Algebr Represent Theor (2016) 19:171–180
DOI 10.1007/s10468-015-9568-1

On Homomorphisms from Ringel-Hall Algebras to Quantum Cluster Algebras

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Received: 11 February 2015 / Accepted: 11 August 2015 / Published online: 27 August 2015

Abstract In Berenstein and Rupel (2015), the authors defined algebra homomorphisms from the dual Ringel-Hall algebra of certain hereditary abelian category $\mathcal{A}$ to an appropriate q-polynomial algebra. In the case that $\mathcal{A}$ is the representation category of an acyclic quiver, we give an alternative proof by using the cluster multiplication formulas in (Ding and Xu, Sci. China Math. **55**(10) 2045–2066, 2012). Moreover, if the underlying graph of Q associated with $\mathcal{A}$ is bipartite and the matrix B associated to the quiver Q is of full rank, we show that the image of the algebra homomorphism is in the corresponding quantum cluster algebra.

Keywords Ringel-Hall algebra · Quantum cluster algebra · Cluster variable · Bipartite graph

Mathematics Subject Classification (2010) 16G20 · 17B67 · 17B35 · 18E30

Presented by Vlastimil Dlab.

Ming Ding was supported by NSF of China (No. 11301282) and Specialized Research Fund for the Doctoral Program of Higher Education (No. 20130031120004) and Fan Xu was supported by NSF of China (No. 11071133).

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1 Background

The Ringel-Hall algebra $\mathcal{H}(\mathcal{A})$ of a (small) finitary abelian category $\mathcal{A}$ was introduced by Ringel ([14]). When $\mathcal{A}$ is the category $\mathrm{Rep}_{\mathbb{F}_q} Q$ of finite dimensional representations of a simply-laced quiver Q over a finite field $\mathbb{F}_q$, the Ringel-Hall algebra $\mathcal{H}(\mathcal{A})$ is isomorphic to the positive part $U_q(\mathfrak{n})$ of the corresponding quantum group $U_q(\mathfrak{g})$ ([14]). Lusztig ([12]) constructed the canonical basis of the quantum group $U_q(\mathfrak{n})$ under the context of Ringel-Hall algebras. In order to study the canonical basis algebraically and combinatorially, Berenstein and Zelevinsky ([2]) defined quantum cluster algebras as a noncommutative analogue of cluster algebras (see [9, 10]). A quantum cluster algebra is a subalgebra of a skew field of rational functions in q-commuting variables and generated by a set of generators called the *cluster variables*.

A natural question is to study the relations between Ringel-Hall algebras and quantum cluster algebras. Geiss, Leclerc and Schröer ([11]) showed that quantum groups of type A, D and E have quantum cluster structures. Recently, Berenstein and Rupel [1] constructed algebra homomorphisms from Ringel-Hall algebras to quantum cluster algebras. Let $\mathcal{A}$ be a finitary hereditary abelian category and $\mathbf{i} = (i_1, \cdots, i_m)$ be a sequence of simple objects in $\mathcal{A}$. Berenstein and Rupel [1] showed that, under certain co-finiteness conditions, the assignment $[V]^* \to X_{V,\mathbf{i}}$ defines a homomorphism of algebras

$$\Psi_{\mathbf{i}} : \mathcal{H}^*(\mathcal{A}) \to P_{\mathbf{i}}$$

where $\mathcal{H}^*(\mathcal{A})$ is the dual Ringel-Hall algebra and $X_{V,\mathbf{i}}$ is the quantum cluster $\mathbf{i}$-character of V in an appropriate q-polynomial algebra $P_{\mathbf{i}}$. Moreover, for an appropriate $\mathbf{i}$, the image restricting to the composition algebra of $\mathcal{H}^*(\mathcal{A})$ is in the corresponding upper cluster algebra.

The aim of this note is to give an alternative proof of the above result when $\mathcal{A}$ is the representation category of an acyclic quiver. Different from [1], a key ingredient of our proof is to apply the cluster multiplication formulas proved in [8] (see also Theorem 3.3). We show that if the underlying graph of Q is bipartite (i.e, we can associate this graph an orientation such that every vertex is a sink or a source) and the matrix B associated to the quiver Q is of full rank, then the algebra $\mathcal{AH}_{|k|}(Q)$ generated by all quantum cluster characters is exactly the quantum cluster algebra $\mathcal{A}_{|k|}(Q)$ (see Theorem 4.5). As a corollary, the image of the algebra homomorphism is in the quantum cluster algebra $\mathcal{A}_{|k|}(Q)$ (see Corollary 4.6). We expect that the approach in this note can be extended to construct algebra homomorphisms from derived Hall algebras to quantum cluster algebras.

