

Laplacian simplices associated to digraphs

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Abstract. We associate to a finite digraph D a lattice polytope P_D whose vertices are the rows of the Laplacian matrix of D . This generalizes a construction introduced by Braun and the third author. As a consequence of the Matrix-Tree Theorem, we show that the normalized volume of P_D equals the complexity of D , and P_D contains the origin in its relative interior if and only if D is strongly connected. Interesting connections with other families of simplices are established and then used to describe reflexivity, the h^* -polynomial, and the integer decomposition property of P_D in these cases. We extend Braun and Meyer’s study of cycles by considering cycle digraphs. In this setting, we characterize reflexivity and show there are only four non-trivial reflexive Laplacian simplices having the integer decomposition property.

1. Introduction

The use of linear algebra to study properties of graphs is an established technique in combinatorics and consequently has led to the development of what has come to be known as *spectral graph theory*. There is an extensive literature on algebraic aspects of spectral graph theory and on how combinatorial properties are encoded in characteristic polynomials, eigenvalues and eigenvectors of adjacency or Laplacian matrices of graphs (see [6] for a survey). It is tempting to take a step forward and associate a polytope P_G to any graph G by interpreting the rows of a matrix encoding the data of G as the vertices of P_G . This is the case of the *edge polytope* [14], [19], the convex hull of the columns of the unsigned vertex-edge incidence matrix of a graph, whose geometric and combinatorial properties have been extensively studied in the last two decades (see e.g. [12], [18]) and used to build counterexamples [13]. Recently, the third author and Braun [5] took a similar direction by associating to any graph G the simplex T_G (called the *Laplacian sim-*

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plex) whose vertices are the rows of the Laplacian matrix of G . They established basic properties of T_G and study reflexivity, the integer decomposition property, and unimodality of the Ehrhart h^* -vectors of T_G for some special classes of graph.

Our contribution is to provide a more general setting for the investigation of Laplacian simplices. We do this by allowing G to be a directed multigraph. In this way the objects studied in [5] can be seen as a special case of our setting (see Remark 2). We have reasons to believe this generalization is the correct direction to take. Indeed, in the undirected and simple case, the origin of a Laplacian simplex coincides with the barycenter of its vertices, which is an uncommon property for a lattice simplex. In our setting, it is clarified (Proposition 5 and Corollary 6) this happens only for special digraphs, i.e. they need to be strongly connected and have the same number of spanning trees converging to each vertex. As the intuition suggests, a spanning tree converging to a vertex is a spanning tree such that, starting from any other vertex and moving along the directions of the edges, one always ends on a unique final vertex. Moreover, in the original setting the volume of a Laplacian simplex associated to a graph with n vertices equals n times the number of spanning trees of the graph G . Extending to digraphs, it turns out (Proposition 7) the factor n appears because in the undirected case each vertex has the same number of spanning trees converging to it.

Main results and organization of the paper

In Section 2 we set notation, basic definitions and prove the first important properties of Laplacian simplices in this new setting. In particular, we prove that the Laplacian simplex P_D associated to a digraph D with n vertices satisfies the following properties.

(1) P_D is a $(n-1)$ -simplex if and only if D has positive complexity, i.e. if D has at least a spanning converging tree (Proposition 4).

Now assume that D has positive complexity.

(2) The numbers of spanning trees converging to each of the vertices of D encode the barycentric coordinates of the origin with respect to the vertices of P_D (Proposition 5).

(3) P_D contains the origin, which is in the strict relative interior of P_D if and only if D is strongly connected (Corollary 6).

(4) The normalized volume of P_D equals the total complexity of D , i.e. the total number of spanning converging trees (Proposition 7).

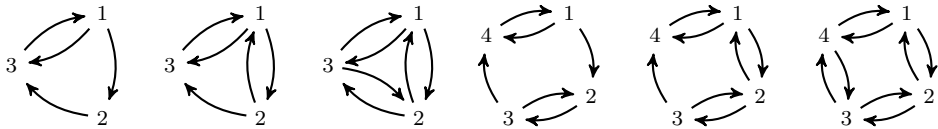
Moreover, in Section 3, we prove that under some assumptions on D , P_D is unimodularly equivalent to the simplex associated to a weighted projective space (Proposition 8). Under even more restrictive assumptions, P_D is equivalent to one

of the $\Delta_{(1,q)}$ simplices described in [4]. In such cases, we use this equivalence to characterize reflexivity (Corollary 10) and describe the Ehrhart h^* -polynomial as well as the integer decomposition property of P_D in terms of the number of spanning converging trees of D .

In Section 4, we use these descriptions to extend the study of cycle graphs of Braun–Meyer [5] to *cycle digraphs*, i.e. strongly connected simple digraphs whose underlying graph is a cycle (see Definition 16). We prove the following results.

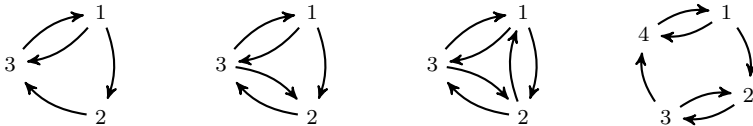
(1) In Proposition 14, we prove that a Laplacian simplex associated to a simple digraph has at most one interior lattice point.

(2) In Theorem 17, we prove that a Laplacian simplex associated to a cycle digraph is terminal Fano, i.e. it contains no lattice points other than the origin and its vertices, unless D is one of the following six digraphs.



(3) In Theorem 18, we characterize the reflexivity of P_D in terms of combinatorial properties of the cycle digraph D .

(4) In Theorem 19, we prove that a reflexive Laplacian simplex P_D has the integer decomposition property if and only if D is the oriented cycle $1 \rightarrow 2 \rightarrow \dots \rightarrow n-1 \rightarrow n \rightarrow 1$ for any n , or one of the following four exceptional digraphs.



(5) In Theorem 20, we construct a family of reflexive Laplacian simplices with symmetric but non-unimodal h^* -vector $(1, \dots, 1, 2, \dots, 2, 1, \dots, 1, 2, \dots, 2, 1, \dots, 1, 2, \dots, 2, 1, \dots, 1)$.

In the final section, we try to understand how the structure of the underlying simple and undirected graph of a digraph affects the reflexivity of its Laplacian simplex. In particular, we show there is a graph which is not the underlying graph of any simple directed graph whose Laplacian simplex is reflexive. More general versions of this problem (Questions A–C) remain open.

