Positivity-preserving high order finite difference WENO schemes for compressible Euler equations[1]

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Abstract

In [20, 21, 23], we constructed uniformly high order accurate discontinuous Galerkin (DG) and finite volume schemes which preserve positivity of density and pressure for the Euler equations of compressible gas dynamics. In this paper, we present an extension of this framework to construct positivity-preserving high order essentially non-oscillatory (ENO) and weighted essentially non-oscillatory (WENO) finite difference schemes for compressible Euler equations. General equations of state and source terms are also discussed. Numerical tests of the fifth order finite difference WENO scheme are reported to demonstrate the good behavior of such schemes.

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1 Introduction

In this paper we are interested in the Euler equations, the one dimensional version for the perfect gas being given by

$$\mathbf{w}_t + \mathbf{f}(\mathbf{w})_x = 0, \quad t \geq 0, \ x \in \mathbb{R}, \tag{1.1}$$

$$\mathbf{w} = \begin{pmatrix} \rho \\ m \\ E \end{pmatrix}, \quad \mathbf{f}(\mathbf{w}) = \begin{pmatrix} m \\ \rho u^2 + p \\ (E+p)u \end{pmatrix} \tag{1.2}$$

with

$$m = \rho u, \quad E = \frac{1}{2}\rho u^2 + \rho e, \quad p = (\gamma - 1)\rho e,$$

where ρ is the density, u is the velocity, m is the momentum, E is the total energy, p is the pressure, e is the internal energy, and $\gamma > 1$ is a constant ($\gamma = 1.4$ for the air). The speed of sound is given by $c = \sqrt{\gamma p/\rho}$ and the three eigenvalues of the Jacobian $\mathbf{f}'(\mathbf{w})$ are $u - c$, u and $u + c$.

In a conservative numerical scheme, the internal energy is obtained by subtracting the kinetic energy from the total energy, thus the resulting pressure may be negative, for example, for problems in which the dominant energy is kinetic. Negative density may often emerge in computing blast waves. Physically, the density ρ and the pressure p should both be positive. The eigenvalues of the Jacobian will become imaginary if density or pressure is negative so the initial value problem for the linearized system will be ill-posed. This explains why failure of preserving positivity of density or pressure may cause blow-ups of the numerical algorithm.

Replacing the negative density or negative pressure by positive ones is neither a conservative cure nor a stable solution. Therefore, it is highly important to design a conservative positivity-preserving scheme. First order and second order positivity-preserving schemes were well studied, e.g., [4, 11]. A general framework for constructing arbitrarily high order positivity-preserving discontinuous Galerkin (DG) and finite volume schemes was proposed recently in [20]. This framework can be easily generalized, for instance, to unstructured meshes [23], and to general equations of state and Euler system with source terms [21].

Generalization of the positivity-preserving method in [20] to high order finite difference schemes is not straightforward. However, in some applications where high order schemes are preferred, for example, cosmological simulation [5], finite difference WENO schemes [10] is more favored than DG schemes [2, 3] and the finite volume WENO scheme [12, 15] due to their smaller memory cost (compared to DG) and smaller computational cost (compared both to finite volume schemes and to DG schemes) for multi-dimensional problems, see for example a comparison in [1] in the context of ENO schemes.

In this paper, we will follow the idea in [20] to construct positivity-preserving high order finite difference WENO schemes. We will show that by adopting the same simple limiter as in [20], to a slightly different version of finite difference WENO schemes from the one in [10], the final scheme will keep the positivity of density and pressure without losing conservation. A conservative positivity-preserving scheme is L^1-stable, see [22]. The limiter will not destroy the high order accuracy of the WENO scheme for smooth solutions without vacuum. All the results also hold for finite difference ENO schemes [16].

The paper is organized as follows. In Section 2, we briefly review the positivity-preserving finite volume schemes in [20] and the finite difference WENO scheme in [10]. Then we introduce positivity-preserving finite difference schemes in one space dimension for the perfect gas in Section 3. In Section 4, we discuss a straightforward extension to multi-dimensions, general equations of state and source terms. In Section 5, numerical tests of the fifth order WENO schemes for some very demanding problems are shown. Concluding remarks are given in Section 6.

2 Preliminaries

2.1 Review of positivity-preserving high order finite volume WENO schemes

We first briefly review the basic idea in [20, 22] for finite volume WENO schemes. Consider the Euler equations (1.1) in more detail. Let $p(\mathbf{w}) = (\gamma - 1)(E - \frac{1}{2}\frac{m^2}{\rho})$ be the pressure function. It can be easily verified that p is a concave function of $\mathbf{w} = (\rho, m, E)^T$ if $\rho > 0$. For $\mathbf{w}_1 = (\rho_1, m_1, E_1)^T$ and $\mathbf{w}_2 = (\rho_2, m_2, E_2)^T$, Jensen's inequality implies, for $0 \le s \le 1$,

$$p\left(s\mathbf{w}_1 + (1-s)\mathbf{w}_2\right) \ge sp\left(\mathbf{w}_1\right) + (1-s)p\left(\mathbf{w}_2\right), \quad \text{if} \quad \rho_1 > 0, \quad \rho_2 > 0. \tag{2.1}$$

Define the set of admissible states by

$$G = \left\{ \mathbf{w} = \left(\begin{array}{c} \rho \\ m \\ E \end{array} \right) \middle| \rho > 0 \quad \text{and} \quad p = (\gamma - 1)\left(E - \frac{1}{2}\frac{m^2}{\rho}\right) > 0 \right\},$$

then G is a convex set. We want to construct finite volume WENO schemes producing solutions in the set G. Notice that the condition $p > 0$ in the definition of set G can be changed to $p \ge 0$ without affecting convexity.

The time discretization is chosen as the high order strong stability preserving (SSP) methods [14, 16, 8, 9] which are convex combinations of Euler forward. Thus we only need to discuss the Euler forward since G is convex.

A general high order finite volume scheme has the following form

$$\overline{\mathbf{w}}_i^{n+1} = \overline{\mathbf{w}}_i^n - \lambda \left[\widehat{\mathbf{f}}\left(\mathbf{w}^-_{i+\frac{1}{2}}, \mathbf{w}^+_{i+\frac{1}{2}}\right) - \widehat{\mathbf{f}}\left(\mathbf{w}^-_{i-\frac{1}{2}}, \mathbf{w}^+_{i-\frac{1}{2}}\right) \right], \tag{2.2}$$

where $\widehat{\mathbf{f}}$ is a positivity preserving flux, for instance, Lax-Friedrichs flux, $\overline{\mathbf{w}}_i^n$ is the approximation to the cell average of the exact solution $\mathbf{v}(x,t)$ in the cell $I_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ at time level n, and $\mathbf{w}^-_{i+\frac{1}{2}}$, $\mathbf{w}^+_{i+\frac{1}{2}}$ are the high order approximations of the point values $\mathbf{v}(x_{i+\frac{1}{2}}, t^n)$ within the cells I_i and I_{i+1} respectively. These values are reconstructed from the cell averages $\overline{\mathbf{w}}_i^n$ by the WENO reconstruction. We assume that there is a polynomial vector $\mathbf{q}_i(x) = (\rho_i(x), m_i(x), E_i(x))^T$ with degree k which are $(k+1)$-th order accurate approximations to smooth exact solutions $\mathbf{v}(x,t)$ on I_i, and satisfies that $\overline{\mathbf{w}}_i^n$ is the cell average of $\mathbf{q}_i(x)$ on I_i, $\mathbf{w}^+_{i-\frac{1}{2}} = \mathbf{q}_i(x_{i-\frac{1}{2}})$ and $\mathbf{w}^-_{i+\frac{1}{2}} = \mathbf{q}_i(x_{i+\frac{1}{2}})$. The existence of such polynomials can be established by interpolation for WENO schemes. For example, for the fifth order WENO scheme, there is a unique vector of polynomials of degree four $\mathbf{q}_i(x)$ satisfying $\mathbf{q}_i(x_{i-\frac{1}{2}}) = \mathbf{w}^+_{i-\frac{1}{2}}$, $\quad \mathbf{q}_i(x_{i+\frac{1}{2}}) = \mathbf{w}^-_{i+\frac{1}{2}}$ and

$$\frac{1}{\Delta x} \int_{I_j} \mathbf{q}_i(x) dx = \overline{\mathbf{w}}_j^n, \quad \text{for} \quad j = i-1, i, i+1.$$

We need the N-point Legendre Gauss-Lobatto quadrature rule on the interval $I_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$, which is exact for the integral of polynomials of degree up to $2N-3$. We would need to choose N such that $2N-3 \geq k$. Denote these quadrature points on I_i as

$$S_i = \{x_{i-\frac{1}{2}} = \widehat{x}_i^1, \widehat{x}_i^2, \cdots, \widehat{x}_i^{N-1}, \widehat{x}_i^N = x_{i+\frac{1}{2}}\}. \tag{2.3}$$

In [20], we have shown that $\mathbf{q}_i(\widehat{x}_i^\alpha) \in G$ for all j and α is a sufficient condition for $\overline{\mathbf{w}}_i^{n+1} \in G$ under suitable CFL conditions. The limiter in [20] can enforce this sufficient condition without destroying conservation. Moreover, the limiter is an accurate modification for smooth solutions if there is no vacuum region in the exact solution. See [22, 17] for simpler implementations of the limiter.

