ON CENTRAL CRITICAL VALUES OF RANKIN-TYPE L-FUNCTIONS OVER GLOBAL FUNCTION FIELDS

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Abstract. The aim of this paper is to study the central critical value of Rankin-type L-functions coming from "Drinfeld-type" automorphic cusp forms convolved with "imaginary" quadratic characters. Rankin-Selberg method provides us with a very explicit functional equation for these Rankin-type L-functions. When the "root number" in question is positive, we derive a Gross-type formula over arbitrary global function field. Via the theta series constructed from definite pure quaternions, we then establish a Shimura correspondence between Drinfeld-type forms and metaplectic forms on $\widetilde{\mathrm{SL}}_2$. Having this correspondence at hand leads us to an explicit Waldspurger-type formula in this setting.

Introduction

In 1987, Gross [6] presented a formula for the central critical values of certain Rankin-type L-functions associated to a weight 2 cusp form of prime level, which connects the L-values in question to the height of certain special points on the corresponding "definite" Shimura curve. Reformulating this formula via the Shimura correspondence (after Waldspurger [14]), the cusp form gives rise to a specific weight 3/2 form whose Fourier coefficients account for the above central critical values. There are now many arithmetic consequences of these formulas. The purpose of this paper is to study analogous phenomena in the function field setting, with "Drinfeld-type" automorphic forms playing the role of classical modular forms.

Let k be a global function field with finite constant field $\mathbb{F}_q$. Throughout this paper, we always assume that q is **odd**. Fix a place ∞ of k, regarded as the place at infinity. By a Drinfeld-type automorphic form F we mean that F is an automorphic form on $\mathrm{GL}_2(\mathbb{A})$ (where $\mathbb{A}$ is the adele ring of k) satisfying a "harmonicity" condition at ∞ (cf. Section 2). In other words, the ∞-component of the automorphic representation associated to F is isomorphic to the special representation $\mathrm{sp}(|\cdot|_\infty^{-1/2}, |\cdot|_\infty^{1/2})$. Drinfeld-type cusp forms, first introduced in Drinfeld's paper [3], are very useful tools in function field arithmetic (cf. [4], [12], [16] and [18] for further details).

Let A be the ring of functions in k regular outside ∞. The non-zero prime ideals of A correspond bijectively to the places of k distinct from ∞. Let F be a "normalized" Drinfeld-type newform of square-free level. Here being a newform means that F is a Hecke eigenform and is orthogonal to all the old forms (cf. Section 2.1 and 2.2); the level will be denoted by $\mathfrak{n}_F$, which is a square-free ideal of A. Then F always admits a central character denoted by ω_F.

Let ϱ_F be the degree two Galois representation associated to F under the Langlands correspondence. We are interested in the L-function associated to the Galois representation $\varrho_F \otimes \mathrm{Ind}_K^k(\eta)$ of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$, where K is an imaginary quadratic extension over k (i.e. ∞ does not split in K/k), and $\eta : \mathrm{Gal}(k^{\mathrm{sep}}/K) \to \mathbb{C}^\times$ is a continuous character. We take Langlands' functorial point of view, and look at this type of automorphic L-functions. Under

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class field theory, we identify η with a Hecke character on the idele class group of K. Via the theta correspondence, one can also view η as an automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ with a central character denoted by ω_η. Then it is known that the L-function in question coincides with the *Rankin-type L-function* $L(F \times \eta, s - 1/2)$ associated to the pair (F, η).

The main result of this paper is to show a function field analogue of Gross formula for the central critical value of $L(F \times \eta, s)$, under the following two conditions on η:

- C.1 the character η is unramified everywhere and trivial at ∞, i.e. $\eta(\mathrm{Frob}_{\infty_K}) = 1$ where Frob_{∞_K} is the Frobenius at the place ∞_K of K lying above ∞;
- C.2 $\omega_F \cdot \omega_\eta = \chi_K$, where χ_K is the quadratic Hecke character associated to K (cf. Section 1.2).

By the Rankin-Selberg method, we express $L(F \times \eta, s)$ as a *zeta integral* (cf. Theorem 3.10). From the meromorphic continuation and the functional equation of the Eisenstein series in question, the L-function $L(F \times \eta, s)$ extends to a meromorphic function on the complex s-plane which is holomorphic at $s = 0$ and satisfies (cf. Theorem 3.12)

$$L^*(F \times \eta, s) = (-1)^{\#\Sigma} \cdot L^*(F \times \eta, -s),$$

where:

- $L^*(F \times \eta, s)$ is a modified L-function (cf. Section 3.3),
- $\Sigma = \Sigma(F, \eta)$
$$:= \{\infty\} \cup \big\{\text{prime } \mathfrak{p} \text{ dividing } \mathfrak{n}_F \;\big|\; \mathfrak{p} \text{ is ramified in } K/k \text{ and } \lambda_\mathfrak{p}(F) \cdot \eta(\mathrm{Frob}_\mathfrak{P}) = 1\big\}$$
$$\cup \{\text{prime } \mathfrak{p} \text{ dividing } \mathfrak{n}_F \;\big|\; \mathfrak{p} \text{ is inert in } K/k\},$$
- $\lambda_\mathfrak{p}(F)$ is the Hecke eigenvalue of F corresponding to $\mathfrak{p}$,
- $\mathrm{Frob}_\mathfrak{P}$ is the Frobenius at the prime $\mathfrak{P}$ lying above $\mathfrak{p}$.

The value $(-1)^{\#\Sigma}$ is called the *root number* of $L(F \times \eta, s)$. When $\#\Sigma$ is odd, the central critical value $L(F \times \eta, 0)$ vanishes. Assuming that $\#\Sigma$ is even, we present our formula for the central critical value $L(F \times \eta, 0)$ as follows (cf. Theorem 4.16):

Theorem 0.1. *Let F be a normalized Drinfeld-type newform of square-free level $\mathfrak{n}_F$ with central character ω_F. Given an imaginary quadratic extension K over k, let η be a character on* $\mathrm{Gal}(k^{\mathrm{sep}}/K)$ *satisfying the conditions* C.1 *and* C.2 *with the root number $(-1)^{\#\Sigma(F,\eta)} = 1$. Let $\mathfrak{n}_{F,\eta}^- := \prod_{\mathfrak{p} \in \Sigma - \{\infty\}} \mathfrak{p}$ and X be the definite Shimura curve of type $(\mathfrak{n}_F/\mathfrak{n}_{F,\eta}^-, \mathfrak{n}_{F,\eta}^-)$. Then we have*

$$L(F \times \eta, 0) = \mathcal{P}(F, K) \cdot \frac{4 \cdot (F, F)_{\mathfrak{n}_F}}{f_K(\infty)^2 \cdot \# \operatorname{Pic}(A)} \cdot \frac{|\langle e_\eta, e_F \rangle|^2}{\langle e_F, e_F \rangle}.$$

Here

- $\mathcal{P}(F, K)$ *is a "period constant" defined in* Theorem 4.8*.*
- $f_K(\infty)$ *is the residue degree of* ∞ *in* K/k
- $\operatorname{Pic}(A)$ *is the ideal class group of* A*.*
- $e_F \in \operatorname{Pic}(X) \otimes_\mathbb{Z} \mathbb{C}$ *generates the one dimensional "F-eigenspace" via the Jacquet-Langlands correspondence (cf.* Theorem 4.14*).*
- $(F, F)_{\mathfrak{n}_F}$ *is the Petersson norm of F (cf.* Section 2.3*).*
- $e_\eta \in \operatorname{Pic}(X) \otimes_\mathbb{Z} \mathbb{C}$ *is a divisor class coming from a "Gross point" with trivial conductor on X and the character η. (cf.* Section 4.4*).*
- $\langle \cdot, \cdot \rangle : \operatorname{Pic}(X) \times \operatorname{Pic}(X) \to \mathbb{C}$ *is the Gross height pairing on* $\operatorname{Pic}(X)$ *(cf.* Section 4.3*).*

Although the choice of e_F is unique up to a non-zero scalar multiple, the formula in the above theorem is invariant under rescaling. The divisor class $e_\eta = e_\eta(x)$ actually depends on the chosen Gross point x on X; however, the Gross height $\frac{|\langle e_\eta(x), e_F\rangle|^2}{\langle e_F, e_F\rangle}$ is invariant as x varies.

When the base field k is rational, this formula was first proved by Papikian [10] with the assumption that the level $\mathfrak{n}_F$ is a prime and inert in K (based upon the calculations in Rück-Tipp [12]).

To derive the formula in Theorem 0.1, one crucial step is to express the central critical value of the zeta integral in question as a Petersson inner product of the given newform F and a linear combination of "quaternionic" theta series (cf. Proposition 4.6 and Theorem 4.8). Note that these theta series are also Drinfeld type forms. To accomplish this, our strategy is outlined as follows. Suppose the root number is positive. Let $\mathcal{D}$ be the quaternion algebra over k which is ramified precisely at the places in $\Sigma(F,\eta)$. We take $\gamma \in k^\times$ so that the quadratic space $(\mathcal{D}, \mathrm{Nr}_{\mathcal{D}/k})$ (where $\mathrm{Nr}_{\mathcal{D}/k}$ is the reduced norm on $\mathcal{D}$) decomposes into

$$(\mathcal{D}, \mathrm{Nr}_{\mathcal{D}/k}) = (K, \mathbf{N}_{K/k}) \oplus (K, \gamma\mathbf{N}_{K/k}).$$

Here $\mathbf{N}_{K/k}$ is the norm on K/k. Applying a Siegel-Weil formula over function fields (cf. Theorem 4.1), the central critical value of the Eisenstein series appearing in the zeta integral becomes a "theta integral" I with respect to the quadratic space $(K, \gamma\mathbf{N}_{K/k})$. Note that the "newform" Θ_K^η associated to η can be realized by quadratic theta series with respect to $(K, \mathbf{N}_{K/k})$. Combining I with Θ_K^η, we are able to connect the "kernel function" in our zeta integral directly with quaternionic theta series having level higher than $\mathfrak{n}_F$ (cf. Lemma 4.5). The final step is to "lower" the level of these quaternionic theta series to $\mathfrak{n}_F$, which amounts to "enlarge" the corresponding Eichler A-order (cf. Lemma 4.7).

Let $\mathcal{D}$ be the definite quaternion algebra over k ramified precisely at $v \in \Sigma$. One can also view e_F as an automorphic form on $\mathcal{D}^\times(\mathbb{A}) := (\mathcal{D} \otimes_k \mathbb{A})^\times$. Normalizing the Haar measure dg on $\mathcal{D}^\times(\mathbb{A})$ such that

$$\langle\!\langle e_F, e_F\rangle\!\rangle := \int_{\mathcal{D}^\times\mathbb{A}^\times\backslash\mathcal{D}^\times(\mathbb{A})} e_F(g)\overline{e_F(g)}dg = \frac{2}{\#\,\mathrm{Pic}(A)}\langle e_F, e_F\rangle,$$

the above formula can be rewritten in an integral form:

$$L(F\times\eta, 0) = \mathcal{P}(F,K)' \cdot \frac{2\cdot (F,F)_{\mathfrak{n}_F}}{\langle\!\langle e_F, e_F\rangle\!\rangle} \cdot \Big|\int_{K^\times\mathbb{A}^\times\backslash\mathbb{A}_K^\times} e_F(a)\eta(a)d^\times a\Big|^2,$$

where $\mathcal{P}(F,K)'$ is defined in *Remark* 4.17 (3). This is parallel to the classical formula in [15, Proposition 7]. In particular, suppose that another place $\infty' \mid \mathfrak{n}_F$ is non-split in K such that F, as a function on $\mathrm{GL}_2(\mathbb{A})$, is also of Drinfeld-type with respect to ∞' and $\eta(\mathrm{Frob}_{\mathfrak{P}'}) = 1$, where $\mathfrak{P}'$ is the prime above ∞'. Replacing ∞ by ∞' gives the same Σ and $\mathcal{D}$, and the level of F becomes $\mathfrak{n}_F\infty/\infty'$. From the Jacquet-Langlands correspondence, it is clear that the corresponding e_F, as an automorphic form on $\mathcal{D}^\times(\mathbb{A})$, remains the same. Thus, both sides of the equality in Theorem 0.1 are invariant when switching ∞ and ∞'.

One application of Theorem 0.1 is to derive a Waldspurger-type formula, which connects the central critical values in question with the Fourier coefficients of the corresponding metaplectic forms. Following the classical story, we first establish a Shimura-type correspondence **Sh** between Drinfeld-type forms and metaplectic forms via theta series from pure quaternions (cf. Theorem 5.5). The Fourier coefficients of these metaplectic forms can be interpreted by Gross heights (cf. Lemma 5.7). Supposing that for the given F and η, the central character

ω_F and the character $\eta = \mathbf{1}_K$ are both trivial. The Rankin-type L-function $L(F \times \mathbf{1}_K, s)$ can then be written as (cf. Lemma 3.2)

$$L(F \times \mathbf{1}_K, s) = L(F, s) \cdot L(F \otimes \chi_K, s),$$

where $L(F \otimes \chi_K, s)$ is the twisted L-function of F by the quadratic Hecke character χ_K. Applying Theorem 0.1, we then arrive at:

Theorem 0.2. *Let F be a normalized Drinfeld-type newform of square-free level $\mathfrak{n}_F$ with trivial central character. Suppose the cardinality of $\Sigma' := \{\text{prime } \mathfrak{p} \text{ dividing } \mathfrak{n}_F \mid \lambda_{\mathfrak{p}}(F) = 1\}$ is odd. For each imaginary quadratic field K satisfying $\Sigma(F, \mathbf{1}_K) = \Sigma' \cup \{\infty\}$, we have*

$$L(F,0)L(F \otimes \chi_K, 0) = \mathcal{P}(F,K) \cdot \frac{4 \cdot (F,F)_{\mathfrak{n}_F}}{f_K(\infty)^2 \cdot \# \operatorname{Pic}(A)} \cdot \left(\prod_{\substack{\mathfrak{p} \mid \mathfrak{n}_F, \\ \mathfrak{p} \text{ is unramified in } K/k}} 4 \right)^{-1} m(F,K)^2,$$

where $m(F,K)$ is the "K-th" Fourier coefficient of the corresponding metaplectic form $\mathbf{Sh}(F)$ (cf. Theorem 5.8*).*

The contents of this paper are as follows. Section 1 contains the basic notations used throughout this paper. In Section 2, we discuss the needed properties of Drinfeld-type automorphic forms. The Rankin-type L-function $L(F \times \eta, s)$ is introduced in Section 3. We choose a particular "flat section" in Section 3.1, and use the associated Eisenstein series to express $L(F \times \eta, s)$ as a zeta integral in Section 3.2. The explicit functional equation is verified in Section 3.3. Supposing that the root number is positive, our Gross-type formula is derived in Section 4. Making use of the Siegel-Weil formula stated in Section 4.1, the central critical value $L(F \times \eta, 0)$ can be written as the Petersson inner product of F and a specific quaternionic theta series in Section 4.2. In Section 4.3 and 4.4, we briefly recall definite Shimura curves, the Gross height pairing and Gross points. In Section 4.5, we make an appeal to the key Hecke module homomorphism from the Picard group of definite Shimura curves to the space of Drinfeld type forms of the corresponding level. Also given there is a version of Jacquet-Langlands correspondence we rely on. In Section 5.2, we construct metaplectic theta series from pure quaternions. The Shimura-type correspondence between Drinfeld-type forms and metaplectic forms is established in Section 5.3. Finally, we prove our Waldspurger-type formula in Section 5.4. For completeness, we record the construction of quadratic theta series in the function field setting, with all the needed properties stated in the appendix.

1. Preliminaries

1.1. **Basic setting.** We fix the following notations:

- (*Global*)

k : a global function field with finite constant field $\mathbb{F}_q$ where q is odd, and $\mathbb{F}_q$ is algebraically closed in k.

- (*Local*)

k_v : the completion of k at a place v of k.
ord_v : the valuation map on k_v.
O_v : the valuation ring in k_v, i.e. $O_v = \{a \in k_v \mid \operatorname{ord}_v(a) \geq 0\}$.
π_v : a chosen uniformizer in O_v.
$\mathbb{F}_v$: the residue field of k_v, i.e. $\mathbb{F}_v = O_v/\pi_v O_v$.
$\deg v$: the degree of v, i.e. $\deg v = [\mathbb{F}_v : \mathbb{F}_q]$.
q_v : the cardinality of $\mathbb{F}_v$.

$|\cdot|_v$: the absolute value on k_v, normalized by $|a|_v := q_v^{-\operatorname{ord}_v(a)}$ for every $a \in k_v$.

- (*Adelic*)

$\mathbb{A}$: the adele ring of k, i.e. $\mathbb{A} = \prod'_v k_v$, the restricted direct product with respect to the O_v.
$O_{\mathbb{A}}$: the maximal compact subring of $\mathbb{A}$, i.e. $O_{\mathbb{A}} = \prod_v O_v$.
$\mathbb{A}^\times$: the idele group of k, i.e. $\mathbb{A}^\times = \prod'_v k_v^\times$, the restricted direct product with respect to the $O_v^\times$.
$|\cdot|_{\mathbb{A}}$: the idele norm on $\mathbb{A}^\times$, i.e. $|a|_{\mathbb{A}} := \prod_v |a_v|_v$ for every $a = (a_v)_v \in \mathbb{A}^\times$.

- (*Divisors*)

$\operatorname{Div}(k)$: the divisor group of k. We adopt the multiplicative notation so that every element $\mathfrak{m}$ in $\operatorname{Div}(k)$ is written as

$$\mathfrak{m} = \prod_v v^{\operatorname{ord}_v(\mathfrak{m})}.$$

$\operatorname{Div}_{\geq 0}(k)$: the monoid of effective divisors of k.
$\deg \mathfrak{m}$: the degree of $\mathfrak{m} \in \operatorname{Div}(k)$, i.e. $\deg \mathfrak{m} = \sum_v \deg v \cdot \operatorname{ord}_v(\mathfrak{m})$.
$\|\mathfrak{m}\|$: the norm of $\mathfrak{m} \in \operatorname{Div}(k)$, i.e. $\|\mathfrak{m}\| = q^{\deg \mathfrak{m}}$.
div : the group epimorphism from $\mathbb{A}^\times$ onto $\operatorname{Div}(k)$ defined by

$$a = (a_v)_v \longmapsto \operatorname{div}(a) := \prod_v v^{\operatorname{ord}_v(a_v)}.$$

$\pi_{\mathfrak{m}}$: the idele $(\pi_v^{\operatorname{ord}_v(\mathfrak{m})})_v \in \mathbb{A}^\times$ for $\mathfrak{m} \in \operatorname{Div}(k)$.

Given divisors $\mathfrak{m}, \mathfrak{n} \in \operatorname{Div}(k)$, we set

$$(\mathfrak{m}, \mathfrak{n}) := \prod_v v^{\min\{\operatorname{ord}_v(\mathfrak{m}), \operatorname{ord}_v(\mathfrak{n})\}} \quad \text{and} \quad [\mathfrak{m}, \mathfrak{n}] := \prod_v v^{\max\{\operatorname{ord}_v(\mathfrak{m}), \operatorname{ord}_v(\mathfrak{n})\}}.$$

- (*v-component*) Let v be a place of k. Let $\mathbb{A}^v := \prod'_{v' \neq v} k_{v'}$. Since $\mathbb{A} = \mathbb{A}^v \times k_v$, one has the following natural embeddings

$$\begin{array}{ccc} k_v & \hookrightarrow & \mathbb{A} \\ a_v & \mapsto & (0, a_v) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbb{A}^v & \hookrightarrow & \mathbb{A} \\ a^v = (a_{v'})_{v' \neq v} & \mapsto & (a^v, 0). \end{array}$$

Let G be a general linear group GL_n for $n \geq 1$ or an orthogonal similitude group $\mathrm{GO}(V)$ of a quadratic space V over k. Expressing $G(\mathbb{A})$ as $G(\mathbb{A}^v) \times G(k_v)$, one also gets two embeddings:

$$\begin{array}{ccc} G(k_v) & \hookrightarrow & G(\mathbb{A}) \\ g_v & \mapsto & (1, g_v), \end{array} \quad \text{and} \quad \begin{array}{ccc} G(\mathbb{A}^v) & \hookrightarrow & G(\mathbb{A}) \\ g^v & \mapsto & (g^v, 1). \end{array}$$

Every $g \in G(\mathbb{A})$ can be written as $g = (g^v, g_v)$ where $g^v \in G(\mathbb{A}^v)$ and $g_v \in G(k_v)$.

- (*Ring of integers*) Fix a place ∞ of k, referred to as the place at infinity; and others are referred to as finite places of k. Let A be the ring of integers of k with respect to ∞, i.e.

$$A = \{a \in k \mid \operatorname{ord}_v(a) \geq 0, \ \forall v \neq \infty\}.$$

Identifying finite places of k with non-zero prime ideals of A, every fractional (resp. non-zero integral) ideal of A corresponds bijectively to a (resp. an effective) divisor of k with support away from ∞. Thus for every fractional ideal $\mathfrak{m}$ of A, we still denote by $\mathfrak{m}$ the corresponding divisor of k (and vice versa) by abuse of notations.

Set $O_{\mathbb{A}^\infty} := \prod_{v\neq\infty} O_v$, the maximal compact subring of $\mathbb{A}^\infty$, and $O_{\mathbb{A}^\infty}^\times := \prod_{v\neq\infty} O_v^\times$, the maximal compact subgroup of $\mathbb{A}^{\infty,\times}$. The map div induces the following natural isomorphisms

$$\mathbb{A}^{\infty,\times}/O_{\mathbb{A}^\infty}^\times \cong \mathcal{I}(A) \quad \text{and} \quad k^\times\backslash\mathbb{A}^{\infty,\times}/O_{\mathbb{A}^\infty}^\times \cong \operatorname{Pic}(A),$$

where $\mathcal{I}(A)$ is the group of fractional ideals of A, and $\operatorname{Pic}(A)$ is the ideal class group of A. Here $k^\times$ embeds diagonally into $\mathbb{A}^{\infty,\times}$. Throughout this paper, every function on $\operatorname{Pic}(A)$ is also identified as a function on $\mathcal{I}(A)$ via the surjection $\mathcal{I}(A) \twoheadrightarrow \operatorname{Pic}(A)$. For convenience, we denote by $\mathfrak{a} \lhd A$ if $\mathfrak{a}$ is a non-zero integral ideal of A.

- (*Additive character*) Fix a non-trivial continuous additive character $\psi = \mathbb{A} \to \mathbb{C}^\times$ which is trivial on k. Here k embeds diagonally into $\mathbb{A}$. For each place v of k, we denote by $\psi_v : k_v \to \mathbb{C}^\times$ the restriction of ψ on k_v, and let δ_v be the "conductor of ψ at v," i.e. the maximal integer r such that $\pi_v^{-r}O_v$ is contained in the kernel of ψ_v. It is known that $\sum_v \delta_v \deg v = 2g_k - 2$, where g_k is the genus of k. By [21, Theorem 13, Section 12, chapter XIII], we may assume that the chosen additive character ψ has even conductor at every place v of k. Put $\delta = (\pi_v^{\delta_v})_v \in \mathbb{A}^\times$, and call δ a *differential idele belonging to* ψ. It depends upon the choice of the uniformizers π_v.

1.2. **Imaginary quadratic function fields.** Let K be a quadratic field extension over k. We say that K is imaginary (with respect to the chosen ∞) if ∞ does not split in K/k. Let χ_K be the quadratic character associated to K, i.e. χ_K is the non-trivial character on $\mathbb{A}^\times$ with kernel $k^\times \cdot \mathbf{N}_{K/k}(\mathbb{A}_K^\times)$. Here $\mathbb{A}_K$ (resp. $\mathbb{A}_K^\times$) is the adele ring (resp. idele group) of K, and $\mathbf{N}_{K/k}$ is the norm of K/k. For each place v of k, put $K_v := K \otimes_v k_v$, and let $\chi_{K,v} : k_v^\times \to \{\pm 1\}$ be the character defined by:

$$\chi_{K,v}(a_v) = \begin{cases} 1, & \text{if } a_v \in \mathbf{N}_{K/k}(K_v^\times), \\ -1, & \text{otherwise.} \end{cases}$$

Then for every $a = (a_v)_v \in \mathbb{A}^\times$, one has $\chi_K(a) = \prod_v \chi_{K,v}(a_v)$.

Let $\mathfrak{d}_K \in \operatorname{Div}_{\geq 0}(k)$ be the discriminant divisor of K over k. Since K/k is quadratic and q is odd, one has

$$\mathfrak{d}_K = \prod_{\substack{\text{place } v \text{ of } k: \\ v \text{ is ramified in } K}} v.$$

Let $L(s,\chi_K)$ be the Hecke L-function associated to χ_K, i.e. $L(s,\chi_K) = \prod_v L_v(s,\chi_{K,v})$ where

$$L_v(s,\chi_{K,v}) := \begin{cases} (1-\chi_{K,v}(\pi_v)q_v^{-s})^{-1}, & \text{if } v \nmid \mathfrak{d}_K; \\ 1, & \text{otherwise.} \end{cases}$$

It is known that $L(s,\chi_K)$ has meromorphic continuation and functional equation with the symmetry between s and $1-s$. Let O_K be the integral closure of A in K. The ideal class group of O_K is denoted by $\operatorname{Pic}(O_K)$. Set $\mathbb{F}_K$ to be the constant field of K and $f_K(\infty)$ to be the residue degree of ∞ in K/k. Then the class number formula (cf. [11, Theorem 5.9]) assures that

$$L(1,\chi_K) = \frac{\#\left(\mathbb{F}_q^\times\right)}{\#\left(\mathbb{F}_K^\times\right)} \cdot q^{1-g_k} \cdot \|\mathfrak{d}_K\|^{-\frac{1}{2}} \cdot \frac{\#\operatorname{Pic}(O_K)}{f_K(\infty)\cdot\#\operatorname{Pic}(A)}.$$

1.3. **Definite quaternion algebras over function fields.** A definite quaternion algebra $\mathcal{D}$ is a central simple algebra over k with $\dim_k \mathcal{D} = 4$ and ramified at ∞. Let $\mathfrak{n}^- \lhd A$ be the product of primes of A where $\mathcal{D}$ is ramified, called the (finite) discriminant of $\mathcal{D}/k$. Let $\mathfrak{n}^+ \lhd A$ be coprime to $\mathfrak{n}^-$. We call a ring R an *Eichler A-order of type* $(\mathfrak{n}^+, \mathfrak{n}^-)$ if R is an A-order of $\mathcal{D}$ such that $R_\mathfrak{p} := R \otimes_A O_\mathfrak{p}$ is a maximal $O_\mathfrak{p}$-order in $\mathcal{D}_\mathfrak{p} := \mathcal{D} \otimes_k k_\mathfrak{p}$ for each prime $\mathfrak{p} \lhd A$ with $\mathfrak{p} \nmid \mathfrak{n}^+$; and when $\mathfrak{p} \mid \mathfrak{n}^+$, there exists an isomorphism $i : \mathcal{D}_\mathfrak{p} \cong \mathrm{Mat}_2(k_\mathfrak{p})$ such that

$$i(R_\mathfrak{p}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_2(O_\mathfrak{p}) \;\middle|\; c \in \pi_\mathfrak{p}^{\mathrm{ord}_\mathfrak{p}(\mathfrak{n}^+)} O_\mathfrak{p} \right\}.$$

A *locally-principal* (*fractional*) *right ideal I of R* is an A-lattice in $\mathcal{D}$ such that $I \cdot R = I$, and for each prime $\mathfrak{p} \lhd A$, there exists $\alpha_\mathfrak{p}$ in $\mathcal{D}_\mathfrak{p}^\times$ such that

$$I_\mathfrak{p} (:= I \otimes_A O_\mathfrak{p}) = \alpha_\mathfrak{p} R_\mathfrak{p}.$$

Let $R_I := \{b \in \mathcal{D} : bI \subset I\}$ be the left order of I, which is also an Eichler A-order of type $(\mathfrak{n}^+, \mathfrak{n}^-)$. Two locally-principal right ideals I and I' are called equivalent if there exists an element b in $\mathcal{D}^\times$ such that $I = b \cdot I'$. The left orders R_I only depend on the ideal classes of R up to isomorphism. Let $\mathrm{Cl}(R)$ be the set of locally-principal right ideal classes of R. For each class $\mathcal{I} \in \mathrm{Cl}(R)$, we set $\mathrm{w}_\mathcal{I}^R := \frac{\#(R_I^\times)}{q-1}$ for any representative $I \in \mathcal{I}$.