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2. The Laplacian polytope construction and properties of Laplacian simplices

2.1. The Laplacian of a digraph

Let D be a finite directed graph (*digraph*) on the vertex set $V(D)=[n]$, where $[n]:=\{1, \dots, n\}$. Let $E(D)$ be the set of the *directed edges* of D . A directed edge $e=(i, j) \in E(D)$ points from a vertex i (called the *tail* of e) to another vertex j (called the *head* of e). Multiple directed edges between vertices are allowed, and we denote by $a_{i,j}$ the number of directed edges having tail on the vertex i and head on the vertex j of D , with $i, j \in [n]$ and $i \neq j$. Since loops will not affect the Laplacian matrix, we assume D to be without loops, and thus $a_{i,i}=0$ for all $i \in [n]$. The number of edges with vertex i as a tail is called the *outdegree* of i and is denoted by $\text{outdeg}(i)$, while the number of edges with vertex i as a head is called the *indegree* of i and is denoted by $\text{indeg}(i)$. We call D *strongly connected* if it contains a directed path from i to j for every pair of distinct vertices $i, j \in [n]$ and *weakly connected* if there exists a path (not necessarily directed) between i and j for every pair of distinct vertices $i, j \in [n]$. In this paper, we assume D has no isolated vertices, i.e. vertices with indegree and outdegree equal to zero. A *converging tree* is a weakly connected digraph having one vertex with outdegree zero, called the *root* of the tree, while all other vertices have outdegree one. We say that a subgraph D' of D is *spanning* if the vertex set of D' is $[n]$.

All the data of D can be encoded in the *adjacency matrix* of D , that is, the $n \times n$ matrix $A(D):=(a_{i,j})_{1 \leq i, j \leq n}$. We define the *outdegree matrix* of D to be $O(D):=(d_{i,j})_{1 \leq i, j \leq n}$, the $n \times n$ matrix with $d_{i,j}=\text{outdeg}(i)$, if $i=j$, and $d_{i,j}=0$ otherwise. We define the *Laplacian matrix* of D to be the matrix $L(D):=O(D)-A(D)$.

Observe the sum of the entries in each row of $L(D)$ is zero. Thus the rank of the Laplacian matrix is never maximal, i.e.

$$(1) \quad \text{rk}(L(D)) \leq n-1.$$

A combinatorial interpretation for having equality in (1) is given by the Matrix-Tree Theorem, which is presented here in its generalized version for digraphs. The

interpretation is given in terms of *spanning converging trees* of D . For any $i \in [n]$, we denote by c_i the number of spanning trees which converge to i , i.e. the converging trees of D with n vertices having i as the root. We denote by $c(D)$ the total number of spanning converging trees of D , i.e. $c(D) := \sum_{i=1}^n c_i$. The number $c(D)$ is usually referred to as the *complexity* of the digraph D .

Theorem 1. (Matrix-Tree Theorem [17, Theorem 5.6.4]) *Let D be a digraph without loops on the vertex set $[n]$. Let $i, j \in [n]$, and $L(D)_{i,j}$ the matrix obtained from $L(D)$ by removing its i -th row and j -th column. Then the determinant of $L(D)_{i,j}$ equals, up to a change of sign, the number of spanning trees of D converging to i , i.e.*

$$(-1)^{i+j} \det L(D)_{i,j} = \det L(D)_{i,i} = c_i.$$

In particular, the complexity of D is

$$c(D) = \sum_{i=1}^n \det L(D)_{i,i}.$$

2.2. The Laplacian polytope associated to a digraph

Let D be a digraph on the vertex set $[n]$. To D we associate a convex polytope in \mathbb{R}^n having vertices in the integer lattice \mathbb{Z}^n . We call the *Laplacian polytope* associated to D the polytope $P_D := \text{conv}(\{\mathbf{v}_1, \dots, \mathbf{v}_n\}) \subseteq \mathbb{R}^n$, where \mathbf{v}_i is the i -th row of the Laplacian matrix of D . The polytope P_D is not full-dimensional; since the sum of the entries in each row of $L(D)$ vanishes, P_D is contained in the hyperplane $H := \{\mathbf{x} = (x_1, \dots, x_n) \mid \sum_{i=1}^n x_i = 0\}$ of \mathbb{R}^n . In particular, the dimension of the Laplacian polytope, $\dim(P_D)$, equals the rank of the Laplacian matrix $L(D)$. When the rank of $L(D)$ is equal to $n - 1$, then P_D is a simplex, called the *Laplacian simplex* associated to D .

Remark 2. The Laplacian simplex in this context is a generalization of the Laplacian simplex introduced by Braun-Meyer in [5]. For a connected simple graph G , they define the simplex T_G as the convex hull of the rows of the graph Laplacian matrix of G . The Laplacian $L(G)$ of G can be interpreted as the Laplacian of a digraph D_G , and thus the resulting simplices are equal, that is, $T_G = P_{D_G}$.

Two lattice polytopes $P \subseteq \mathbb{R}^n$ and $P' \subseteq \mathbb{R}^{n'}$ are said to be *unimodularly equivalent* if there exists an affine map from the affine span $\text{aff}(P)$ of P to the affine span $\text{aff}(P')$ of P' that maps $\mathbb{Z}^n \cap \text{aff}(P)$ bijectively onto $\mathbb{Z}^{n'} \cap \text{aff}(P')$ and maps P to P' . Sometimes it is convenient to work with full-dimensional lattice polytopes, i.e. lattice polytopes embedded in a space of their same dimension. Given a Laplacian

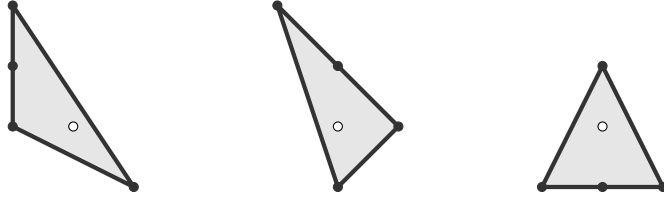
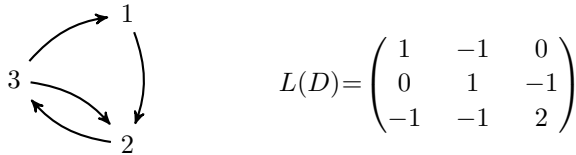


Figure 1. Three unimodularly equivalent full-dimensional copies of P_D obtained by deleting the first, second, and third columns of $L(D)$, respectively.

simplex P_D , one can easily get a full-dimensional unimodularly equivalent copy of P_D by considering the lattice polytope defined as the convex hull of the rows of $L(D)$ with one column deleted. An example of this can be observed in Example 3.

Example 3. Let D be the following digraph with its Laplacian matrix $L(D)$.



