Journal of Algebra 414 (2014) 241–263

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Canonical bases and quantum coordinate algebras

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A R T I C L E I N F O

Article history:
Received 30 January 2012
Available online 13 June 2014
Communicated by Gus I. Lehrer

Keywords:
Canonical basis
Crystal basis
Quantum coordinate algebra

A B S T R A C T

Some filtrations of the tensor product of a highest weight module and a lowest weight module over quantum group $U_q(\mathfrak{g})$ are constructed in [1] and one can use them to define a two-sided ideal of the modified quantized enveloping algebra. It is shown that the quotient algebra inherits a canonical basis from the modified quantized enveloping algebra and is dual to the quantum coordinate algebra defined by Kashiwara for a symmetrizable Kac–Moody algebra $\mathfrak{g}$.

1. Introduction

A quantum coordinate algebra, or a quantum function algebra, is the q-deformed version of the coordinate algebra associated with a Lie group G. It can be viewed, in some sense, as an algebra dual to the quantized enveloping algebra $U = U_q(\mathfrak{g})$ and thus it is natural to study its structure and representations as well as its integral form.

There are various ways to define the quantum coordinate algebra C. For any Kac–Moody algebra $\mathfrak{g}$ with a symmetrizable generalized Cartan matrix, M. Kashiwara defined in [2] C as the algebra generated by all coordinate functions of the U-modules in the category $\mathcal{O}_{int}$ and moreover, there is an analogue of Peter–Weyl theorem

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http://dx.doi.org/10.1016/j.jalgebra.2014.05.023

$$C \cong \bigoplus_{\lambda \in P+} V(\lambda) \otimes V(\lambda)^{\circ}$$

where $V(\lambda)$ is the irreducible integrable highest weight U-module with highest weight λ and $V(\lambda)^{\circ}$ is its graded dual. In particular, for $\mathfrak{g}$ of finite type, Lusztig [3] gave another equivalent definition of the quantum coordinate algebra. In [3], canonical bases for the tensor products $V(\lambda) \otimes V(-\mu)$ as well as for the modified quantized enveloping algebra $\widetilde{U}$ are constructed, where $V(\lambda)$ and $V(-\mu)$ are irreducible integrable highest and lowest weight modules over U with highest weight vector u_λ and lowest weight vector $u_{-\mu}$ respectively. It is known that there is a surjective linear map

$$\varphi_{\lambda,-\mu} : U \longrightarrow V(\lambda) \otimes V(-\mu)$$

which takes $u \in U$ to $u(u_\lambda \otimes u_{-\mu})$. By taking the image $\mathrm{Im}(\varphi^*_{\lambda,-\mu})$ of the dual map

$$\varphi^*_{\lambda,-\mu} : \big(V(\lambda) \otimes V(-\mu)\big)^* \longrightarrow U^*,$$

Lusztig defined the quantum coordinate algebra as $\sum_{\lambda,\mu} \mathrm{Im}(\varphi^*_{\lambda,-\mu})$, in which the multiplication is defined through the comultiplication in U. Apart from this, another approach to define the quantum coordinate algebra for $\mathfrak{g}$ of finite type was also presented by Lusztig [4]. He considered the subspace $\widetilde{U}^{\circ}$ of the dual space of $\widetilde{U}$ spanned by the dual basis of the canonical basis of $\widetilde{U}$ in [4]. The multiplication in $\widetilde{U}^{\circ}$ is defined through the comultiplication in $\widetilde{U}$ to make it become an associative algebra which is proved later to be isomorphic to the quantum coordinate algebra. In this way, the integral form of this algebra is naturally defined [4]. In the present paper, we will follow Lusztig's approachs to define the quantum coordinate algebra for any symmetrizable Kac–Moody algebra $\mathfrak{g}$.

In [3] Lusztig conjectured that for $\mathfrak{g}$ of finite type, there is a composition series of $V(\lambda) \otimes V(-\mu)$ compatible with the canonical basis. In [5], Lusztig gave an inductive method to construct the composition series of any integrable module in category $\mathcal{O}_{int}$. A different approach to construct the composition series is given in [1] based on the theory of crystal basis. With this method we can also construct a nice filtration of $V(\lambda) \otimes V(-\mu)$ for $\mathfrak{g}$ of any type such that the quotient of any two neighbors is either zero or an irreducible integrable highest weight module. Using these filtrations, we define a subspace $\widetilde{U}'$ of $\widetilde{U}$ spanned by all canonical base elements $G(b)$ such that $b \in \widetilde{B}$ is contained in a connected component not isomorphic to a highest weight crystal. It is proved that $\widetilde{U}'$ is a two-sided ideal of $\widetilde{U}$. The quotient $\mathfrak{U} \triangleq \widetilde{U}/\widetilde{U}'$ is an associative algebra which inherits from $\widetilde{U}$ a canonical basis. Let $\mathfrak{U}$ take the place of $\widetilde{U}$ and then we define, similar to what Lusztig did in [4], an algebra which is proved to be isomorphic to the quantum coordinate algebra. Besides, following [3], a quotient module of $V(\lambda) \otimes V(-\mu)$ is constructed and used to define the quantum coordinate algebra.

The quantum coordinate algebra considered in this paper involves only integrable representations in category $\mathcal{O}_{int}$. Thus it is exactly the algebra of strongly regular functions on symmetrizable Kac–Moody group [6] when $q = 1$. It would be interesting if $\mathcal{O}_{int}$ is replaced by some larger categories. Namely, if there are more generators besides coordinate functions of highest weight modules, say, those of lowest weight modules, the generated subalgebra of U^* will be much more interesting. For $\mathfrak{g}$ of affine type, the structure of level zero part of $\widetilde{U}$ was studied by Beck and Nakajima [7]. The authors of the present paper believe that their work is helpful to understand this coordinate algebra of affine type though this is not included in the present paper.

The paper is organized as follows. In Section 2, we recall some definitions and facts about the crystal and canonical bases of $\widetilde{U}$. In particular, the action of Cartan involution on canonical basis of $\widetilde{U}$ is studied through the bilinear form on $\widetilde{U}$. In Sections 3 and 4, some nice filtrations of the tensor product $V(\lambda)\otimes V(-\mu)$ are constructed. We then define $\mathfrak{U}$ to be the quotient of $\widetilde{U}$ and investigate its cell modules as in [5]. In the last section, an algebra dual to $\mathfrak{U}$ is defined and proved to be isomorphic to quantum coordinate algebra.

2. Preliminaries

2.1. Modified quantized enveloping algebra $\widetilde{U}$

We denote by $\mathfrak{g} = \mathfrak{g}(A)$ any symmetrizable Kac–Moody algebra associated with an $n\times n$ generalized Cartan matrix A. The set of simple roots is indexed by $I = \{1, \cdots, n\}$. Let Q be its root lattice, i.e. $Q = \bigoplus_{i\in I} \mathbb{Z}\alpha_i \subset \mathfrak{h}$ where $\mathfrak{h}$ is the Cartan subalgebra and α_i are the simple roots. Let $\Pi^\vee = \{h_i \in \mathfrak{h} \mid i \in I\}$ be the set of simple coroots. We choose $d_j \in \mathfrak{h}$, $1 \leqslant j \leqslant n\text{-rank}(A)$ such that $\Pi^\vee \bigcup\{d_j \in \mathfrak{h} \mid 1 \leqslant j \leqslant n\text{-rank}(A)\}$ forms a basis of $\mathfrak{h}$. Set

$$P^\vee = \left(\bigoplus_{i\in I}\mathbb{Z}h_i\right) \oplus \left(\bigoplus_{1\leqslant j\leqslant n\text{-rank}(A)} \mathbb{Z}d_j\right) \subset \mathfrak{h}.$$

The weight lattice P is defined as $P = \{\lambda \in \mathfrak{h}^* \mid \lambda(h) \in \mathbb{Z}\ \forall h \in P^\vee\}$. Let Q^+ and P^+ be the positive root lattice and the set of dominant weights respectively. We define P_0 to be the subset of P^+ consisting of weights μ such that $\mu(h_i) = 0$ for all $i \in I$. Let W be the Weyl group associated with $\mathfrak{g}$. There is a W-invariant symmetric bilinear form $(\ ,\ )$ on $P \times P$ such that

$$2\frac{(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)} = \lambda(h_i).$$

Let $U_q(\mathfrak{g})$ be the quantized enveloping algebra generated over $k = \mathbb{Q}(q)$ by E_i, F_i and q^h for $i \in I$, $h \in P^\vee$ [8], which is denoted also by U for simplicity. The subalgebras U^+, U^- and the integral form $U_\mathbb{Z}$ are defined in the same way as in [8]. For $\xi = \sum_{i\in I} n_i\alpha_i \in Q$, define the height of ξ to be $\sum_{i\in I} |n_i|$, denoted by $ht(\xi)$. Set $U_\xi = \{u \in U \mid q^h u q^{-h} = q^{\xi(h)}u\}$. The filtration $F = (F_n)_{n\in\mathbb{Z}_+}$ of $U^\pm$ is defined by

$$F_n(U^{\pm}) = \bigoplus_{ht(\xi)\leqslant n} U_{\xi}^{\pm}.$$

Denote by $\widetilde{U}_q(\mathfrak{g})$ or simply $\widetilde{U}$ the modified quantized enveloping algebra generated by $U_q(\mathfrak{g})a_\lambda$ for all $\lambda \in P$ subject to the relations:

$$q^h a_\lambda = q^{\lambda(h)} a_\lambda, \qquad a_\lambda a_\mu = \delta_{\lambda,\mu} a_\lambda, \qquad u a_\lambda = a_{\lambda+\xi} u \quad \text{for } u \in U_\xi.$$

Note that $\widetilde{U} = \bigoplus_{\lambda\in P} U a_\lambda$. The integral form of $\widetilde{U}$ is defined as

$$\widetilde{U}_{\mathbb{Z}} \triangleq \bigoplus_{\lambda\in P} U_{\mathbb{Z}} a_\lambda.$$

There is an anti-automorphism (resp. automorphism) of U, denoted by $*$ (resp. ω), such that

$$E_i^* = E_i, \qquad F_i^* = F_i, \qquad (q^h)^* = q^{-h}$$
$$\big(\text{resp. } \omega(E_i) = F_i,\ \omega(F_i) = E_i,\ \omega(q^h) = q^{-h}\big).$$

One can see that $*$ and ω can be extended to involutions on $\widetilde{U}$, denoted by the same symbols, with $(a_\lambda)^* = \omega(a_\lambda) = a_{-\lambda}$.