Let $\mathcal{D}_{\mathbb{A}^\infty} := \mathcal{D} \otimes_k \mathbb{A}^\infty$ and $\widehat{R} := R \otimes_A O_{\mathbb{A}^\infty}$. For each locally-principal right ideal I of R, there exists an element $b_I \in \mathcal{D}_{\mathbb{A}^\infty}^\times$ such that $I = \mathcal{D} \cap b_I \widehat{R}$. This induces a bijection between the set $\mathrm{Cl}(R)$ with the finite double coset space $\mathcal{D}^\times \backslash \mathcal{D}_{\mathbb{A}^\infty}^\times / \widehat{R}^\times$.

Given an element b in $\mathcal{D}$, its reduced trace and the reduced norm are denoted by $\mathrm{Tr}_{\mathcal{D}/k}(b)$ and $\mathrm{Nr}_{\mathcal{D}/k}(b)$, respectively. We have an involution $b \mapsto \bar{b} := \mathrm{Tr}_{\mathcal{D}/k}(b) - b$ on $\mathcal{D}$, which induces an order 2 permutation τ on $\mathrm{Cl}(R)$ defined by $\tau([I]) := [\overline{I}^{-1}]$ for each class $[I] \in \mathrm{Cl}(R)$. Here $\overline{I}^{-1} := \{\bar{b} \mid b \in \mathcal{D} \text{ such that } bI \subset R\}$.

2. Drinfeld-type automorphic forms

Given $\mathfrak{m} \in \mathrm{Div}_{\geq 0}(k)$, let $\mathcal{K}_0(\mathfrak{m}) := \prod_v \mathcal{K}_{v^{\mathrm{ord}_v(\mathfrak{m})}}$ where, for $\ell \geq 0$,

$$\mathcal{K}_{v^\ell} := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(O_v) \;\middle|\; c \in \pi_v^\ell O_v \right\}.$$

For $\mathfrak{n} \lhd A$, let

$$\mathbb{Y}_0(\mathfrak{n}) := \mathrm{GL}_2(k) \backslash \mathrm{GL}_2(\mathbb{A}) / k_\infty^\times \mathcal{K}_0(\mathfrak{n}\infty).$$

Here the elements in $k_\infty^\times$ are identified with the scalar matrices in $\mathrm{GL}_2(k_\infty)$, and we embed $\mathrm{GL}_2(k_\infty)$ into $\mathrm{GL}_2(\mathbb{A})$ as in Section 1.1.

Definition 2.1. By a *Drinfeld-type automorphic form F of level* $\mathfrak{n} \lhd A$, we mean a $\mathbb{C}$-valued function F on the double coset space $\mathbb{Y}_0(\mathfrak{n})$ satisfying the following *harmonicity property*: viewing F as a function on $\mathrm{GL}_2(\mathbb{A})$, we have

$$F\left(g \begin{pmatrix} 0 & 1 \\ \pi_\infty & 0 \end{pmatrix}\right) = -F(g) \quad \text{and} \quad \sum_{\kappa \in \mathrm{GL}_2(O_\infty)/\mathcal{K}_\infty} F(g\kappa) = 0, \quad \forall g \in \mathrm{GL}_2(\mathbb{A}).$$

Drinfeld-type automorphic forms can be viewed as the function field analogue of weight 2 modular forms (cf. [4] and [16]). In the following, we recall the analytic properties of these forms, and refer the reader to [2, Chapter III Section 3] for further details.

2.1. **Fourier expansions.** Let F be a Drinfeld-type automorphic form of level $\mathfrak{n}$. For each divisor $\mathfrak{m} \in \operatorname{Div}(k)$, the $\mathfrak{m}$-th Fourier coefficient $F^*(\mathfrak{m})$ of F is given by

$$F^*(\mathfrak{m}) := \int_{k\backslash\mathbb{A}} F\left(\begin{pmatrix} \delta^{-1}\pi_{\mathfrak{m}} & u \\ 0 & 1 \end{pmatrix}\right)\psi(-u)du.$$

Here $\pi_{\mathfrak{m}} \in \mathbb{A}^\times$ is taken in Section 1.1, and δ is the chosen differential idele of k; the Haar measure du is normalized such that the volume of $k\backslash\mathbb{A}$ is one. Let

$$F_0^*(\mathfrak{m}) := \int_{k\backslash\mathbb{A}} F\left(\begin{pmatrix} \delta^{-1}\pi_{\mathfrak{m}} & u \\ 0 & 1 \end{pmatrix}\right)du.$$

Then it is clear that $F_0^*(\operatorname{div}(a)\mathfrak{m}) = F_0^*(\mathfrak{m})$ for every $a \in k^\times$. For any $(x,y) \in \mathbb{A}^\times \times \mathbb{A}$, one has the following Fourier expansion

$$F\left(\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}\right) = F_0^*(\operatorname{div}(\delta x)) + \sum_{\lambda \in k^\times} F^*(\operatorname{div}(\delta\lambda x))\psi(\lambda y).$$

It is known that $F^*(\mathfrak{m}) = 0$ unless $\mathfrak{m}$ is effective (cf. [20, Chapter III Proposition 1]). Moreover, put

$$\mathfrak{m}_\infty := \infty^{\operatorname{ord}_\infty(\mathfrak{m})}, \quad \text{and} \quad \mathfrak{m}^\infty := \frac{\mathfrak{m}}{\mathfrak{m}_\infty}.$$

Then the harmonicity of F gives

$$F^*(\mathfrak{m}) = \|\mathfrak{m}_\infty\|^{-1}F^*(\mathfrak{m}^\infty), \quad \text{and} \quad F_0^*(\mathfrak{m}) = \|\mathfrak{m}_\infty\|^{-1}F_0^*(\mathfrak{m}^\infty).$$

Thus taking representatives $\mathfrak{a}_1, ..., \mathfrak{a}_h$ for the ideal classes of A, the values of F on $\begin{pmatrix} \mathbb{A}^\times & \mathbb{A} \\ 0 & 1 \end{pmatrix}$ are determined by $F_0^*(\mathfrak{a}_i)$ and $F^*(\mathfrak{m})$ for $i = 1, ..., h$ and $\mathfrak{m} \lhd A$.

Given a fractional ideal $\mathfrak{a}$ of A, set

$$\big(\rho(\mathfrak{a})F\big)(g) := F\left(\begin{pmatrix} \pi_{\mathfrak{a}} & 0 \\ 0 & \pi_{\mathfrak{a}} \end{pmatrix} g\right), \quad \forall g \in \mathrm{GL}_2(\mathbb{A}).$$

This action only depends on the ideal class of A represented by $\mathfrak{a}$. Let $\mathrm{B} := \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \mathrm{GL}_2\right\}$, the standard Borel subgroup of GL_2. Then the values of F on $\mathrm{B}(\mathbb{A})$ are completely determined by the Fourier coefficients

$$\big(\rho(\mathfrak{a}_i)F\big)^*(\mathfrak{m}),\ \mathfrak{m} \lhd A, \quad \text{and} \quad \big(\rho(\mathfrak{a}_i)F\big)_0^*(\mathfrak{a}_j).$$

Here $\mathfrak{a}_1, ..., \mathfrak{a}_h$ are representatives for the ideal classes of A. From the surjectivity of the canonical map $\mathrm{B}(\mathbb{A}) \twoheadrightarrow \mathbb{Y}_0(\mathfrak{n})$, we have that F is also uniquely determined by these Fourier coefficients.

A Drinfeld-type automorphic form F of level $\mathfrak{n}$ is said to have a central character ω_F if $\omega_F : \mathbb{A}^\times \to \mathbb{C}^\times$ is a character so that

$$F\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} g\right) = \omega_F(a)F(g), \text{ for all } a \in \mathbb{A}^\times.$$

We may identify ω_F with a character on $k^\times\backslash\mathbb{A}^\times/k_\infty^\times O_{\mathbb{A}^\infty}^\times \cong \operatorname{Pic}(A)$ (and also a character on $\mathcal{I}(A)$). Then $\rho(\mathfrak{a})F = \omega_F(\mathfrak{a})F$ for every $\mathfrak{a} \in \mathcal{I}(A)$.

2.2. **Hecke operators.** Let $\mathcal{M}_0(\mathfrak{n})$ be the space of Drinfeld-type automorphic forms of level $\mathfrak{n}$. For each place v of k, the Hecke operator T_v on $\mathcal{M}_0(\mathfrak{n})$ is defined by the following: for $F \in \mathcal{M}_0(\mathfrak{n})$ and $g \in \mathrm{GL}_2(\mathbb{A})$

$$(T_v F)(g) := \sum_{u \in \mathbb{F}_v} F\left(g \begin{pmatrix} \pi_v & u \\ 0 & 1 \end{pmatrix}\right) + \mu_{\mathfrak{n}\infty}(v) \cdot F\left(g \begin{pmatrix} 1 & 0 \\ 0 & \pi_v \end{pmatrix}\right).$$

Here $\mu_{\mathfrak{n}\infty}(v) = 1$ if $v \nmid \mathfrak{n}\infty$ and 0 otherwise.

The harmonicity of F implies that $T_\infty F = F$ for all $F \in \mathcal{M}_0(\mathfrak{n})$. Since T_v and $T_{v'}$ commute with each other for any places v and v', we define the Hecke operator $T_\mathfrak{m}$ recursively for all $\mathfrak{m} \lhd A$ by the following:

$$\begin{cases} T_{\mathfrak{m}\mathfrak{m}'} := T_\mathfrak{m} \cdot T_{\mathfrak{m}'}, & \text{for } \mathfrak{m} \text{ and } \mathfrak{m}' \text{ relatively prime;} \\ T_{\mathfrak{p}^{l+2}} := T_\mathfrak{p} T_{\mathfrak{p}^{l+1}} - \mu_{\mathfrak{n}\infty}(\mathfrak{p}) \|\mathfrak{p}\| \cdot \rho(\mathfrak{p}) T_{\mathfrak{p}^l}, & \text{for any prime } \mathfrak{p} \lhd A \text{ and } l \in \mathbb{Z}_{\geq 0}. \end{cases}$$

Suppose that F is a *Hecke eigenform*, i.e. for each $\mathfrak{m} \lhd A$, there exists $\lambda_\mathfrak{m}(F) \in \mathbb{C}$ such that $T_\mathfrak{m} F = \lambda_\mathfrak{m}(F) \cdot F$. Then F always has central character, and the eigenvalues $\lambda_\mathfrak{m}(F)$ can be read off by the Fourier coefficients of F:

$$\|\mathfrak{m}\| \cdot F^*(\mathfrak{m}) = \lambda_\mathfrak{m}(F) \cdot F^*(1).$$

We call F *normalized* if $F^*(1) = 1$.

2.3. **Petersson inner product.** Given $\mathfrak{n} \lhd A$, a Drinfeld-type automorphic form F of level $\mathfrak{n}$ is called a *cusp form* if

$$\int_{k \backslash \mathbb{A}} F\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g\right) du = 0 \quad \text{for every } g \in \mathrm{GL}_2(\mathbb{A}).$$

Given two Drinfeld-type automorphic forms F_1 and F_2 of level $\mathfrak{n}$, suppose one of them is a cusp form. The *Petersson inner product of* F_1 and F_2 is given by:

$$(F_1, F_2)_\mathfrak{n} := \int_{k_\infty^\times \mathrm{GL}_2(k) \backslash \mathrm{GL}_2(\mathbb{A})} F_1(g) \overline{F_2(g)} dg = \sum_{[g] \in \mathbb{Y}_0(\mathfrak{n})} F_1(g) \overline{F_2(g)} \mu([g]),$$

where for each double coset $[g] \in \mathbb{Y}_0(\mathfrak{n})$ represented by $g \in \mathrm{GL}_2(\mathbb{A})$, the measure $\mu([g])$ is normalized to be

$$\mu([g]) := \frac{q-1}{2 \cdot \#(\mathrm{Pic}(A))} \cdot \frac{1}{\#(\mathrm{GL}_2(k) \cap g \mathcal{K}_0(\mathfrak{n}\infty) g^{-1})}.$$

Suppose F_i has the central character ω_i. We remark that

$$(F_1, F_2)_\mathfrak{n} = \begin{cases} 0, & \text{if } \omega_1 \neq \omega_2; \\ \frac{1}{2} \cdot \displaystyle\int_{\mathbb{A}^\times \mathrm{GL}_2(k) \backslash \mathrm{GL}_2(\mathbb{A})} F_1(g) \overline{F_2(g)} d'g, & \text{if } \omega_1 = \omega_2. \end{cases}$$

Here the measure $d'g$ is induced by the Haar measure on $\mathrm{GL}_2(\mathbb{A})$ normalized so that $\mathcal{K}_0(\mathfrak{n}\infty)$ has volume 1.

A Drinfeld-type cusp form F of level $\mathfrak{n}$ is called an *old form* if F is a $\mathbb{C}$-linear combination of

$$F'\left(g \begin{pmatrix} 1 & 0 \\ 0 & \pi_{\mathfrak{n}''} \end{pmatrix}\right), \quad \text{for } g \in \mathrm{GL}_2(\mathbb{A}),$$

where F' is a Drinfeld-type cusp form of level $\mathfrak{n}'$ with $\mathfrak{n}'\mathfrak{n}'' \mid \mathfrak{n}$ and $\mathfrak{n}' \neq \mathfrak{n}$. Let F be a Drinfeld-type cusp form of level $\mathfrak{n}$ which is also a Hecke eigenform. In addition, if F is also orthogonal (with respect to Petersson inner product) to all the old forms of level $\mathfrak{n}$, then we call F a **newform** of level $\mathfrak{n}$.

2.4. ***L*-function associated to Drinfeld-type newforms.** Given a newform F of level $\mathfrak{n}$ with central character denoted by ω_F, it is known that:

(1) For prime $\mathfrak{p} \nmid \mathfrak{n}$, the Hecke eigenvalue $\lambda_\mathfrak{p}(F)$ satisfies the so-called Ramanujan bound: $|\lambda_\mathfrak{p}(F)| \leq 2 \cdot \|\mathfrak{p}\|^{\frac{1}{2}}$ (cf. [3]).
(2) For prime $\mathfrak{p} \mid \mathfrak{n}$, one has $\lambda_\mathfrak{p}(F)^2 = \omega_F(\mathfrak{p})$ if $\mathrm{ord}_\mathfrak{p}(\mathfrak{n}) = 1$ and 0 otherwise.

The L-function associated to a normalized newform F of level $\mathfrak{n}$ with central character ω_F is defined by

$$L(F,s) := \sum_{\mathfrak{m} \lhd A} \frac{F^*(\mathfrak{m})}{\|\mathfrak{m}\|^s}$$

which is absolutely convergent on $\mathrm{Re}(s) > 1/2$. Since F is a Hecke eigenform, we can write $L(F,s)$ as the following Euler product:

$$L(F,s) = \prod_{\text{places } v \text{ of } k} L_v(F,s),$$

where $L_\infty(F,s) := (1 - q_\infty^{-(s+1)})^{-1}$, and for every prime $\mathfrak{p} \lhd A$,

$$L_\mathfrak{p}(F,s) := (1 - \lambda_\mathfrak{p}(F)\|\mathfrak{p}\|^{-(s+1)} + \mu_\mathfrak{n}(\mathfrak{p}) \cdot \omega_F(\mathfrak{p}) \cdot \|\mathfrak{p}\|^{1-2(s+1)})^{-1}.$$

Moreover, it has analytic continuation to the whole s-plane, (in fact, a polynomial in q^{-s}) and satisfies the following functional equation (cf. [20, Theorem 2 in Chapter 7]):

$$L(F,s) = w(F) \cdot q^{(4-4g_k)s} \|\mathfrak{n}\infty\|^{-s} L(F,-s),$$

where $w(F)$, called the *root number associated to* $L(F,s)$, satisfies that $w(F)^2 = \omega_F(\mathfrak{n})$. In particular, when $\mathfrak{n}$ is square-free, one has $w(F) = -\prod_{\mathfrak{p}\mid\mathfrak{n}}\big(-\lambda_\mathfrak{p}(F)\big)$.

2.5. **Whittaker functions.** For each Drinfeld-type cusp form F of level $\mathfrak{n}$, the Whittaker function W_F associated to F is the following function on $\mathrm{GL}_2(\mathbb{A})$:

$$W_F(g) := \int_{k\backslash\mathbb{A}} F\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g\right) \psi(-u)du, \quad \forall g \in \mathrm{GL}_2(\mathbb{A}).$$

One sees that

$$W_F\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) = \psi(x) \cdot W_F(g) \quad \forall x \in \mathbb{A}, \quad \text{and} \quad F(g) = \sum_{\alpha \in k^\times} W_F\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right).$$

Suppose F is a normalized nerform. Let $W_{F,v} := W_F|_{\mathrm{GL}_2(k_v)}$ (here $\mathrm{GL}_2(k_v)$ is embedded into $\mathrm{GL}_2(\mathbb{A})$ as in Section 1.1). Then

$$W_F(g) = \prod_v W_{F,v}(g_v), \quad \forall g = (g_v)_v \in \mathrm{GL}_2(\mathbb{A}).$$

Lemma 2.2. (cf. [1] Exercise 4.6.2) *Let F be a normalized newform of square-free level $\mathfrak{n}$ with central character ω_F. (1) For $v \nmid \mathfrak{n}\infty$, one has*

$$W_{F,v}\begin{pmatrix} \pi_v^{r-\delta_v} & 0 \\ 0 & 1 \end{pmatrix} = \begin{cases} q_v^{-r} \cdot \dfrac{\alpha_v^{(1)}(F)^{r+1} - \alpha_v^{(2)}(F)^{r+1}}{\alpha_v^{(1)}(F) - \alpha_v^{(2)}(F)}, & \text{for } r \geq 0; \\ 0, & \text{for } r < 0; \end{cases}$$

Here $\alpha_v^{(1)}(F)$ and $\alpha_v^{(2)}(F)$ are the two roots of $X^2 - \lambda_v(F)X + \omega_F(v)q_v$.
(2) *For $v \mid \mathfrak{n}\infty$, one has*

$$W_{F,v}\begin{pmatrix} \pi_v^{r-\delta_v} & 0 \\ 0 & 1 \end{pmatrix} = \begin{cases} \lambda_v(F)^r \cdot q_v^{-r}, & \text{for } r \geq 0; \\ 0, & \text{for } r < 0; \end{cases}$$

$$W_{F,v}\left(\begin{pmatrix} \pi_v^{r-\delta_v} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = \begin{cases} -\lambda_v(F)^r \cdot q_v^{-r-1}, & \text{for } r \geq -1; \\ 0, & \text{for } r < -1. \end{cases}$$

3. Rankin-type L-functions

Let K be an imaginary quadratic extension over k, and choose a continuous character $\eta : \mathrm{Gal}(k^{\mathrm{sep}}/K) \to \mathbb{C}^\times$. Via class field theory the character η can be identified with a Hecke character on the idele class group $K^\times\backslash\mathbb{A}_K^\times$ (and also a character on $\mathbb{A}_K^\times$). We embed $\mathbb{A}^\times$ into $\mathbb{A}_K^\times$ by

$$\mathbb{A}^\times = 1 \otimes \mathbb{A}^\times \subset (K \otimes_k \mathbb{A})^\times = \mathbb{A}_K^\times,$$

and put $\omega_\eta := \eta\big|_{\mathbb{A}^\times} \cdot \chi_K$, where χ_K is the quadratic character associated to K (introduced in Section 1.2). Via the theta correspondence, the character η corresponds to an automorphic representation on $\mathrm{GL}_2(\mathbb{A})$ with central character equal to ω_η.

Let F be a normalized Drinfeld-type newform of square-free level $\mathfrak{n}_F$ with central character ω_F. Throughout of this section, we assume that:

C.1 η is unramified everywhere and $\eta(K_\infty^\times) = 1$, where $K_\infty^\times \hookrightarrow \mathbb{A}_K^\times$ is the ∞-component;
C.2 $\omega_F \cdot \omega_\eta = \chi_K$.

Let O_K be the integral closure of A in K. By the condition C.1, we may consider η as a character on $\mathrm{Pic}(O_K)$, and also as a character on the ideal group $\mathcal{I}(O_K)$ of O_K. The Hecke L-function $L(\eta, s)$ associated to η can then be written as the following:

$$L(\eta, s) = (1 - q_\infty^{-f_K(\infty)s})^{-1} \sum_{\mathfrak{M} \lhd O_K} \frac{\eta(\mathfrak{M})}{\|\mathfrak{M}\|^s} = (1 - q_\infty^{-f_K(\infty)s})^{-1} \sum_{\mathfrak{m} \lhd A} \frac{R_\eta(\mathfrak{m})}{\|\mathfrak{m}\|^s}.$$

Here $f_K(\infty)$ is the residue degree of ∞ in K/k, and

$$R_\eta(\mathfrak{m}) := \sum_{\mathcal{A} \in \mathrm{Pic}(O_K)} \eta(\mathcal{A}) \cdot R_{\mathcal{A}}(\mathfrak{m}) \quad \text{with} \quad R_{\mathcal{A}}(\mathfrak{m}) := \#\{\mathfrak{A} \lhd O_K \mid \mathfrak{A} \in \mathcal{A},\ \mathbf{N}_{K/k}(\mathfrak{A}) = \mathfrak{m}\},$$

where $\mathbf{N}_{K/k}(\mathfrak{A})$ is the ideal of A generated by all the norms $\mathbf{N}_{K/k}(x)$ with $x \in \mathfrak{A}$ for all $\mathfrak{A} \lhd O_K$.

Let

$$L(F, \eta, s) := (1 - q_\infty^{-f_K(\infty)(1+s)})^{-1} \cdot \left(\sum_{\mathfrak{m} \lhd A} \frac{F^*(\mathfrak{m}) R_\eta(\mathfrak{m})}{\|\mathfrak{m}\|^s} \right), \quad \mathrm{Re}(s) > \frac{1}{2}.$$

Put $L^{\mathfrak{n}_F}(s, \chi_K) := \prod_{v \nmid \mathfrak{n}_F \infty} L_v(s, \chi_{K,v})$. The *Rankin-type L-function $L(F \times \eta, s)$ associated to F and η* is:

$$L(F \times \eta, s) := L^{\mathfrak{n}_F}(2s+1, \chi_K) \cdot L(F, \eta, s).$$

Remark 3.1. The Rankin-type L-function can be defined in complete generality via "local integrals", cf. Jacquet & Piatetski-Shapiro & Shalika [7, (2.7)].

Note that $L(F, \eta, s) = \prod_v L_v(F, \eta, s)$, where

$$L_v(F, \eta, s) := \begin{cases} (1 - q_\infty^{-f_K(\infty)(1+s)})^{-1}, & \text{if } v = \infty, \\ \displaystyle\sum_{n \geq 0} \frac{F^*(\mathfrak{p}^n) R_\eta(\mathfrak{p}^n)}{\|\mathfrak{p}\|^{ns}}, & \text{if } v = \mathfrak{p} \lhd A. \end{cases}$$

Thus $L(F \times \eta, s)$ can be expressed as an Euler product $\prod_v L_v(F \times \eta, s)$, where

$$L_v(F \times \eta, s) = L_v(F, \eta, s) \cdot \begin{cases} L_v(s, \chi_{K,v}) & \text{if } v \nmid \mathfrak{n}_F \infty, \\ 1 & \text{otherwise.} \end{cases}$$

We want to connect the local factors of $L(F\times\eta,s)$ with the local factors of $L(F,s)$ and $L(\eta,s)$. In Section 2.4, we saw that

$$L_v(F,s)=\big(1-\alpha_v^{(1)}(F)q_v^{-(s+1)}\big)^{-1}\big(1-\alpha_v^{(2)}(F)q_v^{-(s+1)}\big)^{-1},$$

where:

- when $v\nmid\mathfrak{n}_F\infty$, $\alpha_v^{(1)}(F)$ and $\alpha_v^{(2)}(F)$ are two complex conjugate roots of the quadratic polynomial $X^2-\lambda_v(F)X+\omega_F(v)q_v$;
- when $v\mid\mathfrak{n}_F\infty$, $\alpha_v^{(1)}(F):=\lambda_v(F)$ and $\alpha_v^{(2)}(F)=0$.

Also write $L(\eta,s)$ as an Euler product $\prod_v L_v(\eta,s)$ with

$$L_v(\eta,s)=\big(1-c_v^{(1)}(\eta)q_v^{-s}\big)^{-1}\big(1-c_v^{(2)}(\eta)q_v^{-s}\big)^{-1},$$

where (viewing η as a character on $\mathrm{Div}(K)$ via the natural epimorphism $\mathrm{Div}(K)\twoheadrightarrow\mathcal{I}(O_K)$):

- when $v=w_1w_2$ splits in K/k, put $c_v^{(i)}(\eta):=\eta(w_i)$ for $i=1,2$;
- when v is inert in K/k, put $c_v^{(i)}(\eta):=(-1)^i\sqrt{\eta(w)}$ where $w\mid v$;
- when v is ramified in K/k, put $c_v^{(1)}(\eta):=\eta(w)$ where $w\mid v$, and $c_v^{(2)}(\eta):=0$.