2.2 Review of the finite difference WENO scheme for scalar conservation laws

Before describing the finite difference WENO scheme for the Euler system, we first review the relations between finite difference and finite volume WENO schemes in [15] for the scalar linear equation

$$u_t + u_x = 0, \quad u(x,0) = u_0(x). \tag{2.4}$$

Consider a uniform mesh with nodes x_i. Define $x_{i+\frac{1}{2}} = \frac{1}{2}(x_{i+1}+x_i)$, $I_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ and $\Delta x = x_{i+1} - x_i$. A conservative finite difference scheme with high order spatial discretization and Euler forward time discretization solving (2.4) has the form

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}[\widehat{f}_{i+\frac{1}{2}} - \widehat{f}_{i-\frac{1}{2}}], \tag{2.5}$$

where $\frac{1}{\Delta x}[\widehat{f}_{i+\frac{1}{2}} - \widehat{f}_{i-\frac{1}{2}}]$ should be a high order approximation to u_x at $x = x_i$.

If there exists a function $h(x)$ depending on the mesh size Δx such that

$$u(x) = \frac{1}{\Delta x}\int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} h(\xi)d\xi, \tag{2.6}$$

then we call u and h a reconstruction pair. See [7] for a detailed discussion of the reconstruction pair. Denote them by

$$h = R_{\Delta x}(u), \quad u = R_{\Delta x}^{-1}(h).$$

Then $u_x = \frac{1}{\Delta x}\left[h\left(x+\frac{\Delta x}{2}\right) - h\left(x-\frac{\Delta x}{2}\right)\right]$. Thus if $\widehat{f}_{i+\frac{1}{2}}$ is a high order accurate approximation of $h(x_{i+\frac{1}{2}})$ then $\frac{1}{\Delta x}[\widehat{f}_{i+\frac{1}{2}} - \widehat{f}_{i-\frac{1}{2}}]$ will be a high order approximation to u_x at $x = x_i$.

Notice that the cell average of h over I_i is simply the point value $u(x_i)$ by (2.6). So the high order finite difference WENO scheme for (2.4) can be formulated as

- At time level n, obtain the cell averages of h on I_i by $\overline{h}_i^n = u_i^n$.
- Use the fifth order WENO reconstruction based on $\overline{h}_j^n$ (j in a neighborhood of i) to construct nodal values of h at $x_{i+\frac{1}{2}}$, and denote it by $h^-_{i+\frac{1}{2}}$.
- Set the flux as $\widehat{f}_{i+\frac{1}{2}} = h^-_{i+\frac{1}{2}}$ in (2.5).

It is clear that (2.5) is equivalent to

$$\overline{h}_i^{n+1} = \overline{h}_i^n - \frac{\Delta t}{\Delta x}[h^-_{i+\frac{1}{2}} - h^-_{i-\frac{1}{2}}], \tag{2.7}$$

which is a finite volume scheme for the function h. In other words, for (2.4), a finite difference WENO scheme of u is exactly a finite volume WENO scheme of h.

Let $q_i(x)$ denote a polynomial which is a high order accurate approximation to h on I_i and satisfies $q_i(x_{i+\frac{1}{2}}) = h^-_{i+\frac{1}{2}}$. By the result in [19], if $q_i(\widehat{x}_i^\alpha) \in [m, M]$ where m and M are the minimum and maximum of the initial value $u_0(x)$, then $\overline{h}_i^{n+1} \in [m, M]$ under certain CFL conditions. It appears that we can easily construct maximum-principle-satisfying finite difference WENO schemes for scalar conservation laws following [19]. Unfortunately, this method will destroy accuracy. Even though h will converge to u when Δx goes to zero, for a fixed Δx, h has larger maximum and smaller minimum than u does by the analysis in [7]. Since $q_i(x)$ approximates h rather than u, enforcing $q_i(\widehat{x}_i^\alpha) \in [m, M]$ will destroy the high order accuracy.

However, if the minimum of $u(x, t)$ is strictly positive, then $h(x, t) \geq 0$ for small enough Δx, thus enforcing $q_i(\widehat{x}_i^\alpha) \geq 0$ will not destroy high order accuracy. Therefore, we can still take advantage of this relation to construct high order positivity-preserving finite difference WENO scheme in the next section.

3 Positivity-preserving high order finite difference WENO schemes in one dimension

3.1 A finite difference WENO scheme for Euler equations

We now discuss a $(k+1)$-th order finite difference WENO scheme for (1.1). Given the point values $\mathbf{w}_i^n$ at time level n, let $A_{i+\frac{1}{2}}$ denote the Roe matrix [13] of the two states $\mathbf{w}_{i+1}^n$ and $\mathbf{w}_i^n$. Let $L_{i+\frac{1}{2}}$ and $R_{i+\frac{1}{2}}$ be the left and right eigenvector matrices of $A_{i+\frac{1}{2}}$ respectively, i.e., $A = R\Lambda L$ where Λ is the diagonal matrix with eigenvalues of A on the diagonal. Let $\alpha = \max \parallel (|u| + c) \parallel$ where u and c are the velocity and speed of sound of the state $\mathbf{w}_i^n$ and the maximum is taken either globally over all the $\mathbf{w}_i^n$ or locally over the $\mathbf{w}_i^n$ in the WENO reconstruction stencil. For simplicity, we first assume α is a constant for each time step or time stage, i.e., the maximum is taken globally. The case with locally taken maximum will be discussed in the next section. We will use the Lax-Friedrichs splitting,

$$\mathbf{f}^{\pm}(\mathbf{w}) = \frac{1}{2}\left(\mathbf{w} \pm \frac{\mathbf{f}(\mathbf{w})}{\alpha}\right). \tag{3.1}$$

At each fixed $x_{i+\frac{1}{2}}$,

1. Let $\mathbf{h}_{\pm} = R_{\Delta x}(\mathbf{f}^{\pm})$. Then we have the cell averages $\overline{\mathbf{h}}_{\pm i}^{\ n} = \mathbf{f}^{\pm}(\mathbf{w}_i^n)$.

2. Transform all the cell averages $\overline{\mathbf{h}}_{\pm j}^{\ n}$ (j in a neighborhood of i) to the local characteristic fields by setting

$$\overline{\mathbf{u}}_{\pm j}^{\ n} = L_{i+\frac{1}{2}} \overline{\mathbf{h}}_{\pm j}^{\ n}.$$

3. Perform the WENO reconstruction for each component of $\overline{\mathbf{u}}_{+j}^{\ n}$ to obtain approximations of the point value of the function $L_{i+\frac{1}{2}}\mathbf{h}_+$ at the point $x_{i+\frac{1}{2}}$ and denote them as $(\mathbf{u}_+)^{\pm}_{i+\frac{1}{2}}$. Perform the WENO reconstruction for each component of $\overline{\mathbf{u}}_{-j}^{\ n}$ to obtain approximations of the point value of the function $L_{i+\frac{1}{2}}\mathbf{h}_-$ at the point $x_{i+\frac{1}{2}}$ and denote them as $(\mathbf{u}_-)^{\pm}_{i+\frac{1}{2}}$. Here the superscripts $-$ and $+$ denote approximations within the cells I_i and I_{i+1} respectively.

4. Transform back into physical space by

$$(\mathbf{h}_+)^-_{i+\frac{1}{2}} = R_{i+\frac{1}{2}}(\mathbf{u}_+)^-_{i+\frac{1}{2}}, \quad (\mathbf{h}_-)^+_{i+\frac{1}{2}} = R_{i+\frac{1}{2}}(\mathbf{u}_-)^+_{i+\frac{1}{2}}.$$

Form the flux by

$$\widehat{\mathbf{f}}_{i+\frac{1}{2}} = \alpha \left[(\mathbf{h}_+)^-_{i+\frac{1}{2}} - (\mathbf{h}_-)^+_{i+\frac{1}{2}}\right]. \tag{3.2}$$

Let $\lambda = \Delta t/\Delta x$, then we get the conservative scheme

$$\mathbf{w}_i^{n+1} = \mathbf{w}_i^n - \lambda \left(\widehat{\mathbf{f}}_{i+\frac{1}{2}} - \widehat{\mathbf{f}}_{i-\frac{1}{2}}\right). \tag{3.3}$$

3.2 A sufficient condition to keep positivity

Next, we will derive a sufficient condition for the scheme (3.3) to keep $\mathbf{w}_j^{n+1} \in G$ provided $\mathbf{w}_j^n \in G$. (3.1) and (3.2) imply that (3.3) can be written as

$$\begin{aligned}
\mathbf{w}_i^{n+1} &= \mathbf{w}_i^n - \lambda \left(\widehat{\mathbf{f}}_{i+\frac{1}{2}} - \widehat{\mathbf{f}}_{i-\frac{1}{2}}\right) \\
&= \overline{\mathbf{h}}_{+i}^{\ n} + \overline{\mathbf{h}}_{-i}^{\ n} - \lambda\alpha \left((\mathbf{h}_+)^-_{i+\frac{1}{2}} - (\mathbf{h}_-)^+_{i+\frac{1}{2}} - (\mathbf{h}_+)^-_{i-\frac{1}{2}} + (\mathbf{h}_-)^+_{i-\frac{1}{2}}\right) \\
&= \mathbf{H}^+ + \mathbf{H}^-,
\end{aligned}$$

where

$$\mathbf{H}^+ = \overline{\mathbf{h}}_{+i}^{\ n} - \alpha\lambda \left[(\mathbf{h}_+)^-_{i+\frac{1}{2}} - (\mathbf{h}_+)^-_{i-\frac{1}{2}}\right], \tag{3.4}$$

$$\mathbf{H}^- = \overline{\mathbf{h}}_{-i}^{\ n} + \alpha\lambda \left[(\mathbf{h}_-)^+_{i+\frac{1}{2}} - (\mathbf{h}_-)^+_{i-\frac{1}{2}}\right]. \tag{3.5}$$