2.2. Crystal basis and canonical basis of $\widetilde{U}$

For $\lambda, \mu \in P^+$, let $V(\lambda)\otimes V(-\mu)$ be the tensor product of irreducible integrable highest weight U-module $V(\lambda)$ of highest weight λ with irreducible integrable lowest weight U-module $V(-\mu)$ of lowest weight $-\mu$. Note that it is, by the comultiplication in U, also a U-module and we denote it also by $V(\lambda, -\mu)$. Let u_λ (resp. $u_{-\mu}$) be the highest (resp. lowest) weight vector of $V(\lambda)$ (resp. $V(-\mu)$) and set $u_{\lambda,-\mu} = u_\lambda \otimes u_{-\mu} \in V(\lambda, -\mu)$. It is known in [3,8] that $V(\lambda, -\mu)$ is a cyclic U-module generated by $u_{\lambda,-\mu}$ and that it admits a crystal basis

$$B(\lambda, -\mu) \triangleq B(\lambda) \otimes B(-\mu)$$

where $B(\lambda)$ and $B(-\mu)$ are highest and lowest weight crystals respectively. The corresponding global basis of $V(\lambda, -\mu)$ is constructed in [3] and following Lusztig, we call it canonical basis, which is denoted by $\{G(b) \mid b \in B(\lambda, -\mu)\}$.

Definition 2.1.

(i) For a U-module M with a canonical basis, a subspace N of M is called nice or compatible with the canonical basis if N is spanned over k by a subset of the canonical basis of M.

(ii) For U-modules M and N with canonical bases, a homomorphism of U-modules $\phi : M \longrightarrow N$ is called nice or compatible with the canonical bases if it maps a canonical base element of M to that of N or to zero and if the images of two distinct canonical base elements are distinct when they are both nonzero.

(iii) For a U-module M with a canonical basis, a filtration or composition series of M is called nice or compatible with the canonical basis if any submodule involved in the filtration or composition series is nice.

In [3], the following stability property plays a key role in the construction of the canonical basis of $\widetilde{U}$.

Proposition 2.2. *(See [3].) For $\lambda, \mu, \theta \in P^+$, the map $\pi_{\lambda,\mu,\theta} : V(\lambda + \theta, -\theta - \mu) \longrightarrow V(\lambda, -\mu)$ which takes $xu_{\lambda+\theta,-\theta-\mu}$ to $xu_{\lambda,-\mu}$ for all $x \in U$ is a nice surjective homomorphism of U-modules.*

We see from the proposition that there is an embedding of crystals $B(\lambda, -\mu) \hookrightarrow B(\lambda+\theta, -\theta-\mu)$ and note that it is strict [8]. For $\lambda, \mu \in P^+$, let $\Phi : Ua_{\lambda-\mu} \longrightarrow V(\lambda, -\mu)$ be the U-module homomorphism taking $a_{\lambda-\mu}$ to $u_{\lambda,-\mu}$. It is known that $\widetilde{U}$ as well as each Ua_λ have canonical bases and Φ is a nice surjective U-module homomorphism. We denote the crystal basis of $\widetilde{U}$ (resp. Ua_λ) by $\widetilde{B}$ (resp. $B(Ua_\lambda)$). Hence we have the embedding of crystals

$$B(\lambda, -\mu) \hookrightarrow B(Ua_{\lambda-\mu}).$$

It can be viewed as $B(\lambda, -\mu) \subseteq B(\lambda + \theta, -\theta - \mu) \subseteq B(Ua_{\lambda-\mu}) \subseteq \widetilde{B}$. Note that $B(Ua_\lambda)$ can be written as $B(\infty) \otimes T_\lambda \otimes B(-\infty)$ where $B(\pm\infty)$ is the crystal basis of $U^\mp$ and T_λ is a crystal consisting of a single element t_λ. For $b \in B(\lambda, -\mu) \subseteq \widetilde{B}$, we denote by the same $G(b)$ the corresponding canonical base element in $V(\lambda, -\mu)$ or $\widetilde{U}$ if this causes no confusion. It is known that the anti-automorphism $*$ induces a bijection on $\widetilde{B}$ such that $(b_1 \otimes t_\lambda \otimes b_2)^* = b_1^* \otimes t_{-\lambda-wt(b_1)-wt(b_2)} \otimes b_2^*$ and $G(b)^* = G(b^*)$ [8].

For any $\lambda \in P$, there is an extremal weight U-module $V^{max}(\lambda)$ which admits a crystal basis $B^{max}(\lambda)$ consisting of all $b \in B(Ua_\lambda)$ such that b^* is extremal [8]. We have $V^{max}(\lambda) \cong V^{max}(w\lambda)$ for any $w \in W$ and $V^{max}(\lambda) \cong V(\lambda)$ if $\lambda \in \pm P^+$. It is also known that for any connected component B of $\widetilde{B}$, there is an $l > 0$ such that $(wt(b), wt(b)) \leqslant l$ for all $b \in B$. Moreover, B contains an extremal vector and can be embedded into $B^{max}(\mu)$ for some $\mu \in P$ [8].

For $\mathfrak{g}$ of affine type, let $c \in \mathfrak{h}$ be the canonical central element of $\mathfrak{g}$. Given $\lambda \in P$, we define the level of λ to be the integer $\lambda(c)$, denoted by *level*(λ). Since an integral weight λ of positive (resp. negative) level is W-conjugate to a dominant (resp. anti-dominant) weight, it follows from the previous paragraph that $B(Ua_\lambda)$ is a union of highest (resp. lowest) weight crystals.

Denote by $S^{\#}$ the cardinality of the set S. Given two crystals B_1 and B_2 with B_1 connected, denote by $[B_2 : B_1]$ the cardinality of the set which consists of all connected components of B_2 isomorphic to B_1, i.e.

$$[B_2 : B_1] = \{B \subset B_2 \mid B \cong B_1\}^{\#}.$$

The following result was proved in [1].

Proposition 2.3. *For $\lambda \in P^+$ and $\mu \in P$, $[B(Ua_\mu) : B(\lambda)] = \dim V(\lambda)_\mu$.*

2.3. Bilinear form on $\widetilde{U}$

We introduce another anti-automorphism of U, denoted by Ψ [8], such that

$$\Psi(E_i) = q_i^{-1} t_i^{-1} F_i, \qquad \Psi(F_i) = q_i^{-1} t_i E_i, \qquad \Psi\big(q^h\big) = q^h, \qquad \Psi(q) = q,$$

where $q_i = q^{\frac{(\alpha_i,\alpha_i)}{2}}$ and $t_i = q^{\frac{(\alpha_i,\alpha_i)}{2} h_i}$. One can easily check that $\Psi^2 = id$ and Ψ commutes with the automorphism ω introduced above.

For $\lambda \in \pm P^+$, there is a unique non-degenerate symmetric bilinear form (,) on $V(\lambda)$ such that

$$(u_\lambda, u_\lambda) = 1 \quad \text{and} \quad (su, v) = \big(u, \Psi(s)v\big) \quad \text{for all } s \in U,\ u, v \in V(\lambda).$$

The bilinear form on $V(\lambda)$ and that on $V(-\lambda)$ are related by the following equation

$$(su_\lambda, tu_\lambda) = \big(\omega(s)u_{-\lambda}, \omega(t)u_{-\lambda}\big). \tag{2.1}$$

Given $\lambda, \mu \in P^+$, we define a symmetric bilinear form (,) on $V(\lambda, -\mu)$ by

$$(u_1 \otimes v_1, u_2 \otimes v_2) = (u_1, u_2)(v_1, v_2).$$

Since Ψ commutes with the comultiplication Δ, i.e. $(\Psi \otimes \Psi)\Delta = \Delta\Psi$, it implies that $(su, v) = (u, \Psi(s)v)$ for all $s \in U$, $u, v \in V(\lambda, -\mu)$.

Lemma 2.4. *(See [8].) For $s, t \in U$ and $\theta \in P$, there exists a unique polynomial $f_{s,t,\theta}(x)$ in $x = (x_i)_{i\in I}$ such that for any $\lambda, \mu \in P^+$ with $\lambda - \mu = \theta$, $(su_{\lambda,-\mu}, tu_{\lambda,-\mu}) = f_{s,t,\theta}(x)$ with $x_i = q_i^{\lambda(h_i)}$.*

The bilinear form on Ua_θ is then defined by $(sa_\theta, ta_\theta) = f_{s,t,\theta}(0)$ and this extends to a bilinear form on $\widetilde{U}$ such that $(Ua_{\theta_1}, Ua_{\theta_2}) = 0$ for $\theta_1 \neq \theta_2$. It was shown in [8] that (,) on $\widetilde{U}$ is symmetric and it satisfies

$$(u, v) = \big(u^*, v^*\big) \quad \text{and} \quad (su, v) = \big(u, \Psi(s)v\big) \quad \text{for all } s \in U,\ u, v \in \widetilde{U}.$$

Let A_0 be the subring of k consisting of all rational functions regular at $q = 0$. The crystal lattice $L(\widetilde{U})$ over A_0 and the canonical basis of $\widetilde{U}$ are characterized by the bilinear form.

Proposition 2.5. *(See [8].)*

(i) $L(\widetilde{U}) = \{u \in \widetilde{U} \mid (u,u) \in A_0\}$.
(ii) *If* $u \in \widetilde{U}_{\mathbb{Z}}$ *and* $(u,u) \in 1 + qA_0$, *then* $u \equiv G(b)$ *or* $-G(b) \bmod qL(\widetilde{U})$ *for some* $b \in \widetilde{B}$.

We define another bilinear form $((\ ,\ ))$ on $\widetilde{U}$ by

$$\big((u,v)\big) = \big(\omega(u), \omega(v)\big) \quad \text{for all } u, v \in \widetilde{U}.$$

Proposition 2.6. $((u,v)) = (u,v)$ *for all* $u, v \in \widetilde{U}$.

Proof. Assume that $u = sa_\theta$, $v = ta_\theta$ where $s, t \in U$, $\theta \in P$. Let $\Psi(s)t = \sum_j x_j^+ x_j^-$ where $x_j^\pm \in U^\pm \otimes k[q^h : h \in P^\vee]$. We have

$$\begin{aligned}\big((sa_\theta, ta_\theta)\big) &= \big(\omega(s)a_{-\theta}, \omega(t)a_{-\theta}\big) = \big(a_{-\theta}, \Psi\big(\omega(s)\big)\omega(t)a_{-\theta}\big) \\ &= \big(a_{-\theta}, \omega\big(\Psi(s)t\big)a_{-\theta}\big) = \sum_j \big(a_{-\theta}, \omega\big(x_j^+ x_j^-\big)a_{-\theta}\big) \\ &= \sum_j \big(\omega\big(\Psi\big(x_j^+\big)\big)a_{-\theta}, \omega\big(x_j^-\big)a_{-\theta}\big) = \sum_j \big(\big(\Psi\big(x_j^+\big)a_\theta, x_j^- a_\theta\big)\big)\end{aligned}$$

Since $q^h a_\theta = q^{\theta(h)} a_\theta$, it is sufficient to show the equality when $s, t \in U^-$. Given $\lambda, \mu \in P^+$ such that $\lambda - \mu = \theta$,

$$(su_{\lambda,-\mu}, tu_{\lambda,-\mu}) = (su_\lambda \otimes u_{-\mu}, tu_\lambda \otimes u_{-\mu}) = (su_\lambda, tu_\lambda) = f_{s,t,\theta}(x)$$

Meanwhile we have,

$$\begin{aligned}&\big(\omega(s)u_{\mu,-\lambda}, \omega(t)u_{\mu,-\lambda}\big) = \big(u_\mu \otimes \omega(s)u_{-\lambda}, u_\mu \otimes \omega(t)u_{-\lambda}\big) \\ &\quad = \big(\omega(s)u_{-\lambda}, \omega(t)u_{-\lambda}\big) = (su_\lambda, tu_\lambda) = f_{s,t,\theta}(x).\end{aligned}$$

Hence $((sa_\theta, ta_\theta)) = f_{s,t,\theta}(0) = (sa_\theta, ta_\theta)$. □

Corollary 2.7.