2 Quantum Cluster Algebras and Caldero-Chapoton Maps

2.1 Quantum Cluster Algebras

We briefly recall the definition of quantum cluster algebras. Let L be a lattice of rank m and $\Lambda : L \times L \to \mathbb{Z}$ a skew-symmetric bilinear form. We will need a formal variable q and consider the ring of integeral Laurent polynomials $\mathbb{Z}[q^{\pm 1/2}]$. Define the *based quantum torus* associated to the pair (L, Λ) to be the $\mathbb{Z}[q^{\pm 1/2}]$-algebra $\mathcal{T}$ with a distinguished $\mathbb{Z}[q^{\pm 1/2}]$-basis $\{X^e : e \in L\}$ and the multiplication given by

$$X^e X^f = q^{\Lambda(e,f)/2} X^{e+f}.$$

It is easy to see that $\mathcal{T}$ is associative and the basis elements satisfy the following relations:

$$X^e X^f = q^{\Lambda(e,f)} X^f X^e,\ X^0 = 1 \text{ and } (X^e)^{-1} = X^{-e}.$$

It is known that $\mathcal{T}$ is an Ore domain, i.e., is contained in its skew-field of fractions $\mathcal{F}$. The quantum cluster algebra will be defined as a $\mathbb{Z}[q^{\pm 1/2}]$-subalgebra of $\mathcal{F}$.

A *toric frame* in $\mathcal{F}$ is a map $M : \mathbb{Z}^m \to \mathcal{F} \setminus \{0\}$ of the form

$$M(\mathbf{c}) = \varphi(X^{\eta(\mathbf{c})})$$

where φ is an automorphism of $\mathcal{F}$ and $\eta : \mathbb{Z}^m \to L$ is an isomorphism of lattices. By definition, the elements $M(\mathbf{c})$ form a $\mathbb{Z}[q^{\pm 1/2}]$-basis of the based quantum torus $\mathcal{T}_M := \varphi(\mathcal{T})$ and satisfy the following relations:

$$M(\mathbf{c})M(\mathbf{d}) = q^{\Lambda_M(\mathbf{c},\mathbf{d})/2} M(\mathbf{c}+\mathbf{d}),\ M(\mathbf{c})M(\mathbf{d}) = q^{\Lambda_M(\mathbf{c},\mathbf{d})} M(\mathbf{d})M(\mathbf{c}),$$
$$M(\mathbf{0}) = 1 \text{ and } M(\mathbf{c})^{-1} = M(-\mathbf{c}),$$

where Λ_M is the skew-symmetric bilinear form on $\mathbb{Z}^m$ obtained from the lattice isomorphism η. Let Λ_M also denote the skew-symmetric $m \times m$ matrix defined by $\lambda_{ij} = \Lambda_M(e_i, e_j)$ where $\{e_1, \dots, e_m\}$ is the standard basis of $\mathbb{Z}^m$. Given a toric frame M, let $X_i = M(e_i)$. Then we have

$$\mathcal{T}_M = \mathbb{Z}\left[q^{\pm 1/2}\right]\left\langle X_1^{\pm 1}, \dots, X_m^{\pm 1} : X_i X_j = q^{\lambda_{ij}} X_j X_i \right\rangle.$$

Let Λ be an $m \times m$ skew-symmetric matrix and let $\tilde{B}$ be an $m \times n$ matrix with $m \geq n$, whose principal part is denoted by B. We call the pair $(\Lambda, \tilde{B})$ *compatible* if $\tilde{B}^{tr}\Lambda = (D|0)$ is an $n \times m$ matrix with $D = diag(d_1, \cdots, d_n)$ where $d_i \in \mathbb{N}$ for $1 \leq i \leq n$. The pair $(M, \tilde{B})$ is called a *quantum seed* if the pair $(\Lambda_M, \tilde{B})$ is compatible. Define the $m \times m$ matrix $E = (e_{ij})$ by

$$e_{ij} = \begin{cases} \delta_{ij} & \text{if } j \neq k; \\ -1 & \text{if } i = j = k; \\ max(0, -b_{ik}) & \text{if } i \neq j = k. \end{cases}$$