Then:

Lemma 3.2. *For each place v of k, we have*

$$L_v(F\times\eta,s)=\prod_{1\le i_1,i_2\le 2}\big(1-\alpha_v^{(i_1)}(F)c_v^{(i_2)}(\eta)\cdot q_v^{-(s+1)}\big)^{-1}.$$

In particular, when ω_F and $\eta=\mathbf{1}_K$ are both trivial, we get

$$L(F\times\mathbf{1}_K,s)=L(F,s)\cdot L(F\otimes\chi_K,s),$$

where $L(F\otimes\chi_K,s)=\prod_v L_v(F\otimes\chi_K,s)$ with

$$L_v(F\otimes\chi_K,s):=\begin{cases}(1-\lambda_v(F)\chi_{K,v}(\pi_v)q_v^{-(1+s)}+\mu_{\mathfrak{n}\infty}(v)\omega_F(v)q_v^{1-2(s+1)})^{-1}, & \text{if } v\nmid\mathfrak{d}_K,\\ 1, & \text{otherwise.}\end{cases}$$

Proof. The first statement follows from [1, Lemma 1.6.1], and the second statement is then straightforward by comparing the local L-factors of both sides. □

Remark 3.3. Let ϱ_F be the degree two Galois representation associated to F. Then the first statement of Lemma 3.2 shows that $L(F\times\eta,s-1/2)$ is equal to the L-function of the Galois representation $\varrho_F\otimes\mathrm{Ind}_K^k(\eta)$ of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$.

We now study the analytic properties of $L(F\times\eta,s)$ by the Rankin-Selberg method.

3.1. **Eisenstein series.** We first give a brief review of Eisenstein series on $\mathrm{GL}_2(\mathbb{A})$. Further details are referred to [1, Section 3.7]. Let χ_1,χ_2 be two unitary Hecke characters on the idele class group $k^\times\backslash\mathbb{A}^\times$. For $s\in\mathbb{C}$, let $\mathrm{Ind}_{\chi_1,\chi_2}(s)$ be the space consisting of the smooth (i.e. locally constant) functions Φ on $\mathrm{GL}_2(\mathbb{A})$ satisfying

$$\Phi\left(\begin{pmatrix}y_1 & z\\ 0 & y_2\end{pmatrix}g\right)=|y_1|_{\mathbb{A}}^{s+\frac12}|y_2|_{\mathbb{A}}^{-s-\frac12}\chi_1(y_1)\chi_2(y_2)\Phi(g),\quad g\in\mathrm{GL}_2(\mathbb{A}),\ z\in\mathbb{A},\ y_1,y_2\in\mathbb{A}^\times.$$

For each place v of k, put $\chi_{i,v}:=\chi_i\big|_{k_v^\times}$ for $i=1,2$. We can write $\mathrm{Ind}_{\chi_1,\chi_2}(s)$ as a restricted tensor product $\otimes'_v\mathrm{Ind}_{\chi_{1,v},\chi_{2,v},v}(s)$.

Take a smooth function ϕ on $\mathrm{GL}_2(O_{\mathbb{A}})$ which satisfies

$$\phi\left(\begin{pmatrix}u_1 & x\\ 0 & u_2\end{pmatrix}g\right)=\chi_1(u_1)\cdot\chi_2(u_2)\cdot\phi(\kappa),$$

for all $\kappa \in \mathrm{GL}_2(O_{\mathbb{A}})$, $x \in O_{\mathbb{A}}$ and $u_1,\ u_2 \in O_{\mathbb{A}}^{\times}$. We can extend ϕ to a *flat section* $\Phi_\phi(\cdot, s)$: for $g = \begin{pmatrix} y_1 & z \\ 0 & y_2 \end{pmatrix} \kappa \in \mathrm{GL}_2(\mathbb{A})$ where $y_1, y_2 \in \mathbb{A}^\times$, $z \in \mathbb{A}$, and $\kappa \in \mathrm{GL}_2(O_{\mathbb{A}})$, put

$$\Phi_\phi(g, s) := |y_1|_{\mathbb{A}}^{s+\frac{1}{2}} \cdot |y_2|_{\mathbb{A}}^{-s-\frac{1}{2}} \cdot \chi_1(y_1) \cdot \chi_2(y_2) \cdot \phi(\kappa).$$

Then it is clear that for each $s \in \mathbb{C}$, $\Phi_\phi(\cdot, s)$ is a well-defined function in $\mathrm{Ind}_{\chi_1,\chi_2}(s)$ and

$$\Phi_\phi(\kappa, s) = \phi(\kappa), \quad \forall \kappa \in \mathrm{GL}_2(O_{\mathbb{A}}).$$

The Eisenstein series $E(\Phi_\phi, s, \cdot)$ on $\mathrm{GL}_2(k)\backslash \mathrm{GL}_2(\mathbb{A})$ is defined by the following:

$$E(\Phi_\phi, s, g) := \sum_{\gamma \in \mathrm{B}(k)\backslash \mathrm{GL}_2(k)} \Phi_\phi(\gamma \cdot g, s), \quad \forall g \in \mathrm{GL}_2(\mathbb{A}).$$

Here B is the standard Borel subgroup of GL_2 introduced in Section 2.1. It is known that $E(\Phi_\phi, s, g)$ converges absolutely when $\mathrm{Re}(s) > 1/2$. Moreover, $E(\Phi_\phi, s, g)$ has a meromorphic continuation in $s \in \mathbb{C}$ (in fact, a rational function in q^{-s}), and satisfies the following functional equation (cf. [1, Theorem 3.7.2]):

$$E(\Phi_\phi, s, g) \quad = \quad E(M(s)\Phi_\phi, -s, g), \tag{3.1.1}$$

where $M(s) : \mathrm{Ind}_{\chi_1,\chi_2}(s) \to \mathrm{Ind}_{\chi_2,\chi_1}(-s)$ is the intertwining operator

$$M(s)\Phi(g) := \int_{\mathbb{A}} \Phi\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g\right) du.$$

The Haar measure du is self-dual with respect to the fixed additive character ψ. This integral is convergent for $\mathrm{Re}(s) > 1/2$ and also has "meromorphic continuation" to the complex s-plane (cf. [1, p.355]).

For our purposes, we consider the case when $\chi_1 = \mathbf{1}$, the trivial character and $\chi_2 = \chi_K$ where K is a given imaginary quadratic field extension over k. Let F be a normalized Drinfeld-type newform of square-free level $\mathfrak{n}_F$. We shall partition the collection of places of k into the following four subsets (depending on F and K):

$$\begin{aligned}
\mathcal{S} &:= \{\text{ places } v \text{ of } k \mid v \text{ divides neither } \mathfrak{n}_F\infty \text{ nor } \mathfrak{d}_K\}, \\
\mathcal{S}_F &:= \{\text{ places } v \text{ of } k \mid v \text{ divides } \mathfrak{n}_F\infty, \text{ but } v \text{ does not divide } \mathfrak{d}_K\}, \\
\mathcal{S}_K &:= \{\text{ places } v \text{ of } k \mid v \text{ divides } \mathfrak{d}_K, \text{ but } v \text{ does not divide } \mathfrak{n}_F\infty\}, \\
\mathcal{S}_{F,K} &:= \{\text{ places } v \text{ of } k \mid v \text{ divides both } \mathfrak{d}_K \text{ and } \mathfrak{n}_F\infty\}.
\end{aligned}$$

Definition 3.4. For each place v of k, we choose

$$\phi_v^\sharp(x) := \chi_{K,v}(d) \cdot \mathbf{1}_{\mathcal{K}_v}(x) + \xi_v \cdot \chi_{K,v}(c) \cdot \mathbf{1}_{\mathcal{K}_v^c}(x), \quad \forall x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(O_v),$$

where $\chi_{K,v}$ is the character introduced in Section 1.2, $\mathcal{K}_v^c := \mathrm{GL}_2(O_v) - \mathcal{K}_v$ and

$$\xi_v := \begin{cases} 1, & \text{if } v \in \mathcal{S}; \\ -q_v^{-1}, & \text{if } v \in \mathcal{S}_F; \\ q_v^{-1/2} \cdot \overline{\varepsilon_{K,v}(1)}, & \text{if } v \in \mathcal{S}_K; \\ -\lambda_v(F) \cdot q_v^{-1/2} \cdot \eta(w) \cdot \overline{\varepsilon_{K,v}(1)}, & \text{if } v \in \mathcal{S}_{F,K}. \end{cases}$$

Here $\varepsilon_{K,v}(1)$ is the *Weil index* associated to the quadratic space $(K_v, \mathbf{N}_{K/k})$ (cf. Appendix A.1), and w is the place of K lying above v.

We set $\phi^\sharp := \otimes_v \phi_v^\sharp$, which is a function on $\mathrm{GL}_2(O_{\mathbb{A}})$ satisfying

$$\phi^\sharp\left(\begin{pmatrix} u_1 & x \\ 0 & u_2 \end{pmatrix}\kappa\right) = \chi_K(u_2)\cdot \phi^\sharp(\kappa),$$

for all $\kappa \in \mathrm{GL}_2(O_{\mathbb{A}})$, $x \in O_{\mathbb{A}}$ and $u_1,\, u_2 \in O_{\mathbb{A}}^\times$. The associated flat section to $\phi^\sharp$ is denoted by

$$\Phi^\sharp := \otimes_v \Phi_v^\sharp \in \mathrm{Ind}_{\mathbf{1},\chi_K}(s).$$

For each $\mathfrak{m} \in \mathrm{Div}_{\geq 0}(k)$, set $\mathcal{K}_0^{+_K}(\mathfrak{m}) := \mathcal{K}_0(\mathfrak{m}) \cap \mathrm{GL}_2^{+_K}(O_{\mathbb{A}})$ where

$$\mathrm{GL}_2^{+_K}(O_{\mathbb{A}}) := \{\kappa \in \mathrm{GL}_2(O_{\mathbb{A}}) \mid \det(\kappa) \in \mathbf{N}_{K/k}(\mathbb{A}_K^\times)\}.$$

Then:

Lemma 3.5. (1) *For every $g \in \mathrm{GL}_2(\mathbb{A})$ and $\kappa = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{K}_0^{+_K}([\mathfrak{n}_F\infty, \mathfrak{d}_K])$, we have*

$$\Phi^\sharp(g\kappa) = \left(\prod_{v \mid [\mathfrak{n}_F\infty, \mathfrak{d}_K]} \chi_{K,v}(d_v)\right)\Phi^\sharp(g).$$

(2) *For each place v of k, $M_v(s)\Phi_v^\sharp(\cdot, s) = \widetilde{\Phi}_v^\sharp(\cdot, -s)\cdot \epsilon_v^{F,K}(s)$ where*

$$\widetilde{\Phi}_v^\sharp := (\chi_{K,v}\circ \det)\cdot \Phi_v^\sharp \quad \text{and} \quad \epsilon_v^{F,K}(s) := \begin{cases} q_v^{\frac{\delta_v}{2}} \cdot \dfrac{L_v(2s,\chi_{K,v})}{L_v(2s+1,\chi_{K,v})}, & \text{if } v \in \mathcal{S}; \\ -q_v^{\frac{\delta_v}{2}} \cdot \dfrac{L_v(-2s,\chi_{K,v})}{L_v(1-2s,\chi_{K,v})}, & \text{if } v \in \mathcal{S}_F; \\ q_v^{\frac{\delta_v-1}{2}} \cdot \overline{\varepsilon_{K,v}(1)}, & \text{if } v \in \mathcal{S}_K; \\ -\lambda_v(F)\cdot \eta(w)\cdot q_v^{\frac{\delta_v-1}{2}} \cdot \overline{\varepsilon_{K,v}(1)}, & \text{if } v \in \mathcal{S}_{F,K}. \end{cases}$$

Proof. The argument of (1) is straightforward. For (2), we only prove the case when $v \in \mathcal{S}_{F,K}$ since the other cases are similar.

Taking $v \in \mathcal{S}_{F,K}$, we have defined $\phi_v^\sharp$ in Definition 3.4 by: for $\kappa \in \mathrm{GL}_2(O_v)$,

$$\phi_v^\sharp(\kappa) := \chi_{K,v}(d)\cdot \mathbf{1}_{\mathcal{K}_v}(\kappa) - \lambda_v(F)\cdot q_v^{-\frac{1}{2}} \cdot \overline{\varepsilon_{K,v}(1)}\cdot \chi_{K,v}(c)\cdot \eta(w)\cdot \mathbf{1}_{\mathcal{K}_v^c}(\kappa).$$

Here w is the place of K lying above v. Given $g \in \mathrm{GL}_2(k_v)$, write

$$g = \begin{pmatrix} y_1 & z \\ 0 & y_2 \end{pmatrix}\kappa \in \mathrm{GL}_2(k_v), \text{ where } y_1, y_2 \in k_v^\times, z \in k_v \text{ and } \kappa = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(O_v).$$

(i) Suppose that $\kappa \in \mathcal{K}_v$. We have

$$\begin{aligned} M_v(s)\Phi_v^\sharp(g,s) &= \int_{k_v} \Phi_v^\sharp\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & n_v \\ 0 & 1 \end{pmatrix} g\right) dn_v \\ &= |y_2|_v^{s+1/2}\cdot |y_1|_v^{-s+1/2}\cdot \chi_{K,v}(y_1)\chi_{K,v}(a)\cdot \int_{k_v} \Phi_v^\sharp\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & n_v \\ 0 & 1 \end{pmatrix}\right) dn_v \\ &= |y_2|_v^{s+1/2}\cdot |y_1|_v^{-s+1/2}\cdot \chi_{K,v}(y_1)\chi_{K,v}(a)\cdot \int_{k_v} \Phi_v^\sharp\left(\begin{pmatrix} 0 & -1 \\ 1 & n_v \end{pmatrix}\right) dn_v. \end{aligned}$$

It can be verified that

$$\int_{k_v} \Phi_v^\sharp\left(\begin{pmatrix} 0 & -1 \\ 1 & n_v \end{pmatrix}\right) dn_v = -\lambda_v(F)\cdot q_v^{\frac{\delta_v-1}{2}} \cdot \overline{\varepsilon_{K,v}(1)}\cdot \eta(w).$$

Therefore

$$\begin{aligned} M_v(s)\Phi_v^\sharp(g,s) &= -\,|y_2|_v^{s+1/2}\cdot|y_1|_v^{-s+1/2}\cdot\chi_{K,v}(y_1)\chi_{K,v}(a)\cdot\lambda_v(F)\cdot q_v^{\frac{\delta_v-1}{2}}\cdot\overline{\varepsilon_{K,v}(1)}\cdot\eta(w)\\ &= \left(-\lambda_v(F)\cdot\eta(w)\cdot q_v^{\frac{\delta_v-1}{2}}\cdot\overline{\varepsilon_{K,v}(1)}\right)\cdot(\chi_{K,v}\circ\det(g))\cdot\Phi_v^\sharp(g,-s).\end{aligned}$$

(ii) Suppose $\kappa = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathcal{K}_v^c$. Note that

$$\kappa = \begin{pmatrix} 1 & \frac{a}{c} \\ 0 & 1\end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\begin{pmatrix} -c & -d \\ 0 & -c^{-1}(\det(\kappa))\end{pmatrix}.$$

Thus $M_v(s)\Phi_v^\sharp(g,s)$ is equal to

$$|y_2|_v^{s+1/2}\cdot|y_1|_v^{-s+1/2}\cdot\chi_{K,v}(y_1)\cdot\int_{k_v}\Phi_v^\sharp\left(\begin{pmatrix} -c & 0 \\ cn_v & -c^{-1}\det(\kappa)\end{pmatrix}\right)dn_v.$$

Since

$$\int_{k_v}\Phi_v^\sharp\left(\begin{pmatrix} -c & 0 \\ cn_v & -c^{-1}\det(\kappa)\end{pmatrix}\right)dn_v = q_v^{\frac{\delta_v}{2}-1}\cdot\chi_{K,v}(-1)\cdot\chi_{K,v}(c^{-1})\cdot\chi_{K,v}(\det(\kappa)),$$

we obtain that

$$\begin{aligned} M_v(s)\Phi_v^\sharp(g,s) &= |y_2|_v^{s+1/2}|y_1|_v^{-s+1/2}\chi_{K,v}(y_1)\cdot\left(q_v^{\frac{\delta_v}{2}-1}\cdot\chi_{K,v}(-1)\cdot\chi_{K,v}(c^{-1})\cdot\chi_{K,v}(\det(\kappa))\right)\\ &= \left(-\lambda_v(F)\cdot\eta(w)\cdot q_v^{\frac{\delta_v-1}{2}}\cdot\overline{\varepsilon_{K,v}(1)}\right)\cdot(\chi_{K,v}\circ\det(g))\cdot\Phi_v^\sharp(g,-s).\end{aligned}$$

The last equality comes from $\chi_{K,v}(-1) = \varepsilon_{K,v}(1)^2$ and $\lambda_v(F)^2\cdot\eta(w)^2 = 1$. □

Let $\epsilon^{F,K}(s) := \prod_v \epsilon_v^{F,K}(s)$. From the functional equation of $L(s,\chi_K)$:

$$L(s,\chi_K) = q^{(g_k-1+\deg\mathfrak{d}_K/2)(1-2s)}L(1-s,\chi_K),$$

we can express $\epsilon^{F,K}(s)$ as

$$\epsilon^{F,K}(s) = (-1)^{\#\Sigma}\cdot q^{(2-2g_k)(2s)}\cdot\|[\mathfrak{n}_F\infty,\mathfrak{d}_K]\|^{-2s}\cdot\frac{L^{\mathfrak{n}_F}(1-2s,\chi_K)}{L^{\mathfrak{n}_F}(1+2s,\chi_K)},$$

where

$$\begin{aligned}\Sigma &= \Sigma(F,\eta)\\ &:= \{\infty\}\cup\left\{\text{prime } \mathfrak{p} \text{ dividing } \mathfrak{n}_F \;\middle|\; \mathfrak{p} \text{ is ramified in } K/k \text{ and } \lambda_{\mathfrak{p}}(F)\cdot\eta(\mathfrak{P}) = 1\right\}\\ &\qquad \cup\left\{\text{prime } \mathfrak{p} \text{ dividing } \mathfrak{n}_F \;\middle|\; \mathfrak{p} \text{ is inert in } K/k\right\}.\end{aligned} \tag{3.1.2}$$

Here $\mathfrak{P}$ is the prime ideal of O_K lying above $\mathfrak{p}$. Therefore by the equation (3.1.1) and Lemma 3.5 (2), we obtain the explicit functional equation of $E(\Phi^\sharp,s,g)$:

Lemma 3.6.

$$E(\Phi^\sharp,s,g) = \epsilon^{F,K}(s)\cdot E(\widetilde{\Phi}^\sharp,-s,g),\quad \forall g\in\mathrm{GL}_2(\mathbb{A}).$$

3.2. **Zeta integral.** Let Θ_K^η be the quadratic theta series associated to the character η (constructed in Definition A.7). Given a flat section $\Phi(\cdot,s)\in\mathrm{Ind}_{\mathbf{1},\chi_K}(s)$, the *global zeta integral associated to F, η and Φ* is:

$$Z(F,\eta,\Phi,s) := \int_{\mathbb{A}^\times\mathrm{GL}_2(k)\backslash\mathrm{GL}_2(\mathbb{A})} F(g)\overline{\Theta_K^\eta(g)E(\Phi,\bar{s},g)}dg. \tag{3.2.1}$$

Here $\bar{\cdot}$ is the complex conjugation. This integral is a meromorphic function on the complex s-plane.

Set $\mathrm{GL}_2^{+_K}(\mathbb{A}) := \{g \in \mathrm{GL}_2(\mathbb{A}) \mid \det(g) \in k^\times \mathbf{N}_{K/k}(\mathbb{A}_K^\times)\}$. Since the theta series Θ_K^η vanishes outside $\mathrm{GL}_2^{+_K}(\mathbb{A})$ (cf. Lemma A.4), the zeta integral (3.2.1) becomes:

$$Z(F,\eta,\Phi,s) = \int_{\mathbb{A}^\times \mathrm{GL}_2(k)\backslash \mathrm{GL}_2^{+_K}(\mathbb{A})} F(g)\overline{\Theta_K^\eta(g)E(\Phi,\bar{s},g)}dg.$$

Put

$$\mathrm{B}^{+_K}(\mathbb{A}) := B(\mathbb{A}) \cap \mathrm{GL}_2^{+_K}(\mathbb{A}) = \mathrm{U}(\mathbb{A}) \cdot \mathrm{T}_1^{+_K}(\mathbb{A}),$$

where

$$\mathrm{U} := \left\{\begin{pmatrix} 1 & * \\ 0 & 1\end{pmatrix} \in \mathrm{B}\right\}, \quad \mathrm{T}_1 := \left\{\begin{pmatrix} * & 0 \\ 0 & 1\end{pmatrix} \in \mathrm{B}\right\} \text{ and } \mathrm{T}_1^{+_K}(\mathbb{A}) := \mathrm{T}_1(\mathbb{A}) \cap \mathrm{GL}_2^{+_K}(\mathbb{A}).$$

Consider the Iwasawa decomposition for $\mathrm{GL}_2^{+_K}(\mathbb{A})$:

$$\mathrm{GL}_2^{+_K}(\mathbb{A}) = \mathrm{B}^{+_K}(\mathbb{A})\,\mathrm{GL}_2^{+_K}(O_\mathbb{A}),$$

where $\mathrm{GL}_2^{+_K}(O_\mathbb{A}) = \{g \in \mathrm{GL}_2(O_\mathbb{A}) : \det(g) \in \mathbf{N}_{K/k}(\mathbb{A}_K^\times)\}$. We normalize the left Haar measure d_Lb on $\mathrm{B}^{+_K}(\mathbb{A})$ such that $\mathrm{Vol}\Big(\mathrm{B}^{+_K}(O_\mathbb{A}) \cap \mathrm{GL}_2^{+_K}(O_\mathbb{A}), d_Lb\Big) = 1$, and the Haar measure dg on $\mathrm{GL}_2(\mathbb{A})$ such that $\mathrm{Vol}\,(\mathcal{K}_0(\mathfrak{n}_F\infty), dg) = 1$. Let $d\kappa$ be the restriction of dg on $\mathrm{GL}_2^{+_K}(O_\mathbb{A})$. Then for every Schwartz function f on $\mathrm{GL}_2^{+_K}(\mathbb{A})$, we have

$$\int_{\mathrm{GL}_2^{+_K}(\mathbb{A})} f(g)dg = \int_{\mathrm{B}^{+_K}(\mathbb{A})}\int_{\mathrm{GL}_2^{+_K}(O_\mathbb{A})} f(b\kappa)d_Lb\ d\kappa.$$

Lemma 3.7. *For each flat section $\Phi(\cdot,s) \in \mathrm{Ind}_{\mathbf{1},\chi_K}(s)$, the zeta integral (3.2.1) is equal to*

$$Z(F,\eta,\Phi,s) = |\delta|_\mathbb{A}^{-\frac{1}{2}} \cdot \int_{\mathrm{GL}_2^{+_K}(O_\mathbb{A})}\int_{\mathrm{T}_1^{+_K}(\mathbb{A})} W_F(t\kappa)\cdot \overline{W_{\Theta_K^\eta}(t\kappa)\Phi(t\kappa,\bar{s})}|t|_\mathbb{A}^{-1}\ d^\times t d\kappa, \quad \mathrm{Re}(s) > \frac{1}{2}.$$

Here W_F (resp. $W_{\Theta_K^\eta}$) is the Whittaker function associated to F (resp. Θ_K^η), the absolute value $|t|_\mathbb{A} := |y|_\mathbb{A}$ for every $t = \begin{pmatrix} y & 0 \\ 0 & 1\end{pmatrix} \in \mathrm{T}_1^{+_K}(\mathbb{A})$, and the Haar measure $d^\times t$ is normalized so that $\mathrm{Vol}\Big(\mathrm{T}_1(O_\mathbb{A}) \cap \mathrm{GL}_2^{+_K}(O_\mathbb{A}), d^\times t\Big) = 1$.