(i) $\omega(L(\widetilde{U})) = L(\widetilde{U})$.
(ii) $\omega(\widetilde{B}) = \widetilde{B}$.
(iii) $\omega(G(b)) = G(\omega(b))$ *for all* $b \in \widetilde{B}$.

Proof. Similar to Kashiwara's proof of the property of the anti-automorphism $*$ [8], we only show (ii) here. Given $b = b_1 \otimes t_\lambda \otimes b_2 \in \widetilde{B}$ with $ht(wt(b_i)) = l_i$, $i = 1, 2$, we have

$$G(b) \equiv G(b_1)G(b_2)a_\lambda \mod F_{l_1-1}\big(U^-\big)F_{l_2-1}\big(U^+\big)a_\lambda. \tag{2.2}$$

Since $\omega : U^\pm \longrightarrow U^\mp$ induces $\omega : B(\mp\infty) \longrightarrow B(\pm\infty)$ and it maps canonical base elements of $U^\pm$ to those of $U^\mp$, applying ω to (2.2) we have

$$\begin{aligned}\omega\big(G(b)\big) &\equiv \omega\big(G(b_1)\big)\omega\big(G(b_2)\big)a_{-\lambda} = G\big(\omega(b_1)\big)G\big(\omega(b_2)\big)a_{-\lambda} \\ &\equiv G\big(\omega(b_2)\big)G\big(\omega(b_1)\big)a_{-\lambda} \mod F_{l_2-1}\big(U^-\big)F_{l_1-1}\big(U^+\big)a_{-\lambda}.\end{aligned}$$

Since $\omega(G(b)) = G(b')$ or $-G(b')$ for some $b' \in \widetilde{B}$, we obtain that $b' = \omega(b_2) \otimes t_{-\lambda} \otimes \omega(b_1)$ and $\omega(G(b)) = G(b')$. □

3. A quotient algebra of $\widetilde{U}$

Throughout this section, a pair of dominant weights (λ, μ) is fixed.

3.1. Filtration

In this subsection, we recall the construction of some nice filtrations of the U-module $V(\lambda, -\mu)$ in [1]. In order to obtain nice submodules of $V(\lambda, -\mu)$, we need the following lemma due to Kashiwara [9] who proved it in case of $U = U_q(sl_2)$. See also [1] for more details of the proof in general case.

Lemma 3.1.

(i) *Let M be an integrable U-module with a canonical basis. If N is a nice U^+-submodule of M, then $UN = U^-N$ is a nice U-submodule of M. More precisely, $UN = \bigoplus_{b \in B(UN) \subseteq B(M)} kG(b)$.*

(ii) $B(UN) = \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_m} b \mid m \geqslant 0, i_1, \cdots, i_m \in I, b \in B(N)\} \setminus \{0\}$.

There is a total order $<$ on the lowest weight crystal $B(-\mu)$ such that $b_1 < b_2$ if $wt(b_1) < wt(b_2)$. See [1] for the existence of the total order and we note that it is not unique. For $b \in B(-\mu)$, one can define a subspace $V_b(-\mu)$ of $V(-\mu)$ as

$$V_b(-\mu) \triangleq \sum_{c \geqslant b} kG(c)$$

which is easily shown to be a U^+-submodule. Hence $u_\lambda \otimes V_b(-\mu)$ is a U^+-submodule of $V(\lambda, -\mu)$ which has a basis $\{u_\lambda \otimes G(c) \mid c \geqslant b,\ c \in B(-\mu)\}$. Since $u_\lambda \otimes G(c) = G(u_\lambda \otimes c)$, $u_\lambda \otimes V_b(-\mu)$ is actually a nice U^+-submodule. Define $F_\lambda(b)$ to be a U-submodule of $V(\lambda, -\mu)$ generated by $u_\lambda \otimes V_b(-\mu)$, i.e.

$$F_\lambda(b) = U\big(u_\lambda \otimes V_b(-\mu)\big).$$

It follows from Lemma 3.1 that $F_\lambda(b)$ is a nice U-submodule of $V(\lambda, -\mu)$ and

$$B\big(F_\lambda(b)\big) = \big\{\tilde{f}_{i_1} \cdots \tilde{f}_{i_m}(u_\lambda \otimes c) \bigm| i_1, \cdots, i_m \in I,\ c \in B(-\mu),\ c \geqslant b\big\} \setminus \{0\}.$$

Moreover, by comparing the crystal basis, we have the following result.

Theorem 3.2. *(See [1].) For two neighbors $b < c \in B(-\mu)$, $F_\lambda(b)/F_\lambda(c) \cong V(\lambda + wt(b))$ if $\tilde{e}_i(u_\lambda \otimes b) = 0$ for all $i \in I$, otherwise $F_\lambda(b) = F_\lambda(c)$.*

Hence we get a nice descending filtration of $V(\lambda, -\mu)$

$$V(\lambda, -\mu) = F_\lambda(b_1) \supseteq F_\lambda(b_2) \supseteq F_\lambda(b_3) \supseteq \cdots \tag{3.1}$$

where $u_{-\mu} = b_1 < b_2 < b_3 < \cdots$ is a complete list of $B(-\mu)$.

Remark 3.3. Similarly one can also define a total order on $B(\lambda)$ such that $b_1 < b_2$ if $wt(b_1) < wt(b_2)$. Set

$$F^{-\mu}(b) \triangleq \sum_{c \leqslant b} U\big(G(c) \otimes u_{-\mu}\big)$$

with which we can also construct a nice filtration of $V(\lambda, -\mu)$ where the quotient of two neighbors is isomorphic either to an irreducible lowest weight module or to 0.

Let $W(\lambda, -\mu)$ be a subspace of $V(\lambda, -\mu)$ defined by

$$W(\lambda, -\mu) \triangleq \bigcap_{b \in B(-\mu)} F_\lambda(b),$$

and let

$$M(\lambda, -\mu) \triangleq V(\lambda, -\mu)/W(\lambda, -\mu). \tag{3.2}$$

Denote by B' (resp. $B'(\lambda, -\mu)$) the subcrystal of $\widetilde{B}$ (resp. $B(\lambda, -\mu)$) which is a union of all connected components of $\widetilde{B}$ (resp. $B(\lambda, -\mu)$) that are not highest weight crystals. We have the following proposition in [1].

Proposition 3.4.

(i) $W(\lambda, -\mu)$ is a nice U-submodule of $V(\lambda, -\mu)$ and $B(W(\lambda, -\mu)) = B'(\lambda, -\mu)$.
(ii) $M(\lambda, -\mu)$ admits a canonical basis and $B(M(\lambda, -\mu)) = B(\lambda, -\mu) \setminus B'(\lambda, -\mu)$.

Remark 3.5. One can see that $U(\lambda,-\mu) \triangleq \bigcap_{b\in B(\lambda)} F^{-\mu}(b)$ has a crystal basis $B''(\lambda,-\mu)$ as well as a canonical basis where $B''(\lambda,-\mu)$ consists of all connected components of $B(\lambda,-\mu)$ that are not lowest weight crystals. Similarly $N(\lambda,-\mu) \triangleq V(\lambda,-\mu)/U(\lambda,-\mu)$ admits a canonical basis.

Note that when $\mathfrak{g}$ is of finite type, $V(\lambda,-\mu)$ is finite dimensional. Hence there are finitely many terms in the filtration (3.1) and furthermore, we can obtain a nice composition series of $V(\lambda,-\mu)$ [1] by deleting the superfluous terms in (3.1) which provides a complete proof to the conjecture raised by Lusztig [3]. Moreover, $W(\lambda,-\mu)=0$ and $M(\lambda,-\mu)=V(\lambda,-\mu)$ in this case. But when $\mathfrak{g}$ is of affine or indefinite type, the situation is quite different. For $\mathfrak{g}$ of affine type, the following result was shown in [1].

Proposition 3.6.

(*i*) $W(\lambda,-\mu) = N(\lambda,-\mu) = 0$ *and* $M(\lambda,-\mu) = U(\lambda,-\mu) = V(\lambda,-\mu)$ *if* *level*$(\lambda-\mu)>0$.

(*ii*) $W(\lambda,-\mu) = N(\lambda,-\mu) = V(\lambda,-\mu)$ *and* $M(\lambda,-\mu) = U(\lambda,-\mu) = 0$ *if* *level*$(\lambda-\mu)<0$.

(*iii*) $M(\lambda,-\mu)=N(\lambda,-\mu)$ *is a 1-dimensional trivial module if* $\lambda-\mu\in P_0$, *otherwise if* $\lambda-\mu\notin P_0$ *is of level 0*, $W(\lambda,-\mu)=U(\lambda,-\mu)=V(\lambda,-\mu)$ *and* $M(\lambda,-\mu)=N(\lambda,-\mu)=0$.

3.2. $\mathfrak{U}$

We denote by $\mathcal{O}^+$ (resp. $\mathcal{O}^-$) the completely reducible category whose objects are direct sums of irreducible integrable highest (resp. lowest) weight U-modules. Note that $\mathcal{O}^+$ here is often referred to as $\mathcal{O}_{int}$ in the literatures.

Theorem 3.7. *For* $b\in\widetilde{B}$, *the following conditions are equivalent.*

(*i*) $G(b)$ *acts on* $V(\lambda)$ *as zero for all* $\lambda\in P^+$.

(*ii*) $G(b)$ *acts on* M *as zero for any* $M\in ob(\mathcal{O}^+)$.

(*iii*) $b\in B'$.

