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Characterization of Beauville's Numbers via Hodge Theory

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We provide a new Hodge theoretical characterization of the set of complex numbers that arises from the complete list, due to A. Beauville, of semistable families of elliptic curves over $\mathbb{P}^1$ with four singular fibers. The characterization is approached via a detailed analysis of the periodicity of the uniformizing Higgs bundle attached to $\mathbb{P}^1$ minus four points over the field of complex numbers.

1 Introduction

In a beautiful work, Beauville [1] gives a complete list of semistable families of elliptic curves over $\mathbb{P}^1$ with four singular fibers. Based on his classification, it is easy to obtain the complete list of complex numbers λ such that there is a semistable family of elliptic curves over $\mathbb{P}^1$ with four singular fibers along $\{0, 1, \lambda, \infty\}$, that is, the following list of complex numbers:

$$\begin{aligned} \lambda \in \{&-1,\ 2,\ 1/2,\ -8, && 9,\ -1/8,\ 9/8, && 1/9,\ 8/9, \\ &(1-\sqrt{-3})/2, && (1+\sqrt{-3})/2, && \\ &(-123-55\sqrt{5})/2, && (125+55\sqrt{5})/2, && (-123+55\sqrt{5})/2, \\ &(125-55\sqrt{5})/2, && (25-11\sqrt{5})/50, && (25+11\sqrt{5})/50\}. \end{aligned} \tag{1.0.1}$$

We call a complex number in the above list a *Beauville's number*. These numbers have a clear geometric meaning. Namely, these are all possible values such that $\mathbb{P}^1 - \{0, 1, \lambda, \infty\}$

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is a *modular curve*. Consider an arbitrary $\lambda \in \mathbb{C}$ different from $\{0, 1\}$. Then, the fundamental group of $\mathbb{P}^1 - \{0, 1, \lambda, \infty\}$ is freely generated by three loops. As is well known, the uniformization theorem of Riemann surfaces gives rise to a uniformizing representation that does depend on λ

$$\rho_\lambda : \pi_1(\mathbb{P}^1 - \{0, 1, \lambda, \infty\}) \to \mathrm{SL}_2(\mathbb{R}).$$

When λ is a Beauville's number, ρ_λ admits a $\mathbb{Z}$-lattice structure. It is well known that ρ_λ underlies a weight one polarized $\mathbb{R}$-variation of Hodge structure (VHS). Therefore, Beauville's [1] work amounts to the classification of λs such that ρ_λ underlies a weight one polarized $\mathbb{Z}$-VHS. In this paper, we obtain a new characterization of Beauville's numbers.

Theorem 1.1. Let $\lambda \neq 0, 1$ be a complex number. Then, it is a Beauville's number if and only if the associated uniformizing representation ρ_λ satisfies the following properties:

(i) ρ_λ factors through $\mathrm{SL}_2(\mathcal{O}_F) \subset \mathrm{SL}_2(\mathbb{R})$ for some totally real subfield $F \subset \mathbb{R}$;
(ii) $\mathrm{Res}_{F|\mathbb{Q}}\rho_\lambda$ is a VHS;
(iii) there exists one point $x \in \mathbb{P}^1 - \{0, 1, \lambda, \infty\}$ such that the multiplication by any element of $\mathcal{O}_F$ is of Hodge type $(0, 0)$.

Recall that in the Hitchin–Simpson's approach to the uniformization theory of Riemann surfaces, one studies the so-called uniformizing Higgs bundle $(E_\lambda, \theta_\lambda)$, which is constructed as follows:

$$E_\lambda = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1), \quad \theta_\lambda = \theta_\lambda^1 \oplus \theta_\lambda^0,$$

where $\theta_\lambda^0 = 0$ and $\theta_\lambda^1 : \mathcal{O}_{\mathbb{P}^1}(1) \to \mathcal{O}_{\mathbb{P}^1}(-1) \otimes \Omega_{\mathbb{P}^1}(0 + 1 + \lambda + \infty)$ is an isomorphism. Under the correspondence established in [5, 12], the complex local system $\rho_\lambda \otimes \mathbb{C}$ and the logarithmic Higgs bundle $(E_\lambda, \theta_\lambda)$ correspond to each other. In this paper, we aim to link the *$\mathbb{Z}$-lattice structure* of ρ_λ with the *periodicity* of $(E_\lambda, \theta_\lambda)$ under the Hitchin–Simpson correspondence. Theorem 1.1 is a consequence of this exploration.

Definition 1.2. The Higgs bundle $(E_\lambda, \theta_\lambda)$ over $\mathbb{P}^1$ is called *periodic* if there exists *some* spread

$$(\mathbb{P}^1_S, \mathcal{D}, \mathcal{E}_\lambda, \Theta_\lambda) \to S$$

of $(\mathbb{P}^1, D = 0 + 1 + \lambda + \infty, E_\lambda, \theta_\lambda)$ and a positive integer f such that, for all geometric points $s \in S$, the reduction $(\mathcal{E}_s, \Theta_s)$ at s is periodic of period $\leq f$ for *all* $W_2(k(s))$-lifting

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$\tilde{s} \to S$. If $(E_\lambda, \theta_\lambda)$ is periodic, its *period* is defined to be the smallest $f \geq 1$ among all possible spreads.

In the above definition, S is an integral scheme of finite type over $\mathbb{Z}$ and the periodicity in positive characteristic refers to the theory of Higgs–de Rham flows established in [8]. It is meaningful to put forth the notion of a *periodic logarithmic Higgs bundle* over $\mathbb{C}$. We do so in Definition 2.6, and we prove the following *periodicity theorem*.

Theorem 1.3 (Theorem 2.8). Let $f : X \to C$ be a semistable family over a smooth projective curve over $\mathbb{C}$, and let (E, θ) be a logarithmic Kodaira–Spencer system attached to f. Then, any graded Higgs subbundle in (E, θ) of degree zero is periodic.

We remark that polarization possessed by a semistable family plays an essential role in the proof of the above result. Using this theorem, we may deduce immediately the following corollary.

Corollary 1.4 (Proposition 3.2). For any Beauville's number λ, $(E_\lambda, \theta_\lambda)$ is periodic.

Analyzing the periodicity for the logarithmic Higgs bundle $(E_\lambda, \theta_\lambda)$, we obtain the following proposition.

Proposition 1.5 (Propositions 3.1 and 3.4). If $(E_\lambda, \theta_\lambda)$ is periodic, then it must be one-periodic and λ must be algebraic.

Corollary 1.4 and Proposition 1.5 allow us to conclude Theorem 1.1. We call a complex number λ *periodic* if $(E_\lambda, \theta_\lambda)$ is periodic. By Corollary 1.4, the set of periodic numbers contains the Beauville's list 1.0.1. We make the following conjecture.

Conjecture 1.6. The Beauville's numbers are *all* periodic numbers.

A bruteforce calculation on the periodicity of $(E_\lambda, \theta_\lambda)$ reduces the above conjecture to a concrete arithmetic question (see Question 3.7). We shall leave the task of solving the question for a later investigation.

This paper is structured as follows. In Section 2, we introduce the notion of a periodic Higgs bundle over the field of complex numbers and establish the periodicity result for semistable families over curves. Section 3 gives a detailed analysis of the

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periodicity of the uniformizing Higgs bundle in both positive characteristic and zero characteristic. In Section 4, we construct a semistable family of abelian varieties with real multiplication under the Hodge theoretical properties listed in Theorem 1.1. This is a straightforward step. We prove our main result Theorem 1.1 in Section 5.

2 Logarithmic Kodaira–Spencer Systems and Periodicity

Let C be a smooth projective curve over $\mathbb{C}$ and $D \subset C$ be a reduced effective divisor. Let $f: X \to C$ be a semistable family that is smooth over $U = C - D$. Set $B = f^{-1}D$. We attach to f the *logarithmic Kodaira–Spencer system* of degree n for some $0 \le n \le 2d$ (where d is the relative dimension of f) as follows:

$$(E = \bigoplus_{i+j=n} R^i f_* \Omega^j_{X/C}(\log B/D), \theta = \bigoplus_{i+j=n} \theta^{i,j}),$$

where

$$\theta^{i,j} : R^i f_* \Omega^j_{X/C}(\log B/D) \to R^{i+1} f_* \Omega^{j-1}_{X/C}(\log B/D) \otimes \Omega_C(D)$$

is the logarithmic Kodaira–Spencer morphism. Set $E^{i,j} = R^i f_* \Omega^j_{X/C}(\log B/D)$. The pair (E, θ) provides a basic example of a graded logarithmic Higgs bundle over C. These logarithmic Kodaira–Spencer systems are special kinds of graded logarithmic Higgs bundles, as the following result shows.