For $n, k \in \mathbb{Z}$, $k \geq 0$, denote $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n - q^{-n})\cdots(q^{n-r+1} - q^{-n+r-1})}{(q^r - q^{-r})\cdots(q - q^{-1})}$. Let $k \in [1, n]$ and $\mathbf{c} = (c_1, \dots, c_m) \in \mathbb{Z}^m$ with $c_k \geq 0$. Define the toric frame $M' : \mathbb{Z}^m \to \mathcal{F} \setminus \{0\}$ as follows:

$$M'(\mathbf{c}) = \sum_{p=0}^{c_k} \begin{bmatrix} c_k \\ p \end{bmatrix}_{q^{d_k/2}} M(E\mathbf{c} + p\mathbf{b}^k) \text{ and } M'(\text{-}\mathbf{c}) = M'(\mathbf{c})^{-1} \tag{1}$$

where the vector $\mathbf{b}^k \in \mathbb{Z}^m$ is the k-th column of $\tilde{B}$.

Define the $m \times n$ matrix $\tilde{B}' = (b'_{ij})$ by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k; \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{otherwise.} \end{cases}$$

Then the quantum seed $(M', \tilde{B}')$ is defined to be the mutation of $(M, \tilde{B})$ in direction k. Two quantum seeds $(M, \tilde{B})$ and $(M', \tilde{B}')$ are mutation-equivalent if they can be obtained from each other by a sequence of mutations, denoted by $(M, \tilde{B}) \sim (M', \tilde{B}')$. Let $\mathcal{C} := \{M'(e_i) : (M, \tilde{B}) \sim (M', \tilde{B}'), i \in [1, n]\}$. Let $\mathbb{ZP}$ be the ring of integral Laurent polynomials in the (quasi-commuting) variables in $\{q^{1/2}, X_{n+1}, \cdots, X_m\}$. The *quantum cluster algebra* $\mathcal{A}_q(\Lambda_M, \tilde{B})$ is the $\mathbb{ZP}$-subalgebra of $\mathcal{F}$ generated by $\mathcal{C}$.

The following proposition demonstrates the mutation of quantum cluster variables which can be viewed as a quantum analogue of cluster mutation.

Proposition 2.1 *(Mutation of cluster variables)*[2] *The toric frame X' is determined by*

$$X'_k = X\left\{\sum_{1\le i\le m}[b_{ik}]_+e_i - e_k\right\} + X\left\{\sum_{1\le j\le m}[-b_{jk}]_+e_j - e_k\right\},$$
$$X'_i = X_i, \quad 1 \le i \le m, \quad i \ne k.$$

The quantum Laurent phenomenon proved by Berenstein and Zelevinsky is an important result concerning quantum cluster algebras.

Theorem 2.2 *(Quantum Laurent phenomenon)*[2] *The quantum cluster algebra $\mathcal{A}_q(\Lambda_M, \tilde{B})$ is a subalgebra of $\mathcal{T}_M$.*

Set $\mathbf{X} = \{X_1, \cdots, X_n\}$ and $\mathbf{X}_k = \mathbf{X} - \{X_k\} \cup \{X'_k\}$ for any $k \in [1, n]$. Denote by $\mathcal{U}(\Lambda_M, \tilde{B})$ the $\mathbb{ZP}$-subalgebra of $\mathcal{F}$ given by

$$\mathcal{U}(\Lambda_M, \tilde{B}) = \mathbb{ZP}[\mathbf{X}^{\pm 1}] \cap \mathbb{ZP}\left[\mathbf{X}_1^{\pm 1}\right] \cap \cdots \cap \mathbb{ZP}\left[\mathbf{X}_n^{\pm 1}\right].$$

The algebra $\mathcal{U}(\Lambda_M, \tilde{B})$ is called the *quantum upper cluster algebra*. The following result shows that the acyclicity condition closes the gap between the upper bounds and the corresponding quantum cluster algebras.

Theorem 2.3 [2] *If the principal matrix B is acyclic, then $\mathcal{U}(\Lambda_M, \tilde{B}) = \mathcal{A}_q(\Lambda_M, \tilde{B})$.*

2.2 Quantum Caldero-Chapoton Maps

Let k be a finite field with cardinality $|k| = q$ and $m \ge n$ be two positive integers. Let $\widetilde{Q}$ be an acyclic quiver with vertex set $\{1, \ldots, m\}$. Denote $C := \{n+1, \ldots, m\}$. The full subquiver Q on the vertices $\{1, \ldots, n\}$ is called the *principal part* of $\widetilde{Q}$. For $1 \le i \le m$, let S_i be the ith simple module of the path algebra $k\widetilde{Q}$.