Proof. Following [1, Proposition 3.8.2], we have

$$\begin{aligned}
&\int_{\mathbb{A}^\times \mathrm{GL}_2(k)\backslash \mathrm{GL}_2^{+K}(\mathbb{A})} F(g)\overline{\Theta_K^\eta(g)E(\Phi^\sharp,\bar{s},g)}dg \\
&= \int_{\mathrm{GL}_2^{+_K}(O_\mathbb{A})}\int_{\mathbb{A}^\times \mathrm{B}(k)\backslash \mathrm{B}^{+K}(\mathbb{A})} F(b\kappa)\overline{\Theta_K^\eta(b\kappa)\Phi^\sharp(b\kappa,\bar{s})}d_Lb\ d\kappa \\
&= \int_{\mathrm{GL}_2^{+_K}(O_\mathbb{A})}\int_{\mathrm{T}_1(k)\backslash \mathrm{T}_1^{+K}(\mathbb{A})}\int_{\mathrm{U}(k)\backslash \mathrm{U}(\mathbb{A})} F(nt\kappa)\overline{\Theta_K^\eta(nt\kappa)\Phi^\sharp(nt\kappa,\bar{s})}|t|_\mathbb{A}^{-1}dn\ d^\times t\ d\kappa.
\end{aligned}$$

According to the Fourier expansion $F(g) = \sum_{\alpha\in T_1(k)} W_F(\alpha g)$, the above integral equals

$$\begin{aligned}
&\int_{\mathrm{GL}_2^{+_K}(O_\mathbb{A})}\int_{\mathrm{T}_1^{+K}(\mathbb{A})}\int_{\mathrm{U}(k)\backslash \mathrm{U}(\mathbb{A})} W_F(nt\kappa)\overline{\Theta_K^\eta(nt\kappa)\Phi^\sharp(t\kappa,\bar{s})}|t|_\mathbb{A}^{-1}dn\ d^\times t\ d\kappa \\
&= |\delta|_\mathbb{A}^{-\frac{1}{2}} \cdot \int_{\mathrm{GL}_2^{+_K}(O_\mathbb{A})}\int_{\mathrm{T}_1^{+K}(\mathbb{A})} W_F(t\kappa)\cdot \overline{W_{\Theta_K^\eta}(t\kappa)\cdot \Phi^\sharp(t\kappa,\bar{s})}|t|_\mathbb{A}^{-1}d^\times t\ d\kappa.
\end{aligned}$$

□

Suppose $\Phi = \otimes_v \Phi_v$ is a pure tensor. Note that $W_{\Theta_K^\eta} = \otimes_v W_{\Theta_K^\eta, v}$ and $W_{\Theta_K^\eta, v}(g_v) = 0$ unless $\det(g_v) \in \mathbf{N}_{K/k}(K_v^\times)$ (cf. Theorem A.9). Let

$$\mathrm{GL}_2^{+_K}(O_v) := \{\kappa_v \in \mathrm{GL}_2(O_v) \mid \det(\kappa_v) \in \mathbf{N}_{K/k}(K_v^\times)\}$$

and

$$\mathrm{T}_1^{+_K}(k_v) := \left\{ \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{T}_1(k_v) \;\middle|\; y \in \mathbf{N}_{K/k}(K_v^\times) \right\}.$$

Corollary 3.8. *For each pure tensor $\Phi(\cdot, s) \in \mathrm{Ind}_{\mathbf{1},\chi_K}(s)$, the zeta integral can be written as*

$$Z(F, \eta, \Phi, s) = |\delta|_{\mathbb{A}}^{-\frac{1}{2}} \cdot \prod_v Z_v(F, \Theta_K^\eta, \Phi_v, s), \quad \mathrm{Re}(s) > \frac{1}{2},$$

where

$$Z_v(F, \eta, \Phi_v, s) := \int_{\mathrm{GL}_2^{+_K}(O_v)} \int_{\mathrm{T}_1^{+_K}(k_v)} W_{F,v}(t_v\kappa_v) \overline{W_{\Theta_K^\eta, v}(t_v\kappa_v)\Phi_v(t_v\kappa_v, \bar{s})} |t_v|_v^{-1}\, d^\times t_v d\kappa_v.$$

Take $\Phi = \Phi^\sharp = \otimes_v \Phi_v^\sharp$, the flat section associated to the chosen function in Definition 3.4. Then $L_v(F, \eta, s)$ can be expressed as the local zeta integral $Z_v(F, \eta, \Phi_v^\sharp, s)$:

Lemma 3.9. *For each place v of k, the zeta integral $Z_v(F, \eta, \Phi_v^\sharp, s)$ is equal to*

$$q_v^{\delta_v(s-\frac{1}{2})} \cdot L_v(F, \eta, s) \cdot \begin{cases} 1, & \text{if } v \in \mathcal{S}; \\ (1 + q_v^{-1}), & \text{if } v \in \mathcal{S}_F; \\ (q_v + 1)^{-1}, & \text{if } v \in \mathcal{S}_K; \\ \dfrac{(1 + q_v^s)}{2}, & \text{if } v \in \mathcal{S}_{F,K}. \end{cases}$$

Proof. We prove the case when $v \in \mathcal{S}_{F,K}$. The others are similar. Suppose $v \in \mathcal{S}_{F,K}$. Let

$$\mathcal{K}_v^{+_K} = \mathcal{K}_v \cap \mathrm{GL}_2^{+_K}(O_v) \quad \text{and} \quad \mathcal{K}_v^{c,+_K} := \mathrm{GL}_2^{+_K}(O_v) - \mathcal{K}_v^{+_K}.$$

We write

$$\begin{aligned} Z_v(F, \eta, \Phi_v, s) &= \int_{\mathcal{K}_v^{+_K}} \int_{\mathrm{T}_1^{+_K}(k_v)} W_{F,v}(t_v\kappa_v) \overline{W_{\Theta_K^\eta, v}(t_v\kappa_v)\Phi_v(t_v\kappa_v, \bar{s})} |t_v|_v^{-1}\, d^\times t_v d\kappa_v \\ &+ \int_{\mathcal{K}_v^{c,+_K}} \int_{\mathrm{T}_1^{+_K}(k_v)} W_{F,v}(t_v\kappa_v) \overline{W_{\Theta_K^\eta, v}(t_v\kappa_v)\Phi_v(t_v\kappa_v, \bar{s})} |t_v|_v^{-1}\, d^\times t_v d\kappa_v, \end{aligned}$$

Note that $\mathrm{Vol}\left(\mathcal{K}_v^{+_K}, d\kappa_v\right) = 1/2$. By Proposition A.5 and Proposition A.6, the first part of the above sum is equal to

$$\begin{aligned} &\frac{1}{2} \cdot \int_{\mathbf{N}_{K/k}(K_v^\times)} W_{F,v}\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) \overline{W_{\Theta_K^\eta, v}}\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) |y|_v^{s-\frac{1}{2}} d^\times y \\ =\;& \frac{1}{2} \cdot q_v^{\delta_v(s-\frac{1}{2})} \cdot L_v(F, \eta, s). \end{aligned}$$

Choose a uniformizer π_v of k at v so that $\pi_v \in \mathbf{N}_{K/k}(K_v^\times)$ (i.e. $\chi_{K,v}(\pi_v) = 1$). The second part of the above sum can be expressed by

$$(3.2.2) \qquad -\frac{\lambda_v(F)}{2} \cdot \eta(w) \cdot q_v^{\frac{1}{2}} \cdot \varepsilon_{K,v}(1) \cdot \int_{\mathbf{N}_{K/k}(K_v^\times)} W_{F,v}\left(\begin{pmatrix} y\pi_v & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -\pi_v & 0 \end{pmatrix}\right)$$
$$\cdot \overline{W_{\Theta_K^\eta, v}}\left(\begin{pmatrix} y\pi_v & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -\pi_v & 0 \end{pmatrix}\right) \cdot |y|_v^{s-\frac{1}{2}} d^\times y.$$

Here w is the place of K lying above v. By Lemma 2.2 (2) and Proposition A.9, the value (3.2.2) is equal to

$$\begin{aligned} &\frac{1}{2}\cdot q_v^s\cdot \int_{\mathbf{N}_{K/k}(K_v^\times)} W_{F,v}\left(\left(\begin{array}{cc} y\pi_v & 0 \\ 0 & 1 \end{array}\right)\right)\overline{W_{\Theta_K^\eta,v}}\left(\left(\begin{array}{cc} y\pi_v & 0 \\ 0 & 1 \end{array}\right)\right)\cdot |y\pi_v|_v^{s-\frac{1}{2}}d^\times y \\ =& \frac{1}{2}\cdot q_v^s\cdot q_v^{\delta_v(s-\frac{1}{2})}\cdot L_v(F,\eta,s). \end{aligned}$$

□

We conclude that:

Theorem 3.10. *Let F be a normalized newform of square-free level $\mathfrak{n}_F$ with central character ω_F. Given an imaginary quadratic field K/k and a continuous character η on* $\mathrm{Gal}(k^{\mathrm{sep}}/K)$ *satisfying the conditions* C.1 *and* C.2 *(in the beginning of* Section 3*), we have*

$$\begin{aligned} &Z(F,\eta,\Phi^\sharp,s) \\ =& q^{(2g_k-2)s}\cdot\left(\prod_{v\in\mathcal{S}_K}(1+q_v)^{-1}\right)\left(\prod_{v\in\mathcal{S}_F}(1+q_v^{-1})\right)\left(\prod_{v\in\mathcal{S}_{F,K}}\frac{1+q_v^s}{2}\right)\cdot L(F,\eta,s),\quad \mathrm{Re}(s)>\frac{1}{2}, \end{aligned}$$

where g_k is the genus of k. In particular, this gives the meromorphic continuation of $L(F,\eta,s)$ (and so is $L(F\times\eta,s)$) to the complex s-plane.

Remark 3.11. (1) Since $\widetilde{\Phi}^\sharp(g) = (\chi_K\circ\det)(g)\cdot\Phi^\sharp(g) = \Phi^\sharp(g)$ for every $g\in \mathrm{GL}_2^{+_K}(\mathbb{A})$, by Lemma A.4 we have

$$Z(F,\eta,\widetilde{\Phi}^\sharp,s) = Z(F,\eta,\Phi^\sharp,s).$$

(2) For our purposes on the study of the central critical value of $L(F\times\eta,s)$ in Section 4, we choose a different function ϕ_v' for $v\in\mathcal{S}_F^+ := \{\text{place } v \text{ dividing } \mathfrak{n}_F \mid v \text{ is split in } K\}\subset\mathcal{S}_F$ defined by

$$\phi_v^+ = \mathbf{1}_{\mathcal{K}_v}(x) + q_v^{-1}\cdot\mathbf{1}_{\mathcal{K}_v^c}(x),\ \text{for } x\in\mathrm{GL}_2(O_v).$$

Let $\phi^\natural := \otimes_v\phi_v^\natural$ where

$$\phi_v^\natural = \begin{cases} \phi_v^+, & \text{if } v\in\mathcal{S}_F^+; \\ \phi_v^\sharp, & \text{if } v\notin\mathcal{S}_F^+; \end{cases}$$

and the flat section associated to $\phi^\natural$ is denoted by $\Phi^\natural = \otimes_v\Phi_v^\natural\in\mathrm{Ind}_{\mathbf{1},\chi_K}(s)$. Then

$$\begin{aligned} &Z(F,\eta,\Phi^\natural,s) \\ =& \left(\prod_{v\in\mathcal{S}_K}(1+q_v)^{-1}\right)\left(\prod_{v\in\mathcal{S}_F-\mathcal{S}_F^+}(1+q_v^{-1})\right)\left(\prod_{v\in\mathcal{S}_F^+}(1-q_v^{-1})\right)\left(\prod_{v\mid(\mathfrak{n}_F\infty,\mathfrak{d}_K)}\frac{1+q_v^s}{2}\right) \\ &\cdot q^{(2g_k-2)s}\cdot L(F,\eta,s). \end{aligned}$$

In particular, we have

$$Z(F,\eta,\Phi^\natural,0) = \left(\prod_{v\in\mathcal{S}_K}(1+q_v)^{-1}\prod_{v\in\mathcal{S}_F}L_v(1,\chi_{K,v})^{-1}\right)\cdot L(F,\eta,0). \tag{3.2.3}$$

3.3. **Functional equation of $L(F \times \eta, s)$.** Set

$$Z^*(F, \eta, \Phi^\sharp, s) := q^{(2g_k-2)s} \cdot \|[\mathfrak{n}_F\infty, \mathfrak{d}_K]\|^s \cdot L^{\mathfrak{n}_F}(1+2s, \chi_K) \cdot Z(F, \eta, \Phi^\sharp, s).$$

By Lemma 3.6 and Remark 3.11 (1), one has

$$Z^*(F, \eta, \Phi^\sharp, s) = (-1)^{\#\Sigma} \cdot Z^*(F, \eta, \widetilde{\Phi}^\sharp, -s) = (-1)^{\#\Sigma} \cdot Z^*(F, \eta, \Phi^\sharp, -s).$$

Consider the modified L-function

$$L^*(F \times \eta, s) := q^{(4g_k-4)s} \cdot \|[\mathfrak{n}_F\infty, \mathfrak{d}_K]\|^s \cdot \|(\mathfrak{n}_F\infty, \mathfrak{d}_K)\|^{\frac{s}{2}} \cdot L(F \times \eta, s).$$

By Theorem 3.10, we then arrive at:

Theorem 3.12. *Let F be a normalized newform of square-free level $\mathfrak{n}_F$ with central character ω_F. Given an imaginary quadratic field K/k and a continuous character η on* $\mathrm{Gal}(k^{\mathrm{sep}}/K)$ *satisfying the conditions* C.1 *and* C.2 *(in the beginning of* Section 3*), the modified Rankin-type L-function $L^*(F \times \eta, s)$ satisfies*

$$L^*(F \times \eta, s) = (-1)^{\#\Sigma} \cdot L^*(F \times \eta, -s).$$

Here $\Sigma = \Sigma(F, \eta)$ is introduced in (3.1.2).

Remark 3.13.

(1) We call $(-1)^{\#\Sigma}$ the *root number* of $L(F \times \eta, s)$.
(2) When $k = \mathbb{F}_q(t)$ with q odd and $A = \mathbb{F}_q[t]$, let $K = k(\sqrt{D})$ where $D \in A$ is irreducible such that K is an imaginary quadratic extension over k. Then Theorem 3.12 coincides with Rück-Tipp's functional equation in [12].

4. Central critical value of $L(F \times \eta, s)$

Let F be a normalized Drinfeld-type newform F of square-free level $\mathfrak{n}_F$, K/k be an imaginary quadratic field, and a continuous character η on $\mathrm{Gal}(k^{\mathrm{sep}}/K)$ satisfying the conditions C.1 and C.2 (in the beginning of Section 3). In this section we assume that the cardinality of $\Sigma = \Sigma(F, \eta)$ is even, and study the central critical value $L(F \times \eta, 0)$.

Throughout this section, the chosen uniformizer π_v for $v \mid \mathfrak{d}_K$ is assumed to be in $\mathbf{N}_{K/k}(K_v^\times)$. Take $\Pi_v \in K_v^\times$ such that $\mathbf{N}_{K/k}(\Pi_v) = \pi_v$ for all places v not inert in K once for all. For each place w of K lying above a place v of k, we then choose a uniformizer $\varpi_w \in K_w$ so that:

- if K_w/k_v is unramified, then $\varpi_w = \pi_v$;
- if K_w/k_v is ramified, then $\varpi_w = \Pi_v$.

For each divisor $\mathfrak{A} \in \mathrm{Div}(K)$, set $\varpi_{\mathfrak{A}} := (\varpi_w^{\mathrm{ord}_w(\mathfrak{A})})_w \in \mathbb{A}_K^\times$. We also identify fractional ideals of O_K with divisors of K having support away from ∞_K, the place of K lying above ∞. Given a fractional ideal $\mathfrak{A}$ of O_K, we may consider $\varpi_{\mathfrak{A}}$ as in $\mathbb{A}_K^{\infty,\times} := (K \otimes_k \mathbb{A}^\infty)^\times = \prod'_{w \neq \infty_K} K_w^\times$ for convenience's sake.

4.1. **Siegel-Eisenstein series.** Take $\gamma = \gamma_{F,\eta} \in k^\times$ such that $\chi_{K,v}(\gamma) = -1$ if $v \in \Sigma(F, \eta)$ and 1 otherwise. Let $(V_\gamma, Q_{V_\gamma}) := (K, -\gamma \cdot \mathbf{N}_{K/k})$. For each Schwartz function $\varphi \in S(V_\gamma(\mathbb{A}))$, one associates a *Siegel section* $\Phi_\varphi(\cdot, s) \in \mathrm{Ind}_{\mathbf{1},\chi_K}(s)$ to φ: for $g = \begin{pmatrix} a & * \\ 0 & b \end{pmatrix} \kappa \in \mathrm{GL}_2(\mathbb{A})$ with $\kappa \in \mathrm{SL}_2(O_{\mathbb{A}})$, define

$$\Phi_\varphi(g, s) := |a|_{\mathbb{A}}^{s+1/2} |b|_{\mathbb{A}}^{-s-1/2} \chi_K(b) \big(\omega_\gamma(\kappa)\varphi\big)(0).$$

Here ω_γ is the Weil representation of $\mathrm{SL}_2(\mathbb{A})$ on $S(V_\gamma(\mathbb{A}))$. The series $E(\Phi_\varphi, s, g)$ is called *the Siegel-Eisenstein series associated to* φ.

Let $\mathrm{GO}(V_\gamma) = \big\{h \in \mathrm{GL}(V_\gamma) \mid Q_{V_\gamma}(hx) = \nu(h)\cdot Q_{V_\gamma}(x),\ \forall x \in V,\ \text{where } \nu(h) \in \mathbb{G}_m\big\}$ and set

$$\mathrm{G}(\mathrm{SL}_2 \times \mathrm{O}(V_\gamma)) := \{(g,h) \in \mathrm{GL}_2 \times \mathrm{GO}(V_\gamma) \mid \det(g) = \nu(h)\}.$$

We extend ω_γ to a representation ω'_γ of $\mathrm{G}(\mathrm{SL}_2 \times \mathrm{O}(V_\gamma))(\mathbb{A})$ on $S(V_\gamma(\mathbb{A}))$ by the following: for each pair $(g,h) \in \mathrm{G}(\mathrm{SL}_2 \times \mathrm{O}(V_\gamma))$ and $\varphi \in S(V_\gamma)(\mathbb{A})$, define

$$\omega'_\gamma(g,h)\varphi(x) := |v(h)|_{\mathbb{A}}^{-\frac{1}{2}} \cdot \chi_K(\nu(h)) \cdot \left(\omega_\gamma\left(\begin{pmatrix}1 & 0\\ 0 & \det(g)^{-1}\end{pmatrix}\cdot g\right)\varphi\right)(h^{-1}x), \quad \forall x \in V(\mathbb{A}).$$

We shall make use of the following Siegel-Weil formula over function fields (cf. [17, Theorem 0.1]):

Theorem 4.1. *For any* $\varphi \in S(V_\gamma(\mathbb{A}))$ *and* $(g,h) \in \mathrm{G}(\mathrm{SL}_2 \times \mathrm{O}(V_\gamma))(\mathbb{A})$*, we have*

$$E(\Phi_\varphi, 0, g) = 2\cdot I\left(g\begin{pmatrix}1 & 0\\ 0 & \det(g)^{-1}\end{pmatrix}, \varphi_h\right),$$

where

$$\varphi_h := \omega'_\gamma\left(\begin{pmatrix}1 & 0\\ 0 & \nu(h)\end{pmatrix}, h\right)\varphi, \quad \forall h \in \mathrm{GO}(V)(\mathbb{A}),$$

and

$$I(g_1, \varphi) := \int_{\mathrm{O}(V_\gamma)(k)\backslash \mathrm{O}(V_\gamma)(\mathbb{A})} \sum_{x \in V_\gamma(k)} \big(\omega_\gamma(g_1)\varphi\big)(h^{-1}x)dh, \quad \forall g_1 \in \mathrm{SL}_2(\mathbb{A}).$$

Here dh *is normalized so that* $\mathrm{Vol}\big(\mathrm{O}(V_\gamma)(k)\backslash \mathrm{O}(V_\gamma)(\mathbb{A}), dh\big) = 1$.

Let $O_{K_v} := O_K \otimes_A O_v$ for finite place v of k, and O_{K_∞} be the integral closure of O_∞ in K_∞. Take a special Schwartz function $\varphi^\natural = \otimes_v \varphi_v^\natural \in S(V_\gamma(\mathbb{A}))$ as follows:

$$\varphi_v^\natural := \begin{cases} \mathbf{1}_{\pi_v^{-\lfloor \frac{\delta_v + \mathrm{ord}_v(\gamma)}{2}\rfloor} O_{K_v}}, & \text{if } v \text{ is inert in } K;\\ \mathbf{1}_{\Pi_v^{-\delta_v - \mathrm{ord}_v(\gamma) + \mathrm{ord}_v\left(\frac{\mathfrak{n}_F}{(\mathfrak{n}_F, \mathfrak{d}_K)}\right)} O_{K_v}}, & \text{otherwise.}\end{cases}$$

Lemma 4.2. *Let* $\Phi^\natural$ *be the flat section introduced in* Remark 3.11 (2)*. For* $g \in \mathrm{GL}_2(\mathbb{A})$ *and* $s \in \mathbb{C}$*, we have*

$$E(\Phi^\natural, s, g) = E(\Phi_{\varphi^\natural}, s, g).$$

Proof. Note $\phi^\natural = \Phi^\natural\big|_{\mathrm{GL}_2(O_\mathbb{A})}$. It suffices to check that

$$\phi^\natural(\kappa) = \big(\omega_\gamma(\kappa)\varphi^\natural\big)(0), \text{ for all } \kappa \in \mathrm{SL}_2(O_\mathbb{A}),$$

which is straightforward. □

Note that we have the surjective homomorphism $K^\times \rtimes < \varsigma > \twoheadrightarrow \mathrm{O}(V_\gamma)$, where $\varsigma \cdot x := \bar{x}$ and $b \cdot x := (b/\bar{b})x$ for any $x \in V_\gamma$ and $b \in K^\times$.

Lemma 4.3. *For* $g_1 \in \mathrm{SL}_2(\mathbb{A})$ *and* $h \in \mathrm{GO}(V_\gamma)(\mathbb{A})$*, we have*

$$I\left(g_1, \varphi_h^\natural\right) = \frac{1}{\#\,\mathrm{Pic}(O_K)} \sum_{\mathcal{B} \in \mathrm{Pic}(O_K)} \sum_{x \in V_\gamma(k)} \left(\omega_\gamma(g_1)\varphi_h^\natural\right)\left(\frac{\overline{\varpi}_{\mathfrak{B}}}{\varpi_{\mathfrak{B}}} x\right).$$

Here $\varpi_{\mathfrak{B}} \in \mathbb{A}_K^{\infty,\times}$ *is taken in the beginning of* Section 4 *for any chosen representative* $\mathfrak{B} \in \mathcal{B}$*.*

Proof. Note that $\mathrm{SO}(V)(k)\backslash\mathrm{SO}(V)(\mathbb{A}) \cong K^\times\backslash\mathbb{A}_K^\times/\mathbb{A}^\times$ via the surjective homomorphism

$$b \in \mathbb{A}_K^\times \longmapsto \big(x \mapsto (b/\bar{b})\cdot x,\ x \in V(\mathbb{A})\big) \in \mathrm{SO}(V)(\mathbb{A}).$$

From the exact sequence

$$1 \to K_\infty^\times/k_\infty^\times O_\infty^\times \to K^\times\backslash\mathbb{A}_K^\times/O_{\mathbb{A}_K}^\times k_\infty^\times \to \mathrm{Pic}(O_K) \to 1,$$

the integral $I\left(g_1, \varphi_h^\natural\right)$ equals to

$$\begin{aligned} I\left(g_1,\varphi_h^\natural\right) &= \int_{\mathrm{SO(V)(k)\backslash SO(V)(\mathbb{A})}} \sum_{x\in V_\gamma(k)} \left(\omega_\gamma(g_1)\varphi_h^\natural\right)(r^{-1}x)dr \\ &= \frac{1}{\mathrm{Vol}\left(K^\times\backslash\mathbb{A}_K^\times/k_\infty^\times\right)} \int_{K^\times\backslash\mathbb{A}_K^\times/k_\infty^\times} \sum_{x\in V_\gamma(k)} \left(\omega_\gamma(g_1)\varphi_h^\natural\right)\left(\frac{\bar{b}}{b}x\right)db \\ &= \frac{1}{\#\,\mathrm{Pic}(O_K)} \sum_{\mathcal{B}\in\mathrm{Pic}(O_K)} \sum_{x\in V_\gamma(k)} \left(\omega_\gamma(g_1)\varphi_h^\natural\right)\left(\frac{\overline{\varpi_{\mathfrak{B}}}}{\varpi_{\mathfrak{B}}}x\right). \end{aligned}$$

Here the Haar measure db in the second equality is induced by the Haar measure on $\mathbb{A}_K^\times$ such that $\mathrm{Vol}(O_{\mathbb{A}_K}^\times) = 1$. The last equality follows from $\varphi_h^\natural(ux) = \varphi_h^\natural(x)$ for all $u \in O_{\mathbb{A}_K}^\times$. □

Given $g \in \mathrm{GL}_2(\mathbb{A})$ with $\det(g) \in \mathbf{N}_{K/k}(\mathbb{A}_K^\times)$, take any $h = h_g \in \mathrm{GO}(V_\gamma)(\mathbb{A})$ such that $\nu(h) = \det(g)$. By Theorem 4.1 and Lemma 4.2, we obtain

$$\Theta_K^\eta(g)\cdot E(\Phi^\natural,0,g) = 2\cdot\Theta_K^\eta(g)\cdot I\left(g\begin{pmatrix}1 & 0\\ 0 & \det(g)^{-1}\end{pmatrix}, \varphi_h^\natural\right). \tag{4.1.1}$$

Let $\mathcal{D} = \mathcal{D}(F,\eta)$ be the quaternion algebra over k ramified precisely at the places $v \in \Sigma$. We can write $\mathcal{D}$ as $K + Kj$, where $j^2 = \gamma$. In particular, one has

$$\mathrm{Nr}_{\mathcal{D}/k}(a+bj) = \mathbf{N}_{K/k}(a) - \gamma\cdot\mathbf{N}_{K/k}(b), \quad \forall a,b\in K,$$

which says that $(\mathcal{D},\mathrm{Nr}_{\mathcal{D}/k}) = (K,\mathbf{N}_{K/k}) \oplus (K, -\gamma\cdot\mathbf{N}_{K/k})$. Having the expression of $I(\cdot,\varphi_h^\natural)$ given by Lemma 4.3, the above decomposition of the reduced norm form $\mathrm{Nr}_{\mathcal{D}/k}$ enables us to express the "kernel function" $\Theta_K^\eta\cdot I(\cdot,\varphi_h^\natural)$ in terms of the quaternionic theta series introduced in the next subsection.

4.2. **Quaternionic theta series.** Let $\mathcal{D}$ be a definite quaternion algebra over k with discriminant $\mathfrak{n}^- \lhd A$ and R be a fixed Eichler A-order in $\mathcal{D}$ with type $(\mathfrak{n}^+,\mathfrak{n}^-)$, where $\mathfrak{n}^+$ and $\mathfrak{n}^-$ are coprime. Let I and J be two locally-principal right ideals of the Eichler A-order R. There exists $b_I, b_J \in \mathcal{D}_{\mathbb{A}^\infty}^\times$ such that $I = \mathcal{D}\cap b_I\widehat{R}$ and $J = \mathcal{D}\cap b_J\widehat{R}$. Set

$$\tilde{\varphi}_{I,J} := \tilde{\varphi}_{I,J}^\infty \otimes \tilde{\varphi}_{I,J,\infty} \in S(V(\mathbb{A})\times\mathbb{A}^\times),$$

where for $(x^\infty,\alpha^\infty) \in V(\mathbb{A}^\infty)\times\mathbb{A}^{\infty,\times}$, its finite part is given by

$$\tilde{\varphi}_{I,J}^\infty(x^\infty,\alpha^\infty) := \mathbf{1}_{b_I\widehat{R}b_J^{-1}}(x^\infty)\cdot\mathbf{1}_{O_{\mathbb{A}^\infty}^\times}(\alpha^\infty\cdot\beta_{I,J}\delta^\infty),$$

with $\beta_{I,J} = \mathrm{Nr}_{\mathcal{D}/k}(b_Ib_J^{-1}) \in \mathbb{A}^{\infty,\times}$; and for $(x_\infty,\alpha_\infty) \in V(k_\infty)\times k_\infty^\times$, the infinite part is

$$\tilde{\varphi}_{I,J,\infty}(x_\infty,\alpha_\infty) := \mathbf{1}_{O_\infty}(\mathrm{Nr}_{\mathcal{D}/k}(x_\infty)\cdot\alpha_\infty\pi_\infty^{\delta_\infty}).$$

Definition 4.4. Let I, J be two locally-principal right ideals of the given Eichler A-order R. For $g \in \mathrm{GL}_2(\mathbb{A})$, set

$$\Theta_{I,J}^R(g) := \frac{|\delta|_{\mathbb{A}}}{\#(R_J^\times)} \cdot \sum_{(x,\alpha)\in V(k)\times k^\times} \left(\omega_{\mathcal{D}}\left(\begin{pmatrix}\beta_{I,J}\delta & 0\\ 0 & \beta_{I,J}\delta\end{pmatrix}g\right)\tilde{\varphi}_{I,J}\right)(x,\alpha).$$

Here R_J is the left order of J, δ is the fixed differential idele of k, and $\omega_{\mathcal{D}}$ is the Weil representation of GL_2 associated to the quadratic space $(\mathcal{D}, \mathrm{Nr}_{\mathcal{D}/k})$.