Proposition 2.1. Let (E, θ) be a logarithmic Kodaira–Spencer system over C as above. Then, it is polystable of degree zero.

Proof. This conclusion follows from well-known results in nonabelian Hodge theory over a quasi-projective smooth curve [12]. Let f^0 be the smooth part of f. Then, the Hodge metric associated with the weight n VHS attached to f^0 is *tame harmonic*. By Landman's theorem, the local monodromies around D are unipotent. It follows that the filtration structures are absent in the Simpson's correspondence ([12, p. 755, Main Theorem]). The statement follows. ■

The result above has many deep implications in the geometry of fibrations. However, by the transcendental nature of the method of establishing the Simpson's correspondence (which refers to the main theorem in [13]), the lattice structure underlying the $\mathbb{Z}$-VHS associated with a family seems undetectable in the associated logarithmic

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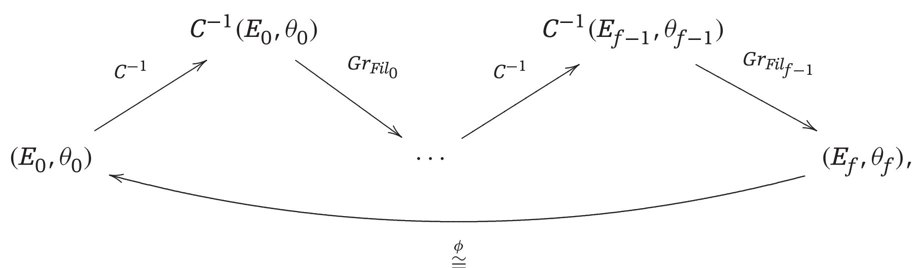
Kodaira-Spencer system. In this note, we argue that some dynamical property associated with various mod p reductions of the logarithmic Kodaira–Spencer system does bear information from the lattice structure. However, our findings regarding this issue are far from being definitive.

Now, let k be a perfect field of positive characteristic p. Let X be a smooth variety over k and $D \subset X$ a simple normal crossing divisor. Set $X_{\log} = (X, D)$ to be the logarithmic variety over k (regarded as a log scheme with trivial log structure) whose log structure is determined by the divisor D. Using the logarithmic generalization of the inverse Cartier transform of Ogus–Vologodsky due to D. Schepler (see also [9, Appendix]), one may generalize [8, Definition 1.1] in this straightforward manner.

Definition 2.2. Let X, D be as above. A *periodic Higgs–de Rham flow* over $X_{\log}$ of period $f \in \mathbb{N}_{>0}$ is a diagram as follows:



where

- the initial term (E_0, θ_0) is a nilpotent logarithmic graded Higgs bundle with pole along D and of level $\leq p-1$;
- for each $i \geq 0$ Fil_i is a Hodge filtration on the logarithmic flat bundle $C^{-1}(E_i, \theta_i)$ making it into a de Rham bundle of level $\leq p-1$;
- $(E_i, \theta_i), i \geq 1$ is the logarithmic graded Higgs bundle associated with the logarithmic de Rham bundle $(C^{-1}(E_{i-1}, \theta_{i-1}), Fil_{i-1})$; and
- ϕ is an isomorphism of logarithmic graded Higgs bundles.

A logarithmic graded Higgs bundle over X with pole along D is said to be *periodic* if it initializes a periodic Higgs–de Rham flow (of certain period f) over $X_{\log}$. We now turn our attention back to the situation over $\mathbb{C}$.

Definition 2.3. Let X be a smooth variety over $\mathbb{C}$ and $D \subset X$ an SNCD. Set $X_{\log} = (X, D)$. Let (E, θ) be a graded Higgs bundle over $X_{\log}$ (namely a graded logarithmic Higgs bundle

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over X with pole along D). A *spread* of (X, D, E, θ) is a four-tuple $(\mathcal{X}, \mathcal{D}, \mathcal{E}, \Theta)$ where $\mathcal{X}$ is a scheme of finite type over $S = \mathrm{Spec}(A)$ with $A \subset \mathbb{C}$ a finitely generated $\mathbb{Z}$-subalgebra, $\mathcal{D}$ an S-relative divisor in $\mathcal{X}$, and

$$\Theta : \mathcal{E} \to \mathcal{E} \otimes \Omega_{\mathcal{X}/S}(\log \mathcal{D})$$

an S-relative logarithmic Higgs bundle over $\mathcal{X}$ such that there is an isomorphism of tuples over $\mathbb{C}$:

$$(\mathcal{X}, \mathcal{D}, \mathcal{E}, \Theta) \otimes_A \mathbb{C} \cong (X, D, E, \theta).$$

We now give two lemmas that will be used later.

Lemma 2.4. Let k be an algebraically closed field of characteristic p. Let C be a smooth projective curve over k and $D \subset C$ a reduced effective divisor. If the initial term of a periodic Higgs–de Rham flow over $C_{\log}$ is stable, then any intermediate Higgs term of the periodic flow is also stable.

Proof. For any given semistable logarithmic Higgs bundle over $C_{\log}$, its Jordan–Hölder filtration may not be unique, but different Jordan–Hölder filtrations have the same set of stable factors. Let $\{E_0, \cdots, E_{f-1}\}$ be the intermediate Higgs terms of an f-periodic Higgs–de Rham flow with $E_f \cong E_0$. Set n_i to be the number of stable factors of a Jordan–Hölder filtration on E_i. We claim that $n_i \leq n_{i+1}$. Indeed, we are going to show that the flow operator $Gr_{Fil} \circ C^{-1}$ maps a Jordan–Hölder filtration on E_i to a filtration of semistable subbundles on E_{i+1} whose length equals n_i, from which the claim follows.

By [8, Proposition 6.3], each E_i is of degree zero. Thus, each member in a Jordan–Hölder filtration of E_i is of degree zero, too. Because the operator C^{-1} is exact and multiplies the degree by p, one consequently obtains a filtration on $C^{-1}(E_i)$ of flat subbundles of degree zero. As Gr_{Fil} preserves the degree, using the induced filtrations and taking the associated gradings, one obtains a filtration of the same length by Higgs subsheaves of degree zero on $Gr_{Fil} \circ C^{-1}(E_i) = E_{i+1}$. It is actually a filtration of Higgs *subbundles* because the saturation of each Higgs subsheave is invariant under the Higgs field and hence of degree ≤ 0 by semistability. The $n_i \leq n_{i+1}$ claim is proved. It follows that

$$n_0 \leq n_1 \leq \cdots \leq n_f = n_0;$$

hence, all the n_is are equal and the flow operator maps a Jordan–Hölder filtration of E_i to a Jordan–Hölder filtration of E_{i+1}. It follows immediately that if E_0 is stable, then each intermediate Higgs term is also stable. This concludes the proof. ■

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Lemma 2.5. Let C, D be as in Lemma 2.4. Let (V, ∇) be a flat bundle over $C_{\log}$. Then, up to a shift of indices, there is at most one Hodge filtration *Fil* on (V, ∇) such that the graded Higgs bundle $Gr_{Fil}(V, \nabla)$ is stable.

Proof. It suffices to note that the proof of [8, Lemma 4.1] for the case of empty D works verbatim for the general case. ■

We introduce the following definition.

Definition 2.6. Let (X, D) be a log pair over $\mathbb{C}$. A graded Higgs bundle (E, θ) over $X_{\log}$ is called *periodic* if there exists *some* spread

$$(\mathcal{X}, \mathcal{D}, \mathcal{E}, \Theta) \to S,$$

of (X, D, E, θ), and a positive integer f such that for all geometric points $s \in S$, the reduction $(\mathcal{E}_s, \Theta_s)$ at s is periodic of period $\leq f$ for *all* $W_2(k(s))$-lifting $\tilde{s} \to S$. The *period* of a periodic Higgs bundle is defined to be the smallest $f \geq 1$ among all possible spreads.