Let $\widetilde{B}$ be the $m \times n$ matrix associated to the quiver $\widetilde{Q}$ whose entry in position (i, j) is given by

$$b_{ij} = |\{\text{arrows } i \longrightarrow j\}| - |\{\text{arrows } j \longrightarrow i\}|$$

for $1 \le i \le m$ and $1 \le j \le n$. Denote by $\widetilde{I}$ the left $m \times n$ submatrix of the identity matrix of size $m \times m$. Assume that there exists some antisymmetric $m \times m$ integer matrix Λ such that

$$\Lambda(-\widetilde{B}) = \begin{bmatrix} \mathrm{I_n} \\ 0 \end{bmatrix}, \tag{2}$$

where $\mathrm{I_n}$ is the identity matrix of size $n \times n$. Let $\widetilde{R} = \widetilde{R}_{\widetilde{Q}}$ be the $m \times n$ matrix with its entry in position (i, j) given by

$$\widetilde{r}_{ij} := \dim_k \mathrm{Ext}^1_{k\widetilde{Q}}(S_j, S_i) = |\{\text{arrows } j \longrightarrow i\}|.$$

for $1 \le i \le m$ and $1 \le j \le n$. Set $\widetilde{R}^{tr} = \widetilde{R}_{\widetilde{Q}^{op}}$. Denote the principal $n \times n$ submatrices of $\widetilde{B}$ and $\widetilde{R}$ by B and R, respectively. Note that $\widetilde{B} = \widetilde{R}^{tr} - \widetilde{R}$ and $B = R^{tr} - R$.

Let $\mathcal{C}_{\widetilde{Q}}$ be the cluster category of $k\widetilde{Q}$, i.e., the orbit category of the derived category $\mathcal{D}^b(\widetilde{Q})$ under the action of the functor $F = \tau \circ [-1]$ (see [3]). Let I_i be the indecomposable injective $k\widetilde{Q}$ module for $1 \le i \le m$. Then the indecomposable $k\widetilde{Q}$-modules and $I_i[-1]$ for

$1 \leq i \leq m$ exhaust all indecomposable objects of the cluster category $\mathcal{C}_{\widetilde{Q}}$. Each object M in $\mathcal{C}_{\widetilde{Q}}$ can be uniquely decomposed as

$$M = M_0 \oplus I_M[-1]$$

where M_0 is a module and I_M is an injective module.

The Euler form on $k\widetilde{Q}$-modules M and N is given by

$$\langle M, N \rangle = \dim_k \mathrm{Hom}(M, N) - \dim_k \mathrm{Ext}^1(M, N).$$

Note that the Euler form only depends on the dimension vectors of M and N.

The quantum Caldero-Chapoton map of the quiver Q has been defined in [6, 8, 13, 15] as

$$X_? : \mathrm{obj}\, \mathcal{C}_{\widetilde{Q}} \longrightarrow \mathcal{T}$$

by the following rules:

(1) If M is a kQ-module, then

$$X_M = \sum_{\underline{e}} |\,\mathrm{Gr}_{\underline{e}}\, M| q^{-\frac{1}{2}\langle \underline{e}, \underline{m}-\underline{e} \rangle} X^{-\widetilde{B}\underline{e}-(\widetilde{I}-\widetilde{R}^{tr})\underline{m}};$$

(2) If M is a kQ-module and I is an injective $k\widetilde{Q}$-module, then

$$X_{M \oplus I[-1]} = \sum_{\underline{e}} |\,\mathrm{Gr}_{\underline{e}}\, M| q^{-\frac{1}{2}\langle \underline{e}, \underline{m}-\underline{e}-\underline{i} \rangle} X^{-\widetilde{B}\underline{e}-(\widetilde{I}-\widetilde{R}^{tr})\underline{m}+\underline{\dim}\,\mathrm{soc}\, I},$$

where $\underline{\dim} I = \underline{i}$, $\underline{\dim} M = \underline{m}$ and $\mathrm{Gr}_{\underline{e}}\, M$ denotes the set of all submodules V of M with $\underline{\dim} V = \underline{e}$. We note that

$$X_{P[1]} = X_{\tau P} = X^{\underline{\dim} P/radP} = X^{\underline{\dim}\,\mathrm{soc}\, I} = X_{I[-1]} = X_{\tau^{-1} I}.$$

for any projective $k\widetilde{Q}$-module P and injective $k\widetilde{Q}$-module I with $\mathrm{soc}\, I = P/\mathrm{rad} P$.

In the following, for convenience, we always use the underlined lower letter $\underline{x}$ to denote the corresponding dimension vector of a kQ-module X and view $\underline{x}$ as a column vector in $\mathbb{Z}^n$.