Proof. The equivalence of (*i*) and (*ii*) is clear. If b satisfies (*ii*), we show that it satisfies (*iii*). Otherwise assume that $b\notin B'$, b is contained in a highest weight subcrystal of $\widetilde{B}$. There exist $\lambda,\mu\in P^+$ such that $b\in B(\lambda,-\mu)\subset\widetilde{B}$. We rewrite the nice filtration (3.1) of $V(\lambda,-\mu)$ as

$$V(\lambda,-\mu)=F_0\supseteq F_1\supseteq\cdots\supseteq F_l\supseteq\cdots. \tag{3.3}$$

There exists an $s\geqslant 0$ such that $G(b)\in F_s$ but $G(b)\notin F_{s+1}$. Hence

$$0 \neq G(b)(u_{\lambda,-\mu} + F_{s+1}) \in V(\lambda,-\mu)/F_{s+1}$$

where $V(\lambda,-\mu)/F_{s+1}$ is an object in $\mathcal{O}^+$. This contradicts (ii). Finally we show that (iii) implies (i). Assume that $G(b)V(\lambda') \neq 0$ for some $\lambda' \in P^+$, then there exists an $m \in V(\lambda')_\xi$ such that $G(b)m \neq 0$ and $b \in B(Ua_\xi) \subset \widetilde{B}$. We can find $\lambda, \mu \in P^+$ with $\lambda - \mu = \xi$ such that $b \in B(\lambda,-\mu) \subset B(Ua_\xi)$ and there exists a homomorphism of U-modules $\phi : V(\lambda,-\mu) \longrightarrow V(\lambda')$ which takes $u_{\lambda,-\mu}$ to m. Since $b \in B'$, $G(b) \in W(\lambda,-\mu)$, that is, $G(b) \in F_s$ for any F_s in the filtration (3.3). Restricting ϕ on F_s, we get a U-module homomorphism

$$\phi|_{F_s} : F_s \longrightarrow V(\lambda').$$

Since the set of generators of F_s is of the form $u_\lambda \otimes V_b(-\mu)$ for some b, the corresponding weights of these generators are not lower than or equal to λ' for a sufficient large s by the construction. Hence $\phi|_{F_s}$ is zero for $s \gg 0$. It follows that $\phi(G(b)) = G(b)m = 0$ which is a contradiction. □

As is known in [10, 7.1.9], if $u \in U$ acts on each $M \in ob(\mathcal{O}^+)$ as zero, then $u = 0$ for $\mathfrak{g}$ of any type. But the above theorem tells us that $u \in \widetilde{U}$ annihilating all objects in $\mathcal{O}^+$ might be nonzero when $\mathfrak{g}$ is of affine or indefinite type.

Proposition 3.8. *For $u = \sum k_b G(b) \in \widetilde{U}$ such that u acts on M as zero for all $M \in ob(\mathcal{O}^+)$ and if $k_b \neq 0$, then $b \in B'$.*

Proof. We assume that $k_{b_0} \neq 0$ for some $b_0 \notin B'$. There exist $\lambda, \mu \in P^+$ such that $b_0 \in B(\lambda,-\mu) \subset \widetilde{B}$. Since $b_0 \notin B'$, there exists an s such that $G(b_0) \in F_s$ but $G(b_0) \notin F_{s+1}$ where F_s and F_{s+1} are in the filtration (3.3) of $V(\lambda,-\mu)$. Hence we have

$$0 \neq u(u_{\lambda,-\mu} + F_{s+1}) \in V(\lambda,-\mu)/F_{s+1}$$

with $V(\lambda,-\mu)/F_{s+1} \in ob(\mathcal{O}^+)$. This is a contradiction. □

By this proposition we know that any $u \in \widetilde{U}$ annihilating all $M \in ob(\mathcal{O}^+)$ is a linear combination of $G(b)$ with $b \in B'$. Denote by $\widetilde{U}'$ the set of all such u. It follows from Theorem 3.7 and Proposition 3.8 that

Theorem 3.9. *$\widetilde{U}'$ is a nice two-sided ideal of $\widetilde{U}$ and it admits a crystal basis B'.*

We define $\mathfrak{U}$ to be the quotient of $\widetilde{U}$ by $\widetilde{U}'$, i.e. $\mathfrak{U} \triangleq \widetilde{U}/\widetilde{U}'$. Hence $\mathfrak{U}$ inherits from $\widetilde{U}$ a canonical basis and we denote by $\mathfrak{B}$ the corresponding crystal basis. One can see from the definition of B' and $\mathfrak{B} = \widetilde{B} \setminus B'$ that $\mathfrak{B}$ is a union of all highest weight subcrystals of $\widetilde{B}$. We know also from Theorem 3.7 that any $M \in ob(\mathcal{O}^+)$ is also a representation of $\mathfrak{U}$. Note that when $\mathfrak{g}$ is of finite type, $\mathfrak{U} = \widetilde{U}$. If $\mathfrak{g}$ is of affine type, it follows from

Proposition 3.6 that $\mathfrak{U}$ is isomorphic to the subalgebra of $\widetilde{U}$ generated by Ua_ξ and a_η for all ξ with a positive level and $\eta \in P_0$.

Remark 3.10. Similarly one can define $\widetilde{U}''$ to be the set of all $u \in \widetilde{U}$ such that u annihilates all $M \in ob(\mathcal{O}^-)$. Then $\widetilde{U}''$ is also a nice ideal of $\widetilde{U}$ with a crystal basis B'' where B'' consists of all connected components of $\widetilde{B}$ that are not lowest weight crystals. We denote by $\mathfrak{V}$ the quotient algebra $\widetilde{U}/\widetilde{U}''$ which admits both a crystal basis and a canonical basis.

4. Cells in $\mathfrak{U}$

Recall that in the previous section, by (3.2) we define $M(\lambda, -\mu)$ which is a representation of U as well as $\widetilde{U}$. Also it can be viewed as a representation of $\mathfrak{U}$. To see that, we need the following lemma.

Lemma 4.1. *For $b \in B' \subset \widetilde{B}$ and $\lambda, \mu \in P^+$, $G(b)M(\lambda, -\mu) = 0$.*

Proof. We only show that $G(b)V(\lambda, -\mu) \subseteq W(\lambda, -\mu)$. Since $V(\lambda, -\mu)/F_s \in ob(\mathcal{O}^+)$ for any F_s in the filtration (3.3), by Theorem 3.7 we have

$$G(b)\bigl(V(\lambda, -\mu)/F_s\bigr) = 0$$

which means $G(b)V(\lambda, -\mu) \subseteq F_s$. Hence we have

$$G(b)V(\lambda, -\mu) \subseteq \bigcap_{s \geqslant 0} F_s = W(\lambda, -\mu). \qquad \square$$

Applying this lemma we have $\widetilde{U}'M(\lambda, -\mu) = 0$ and thus we equip $M(\lambda, -\mu)$ with a $\mathfrak{U}$-action. Quotient by $W(\lambda, -\mu)$, we obtain from (3.3) a filtration of $M(\lambda, -\mu)$ consisting of nice U or $\mathfrak{U}$-submodules

$$M(\lambda, -\mu) = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_l \supseteq \cdots \tag{4.1}$$

where $M_i = F_i/W(\lambda, -\mu)$. Denote by $v_{\lambda,-\mu}$ the image of $u_{\lambda,-\mu}$ in $M(\lambda, -\mu)$. Hence the map

$$\bar{\alpha}_{\lambda,-\mu} : \mathfrak{U} \longrightarrow M(\lambda, -\mu) \qquad x \longmapsto xv_{\lambda,-\mu}$$

takes the canonical base elements of $\mathfrak{U}$ to those of $M(\lambda, -\mu)$ or to zero. For $\xi \in P^+$, let $M(\lambda, -\mu)_{[\xi]}$ (resp. $\mathfrak{U}_{[\xi]}$) be the subspace of $M(\lambda, -\mu)$ (resp. $\mathfrak{U}$) spanned by all $G(b)$ such that b is contained in a subcrystal of $B(M(\lambda, -\mu))$ (resp. $\mathfrak{B}$) isomorphic to $B(\xi)$. Note that $M(\lambda, -\mu)_{[\xi]}$ is usually not a U-submodule of $M(\lambda, -\mu)$. Set $M(\lambda, -\mu)_{[\geqslant \xi]}$, $M(\lambda, -\mu)_{[>\xi]}$, $\mathfrak{U}_{[\geqslant \xi]}$ and $\mathfrak{U}_{[>\xi]}$ as follows

$$M(\lambda,-\mu)_{[\geqslant\xi]} \triangleq \bigoplus_{\eta\geqslant\xi} M(\lambda,-\mu)_{[\eta]}, \qquad \mathfrak{U}_{[\geqslant\xi]} \triangleq \bigoplus_{\eta\geqslant\xi} \mathfrak{U}_{[\eta]},$$
$$M(\lambda,-\mu)_{[>\xi]} \triangleq \bigoplus_{\eta>\xi} M(\lambda,-\mu)_{[\eta]}, \qquad \mathfrak{U}_{[>\xi]} \triangleq \bigoplus_{\eta>\xi} \mathfrak{U}_{[\eta]}.$$

For a U-module $M \in ob(\mathcal{O}^+)$ with a canonical basis, M can be written as $M = \bigoplus_{\lambda\in P^+} M[\lambda]$ where $M[\lambda]$ is the sum of all submodules of M isomorphic to $V(\lambda)$. Here $M[\lambda]$ is usually not nice. But it is known in [5, Proposition 27.1.7] that $M[\xi]$ is a nice U-submodule of M for a maximal ξ, i.e. ξ is maximal in the sense of dominant order among all λ such that $M[\lambda] \neq 0$. Moreover, both $M[\geqslant\xi] \triangleq \bigoplus_{\lambda\geqslant\xi} M[\lambda]$ and $M[>\xi] \triangleq \bigoplus_{\lambda>\xi} M[\lambda]$ are nice. In particular, when $\mathfrak{g}$ is of finite type, $M(\lambda,-\mu)_{[\geqslant\xi]} = M(\lambda,-\mu)[\geqslant\xi]$.

Similar to [5, Lemma 29.1.3] we have the following lemma.