We list several simple properties of periodic Higgs bundles.

Lemma 2.7. The following statements about periodic Higgs bundles hold:

(i) a periodic Higgs bundle is semistable of degree zero;
(ii) a direct sum of periodic Higgs bundles is again periodic;
(iii) a graded Higgs bundle obtained by renumbering the graded structure of a periodic Higgs bundle is again periodic.

Proof. (i) follows from [8, Proposition 6.3]. As for (iii), one notices that the inverse Cartier transform ignores the grading structure, so in char p, if it is periodic for one grading structure, then for another grading structure (as long as its the largest grading is $\leq p-1$), one may simply adjust the indices of the last Hodge filtration to make it periodic as well. It remains to show (ii). Suppose we are given two periodic Higgs bundles $(E_i, \theta_i), i = 1, 2$ over $X_{\log}$. Since the direct sum of two periodic Higgs bundles in positive characteristic is again periodic (the key being the fact that the inverse Cartier transform commutes with direct sum), it suffices to show the existence of spreads $(\mathcal{X}, \mathcal{D}, \mathcal{E}_i, \Theta_i)$ of (X, D, E_i, θ_i) over a *common* base log scheme $(\mathcal{X}, \mathcal{D})$ for $i = 1, 2$. For $i = 1, 2$, we take a spread $(\mathcal{X}_i, \mathcal{D}_i, \tilde{\mathcal{E}}_i, \tilde{\Theta}_i) \to S_i$, $\mathrm{Spec}(S_i) = A_i$, together with an isomorphism

$$(\mathcal{X}_i, \mathcal{D}_i, \mathcal{E}'_i, \Theta'_i) \otimes_{A_i} \mathbb{C} \cong (X, D, E_i, \theta_i)$$

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given by Definition 2.6. Clearly, there exists some finitely generated $\mathbb{Z}$-subalgebra $A_3 \subset \mathbb{C}$ satisfying the following properties:

(a) $A_i \subset A_3, i = 1, 2;$

(b) the isomorphisms $(\mathcal{X}_i, \mathcal{D}_i) \otimes_{A_i} \mathbb{C} \cong (X, D), i = 1, 2$ are defined over A_3;

(c) the natural morphisms $S_3 := \mathrm{Spec}(A_3) \to S_i, i = 1, 2$ are smooth.

Set $(\mathcal{X}, \mathcal{D}) = (\mathcal{X}_1, \mathcal{D}_1) \otimes_{A_1} A_3$, which is isomorphic to $(\mathcal{X}_2, \mathcal{D}_2) \otimes_{A_2} A_3$ as an A_3-scheme and is isomorphic to (X, D) as a $\mathbb{C}$-scheme. Let $\pi_i : (\mathcal{X}, \mathcal{D}) \to (\mathcal{X}_i, \mathcal{D}_i), i = 1, 2$ be the natural morphisms of the log schemes. They are log smooth because of the condition (c). Set $(\mathcal{E}_i, \Theta_i) = \pi_i^*(\mathcal{E}'_i, \Theta'_i)$. We claim that the spread $(\mathcal{X}, \mathcal{D}, \mathcal{E}_i, \Theta_i)$ satisfies the condition in Definition 2.6 for either i. Take any $\tilde{s} \in S_3(W_2(\bar{\mathbb{F}}_p))$, and let $\tilde{s}_i \in S_i(W_2(\bar{\mathbb{F}}_p)), i = 1, 2$ be its image. Let s, s_1, s_2 be the corresponding points by modulo p. Then, π_i is restricted to a log smooth morphism

$$\pi_{i,s} : (\mathcal{X}_{3,\tilde{s}}, \mathcal{D}_{3,\tilde{s}}) \to (\mathcal{X}_{i,\tilde{s}_i}, \mathcal{D}_{i,\tilde{s}_i})$$

over $W_2(\bar{\mathbb{F}}_p)$. At this point, we apply a recent result [10, Theorem 5.3] which shows that there is an isomorphism of functors

$$\pi_{i,s}^* \circ C^{-1}_{(\mathcal{X}_{i,s_i}, \mathcal{D}_{i,s_i}) \subset (\mathcal{X}_{i,\tilde{s}_i}, \mathcal{D}_{i,\tilde{s}_i})} \cong C^{-1}_{(\mathcal{X}_{3,s}, \mathcal{D}_{3,s}) \subset (\mathcal{X}_{3,\tilde{s}}, \mathcal{D}_{3,\tilde{s}})} \circ \pi_{i,s}^*.$$

This allows us to pull back a periodic Higgs–de Rham flow over $(\mathcal{X}_{i,s_i}, \mathcal{D}_{i,s_i})$ initializing $(\mathcal{E}'_{i,s_i}, \Theta'_{i,s_i})$ to a periodic Higgs–de Rham flow over $(\mathcal{X}_{3,s}, \mathcal{D}_{3,s})$ initializing $(\mathcal{E}_{i,s}, \Theta_{i,s})$ of *the same period*. It follows that the periodicity for $(\mathcal{E}_i, \Theta_i)$ at the geometric point $s \in S$ with respect to the W_2-lifting $\tilde{s}$. As $\tilde{s}$ is arbitrary and the boundedness of periods is also maintained, the claim is proved. ■

The main result of this section is the following periodicity result.

Theorem 2.8. Let C be a smooth projective curve and $D \subset C$ a reduced effective divisor. Let $f : X \to C$ be a semistable family and (E, θ) be a logarithmic Kodaira–Spencer system associated with f. Then, any graded Higgs subbundle in (E, θ) of degree zero is periodic in the sense of Definition 2.6.

Proof. We take a spread of the family f as follows. By the standard argument ([4, 8, 11.2, and 17.7]), there exists a sub $\mathbb{Z}$-algebra $A \subset \mathbb{C}$ of finite type and a semistable family $\mathfrak{f} : (\mathfrak{X}, \mathfrak{B}) \to (\mathfrak{C}, \mathfrak{D})$ defined over $S = \mathrm{Spec}(A)$ such that S is integral and regular, $\alpha : \mathfrak{C} \to S$ is smooth and projective and f is the base change of $\mathfrak{f}$ via $\mathrm{Spec}(\mathbb{C}) \to S$.

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Shrinking S if necessary, we may assume that for any closed point $s \in S$,

$$\operatorname{char}(k(s)) > N := \operatorname{rank}(E) + n + d + 1.$$

By Deligne–Illusie [2] and Illusie [6], for any i, j,

$$R^j \mathfrak{f}_* \Omega^i_{\mathfrak{X}/\mathfrak{C}}(\log \mathfrak{B}/\mathfrak{D}),\ R^{i+j} \mathfrak{f}_* \Omega^*_{\mathfrak{X}/\mathfrak{C}}(\log \mathfrak{B}/\mathfrak{D})$$

are locally free of finite type and the spectral sequence

$$E_1^{i,j} = R^j \mathfrak{f}_* \Omega^i_{\mathfrak{X}/\mathfrak{C}}(\log \mathfrak{B}/\mathfrak{D}) \Rightarrow R^{i+j} \mathfrak{f}_* \Omega^*_{\mathfrak{X}/\mathfrak{C}}(\log \mathfrak{B}/\mathfrak{D})$$

degenerates at E_1. The degeneration yields the logarithmic Kodaira–Spencer system associated with $\mathfrak{f}$.

$$E_1^{i,j} = R^j \mathfrak{f}_* \Omega^i_{\mathfrak{X}/\mathfrak{C}}(\log \mathfrak{B}/\mathfrak{D}) \Rightarrow R^{i+j} \mathfrak{f}_* \Omega^*_{\mathfrak{X}/\mathfrak{C}}(\log \mathfrak{B}/\mathfrak{D})$$

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degenerates at E_1. The degeneration yields the logarithmic Kodaira–Spencer system associated with $\mathfrak{f}$.