3 The Dual Ringel-Hall Algebras and the Cluster Multiplication Formulas

Let $\mathcal{A}$ be the representation category of an acyclic quiver Q over a finite field and $K(\mathcal{A})$ the Grothendieck group of $\mathcal{A}$. For an object $V \in \mathcal{A}$, we will write $[V]$ for the isomorphism class of V. Let $\mathcal{H}(\mathcal{A}) = \bigoplus k[V]$ be the k-vector space spanned by the isomorphism classes of objects of $\mathcal{A}$ with the natural grading via class in $K(\mathcal{A})$. For $U, V, W \in \mathcal{A}$ define the *Hall number*

$$g_{UW}^V = |\{R \subset V | R \cong W, V/R \cong U\}|.$$

The assignment $[U][W] = \sum_{[V]} g_{UW}^V [V]$ defines an associative multiplication on $\mathcal{H}(\mathcal{A})$. The algebra $\mathcal{H}(\mathcal{A})$ is known as the Ringel-Hall algebra. Denote by $\mathcal{H}^*(\mathcal{A})$ the dual Ringel-Hall algebra, which is the space of linear functions $\mathcal{H}(\mathcal{A}) \to k$ with a basis of all delta-functions δ_V labeled by isomorphism classes $[V]$ of objects of $\mathcal{A}$.

Proposition 3.1 *Let M and N be kQ-modules, then the product*

$$\delta_M * \delta_N = q^{\frac{1}{2}\Lambda((\widetilde{I}-\widetilde{R}')\underline{m},(\widetilde{I}-\widetilde{R}')\underline{n})+\langle\underline{m},\underline{n}\rangle}\sum_E h_E^{MN}\delta_E$$

defines an associative multiplication on $\mathcal{H}^*(\mathcal{A})$, where $h_E^{MN} = \frac{|\mathrm{Ext}^1_{kQ}(M,N)_E|}{|\mathrm{Hom}_{kQ}(M,N)|}$ and $\mathrm{Ext}^1_{kQ}(M,N)_E$ is the subset of $\mathrm{Ext}^1_{kQ}(M,N)$ consisting of those equivalence classes of short exact sequences with middle term E.

Proof Note that $\sum_E g^E_{MN}g^F_{EL} = \sum_G g^G_{NL}g^F_{MG}$, and the relation between h_E^{MN} and g^E_{MN} is given by the Riedtmann-Peng's formula

$$h_E^{MN} = g^E_{MN}|Aut(M)||Aut(N)||Aut(E)|^{-1}.$$

Thus we have $\sum_E h^E_{MN}h^F_{EL} = \sum_G h^G_{NL}h^F_{MG}$. It is easy to see that $\phi(\underline{m},\underline{n}) := \frac{1}{2}\Lambda((\widetilde{I}-\widetilde{R}')\underline{m},(\widetilde{I}-\widetilde{R}')\underline{n})+\langle\underline{m},\underline{n}\rangle$ is a bilinear form on $\mathbb{Z}^n$. Hence the associativity can be deduced. □

For any $k\widetilde{Q}$−modules M, N and E, denote by ε^E_{MN} the cardinality of the set $\mathrm{Ext}^1_{k\widetilde{Q}}(M,N)_E$ which is the subset of $\mathrm{Ext}^1_{k\widetilde{Q}}(M,N)$ consisting of those equivalence classes of short exact sequences with middle term E. Define

$$\mathrm{Hom}_{k\widetilde{Q}}(M,I)_{BI'} := \{f : M \longrightarrow I | kerf \cong B, cokerf \cong I'\}.$$

Denote

$$[M,N] = \mathrm{dim}_k\mathrm{Hom}_{k\widetilde{Q}}(M,N) \text{ and } [M,N]^1 = \mathrm{dim}_k\mathrm{Ext}^1_{k\widetilde{Q}}(M,N).$$

We have the following cluster multiplication formulas.

Theorem 3.2 [8][6] *Let M and N be any kQ-modules, and I any injective $k\widetilde{Q}$-module, then*

(1) $q^{[M,N]^1}X_MX_N = q^{\frac{1}{2}\Lambda((\widetilde{I}-\widetilde{R}')\underline{m},(\widetilde{I}-\widetilde{R}')\underline{n})}\sum_E \varepsilon^E_{MN}X_E$;

(2) $q^{[M,I]}X_MX_{I[-1]} = q^{\frac{1}{2}\Lambda((\widetilde{I}-\widetilde{R}')\underline{m},-\underline{\mathrm{dim}}SocI)}\sum_{B,I'}|\mathrm{Hom}_{k\widetilde{Q}}(M,I)_{BI'}|X_{B\oplus I'[-1]}$.