Notice that (3.4) and (3.5) are two finite volume WENO schemes for $\mathbf{h}_+$ and $\mathbf{h}_-$. Obviously it suffices to discuss conditions to keep $\mathbf{H}^\pm \in G$. We first discuss (3.4). By interpolation, there exists a vector of polynomials of degree k $\mathbf{q}_i^+(x)$ such that $\mathbf{q}_i^+(x_{i+\frac{1}{2}}) = (\mathbf{h}_+)^-_{i+\frac{1}{2}}$, the cell average of $\mathbf{q}_i^+(x)$ on I_i is $\overline{\mathbf{h}}_{+i}^{\ n}$ and $\mathbf{q}_i^+(x)$ is a $(k+1)$-th order accurate approximation to the function $\mathbf{h}_+$ on the cell I_i if $\mathbf{w}$ is smooth. The exactness of the quadrature rule for polynomials of degree k implies

$$\overline{\mathbf{h}}_{+i}^{\ n} = \frac{1}{\Delta x}\int_{I_j} \mathbf{q}_i^+(x)dx = \sum_{\alpha=1}^{N} \widehat{w}_\alpha \mathbf{q}_i^+(\widehat{x}_i^\alpha)$$

$$= \sum_{\alpha=1}^{N-1} \widehat{w}_\alpha \mathbf{q}_i^+(\widehat{x}_i^\alpha) + \widehat{w}_N (\mathbf{h}_+)^-_{i+\frac{1}{2}} = (1-\widehat{w}_N)\mathbf{q}_i^{+,*} + \widehat{w}_N (\mathbf{h}_+)^-_{i+\frac{1}{2}}, \tag{3.6}$$

where we use $\mathbf{q}_i^{+,*} = \frac{1}{1-\widehat{w}_N} \sum\limits_{\alpha=1}^{N-1} \widehat{w}_\alpha \mathbf{q}_i^+(\widehat{x}_i^\alpha)$ to denote a convex combination of point values of $\mathbf{q}_i^+(x)$. Plugging (3.6) into (3.4), we have

$$\mathbf{H}^+ = (1-\widehat{w}_N)\mathbf{q}_i^{+,*} + (\widehat{w}_N - \alpha\lambda)(\mathbf{h}_+)^-_{i+\frac{1}{2}} + \alpha\lambda(\mathbf{h}_+)^-_{i-\frac{1}{2}}.$$

Therefore, if $\mathbf{q}_i^{+,*}, (\mathbf{h}_+)^-_{i+\frac{1}{2}}, (\mathbf{h}_+)^-_{i-\frac{1}{2}} \in G$, then $\mathbf{H}^+ \in G$ under the CFL condition $\alpha\lambda \le \widehat{w}_N$.

Similarly, for (3.5), there exists a vector of polynomials of degree k $\mathbf{q}_i^-(x)$ such that $\mathbf{q}_i^-(x_{i-\frac{1}{2}}) = (\mathbf{h}_-)^+_{i-\frac{1}{2}}$, the cell average of $\mathbf{q}_i^-(x)$ on I_i is $\overline{\mathbf{h}}_{-i}^{\,n}$ and $\mathbf{q}_i^-(x)$ is a $(k+1)$-th order accurate approximation to the function $\mathbf{h}_-$ on the cell I_i if $\mathbf{w}$ is smooth. Let $\mathbf{q}_i^{-,*} = \frac{1}{1-\widehat{w}_1} \sum\limits_{\alpha=2}^{N} \widehat{w}_\alpha \mathbf{q}_i^-(\widehat{x}_i^\alpha)$, then (3.5) becomes

$$\mathbf{H}^- = (1-\widehat{w}_1)\mathbf{q}_i^{-,*} + (\widehat{w}_1 - \alpha\lambda)(\mathbf{h}_-)^+_{i-\frac{1}{2}} + \alpha\lambda(\mathbf{h}_-)^+_{i+\frac{1}{2}}.$$

Therefore, if $\mathbf{q}_i^{-,*}, (\mathbf{h}_-)^+_{i+\frac{1}{2}}, (\mathbf{h}_-)^+_{i-\frac{1}{2}} \in G$, then $\mathbf{H}^- \in G$ under the CFL condition $\alpha\lambda \le \widehat{w}_1$.

Notice that $\widehat{w}_1 = \widehat{w}_N$, we have

Theorem 3.1 *Under the CFL condition $\alpha\lambda \le \widehat{w}_1$, if $\mathbf{q}_i^{+,*}$, $(\mathbf{h}_+)^-_{i+\frac{1}{2}}$, $(\mathbf{h}_+)^-_{i-\frac{1}{2}}$, $\mathbf{q}_i^{-,*}$, $(\mathbf{h}_-)^+_{i+\frac{1}{2}}$, $(\mathbf{h}_-)^+_{i-\frac{1}{2}} \in G$, the finite difference WENO scheme (3.2) and (3.3) will be positivity-preserving, i.e., $\mathbf{w}_i^{n+1} \in G$, where by (3.6) we have*

$$\mathbf{q}_i^{+,*} = \frac{1}{1-\widehat{w}_N}\left[\overline{\mathbf{h}}_{+i}^{\,n} - \widehat{w}_N(\mathbf{h}_+)^-_{i+\frac{1}{2}}\right], \qquad \mathbf{q}_i^{-,*} = \frac{1}{1-\widehat{w}_1}\left[\overline{\mathbf{h}}_{-i}^{\,n} - \widehat{w}_1(\mathbf{h}_-)^+_{i-\frac{1}{2}}\right]. \tag{3.7}$$

3.3 A simple limiter

At time level n, given $\mathbf{w}_i^n \in G$ for all i, then $\overline{\mathbf{h}}_{\pm i}^{\,n} = \mathbf{f}^\pm(\mathbf{w}_i^n) \in G$, which was proved in Remark 2.4 of [20]. Thus we can enforce the positivity of the two finite volume schemes (3.4) and (3.5) by the techniques for finite volume WENO schemes in [20, 22].

For convenience, we use $\rho(\mathbf{w})$ and $p(\mathbf{w})$ to denote the density and pressure of a state $\mathbf{w}$. Let $\overline{\mathbf{h}}_{+i}^{\,n} = \left(\overline{\rho}_i, \overline{m}_i, \overline{E}_i\right)^T$ $(\mathbf{h}_+)^-_{i+\frac{1}{2}} = \left(\rho^-_{i+\frac{1}{2}}, m^-_{i+\frac{1}{2}}, E^-_{i+\frac{1}{2}}\right)^T$ and $\mathbf{q}_i^{+,*} = (\rho_i^*, m_i^*, E_i^*)^T$. The explicit formula of $\mathbf{q}_i^\pm(x)$ will not be needed. Consider the following limiter for (3.4),

- Setup a small number $\varepsilon = \min_i \{10^{-13}, \overline{\rho}_i\}$.

- For each cell I_i, modify the density first:

$$\widehat{\rho}^{-}_{i+\frac{1}{2}} = \theta_1 \left(\rho^{-}_{i+\frac{1}{2}} - \overline{\rho}_i \right) + \overline{\rho}_i, \quad \theta_1 = \min \left\{ \frac{\overline{\rho}_i - \varepsilon}{\overline{\rho}_i - \rho_{min}} \right\}, \tag{3.8}$$

  where $\rho_{min} = \min \left\{ \rho^{-}_{i+\frac{1}{2}}, \rho^{*}_i \right\}$. Then denote $(\widehat{\mathbf{h}_+})^{-}_{i+\frac{1}{2}} = \left(\widehat{\rho}^{-}_{i+\frac{1}{2}}, m^{-}_{i+\frac{1}{2}}, E^{-}_{i+\frac{1}{2}} \right)^T$ and $\widehat{\mathbf{q}}^{+,*}_i = \frac{1}{1-\widehat{w}_N} \left[\overline{\mathbf{h}}_{+i}^{\,n} - \widehat{w}_N (\widehat{\mathbf{h}_+})^{-}_{i+\frac{1}{2}} \right]$.