$$\Theta^{i,j} : R^j \mathfrak{f}_* \Omega^i_{\mathfrak{X}/\mathfrak{C}}(\log \mathfrak{B}/\mathfrak{D}) \to R^{j+1} \mathfrak{f}_* \Omega^{i-1}_{\mathfrak{X}/\mathfrak{C}}(\log \mathfrak{B}/\mathfrak{D}) \otimes \Omega_{\mathfrak{C}/S}(\log \mathfrak{D})$$

(Another way to obtain $\Theta^{i,j}$ is by taking a suitable edge morphism associated with the higher direct image of the following short exact sequence:

$$0 \to \mathfrak{f}^* \Omega_{\mathfrak{C}/S}(\log \mathfrak{D}) \otimes \Omega^{i-1}_{\mathfrak{X}/\mathfrak{C}}(\log \mathfrak{B}/\mathfrak{D}) \to \Omega^i_{\mathfrak{X}}(\log \mathfrak{B}) \to \Omega^i_{\mathfrak{X}/\mathfrak{C}}(\log \mathfrak{B}/\mathfrak{D}) \to 0.)$$

Set $\mathcal{E}^{i,j} = R^j \mathfrak{f}_* \Omega^i_{\mathfrak{X}/\mathfrak{C}}(\log \mathfrak{B}/\mathfrak{D})$. Thus, $\mathcal{E} = \bigoplus_{i+j=n} \mathcal{E}^{i,j}, \Theta = \bigoplus_{i+j=n} \Theta^{i,j}$ is a spread of (E, θ). By construction, for any geometrically closed point $s \in S$, the base change $(\mathcal{E}_s, \Theta_s)$ is the logarithmic Kodaira–Spencer system associated with the family $\mathfrak{f}_s$, the fiber of $\mathfrak{f}$ at s. By [4, Theorem 6.2] and [9, Proposition 4.1], $(\mathcal{E}_s, \Theta_s)$ is one-periodic with respect to any $W_2(k(s))$-lifting $\tilde{s} : \operatorname{Spec}(W_2(k(s))) \to S$.

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By Proposition 2.1, we may write (E,θ) as a direct sum of stable factors:

$$(E,\theta)=\bigoplus_{i=1}^{l}(E_i,\theta_i)^{\oplus m_i}$$

with (E_i,θ_i) stable. Caution: the above equality is meant to be an equality of Higgs bundles, instead of graded Higgs bundles over $C_{\log}$; for different i and j, (E_i,θ_i) are not isomorphic to (E_j,θ_j) as Higgs bundles.

It suffices to show that (E_i,θ_i) is periodic for each i. This is because any graded Higgs subbundle of degree zero in (E,θ) must be a direct sum of stable factors (E_i,θ_i)s, up to renumbering the graded structure. Then, one applies Lemma 2.7. As the geometric stability is an open condition, shrinking S if necessary, we may assume that

$$(\mathcal{E},\Theta)=\bigoplus_{i=1}^{l}(\mathcal{E}_i,\Theta_i)^{\oplus m_i}$$

such that for each i, $(\mathcal{E}_i,\Theta_i)$ is a spread of (E_i,θ_i) and its base change at s is still stable. We obtain the following decomposition into stable factors:

$$(\mathcal{E}_s,\Theta_s)=\bigoplus_{i=1}^{l}(\mathcal{E}_{i,s},\Theta_{i,s})^{\oplus m_i}.$$

Let *Fil* denote the Hodge filtration of the family in consideration. Then, the one-periodicity of (E,θ) means an isomorphism:

$$Gr_{Fil}\circ C^{-1}(\mathcal{E}_s,\Theta_s)\cong(\mathcal{E}_s,\Theta_s).$$

We argue that the operator $Gr_{Fil}\circ C^{-1}$ induces a self-map on the set $T=\{1,\cdots,l\}$, which represents the set of nonisomorphic stable factors in $(\mathcal{E}_s,\Theta_s)$. Pick any $i\in T$. By Lemma 2.4, $Gr_{Fil}\circ C^{-1}(\mathcal{E}_{i,s},\Theta_{i,s})$ is stable. Therefore, there is a unique $j(i)$ such that

$$Gr_{Fil}\circ C^{-1}(\mathcal{E}_{i,s},\Theta_{i,s})\cong(\mathcal{E}_{j(i),s},\Theta_{j(i),s})$$

as logarithmic Higgs bundles. However, when $m_i\geq 2$, three are more than one factor isomorphic to $(\mathcal{E}_{i,s},\Theta_{i,s})$. We claim that $j(i)$ does not depend on this ambiguity. Let $(\mathcal{E}_{i,s},\Theta_{i,s})_1\cong(\mathcal{E}_{i,s},\Theta_{i,s})_2$ be two stable factors of $(\mathcal{E}_s,\Theta_s)$, which are isomorphic as

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logarithmic Higgs bundles. Since C^{-1} is an equivalence of categories,

$$C^{-1}(\mathcal{E}_{i,s},\Theta_{i,s})_1 \cong C^{-1}(\mathcal{E}_{i,s},\Theta_{i,s})_2$$

as logarithmic flat bundles. As $Gr_{Fil}C^{-1}(\mathcal{E}_{i,s},\Theta_{i,s})_i, i = 1,2$ is stable, it follows from Lemma 2.5 that there is an isomorphism of logarithmic Higgs bundles:

$$Gr_{Fil}\circ C^{-1}(\mathcal{E}_{i,s},\Theta_{i,s})_1 \cong Gr_{Fil}\circ C^{-1}(\mathcal{E}_{i,s},\Theta_{i,s})_2.$$

Hence, the claimed independence holds, and we get a well-defined map

$$Gr_{Fil}\circ C^{-1} : T \to T.$$

This map has to be surjective because the rank is preserved under the flow operator. Because T is a finite set, it is bijective and therefore decomposes into a product of cyclic permutations. Thus, for each i, the Hodge filtration *Fil* induces an f-periodic flow with initial term $(\mathcal{E}_{i,s},\Theta_{i,s})^{\oplus m_i}$ for some $f \leq l!$. This flow induces, in turn, an f-periodic flow with an initial term for any factor

$$(\mathcal{E}_{i,s},\Theta_{i,s}) \subset (\mathcal{E}_{i,s},\Theta_{i,s})^{\oplus m_i} \subset (\mathcal{E}_s,\Theta_s).$$

This completes the proof. ■

3 Periodicity of the Uniformizing Higgs Bundle

In this section, we shall investigate the periodicity of the uniformizing Higgs bundle over $\mathbb{P}^1$ with four simple poles, both in characteristic $p > 0$ and in characteristic zero. As a matter of convention, we use the notation (E_{unif},θ_{unif}) for the uniformizing Higgs bundle over $\mathbb{P}^1$ with four simple poles over an *arbitrary* field.

Let us start with the periodicity over $\mathbb{C}$.

Proposition 3.1. Let (E_{unif},θ_{unif}) be the uniformizing Higgs bundle over $(\mathbb{P}^1, D)$ over $\mathbb{C}$, where D consists of four distinct points. If (E_{unif},θ_{unif}) is periodic, then its period is equal to one.

Proof. Let $(\mathbb{P}^1,\mathcal{D},\mathcal{E}_{unif},\Theta_{unif})$, defined over S, be a spread of $(\mathbb{P}^1,D,E_{unif},\theta_{unif})$. Let $s \in S$ be a geometrically closed point. Shrinking S if necessary, we may assume

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$(\mathcal{E}_{unif,s}, \Theta_{unif,s})$ is isomorphic to $(\mathcal{O}(1) \oplus \mathcal{O}(-1), id)$ and hence stable of trivial determinant. Let $(L \oplus L^{-1}, \theta)$ be a periodic Higgs bundle over $(\mathbb{P}^1_s, \mathcal{D}_s)$ with trivial determinant. We claim that if it is stable, then it must be isomorphic to $(\mathcal{E}_{unif,s}, \Theta_{unif,s})$. Indeed, because the Higgs field induces a nonzero morphism $L^{\otimes 2} \to \Omega_{\mathbb{P}^1_s}(\mathcal{D}_s) \cong \mathcal{O}(2)$, it follows that $\deg L \leq 2$. As $\deg L \geq 0$ in any case, one has

$$\deg L = 0, 1.$$

However, if $\deg L = 0$, we have the Higgs subbundle $(\mathcal{O}_{\mathbb{P}^1}, 0) \subset (L \oplus L^{-1}, \theta)$, which violates the stability. Hence, $\deg L = 1$ and θ must be an isomorphism because of the degree. In other words, $(L \oplus L^{-1}, \theta) \cong (\mathcal{E}_{unif,s}, \Theta_{unif,s})$. By Lemma 2.4, any intermediate Higgs terms of a periodic flow initializing $(\mathcal{E}_{unif,s}, \Theta_{unif,s})$ is periodic and stable and has trivial determinant (which is clear). The proposition follows. ■

The pair $(\mathbb{P}^1, D)$ in Proposition 3.1 is isomorphic to $(\mathbb{P}^1, 0 + 1 + \lambda + \infty)$ for some $\lambda \neq 0, 1$.