Note that Theorem 3.2(1) implies the following result which has been proved by Berenstein and Rupel using generalities on bialgebras in braided monoidal categories.

Theorem 3.3 [1] *The assignment $\delta_V \to X_V$ defines an algebra homomorphism $\Psi : \mathcal{H}^*(\mathcal{A}) \to \mathcal{T}$.*

An alternative proof.

Note that the first cluster multiplication formula in Theorem 3.2 can be rewritten as

$$X_MX_N = q^{\frac{1}{2}\Lambda((\widetilde{I}-\widetilde{R}')\underline{m},(\widetilde{I}-\widetilde{R}')\underline{n})+<\underline{m},\underline{n}>}\sum_E h_E^{MN}X_E.$$

Thus we have

$$\begin{aligned}\Psi(\delta_M * \delta_N) &= \Psi\left(q^{\frac{1}{2}\Lambda((\widetilde{I}-\widetilde{R}')\underline{m},(\widetilde{I}-\widetilde{R}')\underline{n})+\langle\underline{m},\underline{n}\rangle}\sum_E h_E^{MN}\delta_E\right)\\ &= q^{\frac{1}{2}\Lambda((\widetilde{I}-\widetilde{R}')\underline{m},(\widetilde{I}-\widetilde{R}')\underline{n})+\langle\underline{m},\underline{n}\rangle}\sum_E h_E^{MN}X_E\\ &= X_M X_N = \Psi(\delta_M)\Psi(\delta_N).\end{aligned}$$

This completes the proof. □

4 Quantum Cluster Algebras for Bipartite Graphs

In this section, we assume that Q is an acyclic quiver whose underlying graph is bipartite and the matrix B associated to the quiver Q is of full rank. Note that in this case the corresponding quantum cluster algebras are coefficient-free.

Definition 4.1 With respect to the quantum Caldero-Chapoton map, X_L is called *the quantum cluster character* if $L \in \mathcal{C}_Q$.

Definition 4.2 For a quiver Q, denote by $\mathcal{AH}_{|k|}(Q)$ the $\mathbb{Z}$-subalgebra of $\mathcal{F}$ generated by all the quantum cluster characters.

We will show that the algebra $\mathcal{AH}_{|k|}(Q)$ is equal to the quantum cluster algebra $\mathcal{A}_{|k|}(Q)$.

Let Q be an acyclic quiver and i a sink or a source in Q. We define the reflected quiver $\sigma_i(Q)$ by reversing all the arrows ending at i. An *admissible sequence of sinks (resp. sources)* is a sequence $(i_1,\dots,i_l)$ such that i_1 is a sink (resp. source) in Q and i_k is a sink (resp source) in $\sigma_{i_{k-1}}\cdots\sigma_{i_1}(Q)$ for any $2\le k\le l$. A quiver Q' is called *reflection-equivalent* to Q if there exists an admissible sequence of sinks or sources $(i_1,\dots,i_l)$ such that $Q'=\sigma_{i_l}\cdots\sigma_{i_1}(Q)$. Note that mutations can be viewed as generalizations of reflections, i.e, if i is a sink or a source in a quiver Q, then $\mu_i(Q)=\sigma_i(Q)$ where μ_i denotes the quiver mutation in the direction i. From now on, we assume that Q' is quiver mutation-equivalent to Q. Denote by $\Phi_i:\mathcal{A}_{|k|}(Q)\to\mathcal{A}_{|k|}(Q')$ the natural canonical isomorphism of quantum cluster algebras associated to sink or source for any $1\le i\le n$, which sends each initial cluster variable of $\mathcal{A}_{|k|}(Q)$ to its Laurent expansion in the initial cluster of $\mathcal{A}_{|k|}(Q')$.

Let $\Sigma_i^+ : \text{rep}(kQ) \longrightarrow \text{rep}(kQ')$ be the standard BGP-reflection functor and $R_i^+ : \mathcal{C}_Q \longrightarrow \mathcal{C}_{Q'}$ be the extended BGP-reflection functor defined in [16]:

$$R_i^+ : \begin{cases} X \mapsto \Sigma_i^+(X) & \text{if } X \not\cong S_i \text{ is a module}\\ S_i \mapsto P_i[1] & \\ P_j[1] \mapsto P_j[1] & \text{if } j\neq i\\ P_i[1] \mapsto S_i & \end{cases}$$

Rupel proved the following result.