- Then modify the pressure for each cell I_i. For convenience, let $\mathbf{q}^1 = (\widehat{\mathbf{h}_+})^{-}_{i+\frac{1}{2}}$ and $\mathbf{q}^2 = \widehat{\mathbf{q}}^{+,*}_i$. For $m = 1, 2$, if $p(\mathbf{q}^m) < 0$, then solve the following quadratic equation for t^m,

$$p \left[(1 - t^m) \overline{\mathbf{h}}_{+i}^{\,n} + t^m \mathbf{q}^m \right] = 0. \tag{3.9}$$

  Notice that the convexity of G ensures the uniqueness of $t^m \in [0, 1)$. If $p(\mathbf{q}^m) \geq 0$, set $t^m = 1$. Get the modified point value by

$$(\widetilde{\mathbf{h}_+})^{-}_{i+\frac{1}{2}} = \theta_2 \left((\widehat{\mathbf{h}_+})^{-}_{i+\frac{1}{2}} - \overline{\mathbf{h}}_{+i}^{\,n} \right) + \overline{\mathbf{h}}_{+i}^{\,n}, \quad \theta_2 = \min \left\{ t^1, t^2 \right\}, \tag{3.10}$$

The limiter for (3.5) can be defined in the same way. Similarly, we get the revised point value $(\widetilde{\mathbf{h}_-})^{+}_{i-\frac{1}{2}}$. Then we have the modified WENO scheme (3.3) with

$$\widehat{\mathbf{f}}_{i+\frac{1}{2}} = \alpha \left[(\widetilde{\mathbf{h}_+})^{-}_{i+\frac{1}{2}} - (\widetilde{\mathbf{h}_-})^{+}_{i+\frac{1}{2}} \right]. \tag{3.11}$$

It is straightforward to check that the new scheme (3.3) and (3.11) satisfies the sufficient condition in Theorem 3.1.

Suppose $\mathbf{v}(x, t)$ is a smooth solution of (1.1) and it satisfies $\min_{x,t} \rho(\mathbf{v}(x, t)) > 0$ and $\min_{x,t} p(\mathbf{v}(x, t)) > 0$, then $\mathbf{h}^{\pm}(\mathbf{v}(x, t)) = R_{\Delta x}(\mathbf{f}^{\pm}(\mathbf{v}(x, t))) \in G$ for any (x, t) if Δx is small enough. Since the limiter (3.8) and (3.10) is the exactly the same limiter for finite volume scheme (3.4) as in [20, 22], following the same arguments in [20], it is straightforward to show that the accuracy will not be destroyed by the limiter for smooth solutions without vacuum regions when Δx is small.

Remark 3.2 *For a simpler and more robust implementation of the limiter for modifying pressure, see Section 3 in [17], where solving (3.9) is avoided.*

Remark 3.3 *The limiter should be used for each stage in a SSP Runge-Kutta method or each step in a SSP multi-step method. Since* $\widehat{w}_1 = \frac{1}{12}$ *for the fifth order WENO scheme and* $\widehat{w}_1 = \frac{1}{30}$ *for the seventh order WENO scheme, the CFL condition in Theorem 3.1 is much smaller than commonly used ones in practice. But Theorem 3.1 is only a sufficient condition to preserve positivity rather than a necessary one. To save computational cost, the stringent CFL condition* $\alpha\lambda \leq \widehat{w}_1$ *can be strictly enforced only when a precalculation to the next time stage or time step produces negative density or negative pressure. See Section 3 in [17] for implementation details.*

4 Generalizations

4.1 The local Lax-Friedrichs flux splitting

Now consider the case that $\alpha = \max \parallel (|u| + c) \parallel$ where the maximum is taken locally over the $\mathbf{w}_i^n$ in the WENO reconstruction stencil, see [10]. More specifically, (3.1) becomes

$$\mathbf{f}^{\pm}_{i+\frac{1}{2}}(\mathbf{w}) = \frac{1}{2}\left(\mathbf{w} \pm \frac{\mathbf{f}(\mathbf{w})}{\alpha_{i+\frac{1}{2}}}\right).$$

At each fixed $x_{i+\frac{1}{2}}$,

1. Let $\mathbf{h}_{i+\frac{1}{2},\pm} = R_{\Delta x}(\mathbf{f}^{\pm}_{i+\frac{1}{2}})$. Then we have the cell averages $\left(\overline{\mathbf{h}}_{i+\frac{1}{2},\pm}\right)^n_j = \mathbf{f}^{\pm}_{i+\frac{1}{2}}(\mathbf{w}^n_j)$.

2. Transform all the cell averages $\left(\overline{\mathbf{h}}_{i+\frac{1}{2},\pm}\right)^n_j$ (j in a neighborhood of i) to the local characteristic fields by setting

$$\left(\overline{\mathbf{u}}_{i+\frac{1}{2},\pm}\right)^n_j = L_{i+\frac{1}{2}}\left(\overline{\mathbf{h}}_{i+\frac{1}{2},\pm}\right)^n_j.$$

3. Perform the WENO reconstruction for each component of $\left(\overline{\mathbf{u}}_{i+\frac{1}{2},+}\right)^n_j$ to obtain approximations of the point value of the function $L_{i+\frac{1}{2}}\mathbf{h}_{i+\frac{1}{2},+}$ at the point $x_{i+\frac{1}{2}}$ and denote them as $(\mathbf{u}_{i+\frac{1}{2},+})^{\pm}_{i+\frac{1}{2}}$. Similarly, obtain $(\mathbf{u}_{i+\frac{1}{2},-})^{\pm}_{i+\frac{1}{2}}$.

4. Transform back into physical space by

$$(\mathbf{h}_{i+\frac{1}{2},+})^-_{i+\frac{1}{2}} = R_{i+\frac{1}{2}}(\mathbf{u}_{i+\frac{1}{2},+})^-_{i+\frac{1}{2}}, \quad (\mathbf{h}_{i+\frac{1}{2},-})^+_{i+\frac{1}{2}} = R_{i+\frac{1}{2}}(\mathbf{u}_{i+\frac{1}{2},-})^+_{i+\frac{1}{2}}.$$

Form the flux by

$$\widehat{\mathbf{f}}_{i+\frac{1}{2}} = \alpha_{i+\frac{1}{2}}\left[(\mathbf{h}_{i+\frac{1}{2},+})^-_{i+\frac{1}{2}} - (\mathbf{h}_{i+\frac{1}{2},-})^+_{i+\frac{1}{2}}\right]. \tag{4.1}$$

Notice that $\mathbf{w}_i^n = \left(\overline{\mathbf{h}}_{i+\frac{1}{2},+}\right)_i^n + \left(\overline{\mathbf{h}}_{i+\frac{1}{2},-}\right)_i^n = \left(\overline{\mathbf{h}}_{i-\frac{1}{2},+}\right)_i^n + \left(\overline{\mathbf{h}}_{i-\frac{1}{2},-}\right)_i^n$. So the conservative scheme (3.3) can be written as

$$\begin{aligned}
\mathbf{w}_i^{n+1} &= \mathbf{w}_i^n - \lambda\left(\widehat{\mathbf{f}}_{i+\frac{1}{2}} - \widehat{\mathbf{f}}_{i-\frac{1}{2}}\right) \\
&= \frac{1}{2}\left(\overline{\mathbf{h}}_{i+\frac{1}{2},+}\right)_i^n + \frac{1}{2}\left(\overline{\mathbf{h}}_{i+\frac{1}{2},-}\right)_i^n + \frac{1}{2}\left(\overline{\mathbf{h}}_{i-\frac{1}{2},+}\right)_i^n + \frac{1}{2}\left(\overline{\mathbf{h}}_{i-\frac{1}{2},-}\right)_i^n \\
&\quad -\lambda\alpha_{i+\frac{1}{2}}\left((\mathbf{h}_{i+\frac{1}{2},+})^-_{i+\frac{1}{2}} - (\mathbf{h}_{i+\frac{1}{2},-})^+_{i+\frac{1}{2}}\right) + \lambda\alpha_{i-\frac{1}{2}}\left((\mathbf{h}_{i-\frac{1}{2},+})^-_{i-\frac{1}{2}} - (\mathbf{h}_{i-\frac{1}{2},-})^+_{i-\frac{1}{2}}\right) \\
&= \mathbf{H}^+ + \mathbf{H}^-,
\end{aligned}$$

where

$$\mathbf{H}^+ = \frac{1}{2}\left(\overline{\mathbf{h}}_{i+\frac{1}{2},-}\right)_i^n + \frac{1}{2}\left(\overline{\mathbf{h}}_{i+\frac{1}{2},+}\right)_i^n - \lambda\alpha_{i+\frac{1}{2}}(\mathbf{h}_{i+\frac{1}{2},+})^-_{i+\frac{1}{2}} + \lambda\alpha_{i-\frac{1}{2}}(\mathbf{h}_{i-\frac{1}{2},+})^-_{i-\frac{1}{2}}, \tag{4.2}$$

$$\mathbf{H}^- = \frac{1}{2}\left(\overline{\mathbf{h}}_{i-\frac{1}{2},+}\right)_i^n + \frac{1}{2}\left(\overline{\mathbf{h}}_{i-\frac{1}{2},-}\right)_i^n + \lambda\alpha_{i+\frac{1}{2}}(\mathbf{h}_{i+\frac{1}{2},-})^+_{i+\frac{1}{2}} - \lambda\alpha_{i-\frac{1}{2}}(\mathbf{h}_{i-\frac{1}{2},-})^+_{i-\frac{1}{2}}. \tag{4.3}$$

By interpolation, there exists a vector of polynomials of degree k $\mathbf{q}_i^+(x)$ such that $\mathbf{q}_i^+(x_{i+\frac{1}{2}}) = (\mathbf{h}_{i+\frac{1}{2},+})^-_{i+\frac{1}{2}}$, the cell average of $\mathbf{q}_i^+(x)$ on I_i is $\left(\overline{\mathbf{h}}_{i+\frac{1}{2},+}\right)_i^n$ and $\mathbf{q}_i^+(x)$ is a $(k+1)$-th order accurate approximation to the function $\mathbf{h}_{i+\frac{1}{2},+}$ on the cell I_i if $\mathbf{w}$ is smooth. The exactness of the quadrature rule for polynomials of degree k implies

$$\left(\overline{\mathbf{h}}_{i+\frac{1}{2},+}\right)_i^n = \sum_{\alpha=1}^{N-1}\widehat{w}_\alpha\mathbf{q}_i^+(\widehat{x}_i^\alpha) + \widehat{w}_N(\mathbf{h}_{i+\frac{1}{2},+})^-_{i+\frac{1}{2}} = (1-\widehat{w}_N)\mathbf{q}_i^{+,*} + \widehat{w}_N(\mathbf{h}_{i+\frac{1}{2},+})^-_{i+\frac{1}{2}},$$

where we use $\mathbf{q}_i^{+,*} = \frac{1}{1-\widehat{w}_N}\sum\limits_{\alpha=1}^{N-1}\widehat{w}_\alpha\mathbf{q}_i^+(\widehat{x}_i^\alpha) = \frac{1}{1-\widehat{w}_N}\left[\left(\overline{\mathbf{h}}_{i+\frac{1}{2},+}\right)_i^n - \widehat{w}_N(\mathbf{h}_{i+\frac{1}{2},+})^-_{i+\frac{1}{2}}\right]$ to denote a convex combination of point values of $\mathbf{q}_i^+(x)$.