Proposition 3.2. If λ belongs to Beauville's list 1.0.1, then $(E_{unif}, \theta_{unif})$ is periodic.

Proof. Let λ be a value in Beauville's list 1.0.1. Beauville shows that there is a semistable family of elliptic curves over $(\mathbb{P}^1, D = 0 + 1 + \lambda + \infty)$. The family is non-isotrivial, so the associated period map is nonconstant. It follows that the associated logarithmic Kodaira–Spencer system (E, θ) has a nonzero Higgs field (one may also see this by the existence of singular fibers). In fact, (E, θ) is isomorphic to $(E_{unif}, \theta_{unif})$. By Proposition 2.1, (E, θ) takes the form

$$\theta : E^{1,0} \to E^{0,1} \otimes \Omega_{\mathbb{P}^1}(D)$$

with $E^{0,1}$ isomorphic to the dual of $E^{1,0}$ and $\deg E^{1,0} > 0$. On the other hand, since θ is nonzero, we have

$$\deg (E^{1,0})^{\otimes 2} \leq \deg \Omega_{\mathbb{P}^1}(D) = 2.$$

Thus, $\deg E^{1,0} = 1$. By Theorem 2.8, the proposition follows. ■

We conjecture the converse of Proposition 3.2.

Conjecture 3.3. Assume notation as in Proposition 3.2. If $(E_{unif}, \theta_{unif})$ is periodic, then λ must be in Beauville's list 1.0.1.

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We can give a very partial answer to the conjecture.

Proposition 3.4. If $(E_{unif}, \theta_{unif})$ over $(\mathbb{P}^1, 0+1+\lambda+\infty)$ is periodic, then λ is algebraic.

To approach the problem in the conjecture, a detailed study of periodicity in positive characteristic is necessary. To do so, we let $k = \bar{\mathbb{F}}_p$ and do an explicit study of the periodicity condition for $(E_{unif}, \theta_{unif})$ over k.

Let $\lambda = (\lambda_0, \lambda_1) \in W_2(k)$ with λ_0 distinct from $\{0, 1\}$. This λ gives rise to an obvious $W_2(k)$-lifting of the pair $(\mathbb{P}^1, 0+1+\lambda_0+\infty)$ over k. If one fixes a $\lambda_0 \in k$, then any $W_2(k)$-lifting of the pair $(\mathbb{P}^1, 0+1+\lambda_0+\infty)$ is isomorphic to the one given by some $\lambda \in W_2(k)$ as above. We define a $p \times (2p+1)$ matrix T as follows:

$$T_{ij} = \begin{cases} \lambda_1 & \text{if } i = j \\ (-1)^{i-j+1}\frac{\binom{p}{p-i+j}}{p}\left(1-\lambda_0^{i-j}\right) & \text{if } i > j \\ (-1)^{j-i+1}\frac{\binom{p}{p-j+i}}{p}\left(\lambda_0^{p-j+i}-\lambda_0^{p}\right) & \text{if } i < j \leq p \\ (-1)^{i+j-p-1}\frac{\binom{p}{i+j-p-1}}{p}\left(1-\lambda_0^{i+j-p-1}\right) & \text{if } p < j \leq 2p-i \\ 0 & \text{if } j > 2p-i. \end{cases}$$

Let T_m be the $(p-m) \times (p+m)$ submatrix of T containing the first $p-m$ rows and first $p+m$ columns $(0 \leq m \leq p-1)$. We obtain the following result.

Proposition 3.5. Let $k = \bar{\mathbb{F}}_p$ and $\lambda_0 \in k$ be an element distinct from $\{0, 1\}$. Then, $(E_{unif}, \theta_{unif})$ over $\mathbb{P}^1 - \{0, 1, \lambda_0, \infty\}$ with respect to the $W_2(k)$-lifting given by $\lambda := (\lambda_0, \lambda_1) \in W_2(k)$ is periodic if and only if

$$\det(T_0) = 0; \quad \operatorname{rank}(T_1) = p-1.$$

Let us first prove Proposition 3.4.

Proof. Assume the contrary. If $\lambda \in \mathbb{C}$ is transcendental, then any spread of $(\mathbb{P}^1, 0+1+\lambda+\infty)$ must be defined over a noetherian $\mathbb{Z}$-subalgebra $A \supseteq \mathbb{Z}[\lambda]$, which means that the base scheme $S = \operatorname{Spec}(A)$ of any spread is *dominant* over the affine line $\mathbb{A}^1_{\mathbb{Z}}$. By Proposition 3.5, for $\lambda_0 \in \bar{\mathbb{F}}_p$, which is considered as a geometrically closed point of S, there are at most p λ_1s such that $(E_{unif}, \theta_{unif})$ over $(\mathbb{P}^1, 0+1+\lambda_0+\infty)$ is periodic with respect to the W_2-lifting determined by $\lambda = (\lambda_0, \lambda_1)$. Therefore, at any rate, the uniformizing Higgs bundle over $(\mathbb{P}^1, 0+1+\lambda+\infty)$ cannot be periodic. Contradiction. ∎

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Remark 3.6. One may explore further into the implication of Proposition 3.5. Let $\lambda \in \mathcal{O}_K[\frac{1}{N}]$ be an algebraic number, where $K \subset \mathbb{C}$ is an algebraic number field and N a natural number. Then, for almost all places $\nu \in K$, K is unramified at ν and therefore $\mathcal{O}_K/\nu^2 \cong W_2(\mathbb{F}_q) \subset W_2(\bar{\mathbb{F}}_p)$. Fix *any* such morphism $r_\nu = (r_{\nu,0}, r_{\nu,1})$. Conjecture 3.3 amounts to the truth of the following arithmetic question.

Question 3.7. Let $\lambda \in K$ be as above. Assume for almost all places ν of K, the pair $(r_{\nu,0}(\lambda), r_{\nu,1}(\lambda))$ satisfies

$$\det T_0(r_{\nu,0}(\lambda), r_{\nu,1}(\lambda)) = 0, \quad \operatorname{rank} T_1(r_{\nu,0}(\lambda), r_{\nu,1}(\lambda)) = \operatorname{char}(k(\nu)) - 1.$$

Is λ a Beauville's number?

Now, we turn to the proof of Proposition 3.5. It relies on the following analysis.

Lemma 3.8. $(E_{unif}, \theta_{unif})$ over $\mathbb{P}^1 - \{0, 1, \lambda_0, \infty\}$ with respect to some $W_2(k)$-lifting determined by $\lambda = (\lambda_0, \lambda_1)$ is one periodic if and only if the bundle part of the inverse Cartier transform of $(E_{unif}, \theta_{unif})$ with respect to that lifting is isomorphic to $\mathcal{O}(-1) \oplus \mathcal{O}(1)$.