Let $\mathcal{I}, \mathcal{J} \in \mathrm{Cl}(R)$. We remark that for each $I \in \mathcal{I}$ and $J \in \mathcal{J}$, this theta series $\Theta^R_{I,J}$ is independent of the chosen representatives I and J. Therefore, we also adopt the notation $\Theta^R_{\mathcal{I},\mathcal{J}}$. Moreover, for each pair $(\mathcal{I},\mathcal{J}) \in \mathrm{Cl}(R)\times\mathrm{Cl}(R)$, the theta series $\Theta^R_{\mathcal{I},\mathcal{J}}$ is a Drinfeld-type automorphic form of level $\mathfrak{n}^+\mathfrak{n}^-$ whose Fourier coefficients are the $(\mathcal{I},\mathcal{J})$-entry of the *Brandt matrices* (cf. [2, Chapter III Proposition 3.8]).

Now, take $\mathcal{D} = \mathcal{D}(F,\eta)$, and the discriminant of $\mathcal{D}$ is denoted by $\mathfrak{n}^-_{F,\eta}$. Writing $\mathcal{D}$ as $K + Kj$, let $R^\natural$ be the Eichler A-order in $\mathcal{D}$ (chosen with respect to $K \hookrightarrow \mathcal{D}$) such that for each prime $\mathfrak{p} \lhd A$, we have

$$R^\natural_{\mathfrak{p}} = R^\natural \otimes_A O_{\mathfrak{p}} = \begin{cases} O_{K_{\mathfrak{p}}} + O_{K_{\mathfrak{p}}}\pi_{\mathfrak{p}}^{-\lfloor \frac{\mathrm{ord}_{\mathfrak{p}}(\gamma)}{2}\rfloor} j, & \text{if } \mathfrak{p} \text{ is inert in } K,\\ O_{K_{\mathfrak{p}}} + O_{K_{\mathfrak{p}}}\Pi_{\mathfrak{p}}^{-\mathrm{ord}_{\mathfrak{p}}(\gamma)+\mathrm{ord}_{\mathfrak{p}}\left(\frac{\mathfrak{n}_F}{(\mathfrak{n}_F,\mathfrak{d}_K)}\right)} j, & \text{otherwise.}\end{cases}$$

Put $\mathfrak{n}_{F,K} := [\mathfrak{n}_F, \mathfrak{d}_K]^\infty \lhd A$ (where $\mathfrak{m}^\infty$ is the "finite part" of $\mathfrak{m} \in \mathrm{Div}(k)$, cf. Section 2.1). Then $R^\natural$ is of type $\big(\mathfrak{n}_{F,K}/\mathfrak{n}^-_{F,\eta}, \mathfrak{n}^-_{F,\eta}\big)$, and contains O_K. For each class $\mathcal{A} \in \mathrm{Pic}(O_K)$ and $\mathfrak{A} \in \mathcal{A}$, by abuse of notation, we still write $\mathcal{A}$ as the corresponding class $[\mathfrak{A}R^\natural] \in \mathrm{Cl}(R^\natural)$. Thus for $\mathcal{A},\mathcal{B} \in \mathrm{Pic}(O_K)$, we will denote by $\Theta^{R^\natural}_{\mathcal{A},\mathcal{B}}$ the quaternionic theta series $\Theta^{R^\natural}_{[\mathfrak{A}R^\natural],[\mathfrak{B}R^\natural]}$.

From the equation (4.1.1), the zeta integral $Z(F,\eta,\Phi^\natural,0)$ becomes:

$$Z(F,\eta,\Phi^\natural,0) = 2\cdot \int_{\mathbb{A}^\times \mathrm{GL}_2(k)\backslash \mathrm{GL}_2^{+K}(\mathbb{A})} F(g)\overline{\Theta^\eta_K(g) I\left(g\begin{pmatrix}1 & 0\\ 0 & \det(g)^{-1}\end{pmatrix}, \varphi^\natural_{h_g}\right)}dg.$$

Write

$$\Theta^\eta_K = \sum_{\mathcal{A}\in\mathrm{Pic}(O_K)} \eta(\mathcal{A})\Theta_{\mathcal{A}},$$

where $\Theta_{\mathcal{A}}$ is the quadratic theta series attached to $\mathcal{A}$ (cf. Definition A.3). Let $\mathfrak{A} \in \mathcal{A}$ and put

$$\begin{aligned} \mathrm{S}_{\mathcal{A}} &:= k^\times \cdot \delta \cdot \mathbf{N}_{K/k}\big((\varpi_{\mathfrak{A}} O^\times_{\mathbb{A}^\infty_K}) \times K^\times_\infty\big)\\ &\subset k^\times \cdot \delta \cdot \mathbf{N}_{K/k}\Big(\mathbb{A}_K^{\infty,\times} \times K^\times_\infty\Big) = k^\times \cdot \mathbf{N}_{K/k}(\mathbb{A}_K^\times) \subset \mathbb{A}^\times.\end{aligned}$$

Here $\varpi_{\mathfrak{A}} \in \mathbb{A}_K^{\infty,\times}$ is taken in the beginning of Section 4. Then for $g \in \mathrm{GL}_2(\mathbb{A})$, one has that $\Theta_{\mathcal{A}}(g) = 0$ unless $\det(g)^{-1}$ lies in $\mathrm{S}_{\mathcal{A}}$. In the following lemma, we connect $\Theta_{\mathcal{A}} \cdot I(\cdot, \varphi^\natural_h)$ with the quaternionic theta series associated to the chosen A-order $R^\natural$.

Lemma 4.5. *Let $R^\natural$ be the Eichler A-order chosen as above. Take an ideal class $\mathcal{A}$ in $\mathrm{Pic}(O_K)$. Given $g \in \mathrm{GL}_2(\mathbb{A})$ with $\det(g) \in \mathbf{N}_{K/k}(\mathbb{A}_K^\times)$, choose $h_g \in \mathrm{GO}(V_\gamma)(\mathbb{A})$ so that $\nu(h_g) = \det(g)$. We then have*

$$\begin{aligned}&\Theta_{\mathcal{A}}(g)\cdot I\left(g\cdot\begin{pmatrix}1 & 0\\ 0 & \det(g)^{-1}\end{pmatrix}, \varphi^\natural_{h_g}\right)\\ =\ & \frac{\#\big(\mathbb{F}_q^\times \cap \mathbf{N}_{K/k}(K^\times)\big)}{|\delta|_{\mathbb{A}}^{\frac{1}{2}}\cdot\#(\mathbb{F}_K^\times)\cdot\#\mathrm{Pic}(O_K)} \cdot \begin{cases}\displaystyle\sum_{\mathcal{B}\in\mathrm{Pic}(O_K)} \mathrm{w}^{R^\natural}_{\mathcal{B}}\Theta^{R^\natural}_{\mathcal{AB},\mathcal{B}}(g), & \text{if } \det(g)^{-1} \in \mathrm{S}_{\mathcal{A}},\\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

Here the number $\mathrm{w}^{R^\natural}_{\mathcal{B}}$ is defined in Section 1.3.

Proof. It suffices to verify the case when $\det(g)^\infty = (\mathbf{N}_{K/k}(\varpi_{\mathfrak{A}})\cdot\delta^\infty)^{-1} \in \mathbf{N}_{K/k}(\mathbb{A}_K^{\infty,\times})$.

For each $\epsilon \in \mathbb{F}_q$, let

$$g_\epsilon := \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} g \begin{pmatrix} 1 & 0 \\ 0 & \epsilon^{-1}\det(g)^{-1} \end{pmatrix}.$$

By the assumption on g we have that

$$\Theta_{\mathcal{A}}(g) = \frac{|\delta|_{\mathbb{A}}^{\frac{1}{2}}}{\#(\mathbb{F}_K^\times)} |\det(g)|_{\mathbb{A}}^{-\frac{1}{2}} \cdot \sum_{(x,\epsilon)\in K\times\left(\mathbb{F}_q^\times \cap \mathbf{N}_{K/k}(K^\times)\right)} \left(\omega_K(g_\epsilon)\widetilde{\varphi}_{\mathfrak{A}}\right)(x), \tag{4.2.1}$$

where ω_K is the Weil representation associated to the quadratic space $(K, \mathbf{N}_{K/k})$, and

$$\widetilde{\varphi}_{\mathfrak{A}}(x) = \mathbf{1}_{(\overline{\varpi}_{\mathfrak{A}}\delta^\infty)^{-1}\widehat{O_K}}(x^\infty) \cdot \mathbf{1}_{O_\infty}(\mathbf{N}_{K/k}(x_\infty)\det(g)_\infty \pi_\infty^{\delta_\infty}).$$

Here $\widehat{O_K} := O_K \otimes_A O_{\mathbb{A}^\infty}$. Note that for every $\epsilon \in \mathbb{F}_q^\times \cap \mathbf{N}_{K/k}(K^\times)$ and $h_g \in \mathrm{GO}(V_\gamma)(\mathbb{A})$ with $\nu(h_g) = \det(g)$, by the Siegel-Weil formula in Theorem 4.1 we have that

$$E(\Phi_{\varphi^\natural}, 0, g) = E\left(\Phi_{\varphi^\natural}, 0, \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} g \begin{pmatrix} 1 & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}\right) = 2I(g_\epsilon, \varphi^\natural_{h_g}).$$

Let $\delta^{\frac{1}{2}} := (\pi_v^{\frac{\delta_v}{2}})_v \in \mathbb{A}^\times$, and take h_g such that $h_g^\infty = \left(\overline{\varpi}_{\mathfrak{A}}(\delta^{\frac{1}{2}})^\infty\right)^{-1}$. Then for each finite place v of k and $x_v \in V_\gamma(k_v)$, the Schwartz function $\varphi^\natural_{h_g,v}(x_v)$ is written as:

$$|\det(g_v)|_v^{-\frac{1}{2}} \cdot \begin{cases} \mathbf{1}_{\pi_v^{-\lfloor \frac{\mathrm{ord}_v(\gamma)}{2} \rfloor} O_{K_v}}(\overline{\varpi}_{\mathfrak{A},v}\pi_v^{\delta_v}x_v), & \text{if } v \text{ is inert in } K/k; \\ \mathbf{1}_{\Pi_v^{-\mathrm{ord}_v(\gamma)+\mathrm{ord}_v\left(\frac{\mathfrak{n}_F}{(\mathfrak{n}_F,\mathfrak{d}_K)}\right)} O_{K_v}}(\overline{\varpi}_{\mathfrak{A},v}\pi_v^{\delta_v}x_v), & \text{otherwise.} \end{cases}$$

Moreover, one has

$$\varphi^\natural_{h_g,\infty}(x_\infty) = |\det(g)_\infty|_\infty^{-\frac{1}{2}} \cdot 1_{O_\infty}(-\gamma \mathbf{N}_{K/k}(x_\infty)\det(g)_\infty \pi_\infty^{\delta_\infty}),$$

where $\gamma = j^2$. Identifying the quadratic space $(\mathcal{D}, \mathrm{Nr}_{\mathcal{D}/k})$ with $(K, \mathbf{N}_{K/k}) \oplus (K, -\gamma\mathbf{N}_{K/k})$, we obtain that for $x = x_1 + x_2 j \in \mathcal{D} = K + Kj$,

$$\begin{aligned} &\widetilde{\varphi}_{\mathfrak{A}}(x_1) \cdot \varphi^\natural_{h_g}\left(\frac{\overline{\varpi}_{\mathfrak{B}}}{\varpi_{\mathfrak{B}}} x_2\right) \\ = \; &\frac{|\delta|_{\mathbb{A}}^{\frac{1}{2}} |\det(g)|_{\mathbb{A}}^{-1}}{\#(\mathbb{F}_K^\times)} \cdot \mathbf{1}_{\mathfrak{A}\mathfrak{B}\widehat{R^\natural}\mathfrak{B}^{-1}}\left(\mathbf{N}_{K/k}(\varpi_{\mathfrak{A}}) \cdot (\delta^\infty)^{-1} x^\infty\right) \cdot \mathbf{1}_{O_\infty}(\mathrm{Nr}_{\mathcal{D}/k}(x_\infty)\det(g_\infty)\pi_\infty^{\delta_\infty}). \end{aligned}$$

The result then follows from Lemma 4.3, Defintion 4.4 and the identity (4.2.1). □

For any ideal classes $\mathcal{A}, \mathcal{B} \in \mathrm{Pic}(O_K)$, the theta series $\Theta^{R^\natural}_{\mathcal{AB},\mathcal{B}}$ is a Drinfeld-type automorphic form of level $\mathfrak{n}_{F,K}$. In the following, we derive:

Proposition 4.6. *Let F be a normalized newform of square-free level $\mathfrak{n}_F$ with central character ω_F. Let $R^\natural$ be an Eichler A-order chosen as above. Given an imaginary quadratic field K/k and a continuous character η on* $\mathrm{Gal}(k^{\mathrm{sep}}/K)$ *satisfying the conditions* C.1 *and* C.2*, we have*

$$Z(F, \eta, \Phi^\natural, 0) = \frac{\#(\mathbb{F}_q^\times)}{|\delta|_{\mathbb{A}}^{\frac{1}{2}} \cdot \#(\mathbb{F}_K^\times)} \cdot \frac{2}{\#\,\mathrm{Pic}(O_K)} \cdot \left(\prod_{\substack{v \mid \mathfrak{d}_K, \\ v \neq \infty}} 2^{-1}\right) \cdot \left(\prod_{\substack{\mathfrak{p} \lhd A: \\ \mathfrak{p} \mid \frac{\mathfrak{d}_K}{(\mathfrak{n}_F, \mathfrak{d}_K)}}} (1 + \|\mathfrak{p}\|)^{-1}\right)$$

$$\cdot \left(F, \sum_{\mathcal{A},\mathcal{B}\in \mathrm{Pic}(O_K)} \eta(\mathcal{A}) \cdot \mathrm{w}^{R^\natural}_{\mathcal{B}} \cdot \Theta^{R^\natural}_{\mathcal{AB},\mathcal{B}}\right)_{\mathfrak{n}_{F,K}}.$$

Here $(\cdot,\cdot)_{\mathfrak{n}_{F,K}}$ *is the Petersson inner product with respect to the level* $\mathfrak{n}_{F,K}$*.*

Proof. Given $u \in O_{\mathbb{A}^\infty}^\times$, we choose $\kappa_u = (\kappa_u^\infty, \kappa_{u,\infty}) \in \mathrm{GL}_2(O_{\mathbb{A}^\infty}) \times \mathrm{GL}_2(k_\infty) \subset \mathrm{GL}_2(\mathbb{A})$, where $\kappa_u^\infty := \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}$ and

$$\kappa_{u,\infty} := \begin{cases} 1, & \text{if } \chi_K(u) = 1; \\ \begin{pmatrix} 0 & 1 \\ \pi_\infty & 0 \end{pmatrix}, & \text{if } \chi_K(u) = -1 \text{ and } \infty \text{ is inert in } K/k; \\ \begin{pmatrix} 1 & 0 \\ 0 & u_\infty \end{pmatrix} \text{ with } u_\infty \notin \mathbf{N}_{K/k}(O_{K,\infty}^\times), & \text{if } \chi_K(u) = -1 \text{ and } \infty \text{ is ramified in } K/k. \end{cases}$$

We also take $r_u \in \mathrm{GL}_2(k)$ such that $\det(r_u \cdot \kappa_u) \in \mathbf{N}_{K/k}(\mathbb{A}_K^\times)$.

For every $g \in \mathrm{GL}_2(\mathbb{A})$ with $\det(g)^{-1} \in \mathrm{S}_{\mathcal{A}} \cap \mathbf{N}_{K/k}(\mathbb{A}_K^\times)$, put $g_u = r_u \cdot g \cdot \kappa_u$. Then for all $u \in O_{\mathbb{A}^\infty}^\times$ and $\mathcal{B} \in \mathrm{Pic}(O_K)$, one has

$$F(g_u) \cdot \Theta_{\mathcal{AB},\mathcal{B}}^{R^\natural}(g_u) = F(g) \cdot \Theta_{\mathcal{AB},\mathcal{B}}^{R^\natural}(g).$$

On the other hand, Lemma 4.5 says that $I(g_u, \varphi_{h_{g_u}}^\natural) \cdot \Theta_{\mathcal{A}}(g_u)$ is equal to

$$\frac{\#\left(\mathbb{F}_q^\times \cap \mathbf{N}_{K/k}(K^\times)\right)}{|\delta|_{\mathbb{A}}^{\frac{1}{2}} \cdot (\#(\mathbb{F}_K^\times)) \cdot \#\,\mathrm{Pic}(O_K)} \cdot \begin{cases} \sum_{\mathcal{B} \in \mathrm{Pic}(O_K)} \mathrm{w}_{\mathcal{B}}^{R^\natural} \Theta_{\mathcal{AB},\mathcal{B}}^{R^\natural}(g), & \text{if } u \in \mathbb{F}_q^\times \cdot \mathbf{N}_{K/k}(O_{\mathbb{A}_K^\infty}^\times); \\ 0, & \text{otherwise.} \end{cases}$$

Since

$$\#\left(\frac{O_{\mathbb{A}^\infty}^\times}{\mathbb{F}_q^\times \cdot \mathbf{N}_{K/k}(O_{\mathbb{A}_K^\infty}^\times)}\right) = \left(\prod_{\substack{v \mid \mathfrak{d}_K \\ v \neq \infty}} 2\right) \cdot \frac{\#\left(\mathbb{F}_q^\times \cap \ \mathbf{N}_{K/k}(K^\times)\right)}{\#\left(\mathbb{F}_q^\times\right)},$$

we obtain

$$Z(F,\eta,\Phi^\natural,0) = \left(\prod_{\substack{v \mid \mathfrak{d}_K, \\ v \neq \infty}} 2^{-1}\right) \frac{\#(\mathbb{F}_q^\times)}{|\delta|_{\mathbb{A}}^{\frac{1}{2}} \cdot \#(\mathbb{F}_K^\times)} \cdot \frac{2}{\#\,\mathrm{Pic}(O_K)}$$
$$\cdot \sum_{\mathcal{A},\mathcal{B} \in \mathrm{Pic}(O_K)} \int_{\mathbb{A}^\times \mathrm{GL}_2(k) \backslash \mathrm{GL}_2^{+K}(\mathbb{A})} F(g) \cdot \overline{\eta(\mathcal{A}) \mathrm{w}_{\mathcal{B}}^{R^\natural} \cdot \Theta_{\mathcal{AB},\mathcal{B}}^{R^\natural}(g)}\, dg.$$

The proof is completed by the following equality

$$\int_{Z(\mathbb{A}) \mathrm{GL}_2(k) \backslash \mathrm{GL}_2^{+K}(\mathbb{A})} F(g) \cdot \overline{\Theta_{\mathcal{AB},\mathcal{B}}^{R^\natural}(g)}\, dg = \left(\prod_{\substack{\mathfrak{p} \lhd A: \\ \mathfrak{p} \mid \frac{\mathfrak{d}_K}{(\mathfrak{n}_F, \mathfrak{d}_K)}}} (1 + \|\mathfrak{p}\|)^{-1}\right) \cdot \left(F, \Theta_{\mathcal{AB},\mathcal{B}}^{R^\natural}\right)_{\mathfrak{n}_{F,K}}.$$

□

Now, we would like to "lower" the level $\mathfrak{n}_{F,K}$ to $\mathfrak{n}_F$. Given ideals $\mathfrak{n}_1, \mathfrak{n}_2 \lhd A$ with $\mathfrak{n}_1 \mid \mathfrak{n}_2$, the trace map $\mathrm{Tr}_{\mathfrak{n}_1}^{\mathfrak{n}_2} : \mathcal{M}_0(\mathfrak{n}_2) \to \mathcal{M}_0(\mathfrak{n}_1)$ is defined by

$$\mathrm{Tr}_{\mathfrak{n}_1}^{\mathfrak{n}_2}(F_2)(g) := \sum_{\kappa \in \mathcal{K}_0(\mathfrak{n}_1\infty)/\mathcal{K}_0(\mathfrak{n}_2\infty)} F_2(g\kappa), \quad \forall g \in \mathrm{GL}_2(\mathbb{A}).$$

For $F_1 \in \mathcal{M}_0(\mathfrak{n}_1)$ and $F_2 \in \mathcal{M}_0(\mathfrak{n}_2)$, suppose F_1 is a cusp form. One then has

$$(F_1, F_2)_{\mathfrak{n}_2} = \left(F_1, \mathrm{Tr}_{\mathfrak{n}_1}^{\mathfrak{n}_2}(F_2)\right)_{\mathfrak{n}_1}.$$

Choose an Eichler A-order $R^\star$ of type $(\mathfrak{n}_F/\mathfrak{n}_{F,\eta}^-, \mathfrak{n}_{F,\eta}^-)$ with $R^\star \supset R^\natural \supset O_K$. Then:

Lemma 4.7. *For each ideal class $\mathcal{A} \in \operatorname{Pic}(O_K)$, we have*

$$\sum_{\mathcal{B}\in\operatorname{Pic}(O_K)} \mathrm{w}_{\mathcal{B}}^{R^\natural} \cdot \operatorname{Tr}_{\mathfrak{n}_F}^{\mathfrak{n}_{F,K}}(\Theta_{\mathcal{AB},\mathcal{B}}^{R^\natural}) = \left(\prod_{\substack{\mathfrak{p}\lhd A:\\ \mathfrak{p}\mid\frac{\mathfrak{d}_K}{(\mathfrak{n}_F,\mathfrak{d}_K)}}} 2\right) \sum_{\mathcal{B}\in\operatorname{Pic}(O_K)} \mathrm{w}_{\mathcal{B}}^{R^\star} \cdot \Theta_{\mathcal{AB},\mathcal{B}}^{R^\star}.$$

Proof. Choose $\mathfrak{A} \in \mathcal{A}$ and $b_{\mathfrak{A}} \in \mathbb{A}_K^{\infty,\times}$ so that $K \cap b_{\mathfrak{A}} O_{\mathbb{A}_K^\times} = \mathfrak{A}$. For each ideal $\mathfrak{m} \lhd A$ which divides $\mathfrak{d}_K$ and coprime to $\mathfrak{n}_F$, let $\mathfrak{M} \lhd O_K$ be the ideal with $\mathbf{N}_{K/k}(\mathfrak{M}) = \mathfrak{m}$. We then verify

$$\sum_{\kappa\in\mathcal{K}(\mathfrak{n}_F\infty)/\mathcal{K}(\mathfrak{n}_{F,K}\infty)} \omega_{\mathcal{D}}(\kappa)\tilde{\varphi}_{\mathfrak{AB},\mathfrak{B}}^{R^\natural} = \sum_{\substack{\mathfrak{m}\lhd A:\\ \mathfrak{m}\mid\frac{\mathfrak{d}_K}{(\mathfrak{n}_F,\mathfrak{d}_K)}}} \tilde{\varphi}_{\mathfrak{AB}\mathfrak{M}^{-1},\mathfrak{B}\mathfrak{M}^{-1}}.$$

Therefore

$$\begin{aligned}\sum_{\mathcal{B}\in\operatorname{Pic}(O_K)} \mathrm{w}_{\mathcal{B}}^{R^\natural} \cdot \operatorname{Tr}_{\mathfrak{n}_F}^{\mathfrak{n}_{F,K}}(\Theta_{\mathcal{AB},\mathcal{B}}^{R^\natural}) &= \sum_{\substack{\mathfrak{m}\lhd A:\\ \mathfrak{m}\mid\frac{\mathfrak{d}_K}{(\mathfrak{n}_F,\mathfrak{d}_K)}}} \left(\sum_{\mathcal{B}\in\operatorname{Pic}(O_K)} \mathrm{w}_{\mathcal{B}[\mathfrak{M}]^{-1}}^{R^\star} \cdot \Theta_{\mathcal{AB}[\mathfrak{M}]^{-1},\mathcal{B}[\mathfrak{M}]^{-1}}^{R^\star}\right) \\ &= \left(\prod_{\substack{\mathfrak{p}\lhd A:\\ \mathfrak{p}\mid\frac{\mathfrak{d}_K}{(\mathfrak{n}_F,\mathfrak{d}_K)}}} 2\right) \sum_{\mathcal{B}\in\operatorname{Pic}(O_K)} \mathrm{w}_{\mathcal{B}}^{R^\star} \cdot \Theta_{\mathcal{AB},\mathcal{B}}^{R^\star}(g).\end{aligned}$$

Here $[\mathfrak{M}] \in \operatorname{Pic}(O_K)$ denotes the ideal class represented by $\mathfrak{M}$. □

Let

$$\Theta_\eta^{R^\star} := \sum_{\mathcal{A},\mathcal{B}\in\operatorname{Pic}(O_K)} \eta(\mathcal{A}) \cdot \mathrm{w}_{\mathcal{B}}^{R^\star} \cdot \Theta_{\mathcal{AB},\mathcal{B}}^{R^\star} = \sum_{\mathcal{A},\mathcal{B}\in\operatorname{Pic}(O_K)} \eta(\mathcal{AB}^{-1}) \cdot \mathrm{w}_{\mathcal{B}}^{R^\star} \cdot \Theta_{\mathcal{A},\mathcal{B}}^{R^\star}.$$

We thereby arrive at the following formula for the central critical value $L(F \times \eta, 0)$:

Theorem 4.8. *Let F be a normalized newform of square-free level $\mathfrak{n}_F$ with central character ω_F. Given an imaginary quadratic field K/k and a continuous character η on* $\operatorname{Gal}(k^{\mathrm{sep}}/K)$ *satisfying the conditions* C.1 *and* C.2 *(in the beginning of* Section 3*), we have*

$$L(F\times\eta,0) = \mathcal{P}(F,K) \cdot \frac{4\cdot\left(F,\Theta_\eta^{R^\star}\right)_{\mathfrak{n}_F}}{f_K(\infty)^2\cdot\#\operatorname{Pic}(A)},$$

where

$$\mathcal{P}(F,K) := \left[\left(\prod_{v\mid(\mathfrak{d}_K,\mathfrak{n}_F\infty)} 2\right)\|\mathfrak{d}_K\|^{\frac{1}{2}}\right]^{-1} \cdot \left(\frac{\#(\mathbb{F}_q^\times)}{\#(\mathbb{F}_K^\times)}\right)^2,$$

and $f_K(\infty)$ is the residue degree of ∞ in K/k.