Note that $L(D)$ has rank two, which means P_D is a two dimensional simplex in \mathbb{R}^3 . Full-dimensional unimodularly equivalent copies of P_D can be obtained by deleting any of the columns of $L(D)$ and considering the convex hull of the rows as in Figure 1.

2.3. Properties of Laplacian simplices

From the Matrix-Tree Theorem (Theorem 1) the following characterization can be immediately obtained.

Proposition 4. *Let D be a digraph on n vertices. The following are equivalent:*

1. D has positive complexity $c(D)$;
2. $\text{rk}(L(D)) = n - 1$;
3. P_D is an $(n - 1)$ -simplex.

Following the work of Braun-Meyer [5], we focus our attention to the case in which a digraph D on n vertices defines an $(n - 1)$ -simplex. Proposition 4 asserts we will always assume the digraph D has positive complexity. As another consequence of Theorem 1, we deduce that numbers of spanning converging trees encode the barycentric coordinates of $\mathbf{0}$, where $\mathbf{0}$ is the origin of the lattice.

Proposition 5. *Let D be a digraph with positive complexity. Then the numbers of spanning converging trees c_1, \dots, c_n of D encode the unique linear dependence among the vertices $\mathbf{v}_1, \dots, \mathbf{v}_n$ of P_D , i.e.*

$$\sum_{i=1}^n c_i \mathbf{v}_i = \mathbf{0}.$$

Proof. Since the determinant of $L(D)$ is zero, the Laplace expansion along the j -th column of $L(D)$ yields $\sum_{i=1}^n (-1)^{i+j} \det L(D)_{i,j} v_{i,j} = 0$, where $L(D)_{i,j}$ is the matrix of $L(D)$ obtained by removing the i -th row and j -th column of $L(D)$, and $v_{i,j}$ is the j -th entry of \mathbf{v}_i . By Theorem 1, $\det L(D)_{i,i} = c_i$. \square

Corollary 6. *Let D be a digraph on n vertices having positive complexity. Then $\mathbf{0} \in P_D$. Moreover, $\mathbf{0}$ is an interior point of P_D if and only if D is strongly connected.*

Proof. The first statement is a direct consequence of Proposition 5. For the second, it is enough to note that D is strongly connected if and only if each vertex has at least one spanning converging tree. \square

In this setting, we prove a formula for the normalized volume of P_D . If a polytope P is n -dimensional, its *normalized volume* $\text{Vol}(P)$ is defined to be $n!$ times the relative Euclidean volume of P .

Proposition 7. *Let D be a digraph with positive complexity. Then its normalized volume equals the complexity of D , i.e.*

$$\text{Vol}(P_D) = c(D).$$

Proof. In this case, P_D is a $(n-1)$ -simplex by Proposition 4. For $i=1, \dots, n$, we denote by F_i the facet of P_D not containing the vertex i . Let $S_i := \text{conv}(\{\mathbf{0}\} \cup F_i)$ and $I := \{i \in [n] \mid \mathbf{0} \notin F_i\}$. By Proposition 5, $\mathbf{0} \in P_D$, so the set $\{S_i \mid i \in I\}$ forms a triangulation of P_D . In particular

$$\text{Vol}(P_D) = \sum_{i \in I} \text{Vol}(S_i).$$

Let S'_i be the unimodularly equivalent copy of S_i obtained as the convex hull of the rows of $L(D)_{i,i}$ together with the origin, where $L(D)_{i,i}$ is the matrix obtained from

$L(D)$ by removing the i -th row and i -th column. Then

$$\text{Vol}(P_D) = \sum_{i \in I} \text{Vol}(S_i) = \sum_{i \in I} \text{Vol}(S'_i) = \sum_{i \in I} \det L(D)_{i,i} = \sum_{i \in I} c_i = \sum_{i=1}^n c_i,$$

where the fourth equality follows from Theorem 1. \square

3. Connections with other families of simplices

Laplacian simplices associated to strongly connected digraphs have interesting intersections with the study of weighted projective space arising from algebraic geometry as well as the study of other families of simplices. We use these connections to describe properties of Laplacian simplices with particular attention to reflexivity, the integer decomposition property, and h^* -vectors of lattice polytopes. For the convenience of the reader, the next two subsections are a quick introduction to these topics.

3.1. Weighted projective spaces

Given positive integers $\lambda_1, \dots, \lambda_n$ which are coprime, i.e. such that $\gcd\{\lambda_1, \dots, \lambda_n\}=1$, we define the polynomial algebra $S(\lambda_1, \dots, \lambda_n) := \mathbb{C}[x_1, \dots, x_n]$ graded by $\deg x_i := \lambda_i$. A *weighted projective space* with weights $\lambda_1, \dots, \lambda_n$ is the projective variety $\mathbb{P}(\lambda_1, \dots, \lambda_n) := \text{Proj}(S(\lambda_1, \dots, \lambda_n))$. Since $\mathbb{P}(\lambda_1, \dots, \lambda_n)$ is a toric variety, it corresponds to a fan Δ which can be characterized as follows. Let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be primitive lattice points which generate the lattice and satisfy $\sum_{i=1}^n \lambda_i \mathbf{v}_i = \mathbf{0}$, where $\gcd\{\lambda_1, \dots, \lambda_n\}=1$. Then, up to isomorphism, the fan Δ is the fan whose rays are generated by the \mathbf{v}_i . Note the fan Δ identifies uniquely the simplex $S_\Delta := \text{conv}(\{\mathbf{v}_1, \dots, \mathbf{v}_n\})$. With an abuse of terminology, we say a simplex is the weighted projective space $\mathbb{P}(\lambda_1, \dots, \lambda_n)$ if it is unimodularly equivalent to the simplex S_Δ . For details we refer the reader to [8], [10].