Proof. Write $(H, \nabla) = C^{-1}(E_{unif}, \theta_{unif})$, where C^{-1} refers to the inverse Cartier transform with respect to the $W_2(k)$-lifting determined by λ. Assume for this W_2-lifting, $(E_{unif}, \theta_{unif})$ is one-periodic. Then, there is a short, exact sequence

$$0 \to \mathcal{O}(1) \to H \to \mathcal{O}(-1) \to 0.$$

Computing that $\dim \operatorname{Ext}^1(\mathcal{O}(-1), \mathcal{O}(1)) = h^1(\mathcal{O}(2)) = h^0(\mathcal{O}(-4)) = 0$, we find that H must be isomorphic to $\mathcal{O}(-1) \oplus \mathcal{O}(1)$. Conversely, let us assume that $H \cong \mathcal{O}(-1) \oplus \mathcal{O}(1)$. Set $\mathcal{O}(1) \cong Fil^1 \subset H$. Note that Fil^1 cannot be ∇-invariant because of the degree. Thus, the graded Higgs field $Gr_{Fil}\nabla$ must be nonzero. Again, because of the degree, it must be maximal, that is, we have

$$Gr_{Fil}\nabla : Fil^1 \cong H/Fil \otimes \Omega^1_{\mathbb{P}^1}(\log 0 + 1 + \lambda_0 + \infty).$$

This completes the proof of the lemma. ■

The next step toward our proof of Proposition 3.5 is to determine $(H, \nabla) = C^{-1}(E_{unif}, \theta_{unif})$, especially the bundle part H. Note that there exists a unique natural number n such that $H \cong \mathcal{O}(n) \oplus \mathcal{O}(-n)$. Our main goal in the following is to determine the n. We do this via the approach of exponential twisting to the inverse Cartier transform (see

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[9] and [7, Appendix]). Set $\tilde{X} = \mathbb{P}^1 - \{0, 1, \tilde{\lambda}, \infty\}$ over W_2 and X to be its reduction. The curve X has a distinguished open affine covering $\mathcal{U} = \{U_\alpha, U_\beta\}$ with

$$U_\alpha = \mathbb{P}^1 - \{0, \infty\}; \quad U_\beta = \mathbb{P}^1 - \{1, \lambda_0\}.$$

Set $\tilde{U}_\alpha \subset \tilde{X}$ to be the open affine scheme by restricting $\tilde{X}$ to U_α, and similarly define the open subscheme $\tilde{U}_\beta \subset \tilde{X}$. Let z be the affine coordinate of $\mathbb{P}^1 - \{\infty\}$. Then, one may choose the standard log Frobenius lifting determined by

$$\tilde{F}_\alpha(z) = z^p; \quad \tilde{F}_\beta(w) = w^p,$$

where $w = \frac{z-\lambda}{z-1}$ (w is a linear transformation of $\mathbb{P}^1_{W_2}$ that maps λ to zero and 1 to ∞). On the overlap $U_{\alpha\beta} = U_\alpha \cap U_\beta$, we use the coordinate z. Therefore, the 2nd Frobenius lifting $\tilde{F}_\beta$ on $\tilde{U}_\alpha$ is written as

$$z \mapsto \frac{(z-\lambda)^p - F(\lambda)(z-1)^p}{(z-\lambda)^p - (z-1)^p},$$

where $F(\lambda) = (\lambda_0^p, \lambda_1^p)$. Here is a verification: RHS mod p is nothing but z^p. By the definition of a log Frobenius lifting, RHS can be written as $z^p(1 + pa)$ with

$$a = \frac{1}{p} \cdot \frac{(z^p - F(\lambda))(z-1)^p - (z-\lambda)^p(z^p - 1)}{z^p[(z-\lambda)^p - (z-1)^p]} \in k[z, \frac{1}{z(z-1)(z-\lambda_0)}] = \mathcal{O}_{U_{\alpha\beta}}.$$

(Notice that $F(\lambda) = (\lambda_0^p, 0) + (0, \lambda_1^p) = \lambda^p + p(\lambda_1, 0)$, which means the numerator is divisible by p).

Actually, we can write a more precisely as

$$a = \frac{\sum_{i=1}^{p-1}(-1)^i \frac{\binom{p}{i}}{p}(1-\lambda_0^i)z^{2p-i} + \sum_{i=1}^{p-1}(-1)^i \frac{\binom{p}{i}}{p}(\lambda_0^i - \lambda_0^p)z^{p-i} - \lambda_1(z^p - 1)}{(1-\lambda_0^p)z^p},$$

so we have

$$\zeta_\alpha(d\log z \otimes 1) = d\log z; \quad \zeta_\beta(d\log z \otimes 1) = d\log z + da,$$

and

$$dh_{\alpha\beta}(d\log z \otimes 1) = (\zeta_\beta - \zeta_\alpha)(d\log z) = da.$$

Here, ξ_α,ξ_β, and $h_{\alpha\beta}$ appear in Deligne–Illusie's lemma (see [9, Lemma 2.1]).

On the other hand, our Higgs bundle reads

$$E := E_{unif} = \mathcal{O}(1) \oplus \mathcal{O}(-1).$$

Set

$$E_\alpha = E|_{U_\alpha} = \mathcal{O}_{U_\alpha}\{e_\alpha^1, e_\alpha^0\}, \quad E_\beta = E|_{U_\beta} = \mathcal{O}_{U_\beta}\{e_\beta^1, e_\beta^0\}$$

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and the transition is given by

$$\{e_\beta^1, e_\beta^0\} = \{e_\alpha^1, e_\alpha^0\}\begin{pmatrix} z-1 & 0 \\ 0 & (z-1)^{-1} \end{pmatrix}.$$

(The reason is as follows: $\Omega^1 = \mathcal{O}(-2)$, and over U_α, it has basis $e_\alpha = d(\frac{z-\lambda_0}{z-1})$, while over U_β it has basis $e_\beta = dz$. The transition map is given by $e_\beta = \frac{(z-1)^2}{\lambda_0 - 1} e_\alpha$.) Therefore, H is obtained by gluing

$$H_\alpha = F^* E_\alpha, \quad H_\beta = F^* E_\beta$$

via the gluing matrix

$$\{e_\beta^1 \otimes 1, e_\beta^0 \otimes 1\} = \{e_\alpha^1 \otimes 1, e_\alpha^0 \otimes 1\}\begin{pmatrix} (z-1)^p & 0 \\ 0 & (z-1)^{-p} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}.$$

Note that $\mathcal{O}_{U_\alpha} = k[\frac{z-\lambda_0}{z-1}, \frac{z-1}{z-\lambda_0}]$ and $\mathcal{O}_{U_\beta} = k[z, z^{-1}]$. We are computing matrices $P \in \mathrm{GL}_2(\mathcal{O}_{U_\alpha})$ and $Q \in \mathrm{GL}_2(\mathcal{O}_{U_\beta})$ such that

$$P \cdot \begin{pmatrix} (z-1)^p & 0 \\ a(z-1)^{-p} & (z-1)^{-p} \end{pmatrix} \cdot Q$$

is diagonal.

Notice that Proposition 3.5 is only a special case of the following statement.

Proposition 3.9. Let $\lambda = (\lambda_0, \lambda_1)$ and the resulting H be as above. Then, $H \cong \mathcal{O}(n) \oplus \mathcal{O}(-n)$ if and only if T_n is the first full rank matrix in the sequence of matrices $\{T_0, T_1, \cdots, T_{p-1}\}$.

Proof. We show that the transition matrix can be diagonalized to $\mathrm{diag}((z-1)^n, (z-1)^{-n})$ by the following algorithm.

Denote a by A/z^p, and notice that A has degree $2p-1$. Our following argument actually does not require this form to be simplified.

Step 1: Find $f, g \in k[z]$, $\deg(f), \deg(g) \leq p$, such that $f \cdot A + g \cdot z^p$ is divisible by $(z-1)^{2p}$ and $(f, g) = (z-1)^l$ for some $l \geq 0$ by the following algorithm.

Consider the following equations, where $Q_i, R_i \in k[z]$, $\deg(R_i) \leq 2p-1$:

$$A = Q_0 \cdot (z-1)^{2p} + R_0$$

$$z \cdot A = Q_1 \cdot (z-1)^{2p} + R_1$$

$$\cdots$$

$$z^p \cdot A = Q_p \cdot (z-1)^{2p} + R_p,$$

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and denote the coefficient of z^j in R_i by R_{ij}. Since the matrix $(R_{ij})_{0\leq i\leq p, 0\leq j\leq p-1}$ has at most rank p, the following linear system has a nonzero solution:

$$\begin{pmatrix} f_0 & \cdots & f_p \end{pmatrix}_{1\times(p+1)} \cdot \begin{pmatrix} R_{00} & \cdots & R_{0,p-1} \\ \vdots & \ddots & \vdots \\ R_{p,0} & \cdots & R_{p,p-1} \end{pmatrix}_{(p+1)\times p} = \begin{pmatrix} 0 & \cdots & 0 \end{pmatrix}_{1\times p}.$$

By setting $f = f_p z^p + \cdots + f_0$, we have $f \cdot A = h \cdot (z-1)^{2p} + R$, where $h = f_p \cdot Q_p + \cdots + f_0 \cdot Q_0$, $R = R_p \cdot f_p + \cdots + R_0 \cdot f_0$. By the matrix above, we know that the coefficient of $1, z, \cdots, z^{p-1}$ of R are all 0, thus R can be written as $-g \cdot z^p$, and $\deg(g) \leq p-1$. After, we find a pair of f and g, it is not difficult to find a pair that satisfies $(f, g) = (z-1)^l$.

