^2 + \frac{n(n-2)}{(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2}\right). \tag{42}$$

Let $\theta = \theta(n) = 1 - \frac{1}{8n}$. We choose positive constants $\gamma_2(n)$ and $\gamma_3(n)$ such that $3n^{\frac{3}{2}}H + D(n,H)(3 + 3\sqrt{n}H) \le \frac{1}{8}$ for all $H \le \gamma_2(n)$, and $\frac{16n^2(n-2)}{(n-1)}\sqrt{n^2\gamma_3(n)^4 + 4(n-1)\gamma_3(n)^2} \le \frac{9}{16C_2(n,\theta(n))^2}$.

Take $\epsilon_2(n) = \left[\frac{n(n-2)}{(n-1)}\sqrt{n^2\gamma_3(n)^4 + 4(n-1)\gamma_3(n)^2}\right]^{\frac{1}{2}} > 0$. Combining (39), (40) and (42), we obtain

$$\int_M \left[-\frac{1}{2} + O(\epsilon, \theta(n), H)\right]|\nabla\phi|^2 dM \ge 0, \tag{43}$$

for all $H \le \gamma(n) = \min\{\gamma_1(n), \gamma_2(n), \gamma_3(n)\}$ and $\epsilon \le \min\{1, \epsilon_1, \epsilon_2(n)\}$.

For $\epsilon \le 1$, we have

$$\begin{aligned} O(\epsilon,\theta(n),H) \;&\le\; \epsilon D(n,\gamma(n))C_2(n,\theta(n))(a + 1 + 3\sqrt{n}\gamma(n)) \\ &\quad + \epsilon C_2(n,\theta(n))(an + 3n^{\frac{3}{2}}\gamma(n)) \\ &\quad + \epsilon\left(3 - \frac{3\theta(n)}{4} + \frac{C_2(n,\theta(n))}{8} + 3C_2(n,\theta(n))\right) \\ &:= \epsilon\eta(n), \end{aligned} \tag{44}$$

where $a = \frac{\sqrt{17}+1}{2}$.

For $\epsilon \le \epsilon_1(n)$, where $\epsilon_1(n) = \frac{C_2(n,\theta(n))}{8[3\sqrt{n}\gamma(n) + C_2(n,\theta(n))(a + 3\sqrt{n}\gamma(n) - 2)]} > 0, a = \frac{\sqrt{17}+1}{2}$, we have

$$\frac{C_2(n,\theta(n))}{8\epsilon} \ge 3\sqrt{n}\gamma(n) + C_2(n,\theta(n))\left(a + 3\sqrt{n}\gamma(n) - 2\right) - \frac{\theta(n)}{4}. \tag{45}$$

So

$$\frac{\theta(n)}{4}+\frac{C_2(n,\theta(n))}{8\epsilon}-3\sqrt{n}H+\epsilon C_2(n,\theta(n))\left(2-a-3\sqrt{n}H\right)\geq 0.$$

Taking $\delta(n)=\epsilon(n)^2$, where $\epsilon(n)=\min\{1,\epsilon_1(n),\epsilon_2(n),\epsilon_3(n)\}$ and $\epsilon_3(n)=\frac{1}{3\eta(n)}$, we have $\delta(n)>0$. From (43) and the assumption that $\beta(n,H)\leq S\leq\beta(n,H)+\delta(n)$, we obtain $\nabla\phi=0$. This implies $F(\Phi)=0$ and $\Phi=\beta_0(n,H)$.

By Lemma 1, we have

$$\lambda_1=\cdots=\lambda_{n-1}=H-\sqrt{\frac{\beta(n,H)-nH^2}{n(n-1)}},$$
$$\lambda_n=H+\sqrt{\frac{(n-1)(\beta(n,H)-nH^2)}{n}}.$$

Therefore M is the Clifford hypersurface

$$S^1\left(\frac{1}{\sqrt{1+\mu^2}}\right)\times S^{n-1}\left(\frac{\mu}{\sqrt{1+\mu^2}}\right)$$

in S^{n+1}, where $\mu=\frac{nH+\sqrt{n^2H^2+4(n-1)}}{2}$. This completes the proof of Main Theorem.

Finally we would like to propose the following problems.

Open Problem A *Let M be an n-dimensional compact hypersurface with constant mean curvature H in the unit sphere S^{n+1}. Does there exist a positive constant $\delta(n)$ depending only on n such that if $\beta(n,H)\leq S\leq\beta(n,H)+\delta(n)$, then $S\equiv\beta(n,H)$?*

Open Problem B *For an n-dimensional compact hypersurface M^n with constant mean curvature H in S^{n+1}, set $\mu_k=\frac{n|H|+\sqrt{n^2H^2+4k(n-k)}}{2k}$. Suppose that $\alpha(n,H)\leq S\leq\beta(n,H)$. Is it possible to prove that M must be the isoparametric hypersurface $S^k\left(\frac{1}{\sqrt{1+\mu_k^2}}\right)\times S^{n-k}\left(\frac{\mu_k}{\sqrt{1+\mu_k^2}}\right)$, $k=1,2,\ldots,n-1$?*

When $H=0$, the rigidity theorem due to Lawson [8], Chern, do Carmo and Kobayashi [2] provides an affirmative answer for Open Problem B.

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