3.2. Ehrhart theory, reflexivity and integer decomposition properties of lattice polytopes

For a proper introduction to Ehrhart theory and related topics, we refer to the textbook [2]. A classical result by Ehrhart states that the number of lattice points in integer dilations of an n -dimensional lattice polytope $P \subseteq \mathbb{R}^n$ behaves polynomially. In terms of generating series, this translates into the equality

$$1 + \sum_{k \geq 1} |kP \cap \mathbb{Z}^n| z^k = \frac{h_n^* z^n + \dots + h_1^* z + h_0^*}{(1-z)^{n+1}},$$

where $h^*(z) := h_n^* z^n + \dots + h_1^* z + h_0^*$ is a polynomial of degree at most n with non-negative integer coefficients and $h_0^* = 1$. We call this polynomial the h^* -polynomial of P . This is an important invariant as it preserves much information about P . For example, the following relations are well known:

$$h_1^* = |P \cap \mathbb{Z}^n| - n - 1, \quad h_n^* = |P^\circ \cap \mathbb{Z}^n|, \quad 1 + \sum_{i=1}^n h_i^* = \text{Vol}(P),$$

where P° denotes the relative interior of P . The h^* -polynomial of P is often identified with the vector of its coefficient $(h_0^*, h_1^*, \dots, h_n^*)$, called the h^* -vector of P . We call a vector (x_0, x_1, \dots, x_n) *unimodal* if there exists a $1 \leq j \leq n$ such that $x_i \leq x_{i+1}$ for all $0 \leq i < j$ and $x_k \geq x_{k+1}$ for all $j \leq k < n$. An important open problem in Ehrhart theory is to understand under which conditions h^* -vectors are unimodal (see [3] for a survey).

The (*polar*) *dual polytope* P^* of a full-dimensional lattice polytope P which contains the origin in its interior is the rational polytope $P^* := \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \cdot \mathbf{y} \leq 1 \text{ for all } \mathbf{y} \in P\}$. If P is a lattice polytope, we call it *reflexive* if its dual P^* is again a lattice polytope. We extend the definition of reflexive to all the lattice polytopes which are unimodularly equivalent to P . Reflexive polytopes were first introduced in [1]. A well-known result of the second author [9] characterizes reflexive polytopes as lattice polytopes with one interior lattice point which have a *symmetric* h^* -vector, i.e. such that $h_i^* = h_{n-i}^*$ for $0 \leq i \leq \lfloor \frac{n}{2} \rfloor$.

We say that a lattice polytope P has the *integer decomposition property*, if, for every positive integer k and for all $\mathbf{x} \in kP \cap \mathbb{Z}^n$ there exist $\mathbf{x}_1, \dots, \mathbf{x}_k \in P \cap \mathbb{Z}^n$ such that $\mathbf{x} = \mathbf{x}_1 + \dots + \mathbf{x}_k$. A polytope having the integer decomposition property is often called *IDP*.

Many efforts have been made to find sufficient conditions for unimodality. It has been conjectured by Stanley [16] that a standard graded Cohen-Macaulay integral domain has a unimodal h -vector. In the context of lattice polytopes, this can be translated in the following question: does an IDP polytope always have a unimodal h^* -vector? A weaker statement of this question has also been suggested by Ohsugi and the second author [15], who conjectured that being reflexive and IDP is a sufficient condition for a lattice polytope to have a unimodal h^* -vector.

3.3. Connections with weighted projective spaces and $\Delta_{(1,q)}$ -simplices

We now relate Laplacian simplices to weighted projective spaces. Given $(x_1, \dots, x_n) \in \mathbb{Z}_{>0}^n$, we say that the sequence x_1, \dots, x_n is *well-formed* if, for any $i \in [n]$, $\text{gcd}\{x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n\} = 1$.

Proposition 8. *Let D be a strongly connected digraph such that the sequence c_1, \dots, c_n is well-formed. Then P_D is equivalent to the weighted projective space $\mathbb{P}(c_1, \dots, c_n)$.*

Proof. Let $M := \mathbb{Z}^n \cap \text{aff}(P_D)$ be the ambient lattice of P_D . We first prove that all the vertices of P_D are primitive in M . Suppose that there exists $j \in [n]$ such that \mathbf{v}_j can be written as $k\mathbf{u}$ with $\mathbf{u} \in M$ primitive and $k \in \mathbb{Z}_{>0}$. Then $k | \det(L_{i,i}) = c_i$ for any $i \in [n], i \neq j$. Since c_1, \dots, c_n is a well-formed sequence, we get $k = 1$.

Now we prove that the vertices of P_D span the lattice. Let L be the lattice spanned by all the vertices, and L_i the lattice spanned by all the vertices \mathbf{v}_j such that $j \neq i$. Then we have the following inclusions of subgroups of M : $L_i \subseteq L \subseteq \mathbb{Z}^n$. In particular for all i , $|M:L||L:L_i| = |M:L_i| = \det(L_{i,i}) = c_i$, which implies that $L = M$. \square

In [7], [11] characterizations for properties of weighted projective spaces are given in terms of their weights and are used to perform classifications. We use these results to translate properties of D to properties of P_D . Motivated by the open questions mentioned in the previous subsection, we focus on reflexivity, the integer decomposition property, and a description of the h^* -polynomial.

We use the following result of Conrads, presented below in a slightly weaker form.

Proposition 9. ([7, Proposition 5.1]) *Let $S = \text{conv}(\mathbf{v}_1, \dots, \mathbf{v}_n)$ be an $(n-1)$ -simplex such that $\sum_{i=1}^n q_i \mathbf{v}_i = \mathbf{0}$ for some positive integers q_1, \dots, q_n satisfying $\text{gcd}\{q_1, \dots, q_n\} = 1$. Then S is reflexive if and only if*

$$(2) \quad q_i \text{ divides the total weight } \sum_{j=1}^n q_j \text{ for } i = 1, \dots, n.$$

From this, we can derive the following corollary.

Corollary 10. *Let D be a strongly connected digraph such that $\text{gcd}\{c_1, \dots, c_n\} = 1$. Then P_D is reflexive if and only if c_i divides $c(D)$ for all i .*

Proposition 9 is also used by Braun–Davis–Solus [4] to define an interesting class of reflexive simplices. In particular, they are interested in studying the integer decomposition property and unimodality of the h^* -vectors of such simplices constructed the following way. Let $\mathbf{q} = (q_1, \dots, q_n)$ be a nondecreasing sequence of positive integers satisfying the condition $q_j | (1 + \sum_{i \neq j} q_i)$ for all $j \in [n]$. For such a vector \mathbf{q} , the simplex $\Delta_{(1, \mathbf{q})}$ is defined as

$$\Delta_{(1, \mathbf{q})} := \text{conv} \left\{ \mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n, - \sum_{i=1}^n q_i \mathbf{e}_i \right\},$$

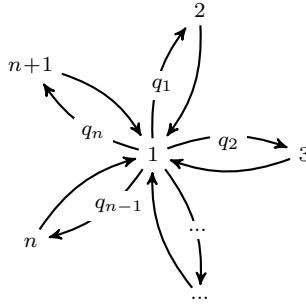


Figure 2. The star shaped digraph D such that $P_D = \mathbb{P}(1, q_1, \dots, q_n)$. The label on an edge from i to j represents the total number of edges from i to j .

where $\mathbf{e}_i \in \mathbb{R}^n$ is the i -th standard basis vector. By Proposition 9, $\Delta_{(1, \mathbf{q})}$ is a reflexive simplex. Note that the condition $q_j | (1 + \sum_{i \neq j} q_i)$ for all $j \in [n]$ implies that the sequence $1, q_1, \dots, q_n$ is well-formed, so $\Delta_{(1, \mathbf{q})}$ is equivalent to the weighted projective space with weights $(1, q_1, \dots, q_n)$.