Step 2: Denote $c = \max(\deg(f), \deg(g))$, and h as above. Find $\beta, \gamma \in k[z]$ such that $\deg(\beta), \deg(\gamma) \leq c - 2p$, and $f \cdot \gamma + g \cdot \beta = (z-1)^{2p}$. Letting $\bar{f} = f/(z-1)^l$, $\bar{g} = g/(z-1)^l$, we have $(\bar{f}, \bar{g}) = 1$.

Case A: $z - 1 \nmid \bar{f}$. In this case, find $\sigma \in k[z]$ such that $\deg(\sigma) \leq 2p - 2c + l$ and $(z-1)^l | h(\beta - \sigma \cdot \bar{f}) - z^p$.

Case B: $z - 1 | \bar{f}$. In this case, we must have $z - 1 \nmid \bar{g}$, thus we can find $\sigma \in k[z]$ such that $\deg(\sigma) \leq 2p - 2c + l$ and $(z-1)^l | A - (\gamma + \sigma \cdot \bar{g})h$.

Let $\gamma' = \gamma + \sigma \cdot \bar{g}$,$\beta' = \beta - \sigma \cdot \bar{f}$. We still have $\deg(\beta'), \deg(\gamma') \leq 2p - c$ and $f \cdot \gamma' + g \cdot \beta' = (z-1)^{2p}$. Let $\alpha = (a \cdot \beta' - \gamma')/(z-1)^{2p}$. Since $\alpha = (A \cdot \beta' - z^p \cdot \gamma')/z^p(z-1)^{2p}$, the fact that $f \cdot (A \cdot \beta' - z^p \cdot \gamma') = (h\beta' - z^p)(z-1)^{2p}$ and $g \cdot (A \cdot \beta' - z^p \cdot \gamma') = (A - \gamma' h)(z-1)^{2p}$ tells us that $A \cdot \beta' - z^p \cdot \gamma'$ is divisible by $(z-1)^{2p}$. Thus, we have $\alpha \in k[z, 1/z]$.

By direct calculation, the following equations holds:

$$\begin{pmatrix} \alpha & \beta' \\ -\frac{h}{z^p} & f \end{pmatrix} \cdot \begin{pmatrix} (z-1)^p & 0 \\ a(z-1)^{-p} & (z-1)^{-p} \end{pmatrix} \cdot \begin{pmatrix} f \cdot (z-1)^{-c} & -\beta' \cdot (z-1)^{c-2p} \\ g \cdot (z-1)^{-c} & \gamma' \cdot (z-1)^{c-2p} \end{pmatrix}$$
$$= \begin{pmatrix} (z-1)^{p-c} & 0 \\ 0 & (z-1)^{c-p} \end{pmatrix}$$

Thus, we have $H \cong \mathcal{O}(p-c) \oplus \mathcal{O}(c-p)$.

Notice that we have $\deg(g) \leq p-1$. By taking the remainder of $(z-1)^{2p}$ on both side of $f \cdot A + g \cdot z^p = h \cdot (z-1)^{2p}$, we get the following equation:

$$R_p \cdot f_p + \cdots + R_0 \cdot f_0 + g \cdot z^p = 0.$$

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We can also write this equation in the following form:

$$\begin{pmatrix} f_0 & \cdots & f_p \end{pmatrix}_{1\times(p+1)} \cdot \begin{pmatrix} R_{00} & \cdots & R_{0,2p-1} \\ \vdots & \ddots & \vdots \\ R_{p,0} & \cdots & R_{p,2p-1} \end{pmatrix}_{(p+1)\times 2p} + \begin{pmatrix} 0 \cdots 0 \, g_0 & \cdots & g_{p-1} \end{pmatrix}_{1\times 2p} = (\, 0 \quad \cdots \quad 0\,)_{1\times 2p}.$$

Since we have the precise expression of a, we can calculate the value of elements of R, and we get $R_{ij} = T_{i+1,j+1}$ for $i,j \le p-1$ and $R_{ij} = T_{i+1,3p-j}$ for $i \le p-1, 2p-i \le j \le 2p-1$.

We can see that $c = \max(\deg(f), \deg(g))$ if and only if the following condition holds.

First, the following linear system has a nonzero solution:

$$\begin{pmatrix} f_0 & \cdots & f_c \end{pmatrix}_{1\times(c+1)} \cdot \begin{pmatrix} R_{00} & \cdots & R_{0,p-1} & R_{0,2p-1} & \cdots & R_{0,p+c+1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ R_{c,0} & \cdots & R_{c,p-1} & R_{c,2p-1} & \cdots & R_{c,p+c+1} \end{pmatrix}_{(c+1)\times(2p-c-1)} = \begin{pmatrix} 0 & \cdots & 0 \end{pmatrix}_{1\times(2p-c-1)}.$$

(Notice that the $(c+1)\times(2p-c-1)$ matrix is exactly T_{p-c-1}.)

Meanwhile, the following linear system does not have a nonzero solution:

$$\begin{pmatrix} f_0 & \cdots & f_{c-1} \end{pmatrix}_{1\times c} \cdot \begin{pmatrix} R_{00} & \cdots & R_{0,p-1} & R_{0,2p-1} & \cdots & R_{0,p+c} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ R_{c-1,0} & \cdots & R_{c-1,p-1} & R_{c-1,2p-1} & \cdots & R_{c-1,p+c} \end{pmatrix}_{c\times(2p-c)} = \begin{pmatrix} 0 & \cdots & 0 \end{pmatrix}_{1\times(2p-c)}.$$

(Notice that the $c\times(2p-c)$ matrix is exactly T_{p-c}.)

Thus, for the sequence $\{T_0, \cdots, T_{p-1}\}$, we know T_{p-c-1} (and all the matrices before T_{p-c-1}) cannot be of full rank, while T_{p-c} (and all the matrices after T_{p-c}) is (are) of full rank. This completes our proof. ■

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4 Semistable Families of Abelian Varieties with Real Multiplication

In this section, we shall construct a semistable family of abelian varieties with $\mathcal{O}_F$ multiplication. The inputs are properties (i)–(iii) in Theorem 1.1. We denote $\mathbb{P}^1 - \{0, 1, \lambda, \infty\}$ by X.

Assume property (i). Let $[F : \mathbb{Q}] = d$, and let $\mathbb{V}_{\mathcal{O}_F}$ be the corresponding $\mathcal{O}_F$ local system of rank 2 to ρ_λ. We define

$$\omega : \mathcal{O}_F^{\oplus 2} \times \mathcal{O}_F^{\oplus 2} \xrightarrow{\det} \mathcal{O}_F \xrightarrow{\mathrm{Tr}} \mathbb{Z}.$$

It is easy to see that ω is skew-symmetric and nondegenerate.

Assume property (ii). Regarding $\mathbb{V}_{\mathcal{O}_F}$ as a $\mathbb{Z}$-local system, it underlies a polarized $\mathbb{Z}$-VHS (with polarization ω). Let $H = \mathbb{V}_{\mathcal{O}_F} \otimes_{\mathbb{Z}} \mathcal{O}_X$ and $Fil^1 \subset H$ be the Hodge filtration. Because the composite $\mathbb{V}_{\mathcal{O}_F} \hookrightarrow H \to H/\mathrm{Fil}^1$ is injective, we get a family of abelian varieties

$$f : \mathcal{A} := \mathbb{V}_{\mathcal{O}_F} \backslash H/Fil^1 \to X.$$

We compactify it to a projective morphism $\bar{f} : \bar{\mathcal{A}} \to \mathbb{P}^1$. By blowing up with centers contained in the singular fibers of $\bar{f}$, we can make $\bar{f}$ into a quasi-semistable family. Since the local monodromies of $\mathbb{V}_{\mathcal{O}_F}$ around D are unipotent, we actually obtain a semistable family $\bar{f}$.