Lemma 4.2. *For $x \in \mathfrak{U}$ and $\xi \in P^+$, the following are equivalent*

- (i) $x \in \mathfrak{U}_{[\geqslant\xi]}$.
- (ii) *For all $\lambda,\mu \in P^+$, $xv_{\lambda,-\mu} \in M(\lambda,-\mu)_{[\geqslant\xi]}$.*
- (iii) *For any $M \in ob(\mathcal{O}^+)$ and $m \in M$, $xm \in M[\geqslant\xi]$.*
- (iv) *If x acts on $V(\eta)$ as a nonzero map for some $\eta \in P^+$, then $\eta \geqslant \xi$.*

Proof. It is clear that the equivalence of (i) and (ii) follows from definitions of $\mathfrak{U}_{[\geqslant\xi]}$ and $M(\lambda,-\mu)_{[\geqslant\xi]}$. (iii) and (iv) are equivalent since any $M \in ob(\mathcal{O}^+)$ can be written as a direct sum of some $V(\eta)$. If x satisfies (iii) we show that it satisfies (ii). Set $x = \sum k_b G(b)$. Assume that (ii) does not hold, then there exists some $b_0 \in B(M(\lambda,-\mu)) \subseteq \mathfrak{B}$ with $k_{b_0} \neq 0$ such that b_0 is contained in a subcrystal of $\mathfrak{B}$ isomorphic to $B(\eta)$ with $\eta \ngeqslant \xi$. It follows that there exists an s such that $G(b_0) \in M_s$ but $G(b_0) \notin M_{s+1}$ where M_s and M_{s+1} are in the filtration (4.1). Thus

$$M_s/M_{s+1} \cong V(\eta).$$

Set $M \triangleq M(\lambda,-\mu)/M_{s+1}$ and $m = v_{\lambda,-\mu} + M_{s+1} \in M$. Then $M \in ob(\mathcal{O}^+)$ and $xm \notin M[\geqslant\xi]$ which contradicts (iii). Conversely we show that (ii) implies (iii). For any $M \in ob(\mathcal{O}^+)$ and $m \in M_\theta$, there exist $\lambda,\mu \in P^+$ with $\lambda-\mu = \theta$ such that $xv_{\lambda,-\mu} \neq 0$ and $\phi: V(\lambda,-\mu) \longrightarrow M,\ u_{\lambda,-\mu} \longmapsto m$ is a nonzero homomorphism of U-modules. As in the proof of Theorem 3.7, one can see that $\phi(W(\lambda,-\mu)) = 0$. Hence we have

$$\bar{\phi}: M(\lambda,-\mu) \longrightarrow M, \qquad v_{\lambda,-\mu} \longmapsto m$$

a homomorphism of both U-modules and $\mathfrak{U}$-modules. As is proved before, there exists an s such that the weights of the generators of M_s are not lower than or equal to any weight in M. Hence $\bar{\phi}(M_s) = 0$ and furthermore, $\bar{\phi}$ factors through the $\mathfrak{U}$-map

$\bar{\phi}' : M(\lambda,-\mu)/M_s \longrightarrow M$, $v_{\lambda,-\mu} + M_s \longmapsto m$. Since $xv_{\lambda,-\mu} \in M(\lambda,-\mu)_{[\geqslant\xi]}$ and $M(\lambda,-\mu)/M_s \in ob(\mathcal{O}^+)$,

$$x(v_{\lambda,-\mu} + M_s) \in \big(M(\lambda,-\mu)/M_s\big)[\geqslant\xi].$$

It follows that $\bar{\phi}'(x(v_{\lambda,-\mu} + M_s)) = xm \in M[\geqslant\xi]$ which proves (*iii*). □

Similarly one can prove the following lemma since

$$\mathfrak{U}_{[>\xi]} = \sum_{\eta>\xi} \mathfrak{U}_{[\geqslant\eta]}, \qquad M_{[>\xi]} = \sum_{\eta>\xi} M_{[\geqslant\eta]}, \qquad M(\lambda,-\mu)_{[>\xi]} = \sum_{\eta>\xi} M(\lambda,-\mu)_{[\geqslant\eta]}.$$

Lemma 4.3. *For $x \in \mathfrak{U}$ and $\xi \in P^+$, the following are equivalent*

(*i*) $x \in \mathfrak{U}_{[>\xi]}$.
(*ii*) *For all $\lambda, \mu \in P^+$, $xv_{\lambda,-\mu} \in M(\lambda,-\mu)_{[>\xi]}$.*
(*iii*) *For any $M \in ob(\mathcal{O}^+)$ and $m \in M$, $xm \in M[>\xi]$.*
(*iv*) *If x acts on $V(\eta)$ as a nonzero map for some $\eta \in P^+$, then $\eta > \xi$.*

The corollary below follows immediately from Lemma 4.2 and Lemma 4.3.

Corollary 4.4. *Both $\mathfrak{U}_{[\geqslant\xi]}$ and $\mathfrak{U}_{[>\xi]}$ are nice two-sided ideals of $\mathfrak{U}$ for any $\xi \in P^+$.*

Remark 4.5. For $\xi \in P^+$, we can define $\mathfrak{V}_{[\leqslant-\xi]}$ (resp. $\mathfrak{V}_{[<-\xi]}$) to be the subset of $\mathfrak{V}$ consisting of all x such that $\eta \geqslant \xi$ (resp. $\eta > \xi$) if x acts on $V(-\eta)$ as a nonzero map. Similarly both of them are nice ideals of $\mathfrak{V}$.

For an integrable left U-module M with finite dimensional weight spaces, let M° denote the graded dual of M, i.e. $M^\circ = \bigoplus_{\theta\in P} M_\theta^*$ where $M = \bigoplus_{\theta\in P} M_\theta$. Then there is a right U-action on M° as

$$(f \cdot x)(v) = f(xv) \quad \text{for } f \in M^\circ,\ v \in M, x \in U.$$

For instance, $V(\lambda)^\circ$ is an irreducible integrable right U-module with highest weight $\lambda \in P^+$. Given a right U-module M, we denote by ${}^*M$ the same k-vector space equipped with a left U-action as

$$x \circ m = m \cdot x^* \quad \text{for } x \in U,\ m \in {}^*M,$$

where x^* is the image of x under the anti-automorphism $*$. It is clear that ${}^*V(\lambda)^\circ \cong V(-\lambda)$ as left U-modules for $\lambda \in \pm P^+$. Given a left U-module N, define ${}^\omega N$ to be the left U-module with the underlying space ${}^\omega N = N$ such that

$$x \circ v = \omega(x) \cdot v \quad \text{for } x \in U,\ v \in {}^\omega N.$$

See that ${}^\omega V(\lambda) \cong V(-\lambda)$ for $\lambda \in \pm P^+$.

Lemma 4.6.

(*i*) *Both* $*$ *and* ω *on* $\widetilde{U}$ *induce bijections* $*, \ \omega : \mathfrak{U} \longleftrightarrow \mathfrak{V}$.
(*ii*) *There are bijections* $*, \ \omega : \mathfrak{U}_{[\geqslant \xi]} \longleftrightarrow \mathfrak{V}_{[\leqslant -\xi]}$ *and* $*, \ \omega : \mathfrak{U}_{[>\xi]} \longleftrightarrow \mathfrak{V}_{[<-\xi]}$ *for* $\xi \in P^+$.

Proof. To prove (*i*), it is sufficient to show that $*(\widetilde{U}') = \widetilde{U}''$ and $\omega(\widetilde{U}') = \widetilde{U}''$. For $b \in B'$, $G(b)$ annihilates all $V(\lambda)$ for $\lambda \in P^+$. Then we have $G(b) \circ {}^*V(-\lambda)^\circ = 0$ for all $\lambda \in P^+$ which implies that $G(b)^* = G(b^*)$ annihilates all $V(-\lambda)$ for $\lambda \in P^+$. Hence $*(\widetilde{U}') \subseteq \widetilde{U}''$. Similarly we have $*(\widetilde{U}'') \subseteq \widetilde{U}'$. It follows from $*^2 = id$ on $\widetilde{U}$ that $*(\widetilde{U}') = \widetilde{U}''$. Given $b \in B'$, $G(b)$ annihilates all ${}^\omega V(-\lambda)$ for $\lambda \in P^+$. It implies that $\omega(G(b))$ annihilates all $V(-\lambda)$ for $\lambda \in P^+$ and thus $\omega(\widetilde{U}') \subseteq \widetilde{U}''$. The proof of the equality is similar to that for $*$. In order to prove (*ii*), we only show $*(\mathfrak{U}_{[\geqslant \xi]}) \subseteq \mathfrak{V}_{[\leqslant -\xi]}$ and $\omega(\mathfrak{U}_{[\geqslant \xi]}) \subseteq \mathfrak{V}_{[\leqslant -\xi]}$. Given $x \in \mathfrak{U}_{[\geqslant \xi]}$, if x^* acts on $V(-\eta)$, $\eta \in P^+$, as a nonzero map, one can see that $x({}^*V(-\eta)^\circ) = xV(\eta) \neq 0$ which implies $\eta \geqslant \xi$. Hence $x^* \in \mathfrak{V}_{[\leqslant -\xi]}$. Similarly if $\omega(x)$ acts on $V(-\eta)$ for some $\eta \in P^+$, as a nonzero map, then $\omega(x)V(-\eta) = x \circ {}^\omega V(-\eta) = xV(\eta) \neq 0$. Hence $\eta \geqslant \xi$ which implies $\omega(x) \in \mathfrak{V}_{[\leqslant -\xi]}$. □

Note that $\mathfrak{U}$ can be written as a direct sum of vector spaces $\bigoplus_{\xi \in P^+} \mathfrak{U}_{[\xi]}$. We have an isomorphism

$$\mathfrak{U}_{[\geqslant \xi]}/\mathfrak{U}_{[>\xi]} \cong \mathfrak{U}_{[\xi]}$$

as k-vector spaces. Furthermore, $\mathfrak{U}_{[\geqslant \xi]}/\mathfrak{U}_{[>\xi]}$ is an algebra as well as a $\mathfrak{U}$-bimodule which we call a *two-sided cell module* of $\mathfrak{U}$ and denote by $\mathfrak{U}(\xi)$ for simplicity. This cell naturally inherits from $\mathfrak{U}$ a canonical basis and its crystal basis is a family of copies of $B(\xi)$. We have the following result similar to [5, Theorem 29.3.3].

Proposition 4.7. *For* $\xi \in P^+$,

(*i*) $\mathfrak{U}(\xi)$ *decomposes into a direct sum of nice irreducible highest weight left* U*-submodules, each summand is isomorphic to* $V(\xi)$.
(*ii*) $\mathfrak{U}(\xi)$ *decomposes into a direct sum of nice irreducible highest weight right* U*-submodules, each summand is isomorphic to* $V(\xi)^\circ$.
(*iii*) $\mathfrak{U}(\xi) \cong V(\xi) \otimes V(\xi)^\circ$ *as* U *or* $\mathfrak{U}$*-bimodules.*

Proof. (*i*) is obvious. Since we have bijections

$$\omega \circ * : \mathfrak{U}_{[\geqslant \xi]} \longleftrightarrow \mathfrak{U}_{[\geqslant \xi]}, \qquad \mathfrak{U}_{[>\xi]} \longleftrightarrow \mathfrak{U}_{[>\xi]}$$

by Lemma 4.6, $\omega \circ *$ induces an anti-automorphism of $\mathfrak{U}(\xi)$. Applying $\omega \circ *$ to any summand V in (*i*), we obtain a nice irreducible right U-module $\omega \circ *(V)$ by Corollary 2.7 and this proves (*ii*). Let ϕ be the restricting map on $\mathfrak{U}_{[\geqslant \xi]}$ of the $\mathfrak{U}$-action on $V(\xi)$, i.e.