The next proposition shows the simplices $\Delta_{(1, \mathbf{q})}$ are a subfamily of Laplacian simplices arising from special star-shaped, strongly connected digraphs.

Proposition 11. *Let $\mathbf{q} = (q_1, \dots, q_n)$ be any nondecreasing sequence of positive integers such that $\gcd\{q_1, \dots, q_n\} = 1$. Then there is a strongly connected digraph D such that P_D is unimodularly equivalent to $\mathbb{P}(1, q_1, \dots, q_n)$. In particular, if \mathbf{q} satisfies the condition $q_j | (1 + \sum_{i \neq j} q_i)$ for all $j = 1, \dots, n$, then P_D is unimodularly equivalent to $\Delta_{(1, \mathbf{q})}$.*

Proof. As in Figure 2, we define D as the star-shaped digraph on the vertices $1, \dots, n+1$ such that

- (1) for $i = 1, \dots, n$ there are q_i many edges directed from 1 to $i+1$;
- (2) for $i = 1, \dots, n$ there is one edge directed from $i+1$ to 1.

It is easy to verify that $c_1 = 1$ and, for $i \geq 2$, $c_i = q_{i-1}$. Proposition 8 concludes the proof. \square

In [4], an explicit formula for the h^* -polynomial of the simplices $\Delta_{(1, \mathbf{q})}$ is given. We remark such formula can be also extracted from [11], where it is proved in the more general setting of weighted projective spaces; however, the formulation given in [4] perfectly fits our needs.

Theorem 12. ([4, Theorem 2.5]) *The h^* -polynomial of $\Delta_{(1, \mathbf{q})}$ is*

$$h^*(z) = \sum_{b=0}^{q_1 + \dots + q_n} z^{w(b)}$$

where

$$w(b) := b - \sum_{i=1}^n \left\lfloor \frac{q_i b}{1 + q_1 + \dots + q_n} \right\rfloor.$$

Finally, in [4], necessary conditions for a $\Delta_{(1, \mathbf{q})}$ simplex to be IDP are given.

Lemma 13. ([4, Corollary 2.7]) *If $\Delta_{(1, \mathbf{q})}$ is IDP, then for all $j=1, 2, \dots, n$*

$$\frac{1}{q_j} + \sum_{i \neq j} \left\{ \frac{q_i}{q_j} \right\} = 1,$$

where $\left\{ \frac{q_i}{q_j} \right\}$ denotes the fractional part of $\frac{q_i}{q_j}$.

4. Laplacian simplices associated to cycle digraphs

We now want to extend the study of Braun–Meyer on simplices associated to cycle graphs. They show that the Laplacian simplex associated to a cycle is reflexive if and only if the cycle has odd length n ; in that case, it has a unimodal h^* -vector and fails to be IDP for $n \geq 5$ [5, Section 5]. We generalize their study by extending the notion of cycle graphs to cycle digraphs. A natural way to extend is to consider digraphs whose underlying simple graphs are cycle graphs. Here, by *underlying simple graph* G_D of a digraph D , we mean the simple undirected graph on the vertex set $V(G_D) := V(D)$ such that the edge $\{i, j\}$ is in $E(G_D)$ if and only if there is at least one directed edge between i and j in D (in either of the two directions). Since we are interested in reflexivity, we know by Corollary 6 that D has to be strongly connected; therefore, D needs to contain a cycle entirely oriented in one of the two possible directions. This generalization of cycle graphs will be made clear later (Definition 16). Moreover, in order to ensure the presence of no more than one interior point, we will assume for each couple of vertices i, j of D , there is at most one oriented edge from i to j .

4.1. Laplacian simplices associated to simple digraphs

In this section, we focus on *simple* digraphs, where by simple we mean there is at most one directed edge from i to j , for any pair of vertices $i, j \in [n]$, $i \neq j$. Note the presence of both a directed edge from i to j and one from j to i is allowed. As in the previous section, we restrict our attention to those digraphs having positive complexity. This case still generalizes the work of Braun–Meyer [5] (see Remark 2) and defines polytopes with at most one interior point. Indeed, we prove that all the Laplacian simplices of a simple digraph on n vertices are subpolytopes of P_{K_n} ,

the Laplacian simplex associated to the complete simple digraph. Observe P_{K_n} is equivalent to the n -th dilation of an $(n-1)$ -dimensional unimodular simplex, and therefore it has exactly one interior lattice point.

Proposition 14. *Let D be a simple digraph on n vertices. Then P_D is a subpolytope of P_{K_n} . In particular, if D is strongly connected, then P_D has exactly one interior lattice point.*

Proof. Corollary 6 implies P_D has at least one interior lattice point, so the second statement follows directly from the first one. In order to prove the first part, we show that any vertex \mathbf{u} of P_D is in P_{K_n} . Up to a relabeling of the vertices, we can assume that $\mathbf{u}=(a, -1, \dots, -1, 0, \dots, 0)$, where a equals the number entries of u which are equal to -1 . We know that the Laplacian $L(K_n)$ is

$$L(K_n) = \begin{bmatrix} n-1 & -1 & \dots & -1 \\ -1 & n-1 & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \dots & n-1 \end{bmatrix}.$$

We denote by \mathbf{v}_i the i -th row of $L(K_n)$, as well as the corresponding vertex of P_{K_n} . It is then enough to prove that \mathbf{u} can be written as a convex combination of the vertices of K_n , i.e. that $\mathbf{u}=\sum_{i=0}^n \lambda_i \mathbf{v}_i$, with $0 \leq \lambda_i \leq 1$ and $\sum_{i=0}^n \lambda_i = 1$. This can be done with the following choice of barycentric coordinates:

$$\lambda_i = \begin{cases} \frac{a+1}{n}, & \text{if } i=1 \\ 0, & \text{if } 2 \leq i \leq a+1 \\ \frac{1}{n}, & \text{if } a+2 \leq i \leq n \end{cases}.$$

This proves P_D is a subpolytope of P_{K_n} . \square

4.2. Lattice simplices associated to generalized cycles

In [5], the authors study the Laplacian simplex associated to the undirected cycle graph C_n , proving the following result.