Theorem 4.3 [15] *For any indecomposable object M in $\mathcal{C}_Q$, we have that* $\Phi_i(X_M^Q) = X_{R_i^+M}^{Q'}$.

The following lemma is well-known.

Lemma 4.4 [4, Lemma 8(b)] *Let*

$$M \longrightarrow E \longrightarrow N \longrightarrow M[1]$$

be a non-split triangle in $\mathcal{C}_Q$. *Then*

$$\dim_k \mathrm{Ext}^1_{\mathcal{C}_Q}(E, E) < \dim_k \mathrm{Ext}^1_{\mathcal{C}_Q}(M \oplus N, M \oplus N).$$

Theorem 4.5 *Assume that Q is an acyclic quiver whose underlying graph is bipartite and the matrix B associated to the quiver Q is of full rank, then $\mathcal{AH}_{|k|}(Q) = \mathcal{A}_{|k|}(Q)$.*

Proof Firstly, we prove that for any indecomposable object $M \in \mathcal{C}_Q$, X_M is in the quantum cluster algebra $\mathcal{A}_{|k|}(Q)$.

Case 1: If Q is an alternating quiver (i.e, whose vertex is either a sink or a source).

By Theorem 4.3, we have that $\Phi_i(X^Q_M) = X^{Q'}_{R^{\pm}_i M}$ for any indecomposable object $M \in \mathcal{C}_Q$. It is easy to see that Q' is again an acyclic quiver. Then we obtain that

$$X_M \in \mathbb{Z}\left[\mathbf{X}^{\pm 1}\right] \cap \mathbb{Z}\left[\mathbf{X}_1^{\pm 1}\right] \cap \cdots \cap \mathbb{Z}\left[\mathbf{X}_n^{\pm 1}\right].$$

Note that the quiver Q is acyclic, thus the corresponding quantum upper cluster algebra associated to Q coincides with the quantum cluster algebra $\mathcal{A}_{|k|}(Q)$ (see Theorem 2.3). Hence X_M is in the quantum cluster algebra $\mathcal{A}_{|k|}(Q)$.

Case 2: If Q is an acyclic quiver whose underlying graph is bipartite.

Note that Q is reflection–equivalent to some alternating quiver Q' and in the *Case 1* we have showed that for any indecomposable object $M \in \mathcal{C}_{Q'}$, X_M is in the corresponding quantum cluster algebra $\mathcal{A}_{|k|}(Q')$. Thus the rest of the proof immediately follows from Theorem 4.3.

Now we need to prove that for any quantum cluster character $X_L \in \mathcal{AH}_{|k|}(Q)$, we have that $X_L \in \mathcal{A}_{|k|}(Q)$. Let $L \cong \bigoplus_{i=1}^{l} L_i^{\oplus n_i}$, $n_i \in \mathbb{N}$ where L_i $(1 \leq i \leq l)$ are indecomposable objects in $\mathcal{C}_Q$. According to Theorem 3.2, we obtain the following equality:

$$X^{n_1}_{L_1} X^{n_2}_{L_2} \cdots X^{n_l}_{L_l} = q^{\frac{1}{2}n_L} X_L + \sum_{\dim_k \mathrm{Ext}^1_{\mathcal{C}_Q}(E,E) < \dim_k \mathrm{Ext}^1_{\mathcal{C}_Q}(L,L)} f_{n_E}(q^{\pm\frac{1}{2}}) X_E$$

where $n_L \in \mathbb{Z}$ and $f_{n_E}(q^{\pm\frac{1}{2}}) \in \mathbb{Z}[q^{\pm\frac{1}{2}}]$. Using Lemma 4.4 and proceeding by induction, it is straightforward to verify that $X_L \in \mathcal{A}_{|k|}(Q)$.

This completes the proof. □

Corollary 4.6 *Assume that Q is an acyclic quiver whose underlying graph is bipartite, and the matrix B associated to the quiver Q is of full rank, then $\Psi(\mathcal{H}^*(\mathcal{A})) \subseteq \mathcal{A}_{|k|}(Q)$.*

Proof By Theorem 3.3, we have that $\Psi(\mathcal{H}^*(\mathcal{A})) \subseteq \mathcal{AH}_{|k|}(Q)$. Hence the proof immediately follows from Theorem 4.5. □

Remark 4.7 It is natural to ask when $\Psi(\mathcal{H}^*(\mathcal{A}))$ is equal to $\mathcal{A}_{|k|}(Q)$. The key point of this problem is to prove that the initial cluster variables can be written as a $\mathbb{Z}[q^{\pm 1/2}]-$combination of some product of cluster characters associated to kQ-modules. In the following, we give an example in this direction.