$\phi : \mathfrak{U}_{[\geqslant\xi]} \longrightarrow End_k(V(\xi))$. Then ϕ is a homomorphism of algebras without 1. It can be seen from Lemma 4.3 that the kernel of ϕ is exactly $\mathfrak{U}_{[>\xi]}$ and thus the induced map $\bar{\phi} : \mathfrak{U}(\xi) \longrightarrow End_k(V(\xi))$ is injective. We view $V(\xi) \otimes V(\xi)^\circ$ as a subset of $End_k(V(\xi))$, i.e.

$$(x \otimes f)(v) = f(v)x \quad \text{for } x,\ v \in V(\xi),\ f \in V(\xi)^\circ.$$

It is easy to see that $V(\xi) \otimes V(\xi)^\circ$ is a U or $\mathfrak{U}$-subbimodule as well as a subalgebra of $End_k(V(\xi))$ where U or $\mathfrak{U}$ acts on $V(\xi) \otimes V(\xi)^\circ$ as

$$\big(x(v \otimes f)y\big)(m) = f(ym)xv \quad \text{for } v, m \in V(\xi),\ f \in V(\xi)^\circ,\ x, y \in U \text{ or } \mathfrak{U}.$$

In fact the $\bar{\phi}$ defined above maps $\mathfrak{U}(\xi)$ injectively into $V(\xi) \otimes V(\xi)^\circ$, and moreover, $\bar{\phi} : \mathfrak{U}(\xi) \longrightarrow V(\xi) \otimes V(\xi)^\circ$ is a homomorphism of U or $\mathfrak{U}$-bimodules. Fixing a right weight $\eta \in P$, $\mathfrak{U}(\xi)a_\eta$ is, by Proposition 2.3, a direct sum of $\dim V(\xi)_\eta$ copies of $V(\xi)$ as a left $\mathfrak{U}$-module, where we denote the image of a_η in $\mathfrak{U}(\xi)$ by the same symbol. Hence $\bar{\phi} : \mathfrak{U}(\xi) \longrightarrow V(\xi) \otimes V(\xi)^\circ$ is surjective and $\mathfrak{U}(\xi) \cong V(\xi) \otimes V(\xi)^\circ$. □

5. Quantum coordinate algebra

5.1. $\mathfrak{U}^\circ$

For $U_1, U_2 \in \{\widetilde{U},\ \widetilde{U}_{\mathbb{Z}},\ \widetilde{U}',\ \mathfrak{U},\ U\text{-modules with canonical bases}\}$, let $U_1 \widehat{\otimes} U_2$ be the set of all formal (possibly infinite) linear combinations

$$\sum k_{b_1,b_2} G(b_1) \otimes G(b_2),$$

where $k_{b_1,b_2} \in k$ or $\mathbb{Z}[q, q^{-1}]$. For $\lambda, \lambda_1, \lambda_2 \in P$, the comultiplication in U induces the map $\Delta_{\lambda,\lambda_1,\lambda_2} : Ua_\lambda \longrightarrow Ua_{\lambda_1} \otimes Ua_{\lambda_2}$ where $\Delta_{\lambda,\lambda_1,\lambda_2}$ is nonzero only if $\lambda = \lambda_1 + \lambda_2$. Set

$$\Delta = \sum_{\lambda,\lambda_1,\lambda_2 \in P} \Delta_{\lambda,\lambda_1,\lambda_2} : \widetilde{U} \longrightarrow \widetilde{U} \widehat{\otimes} \widetilde{U}.$$

For $a, b, c \in \widetilde{B}$, we define $\hat{m}_a^{b,c} \in k$ to satisfy that

$$\Delta\big(G(a)\big) = \sum_{b,c} \hat{m}_a^{b,c} G(b) \otimes G(c).$$

Note that restricting Δ on $\widetilde{U}_{\mathbb{Z}}$, we have in [5, 23.2.3]

$$\Delta : \widetilde{U}_{\mathbb{Z}} \longrightarrow \widetilde{U}_{\mathbb{Z}} \widehat{\otimes} \widetilde{U}_{\mathbb{Z}}.$$

Since $\{G(b) \mid b \in \widetilde{B}\}$ forms a $\mathbb{Z}[q, q^{-1}]$-basis of $\widetilde{U}_{\mathbb{Z}}$, the structure constant $\hat{m}_a^{b,c}$ is actually in $\mathbb{Z}[q, q^{-1}]$. For λ, λ_1, λ_2, μ, μ_1, $\mu_2 \in P^+$ with $\lambda = \lambda_1 + \lambda_2$ and $\mu = \mu_1 + \mu_2$, let τ_1, τ_2 be the U-module homomorphism

$$\tau_1 : V(\lambda) \longrightarrow V(\lambda_1) \otimes V(\lambda_2), \qquad u_\lambda \longmapsto u_{\lambda_1} \otimes u_{\lambda_2},$$
$$\tau_2 : V(-\mu) \longrightarrow V(-\mu_1) \otimes V(-\mu_2), \qquad u_{-\mu} \longmapsto u_{-\mu_1} \otimes u_{-\mu_2}.$$

Set $R_{\lambda_2,-\mu_1}$ to be the unique isomorphism of U-modules (the R-matrix)

$$R_{\lambda_2,-\mu_1} : V(\lambda_2) \otimes V(-\mu_1) \longrightarrow V(-\mu_1) \otimes V(\lambda_2)$$

such that $R_{\lambda_2,-\mu_1}(u_{\lambda_2} \otimes u_{-\mu_1}) = u_{-\mu_1} \otimes u_{\lambda_2}$. Let τ be the composition of $\tau_1 \otimes \tau_2$ and $1 \otimes R_{\lambda_2,-\mu_1} \otimes 1$, i.e.

$$\begin{array}{ccc} V(\lambda, -\mu) & \xrightarrow{\tau_1 \otimes \tau_2} & V(\lambda_1) \otimes V(\lambda_2) \otimes V(-\mu_1) \otimes V(-\mu_2) \\ & \searrow^{\tau} & \downarrow {\scriptstyle 1 \otimes R_{\lambda_2,-\mu_1} \otimes 1} \\ & & V(\lambda_1, -\mu_1) \otimes V(\lambda_2, -\mu_2) \end{array}$$

Let ρ be the map $\rho : \widetilde{U} \,\widehat{\otimes}\, \widetilde{U} \longrightarrow V(\lambda_1, -\mu_1) \,\widehat{\otimes}\, V(\lambda_2, -\mu_2)$ such that

$$\rho\Big(\sum k_{a,b} G(a) \otimes G(b)\Big) = \sum k_{a,b}\big(G(a) u_{\lambda_1,-\mu_1}\big) \otimes \big(G(b) u_{\lambda_2,-\mu_2}\big).$$

One can see that $\widetilde{U}$ acts on $u_{\lambda_1,-\mu_1} \otimes u_{\lambda_2,-\mu_2}$ as a map which can be obtained through Δ. More precisely, we have a commutative diagram

$$\begin{array}{ccc} \widetilde{U} & \xrightarrow{\quad \Delta \quad} & \widetilde{U} \,\widehat{\otimes}\, \widetilde{U} \\ \downarrow {\scriptstyle \gamma} & & \downarrow {\scriptstyle \rho} \\ V(\lambda_1, -\mu_1) \otimes V(\lambda_2, -\mu_2) & \xrightarrow{\quad i \quad} & V(\lambda_1, -\mu_1) \,\widehat{\otimes}\, V(\lambda_2, -\mu_2) \end{array}$$

where $\gamma(x) = x(u_{\lambda_1,-\mu_1} \otimes u_{\lambda_2,-\mu_2})$ and i is the canonical inclusion.

Proposition 5.1. *The following diagram is commutative*

$$\begin{array}{ccc} \widetilde{U} & \xrightarrow{\quad \Delta \quad} & \mathrm{Im}(\Delta) \\ \downarrow {\scriptstyle \alpha_{\lambda,-\mu}} & & \downarrow {\scriptstyle \rho|_{\mathrm{Im}_\Delta}} \\ V(\lambda, -\mu) & \xrightarrow{\quad \tau \quad} & V(\lambda_1, -\mu_1) \otimes V(\lambda_2, -\mu_2) \end{array} \tag{5.1}$$

Proof. We regard $\widetilde{U} \mathbin{\widehat{\otimes}} \widetilde{U}$ as a left $\widetilde{U}$-module through Δ. Hence all the maps are homomorphisms of $\widetilde{U}$-modules. It is easy to check that the two compositions in the diagram coincide when applied to a_ξ for any $\xi \in P$. □

Lemma 5.2. $\Delta(\widetilde{U}') \subseteq \widetilde{U}' \mathbin{\widehat{\otimes}} \widetilde{U} + \widetilde{U} \mathbin{\widehat{\otimes}} \widetilde{U}'$.

Proof. We assume that $\Delta(\widetilde{U}') \nsubseteq \widetilde{U}' \widehat{\otimes} \widetilde{U} + \widetilde{U} \widehat{\otimes} \widetilde{U}'$. Then there exist $a \in B'$ and $b, c \in \widetilde{B} \backslash B'$ such that $\hat{m}_a^{b,c} \neq 0$. We suppose that $b \in B(\lambda_1, -\mu_1)$ and $c \in B(\lambda_2, -\mu_2)$ for some λ_1, λ_2, μ_1, $\mu_2 \in P^+$. By Proposition 5.1, we have

$$\rho\Delta\bigl(G(a)\bigr) = \tau\alpha_{\lambda,-\mu}\bigl(G(a)\bigr) \neq 0$$

where $\lambda = \lambda_1 + \lambda_2$, $\mu = \mu_1 + \mu_2$. Hence $a \in B(\lambda, -\mu)$. It is known that there is a nice filtration of $V(\lambda_i, -\mu_i)$

$$V(\lambda_i, -\mu_i) = F_{i,0} \supseteq F_{i,1} \supseteq \cdots \supseteq F_{i,l} \supseteq \cdots \tag{5.2(i)}$$

for $i = 1, 2$. Since $b, c \notin B'$, there exist s and t such that $G(b) \in F_{1,s}$, $G(b) \notin F_{1,s+1}$ and $G(c) \in F_{2,t}$, $G(c) \notin F_{2,t+1}$. Let π be the canonical map

$$\pi : V(\lambda_1, -\mu_1) \otimes V(\lambda_2, -\mu_2) \longrightarrow \bigl(V(\lambda_1, -\mu_1)/F_{1,s+1}\bigr) \otimes \bigl(V(\lambda_2, -\mu_2)/F_{2,t+1}\bigr)$$

where $(V(\lambda_1, -\mu_1)/F_{1,s+1}) \otimes (V(\lambda_2, -\mu_2)/F_{2,t+1})$ is an object in $\mathcal{O}^+$, denoted by M. Let m be the image of $u_{\lambda_1,-\mu_1} \otimes u_{\lambda_2,-\mu_2}$ in M. It follows from the assumption that $\pi\tau\alpha_{\lambda,-\mu}(G(a)) = G(a)m \neq 0$. This contradicts that $a \in B'$ by Theorem 3.7. □

It follows from the lemma that Δ induces the map $\bar{\Delta} : \mathfrak{U} \longrightarrow \mathfrak{U} \mathbin{\widehat{\otimes}} \mathfrak{U}$. For $a, b, c \in \mathfrak{B}$, similarly we define $\tilde{m}_a^{b,c}$ to satisfy that

$$\bar{\Delta}\bigl(G(a)\bigr) = \sum_{b,c} \tilde{m}_a^{b,c} G(b) \otimes G(c).$$

It is clear that $\tilde{m}_a^{b,c} = \hat{m}_a^{b,c} \in \mathbb{Z}[q, q^{-1}]$ if $a, b, c \in \mathfrak{B} = \widetilde{B} \setminus B'$. The coassociativity of the comultiplication in U implies that

$$\sum_{c \in \mathfrak{B}} \tilde{m}_c^{a,b} \tilde{m}_e^{c,d} = \sum_{c \in \mathfrak{B}} \tilde{m}_e^{a,c} \tilde{m}_c^{b,d} \tag{5.3}$$

for any $a, b, d, e \in \mathfrak{B}$.