Theorem 15. ([5, Theorem 5.1]) *For $n \geq 3$, the simplex T_{C_n} is reflexive if and only if n is odd.*

The rest of this section is aimed to generalize their result to the case of directed cycles. Note that in order to have reflexivity (or, in particular, to have one interior lattice point) we need the digraph to be strongly connected (Corollary 6). Therefore, all cycles we consider will always contain a cycle entirely oriented in one of the

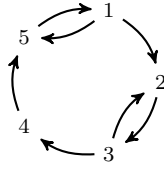


Figure 3. The cycle digraph $C_5^{\{1,3\}}$.

two possible directions and some additional edges directed in the opposite direction. Informally speaking, we define a cycle digraph to have all the edges pointing clockwise and some edges pointing counterclockwise.

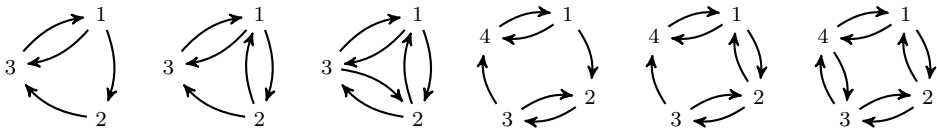
Definition 16. Let $n \geq 3$. We say that a digraph D on the vertex set $[n]$ is a *cycle digraph* if, up to a relabeling of the vertices, $E(D) = \vec{E}(D) \cup \overleftarrow{E}(D)$, where

$$\begin{aligned} \vec{E}(D) &= \{(1, 2), (2, 3), \dots, (n-1, n), (n, 1)\}, \\ \overleftarrow{E}(D) &\subseteq \{(n, n-1), (n-1, n-2), \dots, (2, 1), (1, n)\}. \end{aligned}$$

If such a relabeling exists, D is completely determined by $\overleftarrow{E}(D)$, and we denote it by $D = C_n^S$, where $S \subseteq [n]$ is the set of the tails of the directed edges in $\overleftarrow{E}(D)$. As an example, see Figure 3.

We first prove for most of the directed cycles, the associated Laplacian simplex has no lattice points other than its vertices and the origin. Borrowing some terminology from the algebraic geometers, we call a simplex with this property *terminal Fano*.

Theorem 17. *Let D be any cycle digraph. P_D is terminal Fano if and only if D is not, up to a relabeling of the vertices, one of the following six exceptional directed cycles.*



Proof. We prove that, for $n \geq 5$, $P_{C_n^S}$ is terminal Fano for all $S \subseteq [n]$. The lower dimensional cases are checked individually, leading to the six exceptional cases above. For each $i \in [n]$, we have $\mathbf{v}_i = a_{i-1} \mathbf{e}_{i-1} + b_i \mathbf{e}_i - \mathbf{e}_{i+1}$ where for each

$j \in [n]$ $a_j \in \{-1, 0\}$ and $b_j = 1 - a_j \in \{1, 2\}$, and $a_0 = a_n$, $e_0 = e_n$ and $e_{n+1} = e_1$. Assume that $P_{C_n^S}$ is not terminal Fano. Let $\mathbf{x} = (x_1, \dots, x_n)$ be a lattice point in $P_{C_n^S} \setminus \{\mathbf{v}_1, \dots, \mathbf{v}_n, \mathbf{0}\}$ and set $\mathbf{x} = \sum_{i=1}^n \lambda_i \mathbf{v}_i$ with $0 \leq \lambda_1, \dots, \lambda_n < 1$ and $\lambda_1 + \dots + \lambda_n = 1$. Then one has $x_i = -\lambda_{i-1} + b_i \lambda_i + a_i \lambda_{i+1} \in \{-1, 0, 1\}$ for each i , where $\lambda_0 = \lambda_n$ and $\lambda_{n+1} = \lambda_1$. Suppose that there exists $i \in [n]$ such that $x_i = -1$. We can assume without loss of generality that $x_2 = -1$. Then we obtain $a_2 = -1$, $0 < \lambda_1, \lambda_3 < 1$ and $\lambda_j = 0$ for any $j \neq 1, 3$. This implies that $x_4 = -\lambda_3 + b_4 \lambda_4 + a_4 \lambda_5 = -\lambda_3 \notin \mathbb{Z}$, a contradiction. Hence we have $x_i \in \{0, 1\}$ for each i . Since $\mathbf{x} \neq \mathbf{0}$, we can assume without loss of generality that $x_2 = 1$. Then one has $b_2 = 2$ and $\lambda_2 \geq 1/2$. If $b_3 = 1$, it follows that $x_3 = 0$, $\lambda_2 = \lambda_3 = 1/2$ and $\lambda_j = 0$ for any $j \neq 2, 3$. This implies that $x_4 = -\lambda_3 + b_4 \lambda_4 + a_4 \lambda_5 = -\lambda_3 \notin \mathbb{Z}$, a contradiction. Hence one has $b_3 = 2$. If $x_3 = 1$, then $\lambda_3 \geq 1/2$, hence one has $\lambda_2 = \lambda_3 = 1/2$ and $\lambda_j = 0$ for any $j \neq 2, 3$. However, we obtain $x_4 = -\lambda_3 + b_4 \lambda_4 + a_4 \lambda_5 = -\lambda_3 \notin \mathbb{Z}$, a contradiction. Hence $x_3 = 0$. If $b_1 = 1$, then one has $\lambda_0 = \lambda_1 = 0$. Since $2\lambda_2 - \lambda_3 = 1$ and $-\lambda_2 + 2\lambda_3 - \lambda_4 = 0$, it follows that $3\lambda_2 = \lambda_4 + 2 \geq 2$. Hence one has $\lambda_2 = 2/3$, $\lambda_3 = 1/3$ and $\lambda_j = 0$ for any $j \neq 2, 3$. However, we obtain $x_4 = -\lambda_3 + b_4 \lambda_4 + a_4 \lambda_5 = -\lambda_3 \notin \mathbb{Z}$, a contradiction. Thus, $b_1 = 2$. Then it follows from $\lambda_2 \geq 1/2$ that $\lambda_2 = 1/2$, $\lambda_1 = \lambda_3 = 1/4$ and $\lambda_j = 0$ for $j \in [n] \setminus \{1, 2, 3\}$. This implies that $x_4 = -\lambda_3 + b_4 \lambda_4 + a_4 \lambda_5 = -\lambda_3 \notin \mathbb{Z}$, a contradiction. Therefore, $P_{C_n^S}$ is terminal Fano. \square

Now we characterize reflexivity for Laplacian simplices $P_{C_n^S}$, extending Theorem 15 by Braun–Meyer.