Proof. This follows from the equation (3.2.3), Proposition 4.6, and Lemma 4.7. □

Decomposing $\Theta_\eta^{R^\star}$ as $\Theta_\eta^{R^\star} = c\cdot F + F^\perp$ where

$$c = c(F,\eta) := \frac{\left(F,\Theta_\eta^{R^\star}\right)_{\mathfrak{n}_F}}{(F,F)_{\mathfrak{n}_F}} \quad \text{and} \quad (F,F^\perp)_{\mathfrak{n}_F} = 0,$$

this ratio c is in fact independent of the chosen Eichler A-order $R^\star$ of type $(\mathfrak{n}_F/\mathfrak{n}_{F,\eta}^-, \mathfrak{n}_{F,\eta}^-)$, and

$$L(F\times\eta,0) = \mathcal{P}(F,K)\cdot\frac{4\cdot(F,F)_{\mathfrak{n}_F}}{f_K(\infty)^2\cdot\#\operatorname{Pic}(A)}\cdot c(F,\eta).$$

In the next subsection, we will give a geometric interpretation of this most crucial factor $c(F,\eta)$ in terms of the special points on the corresponding definite Shimura curve.

4.3. **Definite Shimura curve.** Given a definite quaternion algebra $\mathcal{D}/k$ with the discriminant $\mathfrak{n}^-$, let Y be the genus zero curve over k whose points over any k-algebra S are

$$Y(S) = \{x \in \mathcal{D} \otimes_k S : x \neq 0, \mathrm{Tr}_{\mathcal{D}/k}(x) = \mathrm{Nr}_{\mathcal{D}/k}(x) = 0\}/S^\times.$$

The group $\mathcal{D}^\times$ acts on Y (from the left) by conjugation. Taking $\mathfrak{n}^+ \lhd A$ coprime to $\mathfrak{n}^-$, let R be an Eichler A-order of type $(\mathfrak{n}^+, \mathfrak{n}^-)$.

Definition 4.9. The *definite Shimura curve* $X = X_{\mathfrak{n}^+,\mathfrak{n}^-}$ of type $(\mathfrak{n}^+, \mathfrak{n}^-)$ is defined as

$$X = \mathcal{D}^\times \backslash \left(Y \times (\mathcal{D}^\times_{\mathbb{A}^\infty}/\widehat{R}^\times) \right).$$

Here $\mathcal{D}^\times$ acts on $Y \times (\mathcal{D}^\times_{\mathbb{A}^\infty}/\widehat{R}^\times)$ diagonally. For each $x \in X$, write $x = [y, g]$ with representatives $y \in Y$ and $g \in \mathcal{D}^\times_{\mathbb{A}^\infty}/\widehat{R}^\times$.

Up to isomorphism, X is independent of the choice of the Eichler A-order with the given type $(\mathfrak{n}^+, \mathfrak{n}^-)$. The connected components of X correspond canonically to locally-principal right ideal classes of R. Then the Picard group of X can be written as

$$\mathrm{Pic}(X) = \bigoplus_{\mathcal{I} \in \mathrm{Cl(R)}} \mathbb{Z} e_{\mathcal{I}},$$

where $e_{\mathcal{I}}$ is the class of the component corresponding to $\mathcal{I} \in \mathrm{Cl}(R)$. We refer $\{e_{\mathcal{I}} : \mathcal{I} \in \mathrm{Cl}(R)\}$ as a standard basis of $\mathrm{Pic}(X)$.

Remark 4.10.

(1) Let L be a quadratic field of k, then there exists a canonical identification of $Y(L)$ with $\mathrm{Hom}(L, \mathcal{D}) := \{\iota \mid \iota : L \hookrightarrow \mathcal{D}\}$.
(2) Similar to the case of modular curves, one can associate Hecke correspondences $t_{\mathfrak{m}}$ for $\mathfrak{m} \lhd A$ and Atkin-Lehner involution $w_{\mathfrak{p}}$ for prime $\mathfrak{p} \mid \mathfrak{n}^+\mathfrak{n}^-$ on X (cf. [2, Chapter II Section 4.1]). We denote by $\mathbb{T}_{\mathfrak{n}^+}$ the Hecke algebra $\mathbb{Q}[t_{\mathfrak{m}} \mid \mathfrak{m} \lhd A \text{ coprime to } \mathfrak{n}^+]$ on $\mathrm{Pic}(X)$.

Let $\mathcal{I}$, $\mathcal{J} \in \mathrm{Cl}(R)$. The Gross height pairing $\langle \cdot, \cdot \rangle$ on $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ is defined by setting

$$\langle e_{\mathcal{I}}, e_{\mathcal{J}} \rangle := \begin{cases} 0 & \text{if } \mathcal{I} \neq \mathcal{J}, \\ \mathrm{w}^R_{\mathcal{I}}, & \text{if } \mathcal{I} = \mathcal{J}; \end{cases}$$

and extending bi-linearly. Here $\mathrm{w}^R_{\mathcal{I}}$ is defined in Section 1.3. Thus $\mathrm{Pic}(X)^\vee := \mathrm{Hom}(\mathrm{Pic}(X), \mathbb{Z})$ can be viewed as a subgroup of $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ with the dual basis $\{\check{e_{\mathcal{I}}} := e_{\mathcal{I}}/\mathrm{w}^R_{\mathcal{I}} \mid \mathcal{I} \in \mathrm{Cl}(R)\}$ via this pairing.

Note that the permutation τ introduced in Section 1.3 induces an order 2 automorphism on $\mathrm{Pic}(X)$ by setting $\tau e_{\mathcal{I}} := e_{\tau(\mathcal{I})}$.

Proposition 4.11. (cf. [2, Chapter II Proposition 2.7]) *Given classes $e, e' \in \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, we have*

$$\langle t_{\mathfrak{m}} e, e' \rangle = \langle e, t^*_{\mathfrak{m}} e' \rangle$$

for $\mathfrak{m} \lhd A$ coprime to $\mathfrak{n}^+$, where $t^*_{\mathfrak{m}} = \tau^{-1} t_{\mathfrak{m}} \tau$.

4.4. **Gross points.**

Definition 4.12. 1. Let X be the definite Shimura curve of type $(\mathfrak{n}^+, \mathfrak{n}^-)$. Let K be an imaginary quadratic field over k. A point $x = [y, g] \in X$ where $y \in Y(K)$ is called a *Gross point over* K.
2. Let $\iota_y : K \hookrightarrow \mathcal{D}$ be the embedding corresponding to y. Then

$$\iota_y(K) \cap g\hat{R}g^{-1} = \iota_y(O_{K,\mathfrak{c}})$$

for some quadratic order $O_{K,\mathfrak{c}} := A + \mathfrak{c}O_K$, where $\mathfrak{c}$ is an ideal of A. We say that $x = [y, g]$ has conductor $\mathfrak{c}$.

Let $a \in \mathbb{A}_K^{\infty,\times}$ and $x = [y, g] \in X$ be a Gross point of conductor $\mathfrak{c}$ over K. Let $\iota_y : K \hookrightarrow \mathcal{D}$ be the embedding corresponding to y. This also induces a homomorphism

$$\iota_y : \mathbb{A}_K^{\infty} \hookrightarrow \mathcal{D}_{\mathbb{A}^\infty},$$

and for $a \in \mathbb{A}_K^{\infty,\times}$ we define $x_a := [y, \iota_y(a)g]$. Then x_a is also a Gross point of conductor $\mathfrak{c}$. We then have a free action of $\mathrm{Pic}(O_{K,\mathfrak{c}})$ on the set $G_{O_{K,\mathfrak{c}}}$ of Gross points of conductor $\mathfrak{c}$ over K, and denote this action by sending the pair $(\mathcal{A}, x)$ to $x_{\mathcal{A}}$ for each $\mathcal{A} \in \mathrm{Pic}(O_{K,\mathfrak{c}})$ and $x \in G_{O_{K,\mathfrak{c}}}$.

4.5. **Hecke module homomorphism and Jacquet-Langlands correspondence.** Let $\mathcal{M}_0(\mathfrak{n}, \mathbb{Q})$ be the space of $\mathbb{Q}$-valued Drinfeld-type automorphic forms of level $\mathfrak{n} = \mathfrak{n}^+\mathfrak{n}^-$. Define the $\mathbb{Z}$-bilinear map

$$\Phi : \mathrm{Pic}(X) \times \mathrm{Pic}(X)^\vee \to \mathcal{M}_0(\mathfrak{n}, \mathbb{Q})$$

by: for $e = \sum_{\mathcal{I}\in \mathrm{Cl}(R)} a_I e_I \in \mathrm{Pic}(X)$ and $\check{e} = \sum_{\mathcal{J}\in \mathrm{Cl}(R)} a'_{\mathcal{J}} \check{e}_{\mathcal{J}} \in \mathrm{Pic}(X)^\vee$, set

$$\Phi(e, \check{e}) := \sum_{\mathcal{I},\mathcal{J}\in \mathrm{Cl}(R)} a_{\mathcal{I}} a'_{\mathcal{J}} \cdot \Theta^R_{\mathcal{I},\mathcal{J}}.$$

Here the theta series $\Theta^R_{\mathcal{I},\mathcal{J}}$ is introduced in Section 4.2.

Theorem 4.13. (cf. [2, Chapter III Theorem 3.11]) *The map*

$$\Phi : \mathrm{Pic}(X) \times \mathrm{Pic}(X)^\vee \longrightarrow \mathcal{M}_0(\mathfrak{n}, \mathbb{Q})$$

satisfies that for each ideal $\mathfrak{m} \lhd A$ *which is coprime to* $\mathfrak{n}^+$*, we have*

$$T_{\mathfrak{m}}\Phi(e, \check{e}) = \Phi(t_{\mathfrak{m}}e, \check{e}) = \Phi(e, t^*_{\mathfrak{m}}\check{e}).$$

Moreover, this map induces a homomorphism

$$(\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}) \otimes_{\mathbb{T}_{\mathfrak{n}^+}} (\mathrm{Pic}(X)^\vee \otimes_{\mathbb{Z}} \mathbb{Q}) \longrightarrow \mathcal{M}_0(\mathfrak{n}, \mathbb{Q})$$

as Hecke modules.

Set $\mathrm{Pic}(X)_{\mathbb{C}} := \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{C}$ and $\mathrm{Pic}(X)^\vee_{\mathbb{C}} := \mathrm{Pic}(X)^\vee \otimes_{\mathbb{Z}} \mathbb{C}$. Extend $\langle \cdot, \cdot \rangle$ to a pairing on $\mathrm{Pic}(X)_{\mathbb{C}} \times \mathrm{Pic}(X)^\vee_{\mathbb{C}}$ which is linear on the first coordinate and conjugate linear on the second coordinate. Let $\mathcal{S}_0^{(\mathfrak{n}^-)\text{-new}}(\mathfrak{n})$ be the space of Drinfeld-type $(\mathfrak{n}^-)$-new forms of level $\mathfrak{n}$. The Jacquet-Langlands correspondence in this setting says precisely that:

Theorem 4.14. (cf. [2, Chapter III Theorem 3.13]) *There exists an isomorphism (as* $\mathbb{C}$*-vector space)* JL *from* $\mathrm{Pic}_0(X)^\vee_{\mathbb{C}}$ *onto* $\mathcal{S}_0^{(\mathfrak{n}^-)\text{-new}}(\mathfrak{n})$ *such that*

$$\mathrm{JL}^{-1}(T_{\mathfrak{m}}F) = t^*_{\mathfrak{m}}\mathrm{JL}^{-1}(F), \quad \forall \mathfrak{m} \lhd A \text{ and } F \in \mathcal{S}_0^{(\mathfrak{n}^-)\text{-new}}(\mathfrak{n}).$$

Here

$$\mathrm{Pic}_0(X)^\vee_{\mathbb{C}} := \{\check{e} \in \mathrm{Pic}(X)^\vee_{\mathbb{C}} \mid \langle e_{0,\chi}, \check{e}\rangle = 0, \ \forall \chi : \mathrm{Pic}(A) \to \mathbb{C}^\times\}$$

with

$$e_{0,\chi} := \sum_{\mathcal{I} \in \mathrm{Cl}(R)} \chi(\mathrm{Nr}_{\mathcal{D}/k}(\mathcal{I}))^{-1} \cdot \frac{e_{\mathcal{I}}}{\mathrm{w}_{\mathcal{I}}^{R}} \in \mathrm{Pic}(X)_{\mathbb{C}}.$$

Here for each $\mathcal{I} \in \mathrm{Cl}(R)$, $\mathrm{Nr}_{\mathcal{D}/k}(\mathcal{I}) \in \mathrm{Pic}(A)$ *is the class represented by* $\mathrm{Nr}_{\mathcal{D}/k}(I)$ *for any* $I \in \mathcal{I}$.

Let F be a newform of level $\mathfrak{n}$. Define

$$e_F := \overline{\mathrm{JL}^{-1}(F)} \in \mathrm{Pic}_0(X)_{\mathbb{C}}^{\vee}.$$

Here $\bar{\check{e}} := \sum_{\mathcal{I} \in \mathrm{Cl}(R)} \overline{a_{\mathcal{I}}}\check{e}_{\mathcal{I}}$ for any $\check{e} = \sum_{\mathcal{J} \in \mathrm{Cl}(R)} a_{\mathcal{J}}\check{e}_{\mathcal{J}} \in \mathrm{Pic}_0(X)_{\mathbb{C}}^{\vee}$. The Hecke module homomorphism Φ in Theorem 4.14 has the very nice property:

Theorem 4.15. (cf. [2, Chapter III Theorem 3.14]) *Given a newform* F *of level* $\mathfrak{n}$*, one has*

$$\Phi(e, e_F) = \langle e, e_F \rangle \cdot F, \quad \forall e \in \mathrm{Pic}(X)_{\mathbb{C}}.$$

4.6. **Gross-type formula.** Let F be a normalized Drinfeld-type newform of square-free level $\mathfrak{n}_F$ with central character ω_F, K be an imaginary quadratic extension over k with discriminant divisor $\mathfrak{d}_K$, and a continuous character η on $\mathrm{Gal}(k^{\mathrm{sep}}/K)$ satisfying the conditions C.1 and C.2. Suppose the cardinality of $\Sigma = \Sigma(F,\eta)$ is even. We chose the definite quaternion algebra $\mathcal{D} = \mathcal{D}(F,\eta)$ at the end of Section 4.1 and the Eichler A-order $R^\star$ of type $(\mathfrak{n}_F/\mathfrak{n}_{F,\eta}^-, \mathfrak{n}_{F,\eta}^-)$ with a fixed embedding $\iota : O_K \hookrightarrow R^\star$ after Proposition 4.6. Let x be the Gross point on $X = X_{\mathfrak{n}_F/\mathfrak{n}_{F,\eta}^-, \mathfrak{n}_{F,\eta}^-}$ corresponding to ι, which has trivial conductor. For each ideal class $\mathcal{A} \in \mathrm{Pic}(O_K)$, denote $e_{\mathcal{A}}(x)$ to be the divisor class $(x_{\mathcal{A}})$ in $\mathrm{Pic}(X)$. Let

$$e_\eta(x) = \sum_{\mathcal{A} \in \mathrm{Pic}(O_K)} \eta(\mathcal{A}) \cdot e_{\mathcal{A}}(x).$$

We then have

Theorem 4.16. *Let* F *be a normalized newform of square-free level* $\mathfrak{n}_F$ *with central character* ω_F*. Given an imaginary quadratic field* K/k *and a continuous character* η *on* $\mathrm{Gal}(k^{\mathrm{sep}}/K)$ *satisfying the conditions* C.1 *and* C.2 *(in the beginning of* Section 3*), suppose that the root number* $(-1)^{\Sigma}$ *of* $L(F \times \eta, s)$ *is one. Then we have*

$$L(F \times \eta, 0) = \mathcal{P}(F,K) \cdot \frac{4 \cdot (F,F)_{\mathfrak{n}_F}}{f_K(\infty)^2 \cdot \#\mathrm{Pic}(A)} \cdot \frac{|\langle e_\eta(x), e_F \rangle|^2}{\langle e_F, e_F \rangle}.$$

Here $\mathcal{P}(F,K)$ *is introduced in* Theorem 4.8*.*

Proof. The Hecke module homomorphism in Theorem 4.13 gives that

$$\sum_{\mathcal{A},\mathcal{B} \in \mathrm{Pic}(O_K)} \eta(\mathcal{A}\mathcal{B}^{-1}) \cdot \Phi(e_{\mathcal{A}}(x), e_{\mathcal{B}}(x)) = \Phi(e_\eta(x), e_\eta(x)).$$

Furthermore, by Theorem 4.14, the divisor class $e_F = \overline{JL^{-1}(F)} \in \mathrm{Pic}(X)_{\mathbb{C}}^{\vee}$ corresponding to F satisfies that

$$\Phi(e_\eta(x), e_F) = \langle e_\eta(x), e_F \rangle \cdot F.$$

Write $e_\eta(x)$ as

$$e_\eta(x) = \frac{\langle e_\eta(x), e_F \rangle}{\langle e_F, e_F \rangle} e_F + e_F^{\perp}$$

where $\langle e_F, e_F^{\perp} \rangle = 0$. Then

$$\big(F, \Phi(e_\eta(x), e_\eta(x))\big)_{\mathfrak{n}_F} = (F,F)_{\mathfrak{n}_F} \cdot \frac{|\langle e_\eta(x), e_F \rangle|^2}{\langle e_F, e_F \rangle}.$$

The result holds by Theorem 4.8. □

Remark 4.17.

(1) $|\langle e_\eta(x), e_F\rangle|^2$ is actually independent of the chosen Gross point x.
(2) Let $k = \mathbb{F}_q(t)$ with q odd and $A = \mathbb{F}_q[t]$. Let F be a normalized Drinfeld-type newform of square-free level $N \lhd A$. Suppose the number of prime factors of N is odd. Let $K = k(\sqrt{D})$ be an imaginary quadratic field with $D \in A$ irreducible and $\left(\frac{D}{P}\right) = -1$ for every prime factor P of N. Then Theorem 4.16 coincides with Wei-Yu's formula in [18].
(3) For $\check{e} \in \mathrm{Pic}(X)^\vee_{\mathbb{C}}$ we can view $\check{e}$ as a function on $\mathcal{D}^\times\backslash\mathcal{D}^\times(\mathbb{A})$ which is right invariant under $\widehat{R}^{*,\times}O^\times_{\mathcal{D}_\infty}$. Set

$$\langle\!\langle e_F, e_F\rangle\!\rangle := \int_{\mathcal{D}^\times\mathbb{A}^\times\backslash\mathcal{D}^\times(\mathbb{A})} e_F(g)\overline{e_F(g)}dg,$$

where dg is induced from the Haar measure dg on $\mathcal{D}^\times(\mathbb{A})$ with $\mathrm{Vol}(\widehat{R}^{*,\times}O^\times_{\mathcal{D}_\infty}, dg) = 1$. Then

$$\langle\!\langle e_F, e_F\rangle\!\rangle = \frac{2}{\#\,\mathrm{Pic}(A)} \cdot \langle e_F, e_F\rangle.$$

Let $\iota_x : K \hookrightarrow \mathcal{D}$ be the embedding corresponding to x. One has

$$\langle e_\eta(x), e_F\rangle = \frac{f_K(\infty)\cdot \#\,\mathrm{Pic}(A)}{2} \cdot \int_{K^\times\mathbb{A}^\times\backslash\mathbb{A}_K^\times} e_F(a)\eta(a)d^\times a.$$

Here $d^\times a$ is induced from the Haar measure on $\mathbb{A}_K^\times$ with $\mathrm{Vol}(O^\times_{\mathbb{A}_K}) = 1$. Therefore we arrive at the following inner product formula for $L(F\times\eta, 0)$:

$$L(F\times\eta,0) = \mathcal{P}(F,K)' \cdot \frac{2\cdot (F,F)_{\mathfrak{n}_F}}{\langle\!\langle e_F, e_F\rangle\!\rangle} \cdot \Big|\int_{K^\times\mathbb{A}^\times\backslash\mathbb{A}_K^\times} e_F(a)\eta(a)d^\times a\Big|^2,$$

where $\mathcal{P}(F,K)' := \|\mathfrak{d}_K\|^{-1/2} \cdot \prod_{v|(\mathfrak{d}_K,\mathfrak{n}_F\infty)} 2^{-1}$.

4.7. **Example.** Here we provide one example which is not covered by Wei-Yu's formula in [18]. Let $k = \mathbb{F}_3(t)$. Let E be the following elliptic curve over k:

$$E : Y^2 = x^3 + (t^2+1)x^2 + (2t+1)x + t.$$

The conductor of E is $N_0 := t\cdot(t^2+2t+2)$. More precisely, E has split multiplicative reduction at t and ∞, and non-split multiplicative reduction at t^2+2t+2. Observe that $L(E_{/k}, s) = 1$. Let $D = 2t^2+2t+1$ and $K = k(\sqrt{D})$. Then ∞, t^2+2t+2 are inert, and t is split in K/k. Consider the twist E_D of E by K. The conductor of E_D is $D^2\cdot t\cdot(t^2+2t+2)\cdot\infty$. In particular, E_D has non-split multiplicative reduction at ∞, split multiplicative reduction at t^2+2t+2 and t and additive reduction at D. We calculate the L-function of E_D:

$$L(E_{D/k}, s) = 1 - 3q^{-2s} + 81q^{-4s}.$$

Thus $L(E_{/K}, 1) = L(E,1)\cdot L(E_D, 1) = 5/3$.

On the other hand, from a formula of Gekeler (cf. [13] Theorem 1.1) we have that the Petersson norm $(F_E, F_E)_{N_0} = 10$. Let $\mathcal{D}$ be the quaternion algebra over k ramified precisely at ∞ and t^2+t+2. Then

$$\mathcal{D} = k + k\alpha + k\beta + k\alpha\beta,$$

where $\alpha^2 = t$, $\beta^2 = -t^2+t+1$, and $\alpha\beta = -\beta\alpha$. Let $R = A + A\alpha + A\beta + A\alpha\beta$, which is an Eichler A-order of type (t, t^2+2t+2). The class number of the locally-principal right ideal classes of R is 4. We calculate the following Brandt matrices:

$$B(t) = \begin{pmatrix} 1 & 2 & 2 & 2 \\ 2 & 1 & 2 & 2 \\ 2 & 2 & 1 & 2 \\ 2 & 2 & 2 & 1 \end{pmatrix}, B(t+1) = \begin{pmatrix} 0 & 0 & 2 & 2 \\ 0 & 2 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{pmatrix}, B(t+2) = \begin{pmatrix} 2 & 2 & 0 & 0 \\ 2 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 2 & 1 \end{pmatrix}.$$

Therefore, we can choose the corresponding point e_{F_E} to be $[0, 0, 1/\sqrt{2}, -1/\sqrt{2}]^t$.

In this case, $O_K = A[\sqrt{D}]$ with $\# \operatorname{Pic}(O_K) = 2$. Notice that

$$t \cdot O_K = (t, 1 + D)(t, 1 - D) := P_t \overline{P_t}$$

We take the embedding from K into $\mathcal{D}$ by sending $\sqrt{D}$ to $\alpha+\beta$, and denote the corresponding Gross point by x. Take $\eta = \mathbf{1}_K$. Then

$$e_{\mathbf{1}_K}(x) := e_R + e_{P_t R} = [1, 0, 0, 0]^t + [0, 0, 1, 0]^t = [1, 0, 1, 0]^t \in \operatorname{Pic}(X_{\mathfrak{n}^+,\mathfrak{n}^-}) \cong \sum_{i=1}^{4} \mathbb{Z} \cdot e_i.$$

Thus

$$|\langle e_{\mathbf{1}_K}(x), e_{F_E} \rangle|^2 = 1/2 \quad \text{ and } \quad \frac{(F_E, F_E)_{N_0}}{q^{\frac{1}{2}(\deg D)}} \cdot |\langle e_{\mathbf{1}_K}(x), e_{F_E} \rangle|^2 = 5/3 = L(E_{/K}, 1).$$

5. Metaplectic forms and a Waldspurger-type formula

With an infinite place chosen, we establish in this section a Shimura-type correspondence between Drinfeld-type forms and metaplectic forms. A Waldspurger-type formula is derived at the end.

5.1. **Metaplectic group.** We have assumed that q is odd. Let v be a place of k. The *Kubota 2-cocycle* $\sigma'_v : \mathrm{SL}_2(k_v) \times \mathrm{SL}_2(k_v) \to \{\pm 1\}$ is defined by (cf. [8]):

$$\sigma'_v(g_1, g_2) := \left(\frac{x(g_1 g_2)}{x(g_1)}, \frac{x(g_1 g_2)}{x(g_2)} \right)_v, \quad \forall g_1, g_2 \in \mathrm{SL}_2(k_v).$$

Here

$$x \begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{cases} c, & \text{if } c \neq 0, \\ d, & \text{if } c = 0; \end{cases}$$

and $(\cdot, \cdot)_v$ is the Hilbert quadratic symbol at v. Define a map $s_v : \mathrm{SL}_2(k_v) \to \{\pm 1\}$ by setting

$$s_v \begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{cases} (c, d)_v, & \text{if } \operatorname{ord}_v(c) \text{ is odd and } d \neq 0, \\ 1, & \text{otherwise.} \end{cases}$$

Let σ_v be the 2-cocycle defined by

$$\sigma_v(g_1, g_2) := \sigma'_v(g_1, g_2) s_v(g_1) s_v(g_2) s_v(g_1 g_2)^{-1}, \quad \forall g_1, g_2 \in \mathrm{SL}_2(k_v).$$

It is known that (cf. [5, Section 2.3]) $\sigma_v(\kappa_1, \kappa_2) = 1 \; \forall \kappa_1, \kappa_2 \in \mathrm{SL}_2(O_v)$. Hence σ_v induces a central extension $\widetilde{\mathrm{SL}_2(k_v)}$ of $\mathrm{SL}_2(k_v)$ by $\{\pm 1\}$ which splits on the subgroup $\mathrm{SL}_2(O_v)$. More precisely, the extension $\widetilde{\mathrm{SL}_2}(k_v)$ is identified with $\mathrm{SL}_2(k_v) \times \{\pm 1\}$ (as sets) with the following group law:

$$(g_1, \xi_1) \cdot (g_2, \xi_2) = \big(g_1 g_2, \xi_1 \xi_2 \sigma_v(g_1, g_2)\big).$$

Globally, we define a 2-cocycle σ on $\mathrm{SL}_2(\mathbb{A})$ by setting $\sigma := \otimes_v \sigma_v$, and let $\widetilde{\mathrm{SL}_2}(\mathbb{A})$ be the corresponding central extension of $\mathrm{SL}_2(\mathbb{A})$ by $\{\pm 1\}$. The section

$$\begin{array}{ccc} \mathrm{SL}_2(\mathbb{A}) & \longrightarrow & \widetilde{\mathrm{SL}_2}(\mathbb{A}) \\ \kappa & \longmapsto & \tilde{\kappa} := (\kappa, 1) \end{array}$$

becomes a group homomorphism when restricting to $\mathrm{SL}_2(O_{\mathbb{A}})$. The image of $\mathrm{SL}_2(O_{\mathbb{A}})$ in $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ is denoted by $\widetilde{\mathrm{SL}}_2(O_{\mathbb{A}})$. Moreover, for every $\gamma \in \mathrm{SL}_2(k)$, the value $s(\gamma) := \prod_v s_v(\gamma)$ is well-defined, and the embedding

$$\begin{array}{rccc} \tilde{s}\colon & \mathrm{SL}_2(k) & \longrightarrow & \widetilde{\mathrm{SL}}_2(\mathbb{A}) \\ & \gamma & \longmapsto & \tilde{s}(\gamma) := \big(\gamma, s(\gamma)\big) \end{array}$$

preserves the group law. We are interested in functions on $\tilde{s}\big(\mathrm{SL}_2(k)\big)\backslash\widetilde{\mathrm{SL}_2(\mathbb{A})}/\widetilde{\mathcal{K}}$, where $\mathcal{K} \subset \mathrm{SL}_2(O_{\mathbb{A}})$ is a compact subgroup.