Thus (4.2) can be written as

$$\mathbf{H}^+ = \frac{1}{2}\left(\overline{\mathbf{h}}_{i+\frac{1}{2},-}\right)_i^n + \frac{1}{2}(1-\widehat{w}_N)\mathbf{q}_i^{+,*} + (\frac{1}{2}\widehat{w}_N - \alpha_{i+\frac{1}{2}}\lambda)(\mathbf{h}_{i+\frac{1}{2},+})^-_{i+\frac{1}{2}} + \alpha_{i+\frac{1}{2}}\lambda(\mathbf{h}_{i-\frac{1}{2},+})^-_{i-\frac{1}{2}}.$$

Following the previous section, $\mathbf{H}^+ \in G$ provided $\mathbf{w}_i^n, \mathbf{q}_i^{+,*}, (\mathbf{h}_{i+\frac{1}{2},+})^-_{i+\frac{1}{2}}, (\mathbf{h}_{i-\frac{1}{2},+})^-_{i-\frac{1}{2}} \in G$ under the CFL constraint $\alpha_{i+\frac{1}{2}}\lambda \leq \frac{1}{2}\widehat{w}_1, \forall i$. The other formal finite volume scheme (4.3) can be discussed similarly. So the same results as in Theorem 3.1 still hold with the CFL condition halved.

4.2 Source terms

The one dimensional version of the Euler equations with source terms is given by

$$\mathbf{w}_t + \mathbf{f}(\mathbf{w})_x = \mathbf{s}(\mathbf{w}, x), \quad t \geq 0,\ x \in \mathbb{R}, \tag{4.4}$$

The finite difference WENO scheme for (4.4) can be written as

$$\mathbf{w}_i^{n+1} = \mathbf{w}_i^n - \lambda\left(\widehat{\mathbf{f}}_{i+\frac{1}{2}} - \widehat{\mathbf{f}}_{i-\frac{1}{2}}\right) + \Delta t\mathbf{s}(\mathbf{w}_i^n, x_i) = \frac{1}{2}\mathbf{F} + \frac{1}{2}\mathbf{S},$$

with

$$\mathbf{F} = \mathbf{w}_i^n - 2\lambda\left(\widehat{\mathbf{f}}_{i+\frac{1}{2}} - \widehat{\mathbf{f}}_{i-\frac{1}{2}}\right), \tag{4.5}$$

$$\mathbf{S} = \mathbf{w}_i^n + 2\Delta t\mathbf{s}(\mathbf{w}_i^n, x_i). \tag{4.6}$$

For positivity of $\mathbf{w}_i^{n+1}$, it suffices to have $\mathbf{F}, \mathbf{S} \in G$. (4.5) is in the same form as (3.3), thus the positivity of $\mathbf{F}$ is straightforward. It was discussed in [21] that, for four kinds of source terms (geometric, gravity, chemical reaction and radiative cooling), $\mathbf{S} \in G$ under a suitable CFL condition if $\mathbf{w}_i^n \in G$.

4.3 General equations of state

Consider a general equation of state satisfying the following assumption:

$$\text{if} \quad \rho \geq 0, \quad \text{then} \quad e > 0 \Leftrightarrow p > 0. \tag{4.7}$$

The pressure is not necessarily a concave function. However, the internal energy $e = \frac{E - \frac{1}{2}\frac{m^2}{\rho}}{\rho}$ is always a concave function of $\mathbf{w}$, thus $G = \{\mathbf{w} : \rho > 0, p > 0\}$ is still convex.

Therefore, it is straightforward to check that Theorem 3.1 still holds. By setting $\alpha \geq \max \| \, |u| + \frac{p}{\rho\sqrt{2e}} \, \|$, it is still true that $\mathbf{f}^{\pm}(\mathbf{w}) \in G$ if $\mathbf{w} \in G$, see [21]. Replace (3.9) by

$$e\left[(1-t^m)\overline{\mathbf{h}}_{+i}^{\,n} + t^m \mathbf{q}^m\right] = 0,$$

then the same limiter (3.8) and (3.10) can be used to enforce the sufficient condition.

4.4 Extensions to multi-space dimensions

Two dimensional Euler equations are given by

$$\mathbf{w}_t + \mathbf{f}(\mathbf{w})_x + \mathbf{g}(\mathbf{w})_y = 0, \quad t \geq 0, (x,y) \in \mathbb{R}^2, \tag{4.8}$$

$$\mathbf{w} = \begin{pmatrix} \rho \\ m \\ n \\ E \end{pmatrix}, \quad \mathbf{f}(\mathbf{w}) = \begin{pmatrix} m \\ \rho u^2 + p \\ \rho uv \\ (E+p)u \end{pmatrix}, \quad \mathbf{g}(\mathbf{w}) = \begin{pmatrix} n \\ \rho uv \\ \rho v^2 + p \\ (E+p)v \end{pmatrix} \tag{4.9}$$

with

$$m = \rho u, \quad n = \rho v, \quad E = \frac{1}{2}\rho u^2 + \frac{1}{2}\rho v^2 + \rho e, \quad p = (\gamma-1)\rho e,$$

where ρ is the density, u is the velocity in x direction, v is the velocity in y direction, m and n are the momenta, E is the total energy, p is the pressure, e is the internal energy. The speed of sound is given by $c = \sqrt{\gamma p/\rho}$. The eigenvalues of the Jacobian $\mathbf{f}'(\mathbf{w})$ are $u-c$, u, u and $u+c$ and the eigenvalues of the Jacobian $\mathbf{g}'(\mathbf{w})$ are $v-c$, v, v and $v+c$. The pressure function p is still concave with respect to $\mathbf{w}$ if $\rho > 0$ and the set of admissible states

$$G = \left\{ \mathbf{w} = \begin{pmatrix} \rho \\ m \\ n \\ E \end{pmatrix} \middle| \, \rho > 0 \quad \text{and} \quad p = (\gamma-1)\left(E - \frac{1}{2}\frac{m^2}{\rho} - \frac{1}{2}\frac{n^2}{\rho}\right) > 0 \right\}$$

is still convex.

We will use the same Lax-Friedrichs splitting

$$\mathbf{f}^{\pm}(\mathbf{w}) = \frac{1}{2}\left(\mathbf{w} \pm \frac{\mathbf{f}(\mathbf{w})}{\alpha_1}\right), \quad \alpha_1 = \max \| \, (|u| + c) \, \|,$$

$$\mathbf{g}^{\pm}(\mathbf{w}) = \frac{1}{2}\left(\mathbf{w} \pm \frac{\mathbf{g}(\mathbf{w})}{\alpha_2}\right), \quad \alpha_2 = \max \| \, (|v| + c) \, \|,$$

where the maximum is taken either globally or locally. Following Remark 2.4 in [20], it is easy to check that $\mathbf{f}^{\pm}(\mathbf{w}), \mathbf{g}^{\pm}(\mathbf{w}) \in G$ if $\mathbf{w} \in G$.

Assume a uniform mesh with nodes (x_i, y_j), the finite difference WENO scheme is given by

$$\mathbf{w}_{i,j}^{n+1} = \mathbf{w}_{i,j}^{n} - \frac{\Delta t}{\Delta x}\left(\widehat{\mathbf{f}}_{i+\frac{1}{2},j} - \widehat{\mathbf{f}}_{i-\frac{1}{2},j}\right) - \frac{\Delta t}{\Delta y}\left(\widehat{\mathbf{g}}_{i,j+\frac{1}{2}} - \widehat{\mathbf{g}}_{i,j-\frac{1}{2}}\right),$$

where the fluxes $\widehat{\mathbf{f}}_{i+\frac{1}{2},j}$ and $\widehat{\mathbf{g}}_{i,j+\frac{1}{2}}$ are obtained by the same one-dimensional WENO approximation as in the previous section.