Example 4.8 Set $\Lambda = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$. Thus the quiver Q associated to this pair is the Kronecker quiver:

$$1 \bullet \rightrightarrows \bullet\, 2.$$

Let k be a finite field and $q = \sqrt{|k|}$. The category rep(kQ) of finite-dimensional representations can be identified with the category of mod-kQ of finite-dimensional modules over the path algebra kQ. It is well-known (see [5]) that up to isomorphism the indecomposable kQ-module contains three families: the preprojective modules with dimension vector $(n-1, n)$ (denoted by $M(n)$), the indecomposable regular modules with dimension vector (nd_p, nd_p) for $p \in \mathbb{P}^1_k$ of degree d_p (in particular, denoted by $R_p(n)$ for $d_p = 1$) and the preinjective modules with dimension vector $(n, n-1)$ (denoted by $N(n)$) for any $n \in \mathbb{N}$.

For $m \in \mathbb{Z} \setminus \{1, 2\}$, set

$$V(m) = \begin{cases} N(m-2) & \text{if } m \geq 3; \\ M(-m+1) & \text{if } m \leq 0. \end{cases}$$

Now, let $\mathcal{T} = \mathbb{Z}[q^{\pm 1/2}]\langle X_1^{\pm 1}, X_2^{\pm 1} : X_1X_2 = qX_2X_1\rangle$ and $\mathcal{F}$ be the skew field of fractions of $\mathcal{T}$. The quantum cluster algebra of the Kronecker quiver $\mathcal{A}_q(2,2)$ is the $\mathbb{Z}[q^{\pm 1/2}]$-subalgebra of $\mathcal{F}$ generated by the cluster variables in $\{X_k | k \in \mathbb{Z}\}$ defined recursively by

$$X_{m-1}X_{m+1} = qX_m^2 + 1.$$

With the above notation, we have the following results.

Lemma 4.9 [15] *For any $m \in \mathbb{Z} \setminus \{1, 2\}$, the m-th cluster variable X_m of $\mathcal{A}_q(2,2)$ is equal to $X_{V(m)}$.*

Lemma 4.10 [7] *For any $n \in \mathbb{Z}$, we have that*

$$X_n X_{R_p(1)} = q^{-\frac{1}{2}} X_{n-1} + q^{\frac{1}{2}} X_{n+1}.$$

For the Kronecker quiver, we show that the image of the dual Ringel-Hall algebra under the homomorphismΨ coincides with the quantum cluster algebra.

Theorem 4.11 *Assume that Q is the Kronecker quiver, we have that*

$$\Psi(\mathcal{H}^*(\mathcal{A})) = \mathcal{A}_{|k|}(Q).$$

Proof By Corollary 4.6, we know that $\Psi(\mathcal{H}^*(\mathcal{A})) \subseteq \mathcal{A}_{|k|}(Q)$. Note that Ψ is an algebra homomorphism according to Theorem 3.3, thus it is enough to prove that X_1 and X_2 have preimages. By Lemma 4.10, we have $X_0X_{R_p(1)} = q^{-\frac{1}{2}}X_{-1} + q^{\frac{1}{2}}X_1$. This gives $X_1 = q^{-\frac{1}{2}}X_0X_{R_p(1)} - q^{-1}X_{-1}$ which can be rewritten as $X_1 = q^{-\frac{1}{2}}X_{V(0)}X_{R_p(1)} - q^{-1}X_{V(-1)}$ according to Lemma 4.9. Hence we have $X_1 = q^{-\frac{1}{2}}\Psi(\delta_{V(0)})\Psi(\delta_{R_p(1)}) - q^{-1}\Psi(\delta_{V(-1)}) = \Psi(q^{-\frac{1}{2}}\delta_{V(0)} * \delta_{R_p(1)} - q^{-1}\delta_{V(-1)})$. Similarly we have $X_3X_{R_p(1)} = q^{-\frac{1}{2}}X_2 + q^{\frac{1}{2}}X_4$, and using the same method we deduce that $X_2 = \Psi(q^{\frac{1}{2}}\delta_{V(3)} * \delta_{R_p(1)} - q\delta_{V(4)})$. This completes the proof. □

Conflict of Interest Ming Ding was supported by NSF of China (No. 11301282) and Specialized Research Fund for the Doctoral Program of Higher Education (No. 20130031120004) and Fan Xu was supported by NSF of China (No. 11071133).

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