One can see from the proof of Lemma 5.2 that for $b \in B(\lambda_1, -\mu_1) \bigcap \mathfrak{B}$ and $c \in B(\lambda_2, -\mu_2) \bigcap \mathfrak{B}$ with $\tilde{m}_a^{b,c} \neq 0$, we have $a \in B(\lambda_1 + \lambda_2, -\mu_1 - \mu_2)$. Hence when $\mathfrak{g}$ is of finite type, the set $\{a \in \mathfrak{B} \mid \tilde{m}_a^{b,c} \neq 0\}$ is finite for fixed b, c above since $B(\lambda_1 + \lambda_2, -\mu_1 - \mu_2)$ is a finite set. We claim that it is also true for $\mathfrak{g}$ of any type though $B(\lambda_1 + \lambda_2, -\mu_1 - \mu_2)$ is not finite any more in other cases.

Theorem 5.3. *For $b, c \in \mathfrak{B}$, $\{a \in \mathfrak{B} \mid \tilde{m}_a^{b,c} \neq 0\}$ is a finite set.*

Proof. Assume that $b \in B(\lambda_1, -\mu_1)_{\xi_1}$ and $c \in B(\lambda_2, -\mu_2)_{\xi_2}$ with $\tilde{m}_a^{b,c} \neq 0$, then $a \in B(\lambda_1 + \lambda_2, -\mu_1 - \mu_2)_{\xi_1+\xi_2}$ from the above statement. We suppose that $G(a) \in \mathfrak{U}_{[\xi]}$. As in the proof of Lemma 5.2, there exist $s \geqslant 0$ and $t \geqslant 0$ such that $G(b) \in F_{1,s}$, $G(b) \notin F_{1,s+1}$ and $G(c) \in F_{2,t}$, $G(c) \notin F_{2,t+1}$ where $F_{1,s}$, $F_{1,s+1}$ and $F_{2,t}$, $F_{2,t+1}$ are in the filtration (5.2(1)) and (5.2(2)) respectively. Define π, M and $m \in M$ as before. Hence we obtain a homomorphism of U-modules $\pi\tau : V(\lambda, -\mu) \longrightarrow M$ where $\lambda = \lambda_1 + \lambda_2$ and $\mu = \mu_1 + \mu_2$. Since $\tilde{m}_a^{b,c} \neq 0$, it implies that

$$\pi\tau\bigl(G(a)u_{\lambda,-\mu}\bigr) = G(a)m \neq 0.$$

Hence $\xi \leqslant \eta$ for some $\eta \in P^+$ such that $M_\eta \neq 0$ by Lemma 4.2. It follows that there exists an $l > 0$ such that

$$\bigl\{G(a) \bigm| \tilde{m}_a^{b,c} \neq 0\bigr\} \subseteq F_0 \setminus F_l$$

where F_0, F_l is in the filtration (3.3). Furthermore, there is a bijection between $\{G(a) \mid \tilde{m}_a^{b,c} \neq 0\}$ and its image in F_0/F_l under the canonical map $\pi' : F_0 \longrightarrow F_0/F_l$. Since $F_0/F_l \in ob(\mathcal{O}^+)$ and

$$\pi'\bigl\{G(a) \bigm| \tilde{m}_a^{b,c} \neq 0\bigr\} \subseteq (F_0/F_l)_{\xi_1+\xi_2},$$

$\{G(a) \in \mathfrak{U} \mid \tilde{m}_a^{b,c} \neq 0\}$ is a finite set which proves the theorem. □

Let $\mathfrak{U}^*$ be the dual space of $\mathfrak{U}$, that is, the set of all linear functions $\phi : \mathfrak{U} \longrightarrow k$. For $b \in \mathfrak{B}$, set b^* to be the linear function dual to the canonical base element $G(b)$, i.e.

$$b^*\bigl(G(c)\bigr) = \delta_{b,c} \quad \text{for } b, c \in \mathfrak{B}.$$

Let $\mathfrak{U}^\circ$ be the subspace of $\mathfrak{U}^*$ spanned over k by $\{b^* \in \mathfrak{U}^* \mid b \in \mathfrak{B}\}$. We define an algebra structure on $\mathfrak{U}^\circ$ by setting

$$b^* \cdot c^* = \sum_{a \in \mathfrak{B}} \tilde{m}_a^{b,c} a^*.$$

The sum is well-defined by Theorem 5.3 and the associativity of this multiplication follows from (5.3). An integral form of this algebra, denoted by $\mathfrak{U}_{\mathbb{Z}}^\circ$, is then naturally defined by spanning a free $\mathbb{Z}[q, q^{-1}]$-module with the basis $\{b^* \mid b \in \mathfrak{B}\}$.

5.2. Other versions of definition

Let U^* be the dual space of U, that is, the set of all linear functions on U. The comultiplication in U provides a multiplication in U^*, i.e. $(f_1 \cdot f_2)(u) = \sum f_1(u_{(1)}) f_2(u_{(2)})$

where $\Delta(u) = \sum u_{(1)} \otimes u_{(2)}$. For a U-module $M \in ob(\mathcal{O}^+)$, let M° be the graded dual of M as before. For $m \in M$ and $f \in M^\circ$, we define a *coordinate function* $m \otimes f$ on U as

$$(m \otimes f)(u) = f(um) \quad \text{for } u \in U.$$

Definition 5.4. The subalgebra of U^* generated by all coordinate functions $m \otimes f \in M \otimes M^\circ$ for all $M \in ob(\mathcal{O}^+)$ is called the quantum coordinate algebra, denoted by C_1.

For $m_i \otimes f_i \in M_i \otimes M_i^\circ$ with $M_i \in ob(\mathcal{O}^+)$, $i = 1, 2$, since

$$(m_1 \otimes f_1) \cdot (m_2 \otimes f_2) = (m_1 \otimes m_2) \otimes (f_1 \otimes f_2) \in (M_1 \otimes M_2) \otimes (M_1 \otimes M_2)^\circ$$

where $M_1 \otimes M_2 \in ob(\mathcal{O}^+)$, C_1 is actually spanned by all coordinate functions and has a structure of U-bimodule. The complete reducibility of category $\mathcal{O}^+$ implies the following analogue of Peter–Weyl theorem.

Proposition 5.5. *(See [2].)* $C_1 \cong \bigoplus_{\lambda \in P^+} V(\lambda) \otimes V(\lambda)^\circ$ *as U-bimodules and algebras.*

Given $\lambda, \mu \in P^+$, we define a surjective map $\tilde{\alpha}_{\lambda,-\mu} : U \longrightarrow M(\lambda, -\mu)$ which takes x to $xv_{\lambda,-\mu}$. Here $M(\lambda, -\mu)$ inherits from $V(\lambda, -\mu)$ a canonical basis which also has the stability property as in [3], i.e. for $\lambda, \mu, \theta \in P^+$, the homomorphism of U-modules

$$\tau_{\lambda+\theta,-\theta-\mu,\lambda,-\mu} : M(\lambda + \theta, -\theta - \mu) \longrightarrow M(\lambda, -\mu)$$

taking $v_{\lambda+\theta,-\theta-\mu}$ to $v_{\lambda,-\mu}$ is nice and surjective. Note that $M(\lambda, -\mu)$ is usually not in $\mathcal{O}^+$ for $\mathfrak{g}$ of affine or indefinite type and more precisely, its weight space might be infinite dimensional. We define $M(\lambda, -\mu)^\circ$ to be the subspace of $M(\lambda, -\mu)^*$ spanned by the dual basis associated with the canonical basis of $M(\lambda, -\mu)$, i.e.

$$M(\lambda, -\mu)^\circ = \bigoplus_{b \in B(M(\lambda,-\mu))} kb^\circ$$

where $b^\circ(G(c)) = \delta_{b,c}$ for $b, c \in B(M(\lambda, -\mu))$. One can see that $\tilde{\alpha}_{\lambda,-\mu}$ induces the injective map

$$\tilde{\alpha}^*_{\lambda,-\mu} : M(\lambda, -\mu)^\circ \longrightarrow U^*.$$

Denote by $U(\lambda, -\mu)^*$ the image of $\tilde{\alpha}^*_{\lambda,-\mu}$. It follows from the stability property of canonical bases that

$$U(\lambda, -\mu)^* \subseteq U(\lambda + \theta, -\theta - \mu)^*.$$

Indeed, for $b \in B(M(\lambda, -\mu)) \subseteq B(M(\lambda+\theta, -\theta-\mu))$, we have $\tilde{\alpha}^*_{\lambda,-\mu}(b^\circ) = \tilde{\alpha}^*_{\lambda+\theta,-\theta-\mu}(b^\circ)$, still denoted by b° in U^* with no confusion.

Definition 5.6. C_2 is defined to be the subalgebra of U^* generated by $U(\lambda,-\mu)^*$ for all $\lambda,\mu\in P^+$.

Note that the above definition is a generalization of Lusztig's definition of quantum coordinate algebra for $\mathfrak{g}$ of finite type [3].

Lemma 5.7.