For the discussion of positivity, rewrite the scheme as $\mathbf{w}_{i,j}^{n+1} = \frac{1}{2}\mathbf{F} + \frac{1}{2}\mathbf{G}$ with

$$\mathbf{F} = \mathbf{w}_{i,j}^{n} - 2\frac{\Delta t}{\Delta x}\left(\widehat{\mathbf{f}}_{i+\frac{1}{2},j} - \widehat{\mathbf{f}}_{i-\frac{1}{2},j}\right), \tag{4.10}$$

$$\mathbf{G} = \mathbf{w}_{i,j}^{n} - 2\frac{\Delta t}{\Delta y}\left(\widehat{\mathbf{g}}_{i,j+\frac{1}{2}} - \widehat{\mathbf{g}}_{i,j-\frac{1}{2}}\right). \tag{4.11}$$

If $\mathbf{F}, \mathbf{G} \in G$, then $\mathbf{w}_{i,j}^{n+1} \in G$. Notice that (4.10) and (4.11) share the same abstract form as (3.3), it is straightforward to extend the positivity-preserving results for one-dimension to multi-space dimensions.

5 Numerical tests

In this section, we show some results of the fifth order finite difference WENO scheme and the third order Runge-Kutta with the positivity-preserving limiter for several demanding examples. The WENO schemes without the positivity-preserving limiter will blow up for most examples in this section.

Example 5.1 *Accuracy Test.*

Consider the vortex evolution problem for (4.8). The mean flow is $\rho = p = u = v = 1$. Add to the mean flow an isentropic vortex perturbation centered at (x_0, y_0) in (u, v) and $T = p/\rho$, no perturbation in $S = p/\rho^{\gamma}$,

$$(\delta u, \delta v) = \frac{\epsilon}{2\pi}e^{0.5(1-r^2)}(-\bar{y}, \bar{x}), \quad \delta T = \frac{(\gamma-1)\epsilon^2}{8\gamma\pi^2}e^{(1-r^2)},$$

where $(\bar{x}, \bar{y}) = (x - x_0, y - y_0), r^2 = \bar{x}^2 + \bar{y}^2$. The exact solution is the passive convection of the vortex with the mean velocity.

The domain is taken as $[0, 10] \times [0, 10]$ and $(x_0, y_0) = (5, 5)$. The boundary condition is periodic. We set $\gamma = 1.4$ and the vortex strength $\epsilon = 10.0828$ such that the lowest density and lowest pressure of the exact solution are 7.8×10^{-15} and 1.7×10^{-20}. We test the accuracy of the limiter (3.8) and (3.10) on the fifth order finite difference WENO scheme with the third order SSP Runge-Kutta. In order to make the error in spatial discretizations dominant, we take $\Delta t = \Delta x^{\frac{5}{3}}$. See Table 5.1. We clearly observe the fifth order accuracy.

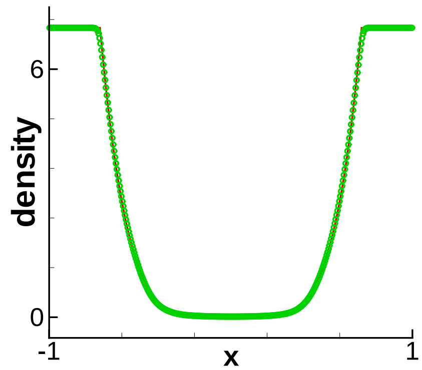
Table 5.1: Example 5.1: Fifth order finite difference WENO scheme with the positivity-preserving limiter, for the vortex evolution problem, $T = 0.01$, and $\Delta x = \Delta y$.

$1/\Delta x$	L^1 error	order	L^∞ error	order
8	6.77e-6	-	5.33e-4	-
16	3.26e-7	4.37	3.77e-5	3.82
32	8.04e-9	5.34	1.01e-6	5.22
64	1.92e-10	5.38	3.62e-8	4.80

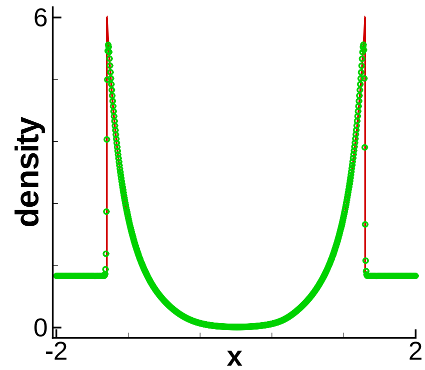
Example 5.2 *1D low density and low pressure problems.*

We consider two one-dimensional low density and low pressure problems of (1.1) for ideal gas. The first one is a one-dimensional Riemann problem, for which the initial condition is $\rho_L = \rho_R = 7$, $u_L = -1$, $u_R = 1$, $p_L = p_R = 0.2$ and $\gamma = 1.4$. The exact solution contains vacuum. It is the same double rarefaction problem in [11]. The results of positivity-preserving fifth order finite difference WENO scheme are shown in Figure 5.1 (left), which are comparable to the results of positivity-preserving DG method [20].

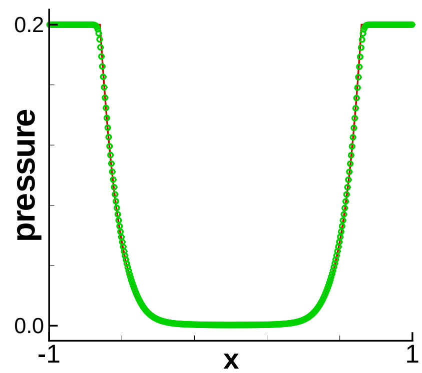
The second one is one-dimensional Sedov blast wave. For the initial condition, the density is 1, velocity is zero, total energy is 10^{-12} everywhere except that the energy in the center cell is the constant $\frac{E_0}{\Delta x}$ with $E_0 = 3200000$ (emulating a δ-function at the center). $\gamma = 1.4$. The results of positivity-preserving fifth order finite difference WENO scheme are shown in Figure 5.1 (right), which are again comparable to the results of positivity-preserving DG method [20].



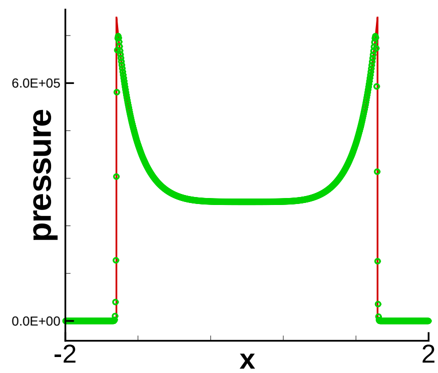
(a) density



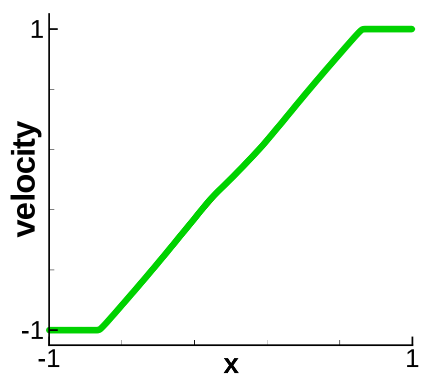
(b) density



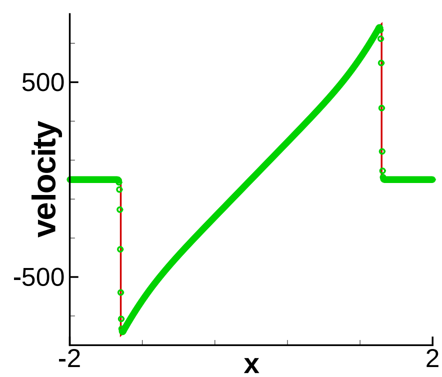
(c) pressure



(d) pressure



(e) velocity



(f) velocity

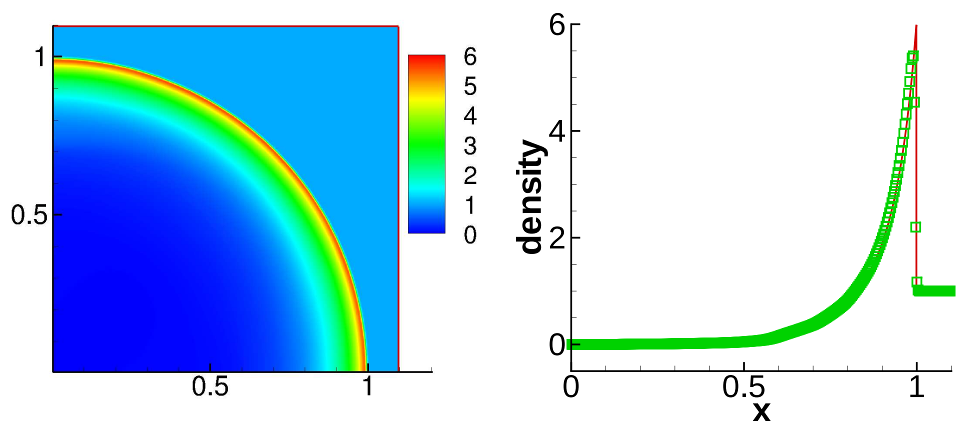
Figure 5.1: Example 5.2. Left: Double rarefaction problem. Right: 1D Sedov blast. $\Delta x = 1/200$. The red curves are exact solutions. Green symbols are numerical solutions of fifth order WENO with the positivity-preserving limiter.

Example 5.3 *2D low density and low pressure problems.*

We consider two two-dimensional low density and low pressure problems of (4.8) for ideal gas. The first one is two-dimensional Sedov blast wave. The computational domain is a square. For the initial condition, the density is 1, velocity is zero, total energy is 10^{-12} everywhere except that the energy in the lower left corner cell is the constant $\frac{0.244816}{\Delta x \Delta y}$. $\gamma = 1.4$. The numerical boundary treatment is reflective for the left and bottom edges. See Figure 5.2.