5.2. **Theta series from pure quaternions.** In Section 1.1, we have chosen an additive character $\psi : k\backslash\mathbb{A} \to \mathbb{C}^\times$ with even conductor at every place of k. Let $\mathcal{D}$ be a definite (i.e. ramified at ∞) quaternion algebra over k with discriminant $\mathfrak{n}^- \lhd A$. Consider the quadratic space $(\mathcal{D}^o, \mathrm{Nr}_{\mathcal{D}/k})$ where $\mathcal{D}^o$ is the set of pure quaternions in $\mathcal{D}$, i.e.

$$\mathcal{D}^o := \{b \in \mathcal{D} \mid \mathrm{Tr}_{\mathcal{D}/k}(b) = 0\}.$$

For each place v of k, put $\mathcal{D}^o(k_v) := \mathcal{D}^o \otimes_k k_v$. The *Weil index* $\varepsilon_{\mathcal{D}^o,v}$ is defined by:

$$\varepsilon_{\mathcal{D}^o,v}(a_v) := \int_{L_3} \psi_v(a_v \,\mathrm{Nr}_{\mathcal{D}/k}(x)) d_{a_v}x, \quad \forall a_v \in k_v^\times,$$

where L_3 is a sufficiently large O_v-lattice in $\mathcal{D}^o(k_v)$. The Haar measure $d_{a_v}x$ is self-dual with respect to the pairing

$$(x, y) \mapsto \psi_v(a_v \cdot \mathrm{Tr}_{\mathcal{D}/k}(x\bar{y})), \quad \forall x, y \in \mathcal{D}^o(k_v).$$

It is clear that $\varepsilon_{\mathcal{D}^o,v}(a_v) = 1$ (resp. -1) if $\mathrm{ord}_v(a_v)$ is even and $v \nmid \mathfrak{n}^-\infty$ (resp. $v \mid \mathfrak{n}^-\infty$). Moreover, let

$$\mathcal{W}_{\mathcal{D}^o,v}(a_v) := \frac{\varepsilon_{\mathcal{D}^o,v}(a_v)}{\varepsilon_{\mathcal{D}^o,v}(1)}.$$

We have

$$\frac{\mathcal{W}_{\mathcal{D}^o,v}(\alpha_v)\mathcal{W}_{\mathcal{D}^o,v}(\beta_v)}{\mathcal{W}_{\mathcal{D}^o,v}(\alpha_v\beta_v)} = (\alpha_v, \beta_v)_v, \quad \forall \alpha_v, \beta_v \in k_v^\times.$$

Theorem 5.1. (Gelbart [5, Section 2.3]) *There is a representation $\omega_{\mathcal{D}^o,v}$ of $\widetilde{\mathrm{SL}}_2(k_v)$ on the space $S(\mathcal{D}^o(k_v))$ satisfying that for every $\phi \in S(\mathcal{D}^o(k_v))$, one has:*

(1) $\omega_{\mathcal{D}^o,v}(1, \xi)\phi(x) := \xi\phi(x), \ \xi \in \{\pm 1\};$

(2) $\omega_{\mathcal{D}^o,v}\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, 1\right)\phi(x) = \psi_v(uQ_3(x))\phi(x), \ u \in k_v;$

(3) $\omega_{\mathcal{D}^o,v}\left(\begin{pmatrix} a_v & 0 \\ 0 & a_v^{-1} \end{pmatrix}, 1\right)\phi(x) = |a_v|_v^{\frac{3}{2}}(a_v, a_v)_v \mathcal{W}_{\mathcal{D}^o,v}(a_v)\phi(a_v x), \ a_v \in k_v^\times;$

(4) $\omega_{\mathcal{D}^o,v}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1\right)\phi(x) = \varepsilon_{\mathcal{D}^o,v}(1) \cdot \widehat{\phi}(x) = \varepsilon_{\mathcal{D}^o,v}(1) \int_{\mathcal{D}^o(k_v)} \phi(y)\psi_v(\mathrm{Tr}_{\mathcal{D}/k}(x\bar{y}))dy.$

We set $\omega^o_{\mathcal{D}} := \otimes_v \omega_{\mathcal{D}^o,v}$, the global Weil representation of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ on $S(\mathcal{D}^o(\mathbb{A}))$, where $\mathcal{D}^o(\mathbb{A}) := \mathcal{D}^o \otimes_k \mathbb{A}$.

Given another ideal $\mathfrak{n}^+ \lhd A$ which is coprime to $\mathfrak{n}^-$, let $R \subset \mathcal{D}$ be an Eichler A-order of type $(\mathfrak{n}^+, \mathfrak{n}^-)$. For each locally-principal right ideal I of R, let R_I denote the left order of I and $R_I^o := R_I \cap \mathcal{D}^o$. Put $O_{\mathcal{D}^o_\infty} := \{b \in \mathcal{D}_\infty \mid \mathrm{Tr}_{\mathcal{D}/k}(b) = 0 \text{ and } \mathrm{ord}_\infty(\mathrm{Nr}_{\mathcal{D}/k}(b)) \geq 0\}$ and for

$v \neq \infty$, set $R^o_{I,v} := R_I \otimes_A O_v$. Take a particular Schwartz function $\varphi^o_I := \otimes_v \varphi^o_{I,v} \in S(\mathcal{D}^o(\mathbb{A}))$ where

$$\varphi^o_{I,v} := \begin{cases} \mathbf{1}_{\pi_v^{-\operatorname{ord}_v(\delta)/2} R^o_{I,v}}, & \text{if } v \neq \infty, \\ \mathbf{1}_{\pi_\infty^{-\operatorname{ord}_\infty(\delta)/2} O_{\mathcal{D}^o_\infty}}, & \text{if } v = \infty. \end{cases}$$

Then the following lemma is straightforward.

Lemma 5.2. *$\omega_{\mathcal{D}^o}(\tilde{\kappa})\varphi^o_I = \varphi^o_I$ for every $\kappa \in \mathcal{K}_0^{(1)}(\mathfrak{n}^+\mathfrak{n}^-\infty)$, where*

$$\mathcal{K}_0^{(1)}(\mathfrak{n}^+\mathfrak{n}^-\infty) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(O_\mathbb{A}) \;\middle|\; c \equiv 0 \bmod \mathfrak{n}^+\mathfrak{n}^-\infty \right\}.$$

Consider the theta series from pure quaternions

$$\widetilde{\Theta}_I(\tilde{g}) \quad := \sum_{x \in \mathcal{D}^o(k)} \big(\omega_{\mathcal{D}^o}(\tilde{g})\varphi^o_I\big)(x), \quad \forall g \in \widetilde{\mathrm{SL}}_2(\mathbb{A}). \tag{5.2.1}$$

It is clear that $\widetilde{\Theta}_I = \widetilde{\Theta}_{I'}$ if I and I' are in the same ideal class $\mathcal{I} \in \mathrm{Cl}(R)$. Therefore, we also adopt the notation $\widetilde{\Theta}_\mathcal{I}$. By Lemma 5.2, the theta series $\widetilde{\Theta}_\mathcal{I}$ can be viewed as a function on $\tilde{s}\big(\mathrm{SL}_2(k)\big)\backslash\widetilde{\mathrm{SL}}_2(\mathbb{A})/\widetilde{\mathcal{K}}_0^{(1)}(\mathfrak{n}^+\mathfrak{n}^-\infty)$.

5.2.1. *Fourier coefficients of $\widetilde{\Theta}_\mathcal{I}$.* For $(z, y) \in \mathbb{A} \times \mathbb{A}^\times$, we have the Fourier expansion of $\widetilde{\Theta}_\mathcal{I}$:

$$\widetilde{\Theta}_\mathcal{I}\left(\begin{pmatrix} y & zy^{-1} \\ 0 & y^{-1} \end{pmatrix}\right) = \sum_{\beta \in k} \widetilde{\Theta}^*_{\mathcal{I},\beta}(y)\psi(\beta z),$$

where

$$\widetilde{\Theta}^*_{\mathcal{I},\beta}(y) = \int_{k\backslash\mathbb{A}} \widetilde{\Theta}_\mathcal{I}\left(\begin{pmatrix} y & xy^{-1} \\ 0 & y^{-1} \end{pmatrix}\right)\psi(-\beta x)dx.$$

It is clear that:

(1) $\widetilde{\Theta}^*_{\mathcal{I},\beta}(y) = 0$ unless $\operatorname{ord}_v(\beta y_v^2) + \operatorname{ord}_v(\delta) \geq 0$ for every place v of k.
(2) $\widetilde{\Theta}^*_{\mathcal{I},\alpha^2\beta}(\alpha^{-1}yu) = \widetilde{\Theta}^*_{\mathcal{I},\beta}(y) \cdot \prod_v(\alpha u_v, y_v)_v$ for every $\alpha \in k^\times$ and $u \in O_\mathbb{A}^\times$.
(3) $\widetilde{\Theta}_\mathcal{I}$ is uniquely determined by its Fourier coefficients $\{\widetilde{\Theta}^*_{\mathcal{I},\beta}(y) \mid (\beta, y) \in k \times \mathbb{A}^\times\}$.

For any $y = (y_v)_v \in \mathbb{A}^\times$, we set

$$\mathcal{W}_{\mathcal{D}^o}(y) := \prod_v \mathcal{W}_{\mathcal{D}^o,v}(y_v).$$

Thus we have the following description of the Fourier coefficients of $\widetilde{\Theta}_\mathcal{I}$.

Lemma 5.3. *Given $(\beta, y) \in k \times \mathbb{A}^\times$ with $\operatorname{ord}_v(\beta y_v^2) + \operatorname{ord}_v(\delta) \geq 0$ for every place v, we have*

$$\widetilde{\Theta}^*_{\mathcal{I},\beta}(y) = \mathcal{W}_{\mathcal{D}^o}(y)|y|_\mathbb{A}^{3/2} \cdot \#\{b \in \mathfrak{y}^{-1}\mathfrak{g}^{-1}R^o_I \mid \mathrm{Nr}_{\mathcal{D}/k}(b) = -b^2 = \beta\}.$$

Here I is a representative of $\mathcal{I} \in \mathrm{Cl}(R)$, and $\mathfrak{y}$, $\mathfrak{g}$ are the fractional ideals of A such that $\operatorname{ord}_v(\mathfrak{y}) = \operatorname{ord}_v(y_v)$ and $\operatorname{ord}_v(\mathfrak{g}) = \operatorname{ord}_v(\delta)/2$ for every finite place v of k.

Proof. It follows from the definition of $\widetilde{\Theta}_\mathcal{I}$. □

5.3. **Shimura-type correspondence.** Recall that we identified the finite places of k with the prime ideals $\mathfrak{p} \lhd A$ in Section 1.1. Given a prime $\mathfrak{p} \lhd A$ with $\mathfrak{p} \nmid \mathfrak{n}^+\mathfrak{n}^-$, one has

$$\widetilde{\mathrm{SL}}_2(O_\mathfrak{p}) \left(\begin{pmatrix} \pi_\mathfrak{p} & 0 \\ 0 & \pi_\mathfrak{p}^{-1} \end{pmatrix}, 1 \right) \widetilde{\mathrm{SL}}_2(O_\mathfrak{p}) = \left[\dot{\bigcup_{u \bmod \pi_\mathfrak{p}^2 O_\mathfrak{p}}} a_u \, \widetilde{\mathrm{SL}}_2(O_\mathfrak{p}) \right] \coprod \left[\dot{\bigcup_{h \in \mathbb{F}_\mathfrak{p}^\times}} b_h \, \widetilde{\mathrm{SL}}_2(O_\mathfrak{p}) \right] \coprod c \, \widetilde{\mathrm{SL}}_2(O_\mathfrak{p}),$$

where

$$a_u := \left(\begin{pmatrix} \pi_\mathfrak{p} & u\pi_\mathfrak{p}^{-1} \\ 0 & \pi_\mathfrak{p}^{-1} \end{pmatrix}, 1 \right), \quad b_h := \left(\begin{pmatrix} 1 & h\pi_\mathfrak{p}^{-1} \\ 0 & 1 \end{pmatrix}, (\pi_\mathfrak{p}, -h)_\mathfrak{p} \right), \quad \text{and } c := \left(\begin{pmatrix} \pi_\mathfrak{p}^{-1} & 0 \\ 0 & \pi_\mathfrak{p} \end{pmatrix}, 1 \right).$$

The Hecke operator at $\mathfrak{p}$ is defined by

$$T_{\mathfrak{p}^2,3/2}\widetilde{\Theta}_\mathcal{I}(\tilde{g}) := \frac{\|\mathfrak{p}\|^{-1/2}}{\mathcal{W}_{\mathcal{D}^o,\mathfrak{p}}(\pi_\mathfrak{p})} \left(\sum_{u \bmod \pi_\mathfrak{p}^2 O_\mathfrak{p}} \widetilde{\Theta}_\mathcal{I}(\tilde{g}a_u) + \sum_{h \in \mathbb{F}_\mathfrak{p}^\times} \widetilde{\Theta}_\mathcal{I}(\tilde{g}b_h) + \widetilde{\Theta}_\mathcal{I}(\tilde{g}c) \right), \quad \forall \tilde{g} \in \widetilde{\mathrm{SL}}_2(\mathbb{A}),$$

Then $T_{\mathfrak{p}^2,3/2}\widetilde{\Theta}_\mathcal{I}$ is also a function on $\tilde{s}\big(\mathrm{SL}_2(k)\big)\backslash\widetilde{\mathrm{SL}}_2(\mathbb{A})/\widetilde{\mathcal{K}}_0^{(1)}(\mathfrak{n}^+\mathfrak{n}^-\infty)$. This definition of $T_{\mathfrak{p}^2,3/2}$ is independent of the chosen uniformizer $\pi_\mathfrak{p}$.

Proposition 5.4. *Let $\mathcal{I}, \mathcal{J} \in \mathrm{Cl}(R)$. For each prime $\mathfrak{p} \lhd A$ with $\mathfrak{p} \nmid \mathfrak{n}^+\mathfrak{n}^-$, we have*

$$T_{\mathfrak{p}^2,3/2}\widetilde{\Theta}_\mathcal{I} = \sum_{\mathcal{J} \in \mathrm{Cl}(R)} B_{\mathcal{I},\mathcal{J}}(\mathfrak{p}) \cdot \widetilde{\Theta}_\mathcal{J}.$$

Here $B_{\mathcal{I},\mathcal{J}}(\mathfrak{p}) \in \mathbb{Z}_{\geq 0}$ is the $(\mathcal{I},\mathcal{J})$-entry of the $\mathfrak{p}$-th Brandt matrix: taking representatives $I \in \mathcal{I}$ and $J \in \mathcal{J}$, one has

$$B_{\mathcal{I},\mathcal{J}}(\mathfrak{p}) := \#\{b \in IJ^{-1} \mid \mathrm{Nr}_{\mathcal{D}/k}(b)\,\mathrm{Nr}_{\mathcal{D}/k}(J) = \mathfrak{p} \cdot \mathrm{Nr}_{\mathcal{D}/k}(I)\} \cdot \#(R_J^\times)^{-1}.$$

Proof. The approach is here similar to Gross [6, Proposition 12.10] (see also [2, Theorem 4.16]). We recall the arguments for the sake of completeness.

Given $(\beta, y) \in k \times \mathbb{A}^\times$ with $\mathrm{ord}_v(\beta y_v^2) + \mathrm{ord}_v(\delta) \geq 0$ for every place v, the Fourier coefficient $T_{\mathfrak{p}^2,3/2}\widetilde{\Theta}^*_{\mathcal{I},\beta}(y)$ is equal to

$$\frac{\|\mathfrak{p}\|^{-1/2}}{\mathcal{W}_{\mathcal{D}^o,\mathfrak{p}}(\pi_\mathfrak{p})} \cdot \left[\|\mathfrak{p}\|^2 (\pi_\mathfrak{p}, y_\mathfrak{p})_\mathfrak{p} \widetilde{\Theta}^*_{\mathcal{I},\beta}(\pi_\mathfrak{p} y) + \Big(\sum_{h \in \mathbb{F}_\mathfrak{p}^\times} \psi(\beta y^2 h/\pi_\mathfrak{p})(\pi_\mathfrak{p}, -h)_\mathfrak{p} \Big) \widetilde{\Theta}^*_{\mathcal{I},\beta}(y) + (\pi_\mathfrak{p}, y_\mathfrak{p})_\mathfrak{p} \widetilde{\Theta}^*_{\mathcal{I},\beta}(\pi_\mathfrak{p}^{-1} y) \right].$$

One checks that

$$\left(\frac{\beta, y}{\mathfrak{p}} \right) := \frac{\|\mathfrak{p}\|^{-1/2}}{\mathcal{W}_{\mathcal{D}^o,\mathfrak{p}}(\pi_\mathfrak{p})} \sum_{h \in \mathbb{F}_\mathfrak{p}^\times} \psi(\beta y^2 h/\pi_\mathfrak{p})(\pi_\mathfrak{p}, -h)_\mathfrak{p} = \begin{cases} 0, & \text{if } \mathrm{ord}_\mathfrak{p}(\beta y^2) + \mathrm{ord}_\mathfrak{p}(\delta) > 0; \\ (\pi_\mathfrak{p}, \beta)_\mathfrak{p}, & \text{otherwise.} \end{cases}$$

We write

$$T_{\mathfrak{p}^2,3/2}\widetilde{\Theta}^*_{\mathcal{I},\beta}(y) = \mathcal{W}_{\mathcal{D}^o}(y)|y|_\mathbb{A}^{3/2} \cdot \left[\#S_I(\mathfrak{p}\mathfrak{y}\mathfrak{g}, \beta) + \left(\frac{\beta, y}{\mathfrak{p}} \right) \#S_I(\mathfrak{y}\mathfrak{g}, \beta) + \|\mathfrak{p}\| \#S_I(\mathfrak{p}^{-1}\mathfrak{y}\mathfrak{g}, \beta) \right],$$

where $\mathfrak{y}$ and $\mathfrak{g}$ are the fractional ideals of A introduced in Lemma 5.3, and for each fractional ideal $\mathfrak{m}$ of A, we put

$$S_I(\mathfrak{m}, \beta) := \{b \in \mathfrak{m}^{-1} R_I^o \mid \mathrm{Nr}_{\mathcal{D}/k}(b) = -b^2 = \beta\}.$$

It remains to show that

$$\#S_I(\mathfrak{p}\mathfrak{y}\mathfrak{g}, \beta) + \left(\frac{\beta, y}{\mathfrak{p}} \right) \#S_I(\mathfrak{y}\mathfrak{g}, \beta) + \|\mathfrak{p}\| \#S_I(\mathfrak{p}^{-1}\mathfrak{y}\mathfrak{g}, \beta) = \sum_{1 \leq j \leq n} B_{\mathcal{I},\mathcal{J}}(\mathfrak{p}) \#S_J(\mathfrak{y}\mathfrak{g}, \beta).$$

For $I \in \mathcal{I}$ and $J \in \mathcal{J}$, set

$$S_{I,J}(\mathfrak{p}) := \{\alpha \in IJ^{-1} \mid \mathrm{Nr}_{\mathcal{D}/k}(\alpha)\,\mathrm{Nr}_{\mathcal{D}/k}(J) = \mathrm{Nr}_{\mathcal{D}/k}(I)\mathfrak{p}\}.$$

Then $\#(S_{I,J}(\mathfrak{p})) = \#(R_J^\times) \cdot B_{\mathcal{I},\mathcal{J}}(\mathfrak{p})$. We consider the following map:

$$\begin{array}{ccccc} \coprod\limits_{\mathcal{J}\in \mathrm{Cl}(R)} \Big(S_{I,J}(v) & \times & S_J(\mathfrak{y}\mathfrak{g},\beta)\Big) & \longrightarrow & S_I(\mathfrak{p}\mathfrak{y}\mathfrak{g},\beta) \\ (\alpha & , & b) & \longmapsto & \alpha b \alpha^{-1}. \end{array}$$

Dividing $S_I(\mathfrak{p}\mathfrak{y}\mathfrak{g},\beta)$ into three parts:

$$S_I(\mathfrak{p}\mathfrak{y}\mathfrak{g},\beta) = \Big(S_I(\mathfrak{p}\mathfrak{y}\mathfrak{g},\beta) - S_I(\mathfrak{y}\mathfrak{g},\beta)\Big) \cup \Big(S_I(\mathfrak{y}\mathfrak{g},\beta) - S_I(\mathfrak{p}^{-1}\mathfrak{y}\mathfrak{g},\beta)\Big) \cup S_I(\mathfrak{p}^{-1}\mathfrak{y}\mathfrak{g},\beta),$$

we analyze the pre-images of each part as follows:

(1) For any $b' \in S_I(\mathfrak{p}\mathfrak{y}\mathfrak{g},\beta) - S_I(\mathfrak{y}\mathfrak{g},\beta)$, there exist unique $\mathcal{J} \in \mathrm{Pic}(A)$ and $\alpha \in S_{IJ}(\mathfrak{p})$ with $J \in \mathcal{J}$, up to the right multiplication by elements in $R_J^\times$, such that
$$b = \alpha^{-1} b' \alpha \in S_J(\mathfrak{y}\mathfrak{g},\beta).$$

(2) Suppose $\mathrm{ord}_\mathfrak{p}(\beta y^2) + \mathrm{ord}_\mathfrak{p}(\delta) = 0$. Let $b' \in S_I(\mathfrak{y}\mathfrak{g},\beta)$. There exist exactly $1 + (\pi_\mathfrak{p},\beta)_\mathfrak{p}$ choices of the pair $(\mathcal{J},\alpha)$ with $J \in \mathcal{J}$ and $\alpha \in S_{I,J}(\mathfrak{p})$, up to the multiplication by elements in $R_J^\times$, such that
$$b = \alpha^{-1} b' \alpha \in S_J(\mathfrak{y}\mathfrak{g},\beta).$$

(3) Suppose $\mathrm{ord}_\mathfrak{p}(\beta y^2) + \mathrm{ord}_\mathfrak{p}(\delta) > 0$. Let $b' \in S_I(\mathfrak{y}\mathfrak{g},\beta) - S_I(\mathfrak{p}^{-1}\mathfrak{y}\mathfrak{g},\beta)$. Then there exist unique $\mathcal{J}$ and $\alpha \in S_{I,J}(v)$ with $J \in \mathcal{J}$, up to the right multiplication by elements in $R_J^\times$, such that
$$b = \alpha^{-1} b' \alpha \in S_J(\mathfrak{y}\mathfrak{g},\beta).$$
Moreover, if $b' \in S_I(\mathfrak{p}^{-1}\mathfrak{y}\mathfrak{g},\beta)$, we have that for every J and $\alpha \in S_{I,J}(\mathfrak{p})$,
$$b = \alpha^{-1} b' \alpha \in S_J(\mathfrak{y}\mathfrak{g},\beta).$$

Therefore by (1), (2), and (3) in the above, we get

$$\begin{aligned} \sum_{\mathcal{J}\in \mathrm{Cl}(R)} B_{\mathcal{I},\mathcal{J}}(\mathfrak{p}) S_J(\mathfrak{y}\mathfrak{g},\beta) &= \Big(\#S_I(\mathfrak{p}\mathfrak{y}\mathfrak{g},\beta) - \#S_I(\mathfrak{y}\mathfrak{g},\beta)\Big) \\ &\quad + (1 + \left(\frac{\beta, y}{\mathfrak{p}}\right))\Big(\#S_I(\mathfrak{y}\mathfrak{g},\beta) - \#S_I(\mathfrak{p}^{-1}\mathfrak{y}\mathfrak{g},\beta)\Big) \\ &\quad + (1 + \|\mathfrak{p}\|)\Big(\#S_I(\mathfrak{p}^{-1}\mathfrak{y}\mathfrak{g},\beta)\Big) \\ &= \#S_I(\mathfrak{p}\mathfrak{y}\mathfrak{g},\beta) + \left(\frac{\beta, y}{\mathfrak{p}}\right) \#S_I(\mathfrak{y}\mathfrak{g},\beta) + \|\mathfrak{p}\| \#S_I(\mathfrak{p}^{-1}\mathfrak{y}\mathfrak{g},\beta), \end{aligned}$$

which completes the proof. □

Let $X = X_{\mathfrak{n}^+,\mathfrak{n}^-}$ be the definite Shimura curve of type $(\mathfrak{n}^+,\mathfrak{n}^-)$. Denote by $\widetilde{\mathcal{M}}_0(\mathfrak{n}^+\mathfrak{n}^-)$ the space of $\mathbb{C}$-valued functions on $\tilde{s}\big(\mathrm{SL}_2(k)\big)\backslash\widetilde{\mathrm{SL}}_2(\mathbb{A})/\widetilde{\mathcal{K}}_0^{(1)}(\mathfrak{n}^+\mathfrak{n}^-\infty)$. We define the map $\Psi : \mathrm{Pic}(X)_\mathbb{C}^\vee \to \widetilde{\mathcal{M}}_0(\mathfrak{n}^+\mathfrak{n}^-)$ by sending $e_\mathcal{I}$ to $\widetilde{\Theta}_{\tau(\mathcal{I})}$, for $\mathcal{I} \in \mathrm{Cl}(R)$. Here τ is the order 2 permutation on $\mathrm{Cl}(R)$ introduced in Section 1.3. Set

$$\mathbf{Sh}(= \mathbf{Sh}_{\mathfrak{n}^+,\mathfrak{n}^-}) := \Psi \circ \mathrm{JL}^{-1} : \mathcal{S}_0^{\mathfrak{n}-\mathrm{new}}(\mathfrak{n}^+\mathfrak{n}^-) \longrightarrow \widetilde{\mathcal{M}}_0(\mathfrak{n}^+\mathfrak{n}^-),$$

where $\mathrm{JL} : \mathrm{Pic}(X)_\mathbb{C}^\vee \to \mathcal{S}_0^{\mathfrak{n}-\mathrm{new}}(\mathfrak{n}^+\mathfrak{n}^-)$ is the Jacquet-Langlands correspondence introduced in Section 4.5. We then arrive at our Shimura-type correspondence:

Theorem 5.5. *For each prime $\mathfrak{p} \lhd A$ with $\mathfrak{p} \nmid \mathfrak{n}^+\mathfrak{n}^-$,*

$$\mathbf{Sh}(T_\mathfrak{p} F) = T_{\mathfrak{p}^2,3/2}\,\mathbf{Sh}(F), \quad \forall F \in \mathcal{S}_0^{\mathfrak{n}-\mathrm{new}}(\mathfrak{n}^+\mathfrak{n}^-).$$

Remark 5.6. We point out that for a Drinfeld-type newform F of level $\mathfrak{n}^+\mathfrak{n}^-$, $\mathbf{Sh}(F) = 0$ if there exists a place $\mathfrak{p} \mid \mathfrak{n}^-$ such that $\lambda_\mathfrak{p}(F) = -1$.