Assume property (iii). We show that the family f admits $\mathcal{O}_F$ multiplication, that is, $\mathcal{O}_F \subset \mathrm{End}_{\mathbb{Z}}(\mathcal{A}/X)$. By the assumption of (iii), there exists one point $x \in X$ such that the multiplication by any element of $\mathcal{O}_F$ is of Hodge type $(0, 0)$. It is clear that the multiplication by $\mathcal{O}_F$ on $\mathbb{V}_{\mathcal{O}_F}$ commutes with the monodromy action. This means that the multiplication by an element of $\mathcal{O}_F$ is a global section of the weight zero $\mathbb{Z}$-VHS $\mathrm{End}_{\mathbb{Z}}(\mathbb{V}_{\mathcal{O}_F}) \cong \mathbb{V}_{\mathcal{O}_F}^{\vee} \otimes_{\mathbb{Z}} \mathbb{V}_{\mathcal{O}_F}$. Applying the rigidity theorem [11, Corollary 7.23], the multiplication by any element of $\mathcal{O}_F$ is of Hodge type $(0, 0)$ everywhere, which means $\mathcal{O}_F \subset \mathrm{End}_{\mathbb{Z}}(\mathcal{A}/X)$. The construction is completed.

5 Proof of the Main Result: Theorem 1.1

For any λ in Beauville's list 1.0.1, Beauville constructs a semistable family $\bar{f}_\lambda$ of elliptic curves over $\mathbb{P}^1$ with singular locus exactly equal to $\{0, 1, \lambda, \infty\}$. Let f_λ be the smooth part of $\bar{f}_\lambda$. We explain why the uniformizing representation ρ_λ over $P^1 - \{0, 1, \lambda, \infty\}$ satisfies the properties in Theorem 1.1. We simply take $F = \mathbb{Q}$ and claim that the monodromy rep-

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resentation associated with the family f_λ underlies the real local system ρ_λ. If this claim holds, then properties (i)–(iii) become obvious. The most efficient way to get this is to use the Simpson correspondence [12]. Set ρ_{geo} to be the monodromy representation of f_λ. The logarithmic Higgs bundle corresponding to $\rho_{geo} \otimes_{\mathbb{Z}} \mathbb{C}$ is the logarithmic Kodaira–Spencer system (E, θ) associated with f_λ. The proof of Proposition 3.2 explains that $(E, \theta) \cong (E_{unif}, \theta_{unif})$. Therefore, $\rho_{geo} \otimes_{\mathbb{Z}} \mathbb{C} \cong \rho_\lambda \otimes_{\mathbb{R}} \mathbb{C}$. Setting $\rho_{geo,\mathbb{R}} = \rho_{geo} \otimes \mathbb{R}$, we have that

$$H^0(U, \rho_{geo,\mathbb{R}} \otimes_{\mathbb{R}} \rho_\lambda^\vee) \otimes_{\mathbb{R}} \mathbb{C} = H^0(U, (\rho_{geo,\mathbb{R}} \otimes_{\mathbb{R}} \rho_\lambda^\vee) \otimes_{\mathbb{R}} \mathbb{C})$$

is nonzero. Thus, $H^0(U, (\rho_{geo,\mathbb{R}} \otimes_{\mathbb{R}} \rho_\lambda^\vee)) \cong \mathbb{R}$ because the complex local systems are irreducible. It follows that $\rho_{geo} \otimes_{\mathbb{Z}} \mathbb{R} \cong \rho_\lambda$ as claimed.

The difficult part of Theorem 1.1 is the converse direction. By the construction of semistable families of abelian varieties in Section 4 and the classification of Beauville [1], it suffices to show that properties (i)–(iii) force F to be $\mathbb{Q}$. In other words, we are going to prove the following negative result.

Claim 5.1. For any λ, there exists *no* totally real subfield F of degree > 1 such that ρ_λ satisfies properties (i)–(iii) simultaneously.

Proof. Let $\bar{f} : \mathcal{A} \to \mathbb{P}^1$ be the semistable family of abelian varieties with singular locus D and with real multiplication $\mathcal{O}_F$. Let ρ be the weight one $\mathbb{Z}$-VHS associated with f and (E, θ) be the corresponding logarithmic Kodaira-Spencer system. By construction, $\rho = Res_{F|\mathbb{Q}} \rho_\lambda$, hence $\rho_\lambda \subset \rho \otimes \mathbb{R}$ as an $\mathbb{R}$-sublocal system. This gives us $(E_{unif}, \theta_{unif}) \subset (E, \theta)$, and by Proposition 2.1, it is in fact a direct factor. We know by Theorem 2.8 that $(E_{unif}, \theta_{unif})$ is periodic. The contradiction arises from the exact analysis of the period. First of all, by Proposition 3.1, the period of $(E_{unif}, \theta_{unif})$ is in every case equal to one. However, we can argue that the existence of the real multiplication $\mathcal{O}_F$ ($d = [F : \mathbb{Q}] > 1$) as an endomorphism of f leads to a contradiction with one-periodicity. The argument goes as follows.

By Čebotarev density theorem, the set of inert primes of F is of positive Dirichlet density. In particular, it is an infinite set. Let $\mathfrak{p}$ be an inert prime over the rational prime p. Thus, there are natural isomorphisms of $\mathbb{Z}_p$-algebras:

$$\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \mathcal{O}_{F_\mathfrak{p}} \cong \mathbb{Z}_{p^d},$$

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where $F_{\mathfrak{p}}$ is the completion of F at the prime $\mathfrak{p}$. Let $\mathfrak{f}$ be a spread of f defined over S, and let $(\mathcal{E}, \Theta)$ be the corresponding logarithmic Kodaira–Spencer system (see the proof of Theorem 2.8). Let $s \in S$ be a geometrically closed point of $\mathrm{char}(k(s)) = p$ (which is assumed to be large enough), and let $\tilde{s} : \mathrm{Spec}(W(k(s))) \to S$ be a closed subscheme lifting $s : \mathrm{Spec}(k(s)) \to S$. Let $\hat{s}$ be the generic point of $\tilde{s}$. Set $\mathfrak{f}_{\tilde{s}}$ to be a base change of $\mathfrak{f}$ over $\tilde{s}$; it is a semistable family over $W(k(s))$ with $\mathcal{O}_F \subset \mathrm{End}(\mathfrak{f}_{\tilde{s}})$.

Let $\mathbb{V}$ be the mod p crystalline representation of the algebraic fundamental group $\pi_1(U_{\hat{s}})$ of the family $\mathfrak{f}_{\tilde{s}}$, which corresponds to the one-periodic flow with the initial Higgs term $(\mathcal{E}_s, \Theta_s)$, the logarithmic Kodaira–Spencer system associated with the special fiber of $\mathfrak{f}_{\tilde{s}}$ ([7, Theorem 1.1]). Now, by transportation of structure, the local system $\mathbb{V}$ contains $\mathcal{O}_F$ as an endomorphism subalgebra. Thus, $\mathbb{V}$ is a $\pi_1(U_{\hat{s}}) - \mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{F}_p$-module and the previous discussion tells us that $\mathbb{V}$ is the restriction of scalar of a $\mathbb{F}_{p^d}$-local system, that is, $\mathbb{V} = Res_{\mathbb{F}_{p^d}|\mathbb{F}_p}\mathbb{W}$ for some rank two $\mathbb{F}_{p^d}$-crystalline representation $\mathbb{W}$. By [8, Corollaries 3.10 and 3.11 (ii)], it follows that any simple factor of $(\mathcal{E}_s, \Theta_s)$ is d-periodic and all its simple factors are pairwisely nonisomorphic. In particular, the factor $(E_{unif}, \theta_{unif})$ cannot be one-periodic. Contradiction. The claim is proved. ■

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