The second one is shock diffraction problem. The setup is the following: the computational domain is the union of $[0, 1] \times [6, 11]$ and $[1, 13] \times [0, 11]$; the initial condition is a pure right-moving shock of $Mach = 5.09$, initially located at $x = 0.5$ and $6 \leq y \leq 11$, moving into undisturbed air ahead of the shock with a density of 1.4 and pressure of 1. The boundary conditions are inflow at $x = 0$, $6 \leq y \leq 11$, outflow at $x = 13$, $0 \leq y \leq 11$, $1 \leq x \leq 13$, $y = 0$ and $0 \leq x \leq 13$, $y = 11$, and reflective at the walls $0 \leq x \leq 1$, $y = 6$ and $x = 1$, $0 \leq y \leq 6$. $\gamma = 1.4$. The density and pressure at $t = 2.3$ are presented in Figures 5.3.

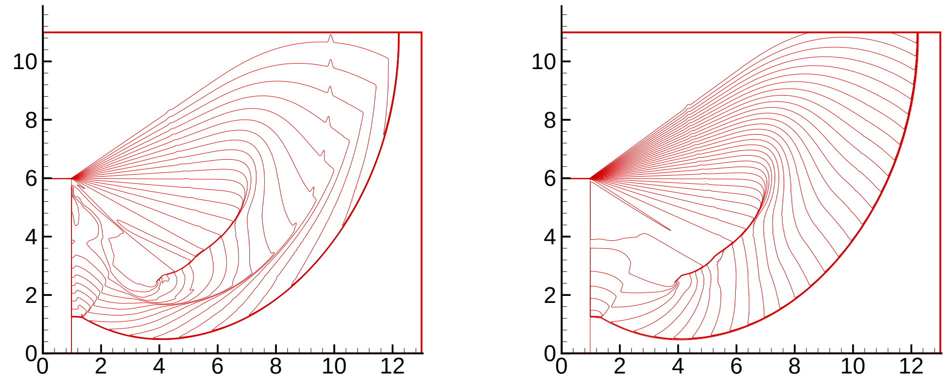
The results are comparable to those of positivity-preserving DG method [20] and finite volume WENO scheme [22].



(a) Color contour of density

(b) Cut along $y = 0$. The solid line is the exact solution. Symbols are numerical solutions.

Figure 5.2: 2D Sedov blast. $T = 1$. $\Delta x = \Delta y = \frac{1.1}{320}$. The fifth order finite difference WENO scheme with positivity-preserving limiter.



(a) Density: 20 equally spaced contour lines from $\rho = 0.066227$ to $\rho = 7.0668$.

(b) Pressure: 40 equally spaced contour lines from $p = 0.091$ to $p = 37$.

Figure 5.3: Shock diffraction problem. $\Delta x = \Delta y = 1/80$. The fifth order finite difference WENO scheme with positivity-preserving limiter.

Example 5.4 *High Mach number flows.*

We consider the Mach 2000 problem in [20]. The equations are (4.8) with $\gamma = 1.2$. The domain is $[0,1]\times[-0.25,0.25]$, initially full of the ambient gas with $(\rho,u,v,p) = (5,0,0,0.4127)$. The boundary conditions for the right, top and bottom are outflow. For the left boundary, $(\rho,u,v,p) = (5,800,0,0.4127)$ if $y \in [-0.05,0.05]$ and $(\rho,u,v,p) = (5,0,0,0.4127)$ otherwise. The terminal time is 0.001. The speed of the jet is 800, which is around Mach 2100 with respect to the sound speed in the jet gas. See Figure 5.4 for the result on a 800×400 mesh, which is comparable to the result in [20].

Example 5.5 *The reactive Euler equations.*

We consider the following equations which are often used to model the detonation waves:

$$\mathbf{w}_t + \mathbf{f}(\mathbf{w})_x + \mathbf{g}(\mathbf{w})_y = \mathbf{s}(\mathbf{w}), \quad t \geq 0, (x,y) \in \mathbb{R}^2, \tag{5.1}$$

$$\mathbf{w} = \begin{pmatrix} \rho \\ m \\ n \\ E \\ \rho Y \end{pmatrix}, \quad \mathbf{f}(\mathbf{w}) = \begin{pmatrix} m \\ \rho u^2 + p \\ \rho uv \\ (E+p)u \\ \rho u Y \end{pmatrix}, \quad \mathbf{g}(\mathbf{w}) = \begin{pmatrix} n \\ \rho uv \\ \rho v^2 + p \\ (E+p)v \\ \rho v Y \end{pmatrix}, \quad \mathbf{s}(\mathbf{w}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \omega \end{pmatrix} \tag{5.2}$$

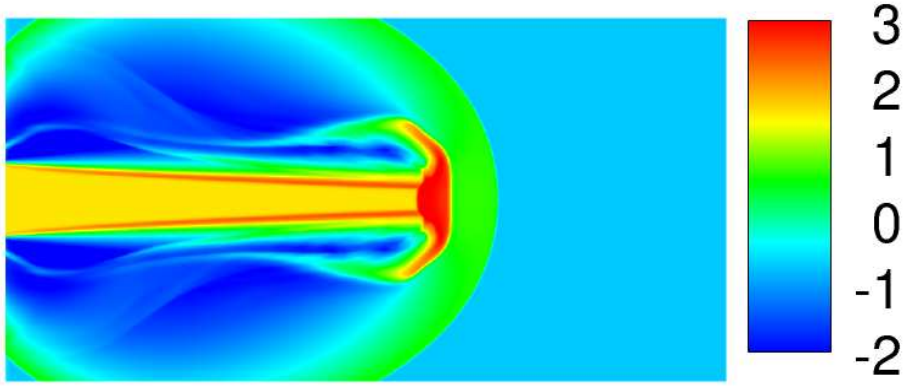


Figure 5.4: Example 5.4. Simulation of Mach 2000 jet. Scales are logarithmic.

with

$$m = \rho u, \quad n = \rho v, \quad E = \frac{1}{2}\rho u^2 + \frac{1}{2}\rho v^2 + \frac{p}{\gamma - 1} + \rho q Y,$$

where q is the heat release of reaction, γ is the specific heat ratio and Y denotes the reactant mass fraction. The source term is assumed to be in an Arrhenius form

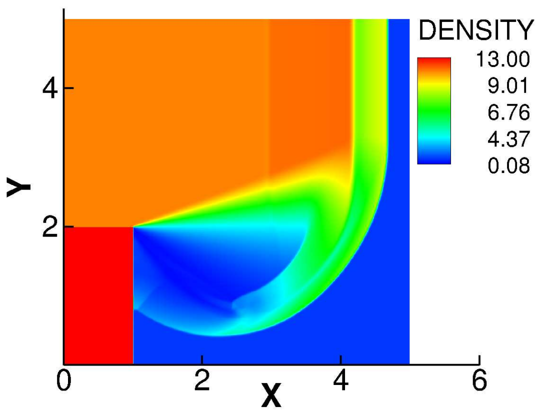
$$\omega = -\widetilde{K}\rho Y e^{-\widetilde{T}/T},$$

where $T = \frac{p}{\rho}$ is the temperature, $\widetilde{T}$ is the activation temperature and $\widetilde{K}$ is a constant. The eigenvalues of the Jacobian $\mathbf{f}'(\mathbf{w})$ are $u-c, u, u, u, u+c$ and the eigenvalues of the Jacobian $\mathbf{g}'(\mathbf{w})$ are $v-c, v, v, v, v+c$, where $c = \sqrt{\gamma \frac{p}{\rho}}$.

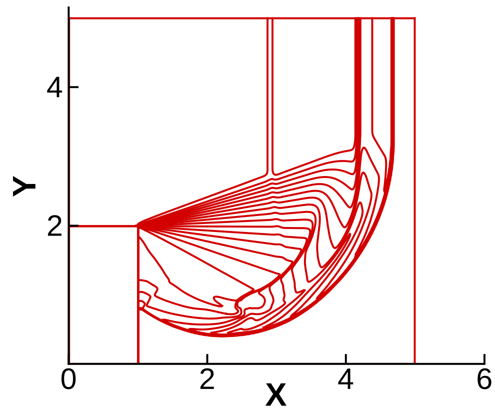
Consider the following problem which is similar to the shock diffraction problem. The computational domain is the union of $[0,1]\times[2,5]$ and $[1,5]\times[0,5]$; The initial conditions are, if $x < 0.5$, $(\rho, u, v, E, Y) = (11, 6.18, 0, 970, 1)$; otherwise, $(\rho, u, v, E, Y) = (1, 0, 0, 55, 1)$. The boundary conditions are reflective except that at $x = 0$, $(\rho, u, v, E, Y) = (11, 6.18, 0, 970, 1)$. The terminal time is $t = 0.6$. The parameters are $\gamma = 1.2$, $q = 50$, $\widetilde{T} = 50$, $\widetilde{K} = 2566.4$.

See Figure 5.5, which is comparable to the result of the third order positivity-preserving DG method in [17].