(i) *$\tau : V(\lambda,-\mu) \longrightarrow V(\lambda_1,-\mu_1)\otimes V(\lambda_2,-\mu_2)$ induces a homomorphism of U or $\mathfrak{U}$-modules $\bar\tau : M(\lambda,-\mu)\longrightarrow M(\lambda_1,-\mu_1)\otimes M(\lambda_2,-\mu_2)$.*

(ii) *Let $\bar\tau^*$ be the map $\bar\tau^* : (M(\lambda_1,-\mu_1)\otimes M(\lambda_2,-\mu_2))^* \longrightarrow M(\lambda,-\mu)^*$ defined by $\bar\tau^*(f)(m)=f(\bar\tau(m))$ for $f\in (M(\lambda_1,-\mu_1)\otimes M(\lambda_2,-\mu_2))^*$, $m\in M(\lambda,-\mu)$. Then $\bar\tau^*(M(\lambda_1,-\mu_1)^\circ\otimes M(\lambda_2,-\mu_2)^\circ)\subseteq M(\lambda,-\mu)^\circ$.*

Proof. Let π be the canonical U-module homomorphism $\pi : V(\lambda_1,-\mu_1)\otimes V(\lambda_2,-\mu_2)\longrightarrow M(\lambda_1,-\mu_1)\otimes M(\lambda_2,-\mu_2)$. Thus we have the composed U-module homomorphism $\pi\tau : V(\lambda,-\mu)\longrightarrow M(\lambda_1,-\mu_1)\otimes M(\lambda_2,-\mu_2)$. Both $M(\lambda_1,-\mu_1)$ and $M(\lambda_2,-\mu_2)$ have nice filtration as in (4.1),

$$M(\lambda_i,-\mu_i)=M_{i,0}\supseteq M_{i,1}\supseteq\cdots\supseteq M_{i,l}\supseteq\cdots \qquad (5.2(i))$$

where $i=1,2$. For any $s,t\geqslant 0$, we define $(\pi\tau)_{s,t}$ to be the composition of $\pi\tau$ with the canonical map

$$\eta_{s,t} : M(\lambda_1,-\mu_1)\otimes M(\lambda_2,-\mu_2)\longrightarrow \big(M(\lambda_1,-\mu_1)/M_{1,s}\big)\otimes\big(M(\lambda_2,-\mu_2)/M_{2,t}\big),$$

i.e. $(\pi\tau)_{s,t} : V(\lambda,-\mu)\longrightarrow (M(\lambda_1,-\mu_1)/M_{1,s})\otimes(M(\lambda_2,-\mu_2)/M_{2,t})$ where the right side is in $\mathcal{O}^+$. Thus for any $b\in B'(\lambda,-\mu)$, $(\pi\tau)_{s,t}(G(b))=0$. It implies $(\pi\tau)(G(b))\in M_{1,s}\otimes M(\lambda_2,-\mu_2)+M(\lambda_1,-\mu_1)\otimes M_{2,t}$ for any $s,t\geqslant 0$. Hence

$$(\pi\tau)\big(G(b)\big)\in\bigcap_{s,t\geqslant 0}\big(M_{1,s}\otimes M(\lambda_2,-\mu_2)+M(\lambda_1,-\mu_1)\otimes M_{2,t}\big)=0.$$

It follows that $\pi\tau$ factors through $\bar\tau : M(\lambda,-\mu)\longrightarrow M(\lambda_1,-\mu_1)\otimes M(\lambda_2,-\mu_2)$ which proves (i). To prove (ii), we only show that $\bar\tau^*(b_1^\circ\otimes b_2^\circ)\subseteq M(\lambda,-\mu)^\circ$ for any $b_i\in B(M(\lambda_i,-\mu_i))$, $i=1,2$. Assume that $\bar\tau^*(b_1^\circ\otimes b_2^\circ)(G(b))\neq 0$ for some $b\in B(M(\lambda,-\mu))$, that is, $(b_1^\circ\otimes b_2^\circ)(\bar\tau G(b))\neq 0$. Then it implies that $\tilde m_b^{b_1,b_2}\neq 0$. We know from Theorem 5.3 that there are only finitely many such $G(b)$. Hence $\bar\tau^*(b_1^\circ\otimes b_2^\circ)\subseteq M(\lambda,-\mu)^\circ$. $\square$

As a consequence, we have the following corollary.

Corollary 5.8.

(i) *$U(\lambda_1,-\mu_1)^*\cdot U(\lambda_2,-\mu_2)^*\subseteq U(\lambda_1+\lambda_2,-\mu_1-\mu_2)^*$.*

(ii) *$C_2=\sum_{\lambda,\mu\in P^+}U(\lambda,-\mu)^*$ and $\{b^\circ\mid b\in\mathfrak{B}\}$ forms a basis of C_2.*

5.3. Equivalence of definitions

For $x, y \in U$, $f \in \mathfrak{U}^\circ$, we define $x \cdot f \cdot y \in \mathfrak{U}^*$ to satisfy that

$$(x \cdot f \cdot y)(u) = f(yux)$$

for any $u \in \mathfrak{U}$. Suppose that $x \in U_{\xi_1}$, $y \in U_{\xi_2}$ and $f = b^*$ such that $G(b) \in \mathfrak{U}_{[\eta]}$ is in the image of Ua_λ with weight μ. For $u = G(b') \in \mathfrak{U}$, it can be seen from the weight that $(x \cdot f \cdot y)(u) \neq 0$ implies u is in the image of $Ua_{\lambda+\xi_1}$ with weight $\mu - \xi_1 - \xi_2$. Also we have, by Corollary 4.4, that $u \in \bigoplus_{\theta \leqslant \eta} \mathfrak{U}_{[\theta]}$ if $(x \cdot f \cdot y)(u) \neq 0$. It follows as in the proof of Theorem 5.3 that $x \cdot f \cdot y$ acts as zero for all but finitely many $G(b') \in \mathfrak{U}$. Hence $x \cdot f \cdot y \in \mathfrak{U}^\circ$ and one can view $\mathfrak{U}^\circ$ as a U-bimodule or similarly as a $\widetilde{U}$ or $\mathfrak{U}$-bimodule. Fixing a left weight μ, that is, taking $x = a_\mu \in \widetilde{U}$, we can see from Proposition 2.3 that the right $\widetilde{U}$-submodule $a_\mu \mathfrak{U}^\circ$ of $\mathfrak{U}^\circ$ corresponds to a crystal which consists of $\dim V(\lambda)_\mu$ copies of $B(\lambda)^\circ$ associated with irreducible integrable highest weight right $\widetilde{U}$-module $V(\lambda)^\circ$ for all $\lambda \in P^+$. Obviously the same happens to C_1 when applying a_μ to the left side.

On the other hand, it is easy to see that the structure constants of the multiplication in C_2 with respect to the basis $\{b^\circ \mid b \in \mathfrak{B}\}$ are exactly the same as those in $\mathfrak{U}^\circ$ since both multiplications are defined through the comultiplication in U. Thus $\mathfrak{U}^\circ$ and C_2 are isomorphic as algebras. All the facts stated above lead us to a belief that the three definitions of the quantum coordinate algebra are equivalent.

Theorem 5.9. *$\mathfrak{U}^\circ \cong C_1 \cong C_2$ as algebras.*

Proof. We only show $C_1 \cong C_2$. Given $f \in U(\lambda, -\mu)^* \subseteq C_2$ with $\lambda, \mu \in P^+$, there is a $g \in M(\lambda, -\mu)^\circ$ such that $f(x) = g(xv_{\lambda,-\mu})$ for all $x \in U$. Since g acts as zero for all but finitely many canonical base elements of $M(\lambda, -\mu)$, $g(M_s) = 0$ for some $s \geqslant 0$ where M_s is in the filtration (4.1). Hence g induces the linear map $\bar{g} : M(\lambda, -\mu)/M_s \longrightarrow k$ where $M(\lambda, -\mu)/M_s$, denoted by M, is an object in $\mathcal{O}^+$. Clearly $\bar{g} \in M^\circ$ and we have

$$f(x) = g(xv_{\lambda,-\mu}) = \bar{g}\big(x(v_{\lambda,-\mu} + M_s)\big) = \big((v_{\lambda,-\mu} + M_s) \otimes \bar{g}\big)(x).$$

We denote $v_{\lambda,-\mu} + M_s \in M$ by m. Thus $f = m \otimes \bar{g} \in M \otimes M^\circ$ which implies that $f \in C_1$. Conversely, assume that $f = m \otimes g \in M \otimes M^\circ \subseteq C_1$ for some $M \in \mathcal{O}^+$ and $m \in M_\xi$. There exist $\lambda, \mu \in P^+$ with $\lambda - \mu = \xi$ such that $\phi : M(\lambda, -\mu) \longrightarrow M$ which takes $xv_{\lambda,-\mu}$ to xm for $x \in U$ is a well-defined U-module homomorphism. Note that ϕ induces an injective map

$$\phi^* : (\operatorname{Im} \phi)^* \longrightarrow M(\lambda, -\mu)^*.$$

As above we have $\phi(M_s) = 0$ for some $s \geqslant 0$ where M_s is in the filtration (4.1) and ϕ factors through a surjective U-module homomorphism $\bar{\phi} : M(\lambda, -\mu)/M_s \longrightarrow \operatorname{Im} \phi$ which induces

$$\bar{\phi}^* : (\operatorname{Im}\phi)^* \longrightarrow \big(M(\lambda, -\mu)/M_s\big)^*.$$

Since $\operatorname{Im}\phi$, $M(\lambda,-\mu)/M_s \in \mathcal{O}^+$ and $g \in M^\circ$, then $g|_{\operatorname{Im}\phi} \in (\operatorname{Im}\phi)^\circ$, $\bar{\phi}^*(g|_{\operatorname{Im}\phi}) \in (M(\lambda,-\mu)/M_s)^\circ$. It implies that $\phi^*(g|_{\operatorname{Im}\phi}) \in M(\lambda,-\mu)^\circ$. One can check easily that

$$f = m \otimes g = \tilde{\alpha}^*_{\lambda,-\mu}\phi^*(g|_{\operatorname{Im}\phi}) \in C_2$$

which completes the proof. □

Remark 5.10.

(i) One may notice that the proof of Theorem 5.9 also implies the equality in Corollary 5.8(*ii*).

(ii) The $\mathbb{Z}[q,q^{-1}]$-algebra $\mathfrak{U}^\circ_{\mathbb{Z}}$ defined in Section 5.1 is the integral form of the quantum coordinate algebra for any symmetrizable Kac–Moody algebra $\mathfrak{g}$. Given any commutative $\mathbb{Z}[q,q^{-1}]$-algebra $\mathcal{A}$, one can define an $\mathcal{A}$-algebra $\mathfrak{U}^\circ_{\mathcal{A}} \triangleq \mathcal{A} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathfrak{U}^\circ_{\mathbb{Z}}$.

Acknowledgment

The authors would like to thank the anonymous referee for his/her valuable comments and suggestions.

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