5.4. **A Waldspurger-type formula.** Let K/k be an imaginary quadratic field such that $\mathfrak{p}$ does not split in K for every prime $\mathfrak{p} \mid \mathfrak{n}^-$. Take $\beta \in k^\times$ such that $K = k(\sqrt{-\beta})$. There exists a unique fractional ideal $\mathfrak{c}_\beta$ of A such that $O_K = A + \mathfrak{c}_\beta\sqrt{-\beta}$. Let $e_K \in \mathrm{Pic}(X)$ be the class representing the divisor

$$\sum_{x \in G_{O_K}} x,$$

where G_{O_K} is the set of Gross points of trivial conductor over K (cf. Section 4.4). Choose $y_\beta \in \mathbb{A}^\times$ so that $\mathrm{ord}_v(y_{\beta,v}) = \mathrm{ord}_v(\mathfrak{c}_\beta) - \mathrm{ord}_v(\delta)/2$ for every place v of k. By Lemma 5.3, we immediately get:

Lemma 5.7. *Let $\mathcal{I} \in \mathrm{Cl}(R)$, then $\widetilde{\Theta}^*_{\mathcal{I},\beta}(y_\beta) = \mathcal{W}_{\mathcal{D}^o}(y_\beta)|y_\beta|_\mathbb{A}^{3/2}\langle e_K, e_\mathcal{I}\rangle$. Moreover, for each $F \in \mathcal{S}_0^{\mathfrak{n}-\mathrm{new}}(\mathfrak{n}^+\mathfrak{n}^-)$, we have*

$$\mathbf{Sh}(F)^*_\beta(y_\beta) = \mathcal{W}_{\mathcal{D}^o}(y_\beta)|y_\beta|_\mathbb{A}^{3/2}\langle e_K, e_F\rangle.$$

Let F be a normalized newform of square-free level $\mathfrak{n}_F$ with trivial central character. Set

$$\mathfrak{n}_F^- := \prod_{\substack{\mathfrak{p} \mid \mathfrak{n}_F \text{ with} \\ \lambda_\mathfrak{p}(F)=1}} \mathfrak{p} \quad \text{and} \quad \mathfrak{n}_F^+ := \mathfrak{n}_F/\mathfrak{n}_F^-.$$

Suppose that the number of prime factors of $\mathfrak{n}_F^-$ is odd. Let K be an imaginary quadratic field K satisfying $\Sigma(F, \mathbf{1}_K) = \{\infty\} \cup \{\text{prime } \mathfrak{p} \lhd A \mid \mathfrak{p} \text{ divides } \mathfrak{n}_F^-\}$. Then for any Gross point $x \in X = X_{\mathfrak{n}_F^+,\mathfrak{n}_F^-}$ of trivial conductor over K, we have

$$\langle e_K, e_F\rangle = \left(\prod_{\mathfrak{p} \mid \frac{\mathfrak{n}_F}{(\mathfrak{n}_F, \mathfrak{d}_K)}} 2\right)\langle e_{\mathbf{1}_K}(x), e_F\rangle.$$

Normalize e_F so that $e_F \in \mathrm{Pic}_0(X)^\vee_\mathbb{R}$ with $\langle e_F, e_F\rangle = 1$, and set

$$m(F, K) := \frac{\mathbf{Sh}(F)^*_\beta(y_\beta)}{\mathcal{W}_{\mathcal{D}^o}(y_\beta)|y_\beta|_\mathbb{A}^{3/2}}.$$

Then by Lemma 5.7, we have $m(F, K) = \langle e_K, e_F\rangle$, which says that $m(F, K)$ is independent of the chosen β and y_β. We call $m(F, K)$ the *K-th Fourier coefficient of* $\mathbf{Sh}(F)$. Note that $m(F, K) \in \mathbb{R}$. Therefore by Theorem 4.16, we arrive at the following Waldspurger-type formula:

Theorem 5.8. *Let F be a normalized Drinfeld-type newform of square-free level $\mathfrak{n}_F$ with trivial central character. Suppose the number of prime factors of $\mathfrak{n}_F^-$ is odd. For each imaginary quadratic field K satisfying $\Sigma(F, \mathbf{1}_K) = \{\infty\} \cup \{\text{prime } \mathfrak{p} \lhd A \mid \mathfrak{p} \text{ divides } \mathfrak{n}_F^-\}$, we have*

$$L(F, 0)L(F \otimes \chi_K, 0) = \mathcal{P}(F, K) \cdot \frac{4 \cdot (F, F)_{\mathfrak{n}_F}}{f_K(\infty)^2 \cdot \# \mathrm{Pic}(A)} \cdot \left(\prod_{\mathfrak{p} \mid \frac{\mathfrak{n}_F}{(\mathfrak{n}_F, \mathfrak{d}_K)}} 4\right)^{-1} m(F, K)^2,$$

where $\mathcal{P}(F, K)$ is introduced in Theorem 4.8.

Appendix A. Quadratic theta series over function fields

Let K be an imaginary quadratic extension over k, and O_K be the integral closure of A in K. Given an ideal class $\mathcal{A}$ of $\mathrm{Pic}(O_K)$, we write down here the quadratic theta series $\Theta_{\mathcal{A}}$. Moreover, the "newform" Θ_K^η associated to a given character $\eta : \mathrm{Gal}(k^{\mathrm{sep}}/K) \to \mathbb{C}^\times$ satisfying the condition C.1 (in the beginning of Section 3) is then simply expressed as:

$$\Theta_K^\eta := \sum_{\mathcal{A}\in\mathrm{Pic}(O_K)} \eta(\mathcal{A})\cdot\Theta_{\mathcal{A}}.$$

Here η is viewed as a character on $\mathrm{Pic}(O_K)$. For the sake of completeness, we also record the properties of $\Theta_{\mathcal{A}}$ and Θ_K^η which are used in the paper.

A.1. **Weil representation.** Let (V, Q_V) be an anisotropic quadratic space over k with even dimension. Then $\dim(V) \leq 4$ by Hasse-Minkowski principle. We focus on the following two cases:

(1) $(V, Q_V) = (K, \gamma\cdot\mathbf{N}_{K/k})$ for some $\gamma \in k^\times$ where K/k is an imaginary quadratic extension, and $\mathbf{N}_{K/k}$ is the norm form on K/k;
(2) $(V, Q_V) = (\mathcal{D}, \mathrm{Nr}_{\mathcal{D}/k})$, where $\mathcal{D}$ is a *definite* (i.e. $\mathcal{D}\otimes_k k_\infty$ is division) quaternion algebra over k, and $\mathrm{Nr}_{\mathcal{D}/k}$ is the reduced norm form on $\mathcal{D}/k$.

For x, y in V, the bilinear form associated with Q_V is denoted by $< x, y >_V$.

For each place v of k, let $V(k_v) := V\otimes_k k_v$, and denote $S(V(k_v))$ to be the space of Schwartz functions on $V(k_v)$. Also denote by $S(V(k_v)\times k_v^\times)$ the space of functions ϕ_v on $V(k_v)\times k_v^\times$ such that for each $\alpha_v \in k_v^\times$, $\phi_v(\cdot,\alpha_v)$ is in $S(V(k_v))$. Choose a basis $\{x_1, ..., x_{\dim(V)}\}$ of V over k. For each place v of k, let

$$\Lambda_v := \oplus_{i=1}^{\dim(V)} O_v x_i \subset V(k_v) \quad \text{and} \quad \phi_v^0 := \mathbf{1}_{\Lambda_v}\otimes\mathbf{1}_{O_v^\times}.$$

We define $S(V(\mathbb{A})\times\mathbb{A}^\times)$ to be the restricted tensor product $\otimes'_v S(V(k_v)\times k_v^\times)$ with respect to $\{\phi_v^0\}_v$.

For each place v of k with an additive character ψ_v fixed, the (local) Weil representation $\omega_{V,v}$ of $\mathrm{GL}_2(k_v)$ on $S(V(k_v)\times k_v^\times)$ is defined by (cf. [5] Proposition 2.31):

$$\left(\omega_{V,v}\begin{pmatrix}1 & b_v\\ 0 & 1\end{pmatrix}\phi\right)(x_v,\alpha_v) := \psi_v(b_v Q_V(x_v)\alpha_v)\phi_v(x_v,\alpha_v), \text{ for } b_v\in k_v;$$

$$\left(\omega_{V,v}\begin{pmatrix}a_v & 0\\ 0 & a_v^{-1}\end{pmatrix}\phi_v\right)(x_v,\alpha_v) := |a_v|_v^{\frac{\dim(V)}{2}}\cdot\chi_{V,v}(a_v)\cdot\phi(a_v x_v,\alpha_v), \text{ for } a_v\in k_v^\times;$$

$$\left(\omega_{V,v}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\phi_v\right)(x_v,\alpha_v) := \varepsilon_{V,v}(\alpha_v)\cdot\mathcal{F}_{V,v}\phi_v(x_v,\alpha_v);$$

$$\left(\omega_{V,v}\begin{pmatrix}1 & 0\\ 0 & a_v\end{pmatrix}\phi_v\right)(x_v,\alpha_v) := |a_v|_v^{-\frac{\dim(V)}{4}}\chi_{V,v}(\alpha_v)\phi(x_v,\alpha_v a_v^{-1}), \text{ for } a_v\in k_v^\times$$

Here:

- $\chi_{V,v} : k_v^\times \to \{\pm 1\}$ is the quadratic character associated with V at v, i.e.
$$\chi_{V,v}(\alpha) := (\alpha, (-1)^{\frac{\dim(V)}{2}}\det(V))_v,$$
where
$$\det(V) := \det\left((< x_i, x_j >_V)_{1\leq i,j\leq\dim(V)}\right) \in k^\times/(k^\times)^2$$
for any k-basis $\{x_1, ..., x_{\dim(V)}\}$ of V. We denote χ_V to be the character
$$\otimes_v\chi_{V,v} : k^\times\backslash\mathbb{A}_k^\times \to \{\pm 1\}.$$

- The Fourier transform $\mathcal{F}_{V,v}\phi_v$ of ϕ_v is the following integral

$$\mathcal{F}_{V,v}\phi_v(x_v,\alpha_v) := \int_{V(k_v)} \phi_v(y_v,\alpha_v)\cdot\psi_v(< x_v, y_v >_V \alpha_v)d_{\alpha_v}y_v,$$

with $d_{\alpha_v}y_v$ the self-dual Haar measure with respect to the additive character $\psi_v(\cdot\,\alpha_v)$.

- $\varepsilon_{V,v}(\alpha_v)$ is the *Weil index of V at v* i.e.

$$\varepsilon_{V,v}(\alpha_v) := \int_L \psi_v(Q_V(y_v)\alpha_v)d_{\alpha_v}y_v$$

for any O_v-submodule L in $V(k_v)$ that is large enough.

We denote by ω_V the (global) Weil representation $\otimes_v\omega_{V,v}$ of $\mathrm{GL}_2(\mathbb{A})$ on $S(V(\mathbb{A})\times\mathbb{A}^\times)$.

Lemma A.1. (cf. [1] Section 4.8, and [19] Proposition 5)
(1) *When $(V,Q_V)=(K,\mathbf{N}_{K/k})$, the Weil index $\varepsilon_{V,v}(\alpha_v)$ has the following properties:*
- (i) $\varepsilon_{V,v}(\alpha_v)=\chi_{K,v}(\alpha_v)\cdot\varepsilon_{V,v}(1)$.
- (ii)

$$\varepsilon_{V,v}(1)=\begin{cases}1, & \text{if } v \text{ is split or inert in } K/k \text{ with } \delta_v \text{ even,}\\ -1, & \text{if } v \text{ is inert in } K/k \text{ with } \delta_v \text{ odd.}\end{cases}$$

- (iii) *When v is ramified in K/k, let π_v be a uniformizer of k at v which is in $\mathbf{N}_{K/k}(K_v)$. Then*

$$\varepsilon_{V,v}(1)=q_v^{-\frac{1}{2}}\cdot\sum_{u\in\mathbb{F}_v^\times}\chi_{K,v}(u)\psi_v(u\pi_v^{-\delta_v-1}).$$

- (iv) $(\varepsilon_{V,v}(1))^2=\chi_{K,v}(-1)$ *for every place v of k.*
- (v) *(Weil's reciprocity)* $\prod_v\varepsilon_{V,v}(1)=1$.

(2) *When $(V,Q_V)=(\mathcal{D},\mathrm{Nr}_{\mathcal{D}/k})$, one has*

$$\varepsilon_{V,v}(\alpha_v) := \begin{cases}-1, & \text{if } \mathcal{D} \text{ is ramified at } v;\\ 1, & \text{if } \mathcal{D} \text{ splits at } v.\end{cases}$$

A.2. **Quadratic theta series.** Consider the quadratic space $(V,Q_V)=(K,\mathbf{N}_{K/k})$ where K is an imaginary quadratic field over k. Let O_K be the integral closure of A in K. Given a fractional ideal $\mathfrak{A}$ of O_K, we choose a finite idele $\varpi_{\mathfrak{A}}=(\varpi_{\mathfrak{A},v})_{v\neq\infty}\in\mathbb{A}_K^{\infty,\times}$ such that $\mathfrak{A}=K\cap b_{\mathfrak{A}}O_{\mathbb{A}_K^\infty}$. We associate $\mathfrak{A}$ a Schwartz function $\varphi_{\mathfrak{A}}=\otimes_v\varphi_{\mathfrak{A},v}\in S(V(\mathbb{A})\times\mathbb{A}^\times)$ defined by the following: Given $(x_v,\alpha_v)\in V(k_v)\times k_v^\times$, set

$$\varphi_{\mathfrak{A},v}(x_v,\alpha_v) := \begin{cases}\mathbf{1}_{O_\infty}\big(\mathbf{N}_{K/k}(x_\infty)\cdot\alpha_\infty\cdot\pi_\infty^{\delta_\infty}\big)\cdot\mathbf{1}_{\mathbf{N}_{K/k}(K_\infty^\times)}(\alpha_\infty), & \text{if } v=\infty,\\ \mathbf{1}_{\varpi_{\mathfrak{A},v}O_{K_v}}(x_v)\cdot\mathbf{1}_{O_v^\times}(\mathbf{N}_{K/k}(\varpi_{\mathfrak{A},v})\cdot\alpha_v\cdot\pi_v^{\delta_v})\cdot\mathbf{1}_{\mathbf{N}_{K/k}(K_v^\times)}(\alpha_v), & \text{otherwise.}\end{cases}$$

Let

$$\mathrm{GL}_2^{+_K}(O_v) := \{g\in\mathrm{GL}_2(O_v)\mid\det(g)\in\mathbf{N}_{K/k}(K_v^\times)\},$$

and

$$\mathcal{K}_{v^\ell}^{+_K} := \mathcal{K}_{v^\ell}\cap\mathrm{GL}_2^{+_K}(O_v).$$

By a straightforward argument, $\varphi_{\mathfrak{A},v}$ satisfies the following invariant property:

Lemma A.2. (1) *If v is unramified in K/k and $\kappa_v\in\mathrm{GL}_2^{+_K}(O_v)$, then we have*

$$\omega_{K,v}(\kappa_v)\varphi_{\mathfrak{A},v}=\varphi_{\mathfrak{A},v}.$$

Here $\omega_{K,v}$ is the Weil representation introduced in Section A.1.

(2) *If v is ramified in K/k and $\kappa_v = \begin{pmatrix} a_v & b_v \\ c_v & d_v \end{pmatrix} \in \mathcal{K}_v^{+_K}$, then we have*

$$\omega_{K,v}(\kappa_v)\varphi_{\mathfrak{A},v} = \chi_{K,v}(d_v)\cdot\varphi_{\mathfrak{A},v}.$$

(3) *Suppose v is ramified in K/k. Choose a uniformizer π_v of k at v with $\pi_v \in \mathbf{N}_{K/k}(K_v)$. Then*

$$\omega_{K,v}\begin{pmatrix} 0 & 1 \\ -\pi_v & 0 \end{pmatrix}\varphi_{\mathfrak{A},v} = \varepsilon_{K,v}(1)\cdot\begin{cases}\varphi_{\mathfrak{A},v}, & \text{if } v=\infty,\\ \varphi_{\mathfrak{P}^{-1}\mathfrak{A},v}, & \text{otherwise.}\end{cases}$$

Here $\mathfrak{P} \lhd O_K$ is the prime ideal lying above v.

(4) *For every $a = (a^\infty, a_\infty) \in \mathbb{A}^{\infty,\times}\times k_\infty^\times = \mathbb{A}^\times$, we have*

$$\omega_K\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\varphi_{\mathfrak{A}} = \chi_K(a)\cdot\varphi_{\mathrm{div}(a^\infty)^{-1}\mathfrak{A}}.$$

Here $\mathrm{div}(a^\infty)$ *is considered as a fractional ideal of A.*

With the Schwartz function $\varphi_{\mathfrak{A}}$ at hand, we may now introduce theta series.

Definition A.3. For $\mathcal{A} \in \mathrm{Pic}(O_K)$ and $\mathfrak{A} \in \mathcal{A}$, the function $\Theta_{\mathfrak{A}}$ on $\mathrm{GL}_2(\mathbb{A}) \to \mathbb{C}$ is defined by

$$\frac{1}{\#(\mathbb{F}_K^\times)|\delta|_{\mathbb{A}}^{\frac{1}{2}}}\cdot\sum_{(x,\alpha)\in V(k)\times k^\times}\left(\omega_K\left(\begin{pmatrix}\mathbf{N}_{K/k}(\varpi_{\mathfrak{A}})\cdot\delta & 0\\ 0 & \mathbf{N}_{K/k}(\varpi_{\mathfrak{A}})\cdot\delta\end{pmatrix}g\right)\varphi_{\mathfrak{A}}\right)(x,\alpha).$$

The function $\Theta_{\mathfrak{A}}$ is independent of the chosen representative $\mathfrak{A}$ of ideal class $\mathcal{A} \in \mathrm{Pic}(O_K)$ Therefore, we may use the notation $\Theta_{\mathcal{A}}$ instead of $\Theta_{\mathfrak{A}}$. Meanwhile, set

$$\mathrm{GL}_2^{+_K}(\mathbb{A}) = \left\{g \in \mathrm{GL}_2(\mathbb{A}) \mid \det(g) \in k^\times\cdot\mathbf{N}_{K/k}(\mathbb{A}_K^\times)\right\},$$

which has index 2 in $\mathrm{GL}_2(\mathbb{A})$. Put

$$\begin{aligned}\mathrm{S}_{\mathcal{A}} &:= k^\times\cdot\delta\cdot\mathbf{N}_{K/k}\big((\varpi_{\mathfrak{A}}O_{\mathbb{A}_K^\infty}^\times)\times K_\infty^\times\big)\\ &\subset k^\times\cdot\delta\cdot\mathbf{N}_{K/k}\Big(\mathbb{A}_K^{\infty,\times}\times K_\infty^\times\Big) = k^\times\cdot\mathbf{N}_{K/k}(\mathbb{A}_K^\times)\subset\mathbb{A}^\times.\end{aligned}$$

Then

Lemma A.4. *Given an ideal class $\mathcal{A} \in \mathrm{Pic}(O_K)$, one has $\Theta_{\mathcal{A}}(g) = 0$ for $g \in \mathrm{GL}_2(\mathbb{A})$ unless* $\det(g)^{-1} \in \mathrm{S}_{\mathcal{A}}$*. In particular, the theta series $\Theta_{\mathcal{A}}$ vanishes outside* $\mathrm{GL}_2^{+_K}(\mathbb{A})$.

Furthermore, set

$$\mathcal{K}_0^{+_K}(\mathfrak{m}) := \prod_v \mathcal{K}_{v^{\mathrm{ord}_v(\mathfrak{m})}}^{+_K}\quad\forall\mathfrak{m}\in\mathrm{Div}_{\geq 0}(k).$$

Lemma A.2 (1) and (2) say that

Proposition A.5. *The function $\Theta_{\mathcal{A}}$ can be viewed as a $\mathbb{C}$-valued function on* $\mathrm{GL}_2(k)\backslash\mathrm{GL}_2(\mathbb{A})$ *satisfying*

$$\Theta_{\mathcal{A}}(g\kappa) = \prod_{v|\mathfrak{d}_K}\chi_{K,v}(d_v)\cdot\Theta_{\mathcal{A}}(g),$$

for all $g \in \mathrm{GL}_2(\mathbb{A})$ and $\kappa = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{K}_0^{+_K}(\mathfrak{d}_K)$.

The theta series $\Theta_{\mathcal{A}}$ has the following Fourier expansion:

$$\Theta_{\mathcal{A}}(g) = C_{\Theta_{\mathcal{A}}}(g) + \sum_{\alpha \in k^\times} W_{\Theta_{\mathcal{A}}}\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right), \quad \forall g \in \mathrm{GL}_2(\mathbb{A}),$$

where

$$C_{\Theta_{\mathcal{A}}}(g) := \int_{k\backslash\mathbb{A}} \Theta_{\mathcal{A}}\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g\right) du \quad \text{and} \quad W_{\Theta_{\mathcal{A}}}(g) := \int_{k\backslash\mathbb{A}} \Theta_{\mathcal{A}}\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g\right) \psi(-u) du.$$

The Fourier coefficients of $\Theta_{\mathcal{A}}$ can be given explicitly. We derive that:

Proposition A.6. *For $y \in \mathbb{A}^\times$, we have*

$$W_{\Theta_{\mathcal{A}}}\left(\begin{pmatrix} \delta^{-1}y & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{cases} \mathbf{1}_{\mathbf{N}_{K/k}(\mathbb{A}_K^\times)}(y) \cdot \dfrac{R_{\mathcal{A}^{-1}}(\mathrm{div}(y)^\infty)}{\|\mathrm{div}(y)\|^{\frac{1}{2}}}, & \text{if } \mathrm{div}(y) \in \mathrm{Div}_{\geq 0}(k); \\ 0, & \text{otherwise.} \end{cases}$$

Here $\mathfrak{m}^\infty := \mathfrak{m} \cdot \infty^{-\mathrm{ord}_\infty(\mathfrak{m})}$ for each divisor $\mathfrak{m} \in \mathrm{Div}(k)$, and $R_{\mathcal{A}}$ is the counting function of O_K ideals in $\mathcal{A}$ introduced in the beginning of Section 3.

Definition A.7. Let $\eta : \mathrm{Gal}(k^{\mathrm{sep}}/K) \to \mathbb{C}^\times$ be a continuous character satisfying the condition C.1 of Section 3. The "newform" Θ_K^η associated to η is defined by:

$$\Theta_K^\eta(g) := \sum_{\mathcal{A} \in \mathrm{Pic}(O_K)} \eta(\mathcal{A}) \cdot \Theta_{\mathcal{A}}(g), \quad \forall g \in \mathrm{GL}_2(\mathbb{A}).$$

On the right hand side of the above equality, we consider η as a character of $\mathrm{Pic}(O_K)$.

Remark A.8. (1) Viewing η as a character on $\mathbb{A}_K^\times = (\mathbb{A} \otimes_k K)^\times$, let $\omega_\eta := \eta\big|_{\mathbb{A}^\times} \cdot \chi_K$. Then for $a \in \mathbb{A}^\times$ and $g \in \mathrm{GL}_2(\mathbb{A})$, on deduces from Lemma A.2 (4) that

$$\Theta_K^\eta\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} g\right) = \overline{\omega_\eta(a)} \cdot \Theta_K^\eta(g).$$

(2) Alternatively, this "newform" Θ_K^η can be also constructed via the converse theorem (cf. [20, Chapter IX]). We refer the readers to [9, §3] for further details.

By Lemma A.2 (3) and Proposition A.6, we then get:

Proposition A.9. *For every $g = (g_v)_v \in \mathrm{GL}_2(\mathbb{A})$, one has*

$$W_{\Theta_K^\eta}(g) = \prod_v W_{\Theta_K^\eta, v}(g_v),$$

where $W_{\Theta_K^\eta, v} = W_{\Theta_K^\eta}\big|_{\mathrm{GL}_2(k_v)}$. Furthermore, if v is ramified in K/k and a uniformizer π_v of k at v is chosen with $\pi_v \in \mathbf{N}_{K/k}(K_v)$, then

$$W_{\Theta_K^\eta}\left(\begin{pmatrix} \pi_v^{-\delta_v} y_v & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -\pi_v & 0 \end{pmatrix}\right) = \eta(\mathrm{Frob}_w) \cdot \varepsilon_{K,v}(1) \cdot W_{\Theta_K^\eta, v}\left(\begin{pmatrix} \pi_v^{-\delta_v} y_v & 0 \\ 0 & 1 \end{pmatrix}\right).$$

Here w is the place of K lying above v.

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