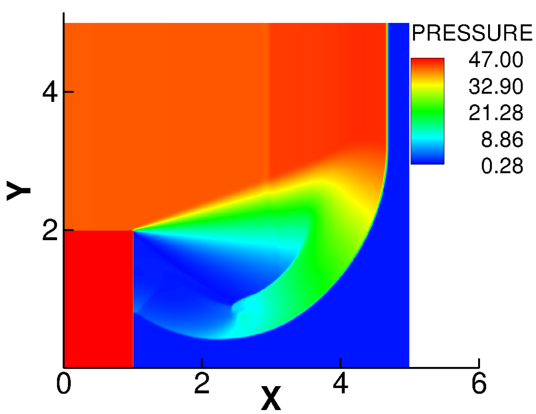
Example 5.6 *A general equation of state.*



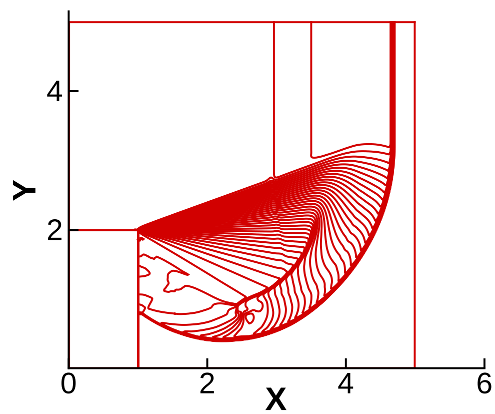
(a) Colored contour of density



(b) Contour line of density



(c) Colored contour of pressure



(d) Contour line of pressure

Figure 5.5: Example 5.5. Detonation diffraction at a 90° corner. $\Delta x = \Delta y = 1/80$. Fifth order WENO scheme with positivity-preserving limiter.

We consider the three species model with a more general equation of state in [18]. The model involves three species, O_2, O and N_2 ($\rho_1 = \rho_O, \rho_2 = \rho_{O_2}$ and $\rho_3 = \rho_{N_2}$) with the reaction:

$$O_2 + N_2 \rightleftharpoons O + O + N_2.$$

The governing equations are

$$\begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho u \\ E \end{pmatrix}_t + \begin{pmatrix} \rho_1 u \\ \rho_2 u \\ \rho_3 u \\ \rho u^2 + p \\ (E+p)u \end{pmatrix}_x = \begin{pmatrix} 2M_1\omega \\ -M_2\omega \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and

$$\rho = \sum_{s=1}^{3} \rho_s, \quad p = RT\sum_{s=1}^{3} \frac{\rho_s}{M_s}, \quad E = \sum_{s=1}^{3} \rho_s e_s(T) + \rho_1 h_1^0 + \frac{1}{2}\rho u^2$$

where the enthalpy h_1^0 is a constant, R is the universal gas constant, M_s is the molar mass of species s, and the internal energy $e_s(T) = 3R/2M_s$ and $5R/2M_s$ for monoatomic and diatomic species respectively. The rate of the chemical reaction is given by

$$\omega = \left(k_f(T)\frac{\rho_2}{M_2} - k_b(T)\left(\frac{\rho_1}{M_1}\right)^2 \right) \sum_{s=1}^{3} \frac{\rho_s}{M_s}, \quad k_f = CT^{-2}e^{-E/T},$$

$$k_b = k_f / \exp\left(b_1 + b_2 \log z + b_3 z + b_4 z^2 + b_5 z^3\right), \quad z = 10000/T$$

where b_i, C and E are constants which can be found in [18, 6].

The eigenvalues of the Jacobian $\mathbf{f}'(\mathbf{w})$ are $(u, u, u, u+c, u-c)$ where $c = \sqrt{\gamma \frac{p}{\rho}}$ with $\gamma = 1 + \frac{p}{T\sum_{s=1}^{3}\rho_s e_s'(T)}$. So if we take $a_0 = ||(|u| + c)||_\infty$ in the Lax-Friedrichs splitting (3.1), then all the results in Section 3 will hold. The following CFL condition will ensure the positivity of $\mathbf{w} + 2\Delta t \mathbf{s}(\mathbf{w})$.

$$\Delta t < \begin{cases} \frac{\rho_2}{2M_2\omega}, & \text{if} \quad \omega > 0 \\ -\frac{\rho_1}{4M_1\omega}, & \text{if} \quad \omega < 0 \end{cases}. \tag{5.3}$$

Consider a shock tube problem for the reactive flows with high pressure on the left and low pressure on the right initially in the chemical equilibrium ($\omega = 0$). The initial conditions are:

$$(p_L, T_L) = (1000N/m^2, 8000K), \quad (p_R, T_R) = (1N/m^2, 8000K)$$

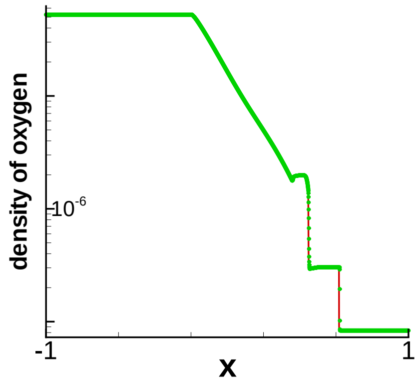
with zero velocity everywhere and the densities satisfying

$$\frac{\rho_O}{2M_O} + \frac{\rho_{O_2}}{M_{O_2}} = \frac{21}{79}\frac{\rho_{N_2}}{M_{N_2}},$$

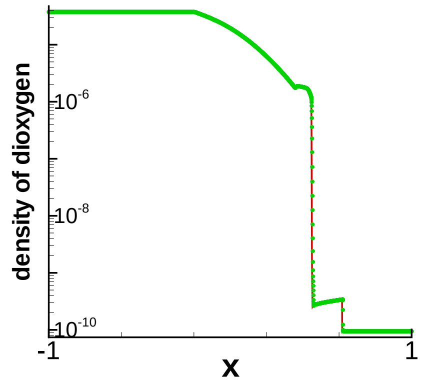
where $M_O = 0.016, M_{O_2} = 0.032$ and $M_{N_2} = 0.028$. The initial densities of O, O_2 and N_2 are $5.251896311257204 \times 10^{-5}$, $3.748071704863518 \times 10^{-5}$ and $2.962489471973072 \times 10^{-4}$ on the left, and $8.341661837019181 \times 10^{-8}$, $9.455418692098664 \times 10^{-11}$ and $2.748909430004963 \times 10^{-7}$ on the right.

High order schemes without the positivity-preserving limiter may blow up due to presence of negative pressure. See Figure 5.6 for the numerical solution of the WENO scheme at $t = 0.0001$. We obtain clean and grid converged solutions for this test case.

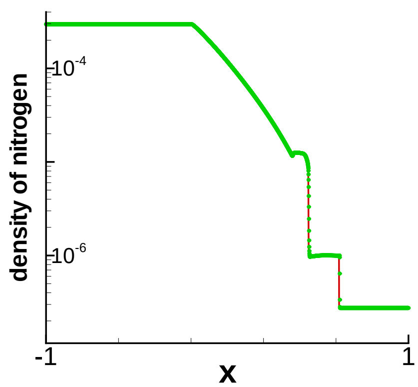
In [10], the positive real number to avoid denominators becoming zero when calculating nonlinear weights in WENO reconstruction was set 10^{-6}. For this particular problem, notice that the lowest density is around 10^{-10}, so we adjust the positive number to 10^{-20}. Otherwise oscillations may emerge.



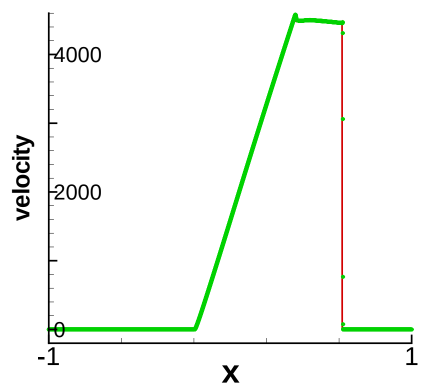
(a) ρ_O



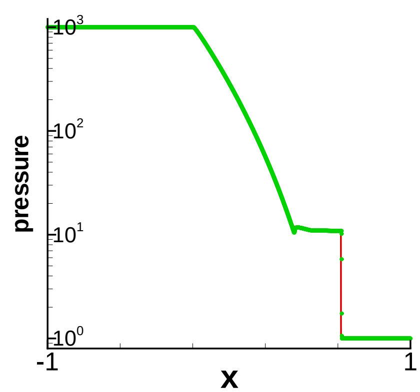
(b) ρ_{O_2}



(c) ρ_{N_2}



(d) Velocity



(e) Pressure

Figure 5.6: Example 5.6. Fifth order WENO scheme with positivity-preserving limiter. The solid line is the numerical solution of $\Delta x = \frac{2}{8000}$. The symbols denote the numerical solution of $\Delta x = \frac{2}{4000}$.

6 Concluding Remarks

In [20], a general framework was established to construct high order DG and finite volume WENO schemes which can preserve the positivity of density and pressure for the compressible Euler equations in the gas dynamics. In this paper, we have shown a generalization to the finite difference WENO scheme. Extensions to multi-space dimensions are straightforward.

With the addition of the positivity-preserving limiter in this paper, which involves small additional computational cost, to the finite difference scheme (e.g. ENO and WENO), the numerical solutions will always have positive density and pressure under suitable CFL condition. We have tested the fifth order finite difference WENO scheme with the positivity-preserving limiter for several tough problems. Future work includes the generalization to Navier-Stokes equations.

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