Theorem 18. *The Laplacian simplex $P_{C_n^S}$ associated to a cycle digraph C_n^S is reflexive if and only if one of the following conditions is satisfied:*

- (1) $S = \emptyset$, or
- (2) $S = [n]$ and $n = 2$, or
- (3) $S = [n]$ and n is odd, or
- (4) $\emptyset \subsetneq S \subsetneq [n]$, such that $k|c(D)$ for each integer $1 \leq k \leq K + 1$, where K is the longest chain of consecutive edges pointing counterclockwise, i.e.

$$K := \max\{j \mid \{a + 1, \dots, a + j\} \subseteq S, \text{ for some } a \in [n]\},$$

where, since $S \subsetneq [n]$, we have assumed without loss of generality, that $1 \notin S$.

Proof. If S satisfies (1) or (2), then thanks to Corollary 10, it is trivial to check that $P_{C_n^S}$ is reflexive. If S satisfies (3), then $P_{C_n^S}$ is reflexive by Theorem 15. Suppose now that S satisfies (4). In particular, we have assumed that $1 \notin S$. This implies vertex n has exactly one spanning converging tree, i.e. $c_n = 1$. As usual, c_i denotes the number of spanning trees which converge to vertex i . Then $\gcd\{c_1, \dots, c_n\} = 1$, and $P_{C_n^S}$ is a weighted projective space by Proposition 8. For each vertex i ,

we denote by K_i the length of the longest chain of consecutive edges pointing counterclockwise ending in i , i.e.

$$K_i := \max\{j \mid \{i+1, \dots, i+j\} \subseteq S\}, \quad \text{for } 1 \leq i \leq n.$$

In particular, $K_n=0$ and $K=\max\{K_i \mid i \in [n]\}$. Given $i \in [n]$, note there are exactly K_i+1 spanning trees converging to i . There are K_i having edge set

$$\{(j, j-1), \dots, (i+1, i), (j+1, j+2), \dots, (n-1, n), (n, 1), \dots, (i-1, i)\},$$

for all $j \in \{i+1, \dots, i+K_i\}$, plus an additional ‘‘clockwise tree’’ with edges

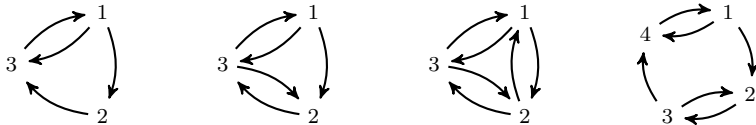
$$\{(i+1, i+2), \dots, (n-1, n), (n, 1), \dots, (i-1, i)\}.$$

By Corollary 10, $P_{C_n^S}$ is reflexive if and only if $c_i \mid c(D)$, for all $i \in [n]$. We conclude by noting that if $c_i > 1$ for some $i \in [n]$, then $c_{i+1} = c_i - 1$, in particular $\{c_i \mid i \in [n]\} = \{1, \dots, K+1\}$. \square

We now have all the tools to completely characterize all reflexive IDP simplices arising from cycle digraphs.

Theorem 19. *Let C_n^S be a cycle digraph on n vertices such that $P_{C_n^S}$ is reflexive. Then $P_{C_n^S}$ possesses the integer decomposition property if and only if D satisfies one of the following conditions:*

- (1) $S = \emptyset$, or
- (2) D is, up to a relabeling of the vertices, one of the following directed cycles.



Proof. If $S = \emptyset$, then C_n^S is known to be a reflexive IDP simplex. If $S = [n]$, from Theorem 15 $P_{C_n^{[n]}}$ is reflexive if and only if n is odd. In this case, it is known $P_{C_n^{[n]}}$ is IDP if and only if $n=3$ [5, Corollary 5.11]. Now, assume that $\emptyset \neq S \neq [n]$ and $P_{C_n^S}$ is IDP. We use the same notation introduced in Theorem 18. Then we can assume $c_1=1, c_2=K+1, c_3=K, \dots, c_{K+1}=2$. Set $\mathbf{q}=(c_2, \dots, c_n)$. It follows that $P_{C_n^S}$ is unimodularly equivalent to $\Delta_{(1,\mathbf{q})}$. By Lemma 13, we know that for each $2 \leq j \leq n$,

$$(3) \quad \frac{1}{c_j} + \sum_{i \neq j} \left\{ \frac{c_i}{c_j} \right\} = 1.$$

But, if $K \geq 3$, by (3) we get

$$\frac{1}{K+1} + \sum_{i=3}^n \left\{ \frac{c_i}{K+1} \right\} \geq \frac{1}{K+1} + \frac{K-1}{K+1} + \frac{K}{K+1} > 1,$$

so $K \in \{1, 2\}$. By applying (3) in these cases, one gets $n \leq 4$. We conclude by checking all the cycle digraphs having up to four vertices. \square

As an application of the tools developed in this section, we build a special family of cycle digraphs whose Laplacian simplices are reflexive and have non unimodal h^* -vectors.

Theorem 20. *Let $\alpha, \beta, k \in \mathbb{Z}_{>0}$ such that $\alpha \leq \beta \leq k-1$ and $\alpha + \beta \leq k+1$. Let $D = C_n^S$ be the cycle digraph with $n := 6(k+1) - 2\alpha - \beta$, and $S := S_1 \cup S_2 \cup S_3$ where*

$$\begin{aligned} S_1 &:= \{1+3h \mid 0 \leq h \leq \alpha-1\}, \\ S_2 &:= \{2+3h \mid 0 \leq h \leq \alpha-1\}, \\ S_3 &:= \{3\alpha+1+2h \mid 0 \leq h \leq \beta-\alpha-1\}. \end{aligned}$$

Then P_D is a reflexive simplex of dimension $6(k+1) - 2\alpha - \beta - 1$ with symmetric and nonunimodal h^* -vector

$$\left(\underbrace{1, \dots, 1}_{2(k+1)-\alpha}, \underbrace{2, \dots, 2}_\alpha, \underbrace{1, \dots, 1}_{(k+1)-\alpha-\beta}, \underbrace{2, \dots, 2}_\beta, \underbrace{1, \dots, 1}_{(k+1)-\alpha-\beta}, \underbrace{2, \dots, 2}_\alpha, \underbrace{1, \dots, 1}_{2(k+1)-\alpha} \right).$$

Proof. An example of the digraph in the statement is represented in Figure 4. The digraph has no more than two consecutive vertices with outdegree two, so the number of spanning trees converging to each of the vertices of D is at most three. Specifically,

$$c_{i-1} = \begin{cases} 3, & \text{if } i \in S_1, \\ 2, & \text{if } i \in S_2 \cup S_3, \\ 1, & \text{if } i \in [n] \setminus S. \end{cases}$$

Above, we set c_0 to be c_n . Since each c_i divides $c(D) = \sum_{i=1}^n c_i = 6(k+1)$, then P_D is reflexive by Theorem 18. Now we use Theorem 12 to describe its h^* -polynomial. In particular,

$$h^*(z) = \sum_{b=0}^{c(D)-1} z^{w(b)}, \quad \text{with } w(b) = b - \sum_{i=1}^n \left\lfloor \frac{c_i b}{6(k+1)} \right\rfloor.$$

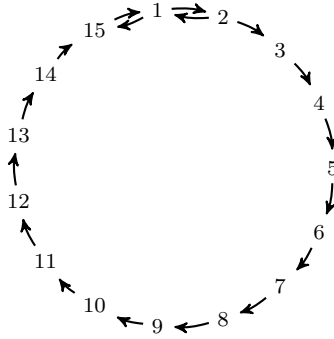


Figure 4. An example of the construction of Theorem 20. In this case $\alpha=\beta=1$ and $k=2$. The Laplacian simplex associated to this digraph has h^* -vector $(1, 1, 1, 1, 1, 2, 1, 2, 1, 2, 1, 1, 1, 1, 1)$.

In our case, this becomes

$$w(b) = b - \alpha \left\lfloor \frac{b}{2(k+1)} \right\rfloor - \beta \left\lfloor \frac{b}{3(k+1)} \right\rfloor,$$

which yields

$$w(b) = \begin{cases} b, & \text{if } 0 \leq b \leq 2(k+1) - 1, \\ b - \alpha, & \text{if } 2(k+1) \leq b \leq 3(k+1) - 1, \\ b - \alpha - \beta, & \text{if } 3(k+1) \leq b \leq 4(k+1) - 1, \\ b - 2\alpha - \beta, & \text{if } 4(k+1) \leq b \leq 6(k+1) - 1. \end{cases}$$

From this, using the condition $\alpha + \beta \leq k + 1$, we deduce the i -th coefficient of the h^* -polynomial:

$$h_i^* = \begin{cases} 2, & \text{if } \begin{cases} 2(k+1) - \alpha \leq i \leq 2(k+1) - 1, \text{ or} \\ 3(k+1) - \alpha - \beta \leq i \leq 3(k+1) - \alpha - 1, \text{ or} \\ 4(k+1) - 2\alpha - \beta \leq i \leq 4(k+1) - \alpha - \beta - 1; \end{cases} \\ 1, & \text{otherwise.} \end{cases} \quad \square$$

5. Further questions

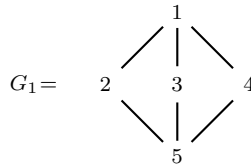
Note that in the case of undirected cycles studied by Braun-Meyer [5], the reflexivity is influenced by the number of vertices of the graph (Theorem 15). On the other hand, when passing to the directed case we discussed in Section 4, it is clear (from Theorem 18) that one can build reflexive Laplacian simplices starting

from cycles of any length. This can be done by orienting a cycle in one of the two directions.

It is natural to wonder how the structure of the underlying simple graph G_D of a digraph D plays a role in determining the reflexivity of P_D . We define an *oriented graph* to be a simple digraph D such that if there is an edge pointing from i to j , then there is no edge pointing from j to i .

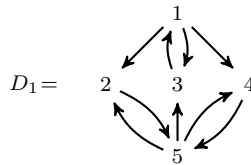
The following examples show that obtaining reflexive Laplacian simplices from digraphs with a fixed underlying simple graph is not an easy task. Example 21 shows there is a simple graph G_1 such that any of its orientations is a digraph whose Laplacian simplex is not a full-dimensional reflexive simplex. However, if we do not require the digraph to be an oriented graph, there is a simple digraph D_1 (Example 22) having G_1 as its underlying graph such that P_{D_1} is a full-dimensional reflexive simplex. On the other hand, in Example 23 we show there is a graph G_2 which is not the underlying graph of any simple digraph whose Laplacian simplex is reflexive. However, if we do not require the digraph to be simple, then there is a digraph D_2 (Example 24) having G_2 as its underlying graph such that P_{D_2} is reflexive.

Example 21. Let G be the following graph.



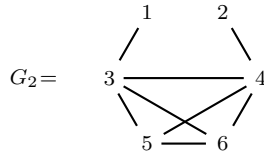
Assume D is an orientation of G_1 such that P_D is a reflexive 4-simplex. Since D must be strongly connected, we may assume, without loss of generality, that $(5, 3), (3, 1), (1, 2), (2, 5)$ are edges of D . It follows that either $(1, 4), (4, 5)$ or $(5, 4), (4, 1)$ are in $E(D_1)$. In both cases, P_D is not reflexive. So none of the orientations of G_1 lead to a reflexive simplex.

Example 22. Let D_1 be the following simple digraph.



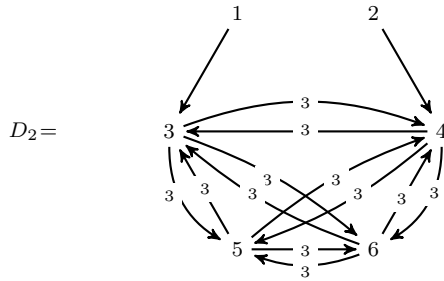
Note that its underlying simple graph is still G_1 of Example 21, but P_{D_1} is reflexive.

Example 23. Let G_2 be the following graph.



Note that there are finitely many possible directed simple graphs having G_2 as an underlying graph. A computer-assisted check shows none of them produces a reflexive Laplacian simplex.

Example 24. Let D_2 be the following digraph (the label on an edge from i to j , if present, represents the total number of edges from i to j).



Then P_{D_2} is a reflexive simplex.

In general, it is still unclear how the underlying graph affects the reflexivity of the Laplacian simplex of a digraph. Examples 21 and 23 show that this is a nontrivial question. We conclude with the following three open questions.

Question A. For which simple graphs G on $[n]$, does there exist an oriented graph D on $[n]$ such that $G_D=G$ and P_D is a reflexive $(n-1)$ -simplex?

Question B. For which simple graphs G on $[n]$, does there exist a simple digraph D on $[n]$ such that $G_D=G$ and P_D is a reflexive $(n-1)$ -simplex?

Question C. For any simple graph G on $[n]$, does there exist a digraph D on $[n]$ such that $G_D=G$ and P_D is a reflexive $(n-1)$ -simplex?

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