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A reverse quasiconformal composition problem for $Q_\alpha(\mathbb{R}^n)$

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Abstract. We give a partial converse to [8, Theorem 1.3] (as a resolution of [2, Problem 8.4] for the quasiconformal Q-composition) for $Q_{0<\alpha<2^{-1}}(\mathbb{R}^{n\geq 2})$, and yet demonstrate that if $f:\mathbb{R}^2\to\mathbb{R}^2$ is a homeomorphism then the boundedness of $u\mapsto u\circ f$ on $Q_{2^{-1}<\alpha<1}(\mathbb{R}^2)\subset BMO(\mathbb{R}^2)$ yields the quasiconformality of f.

1. Introduction

Recall that $Q_{-\infty<\alpha<\infty}(\mathbb{R}^n)$ is the quite-well-known Essén-Janson-Peng-Xiao's space of all measurable functions u on $\mathbb{R}^{n\geq 1}$ with

$$\|u\|_{Q_\alpha(\mathbb{R}^n)} = \sup_{(x_0,r)\in\mathbb{R}^n\times(0,\infty)} \left(r^{2\alpha-n} \int_{|y-x_0|<r} \int_{|x-x_0|<r} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}}\, dx\, dy \right)^{\frac{1}{2}} < \infty.$$

In particular (cf. [2], [5]),

$$Q_{0\leq\alpha<\infty}(\mathbb{R}^n) \subset Q_{-\infty<\alpha<0}(\mathbb{R}^n) = Q_{-\frac{n}{2}}(\mathbb{R}^n) = BMO(\mathbb{R}^n).$$

As a resolution of [2, Problem 8.4] – *Let f be a quasiconformal self-map of $\mathbb{R}^n$. Prove or disprove that $u\mapsto \mathbf{C}_f u = u\circ f$ is bounded on $Q_{0<\alpha<1}(\mathbb{R}^{n\geq 2})$* (which however has an affirmative solution for $BMO(\mathbb{R}^n)$ as proved in [9, Theorem 2] – namely – $\mathbf{C}_f$ is bounded on $BMO(\mathbb{R}^n)$ whenever f is a quasiconformal self-map of $\mathbb{R}^n$), we have

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Theorem 1.1. [8, *Theorem* 1.3] *For* $n-1\in\mathbb{N}$ *let* $f:\mathbb{R}^n\to\mathbb{R}^n$ *be quasiconformal. If there exists a closed set* $E\subseteq\mathbb{R}^n$ *such that*

$\triangleright$ J_f, *the Jacobian determinant of* f, *belongs to the* E*-based Muckenhoupt class* $A_1(\mathbb{R}^n;\,E)$;

$\triangleright$ $\overline{\dim}_L\,E$ (*under* E *being bounded*) *or* $\overline{\dim}_{LG}\,E$ (*under* E *being unbounded*), *the local or global self-similar Minkowski dimension of* E (*bounded or unbounded*), *lies in* $[0,n-2]$, *i.e.*,

$$[0,n-2]\ni\begin{cases}\overline{\dim}_L\,E & \text{as } E \text{ is bounded;}\\ \overline{\dim}_{LG}\,E & \text{as } E \text{ is unbounded,}\end{cases}$$

then $\mathbf{C}_f$ *is bounded on* $Q_{0<\alpha<1}(\mathbb{R}^n)$.

As a partial converse to Theorem 1.1, we here show

Theorem 1.2. *For* $n-1\in\mathbb{N}$ *let* $f:\mathbb{R}^n\to\mathbb{R}^n$ *be a homeomorphism. If*

$\triangleright$ $\mathbf{C}_f$ *and* $\mathbf{C}_{f^{-1}}$ *are bijective and bounded on* $Q_{0<\alpha<2^{-1}}(\mathbb{R}^n)$ *respectively;*

$\triangleright$ f *is not only ACL* (*absolutely continuous on almost all lines parallel to coordinates of* $\mathbb{R}^n$) *but also differentiable almost everywhere on* $\mathbb{R}^n$,

then f *is quasiconformal.*

Remark 1.3. Below are two comments on Theorem 1.2.

(i) Under the above assumptions on f, we have that f^{-1} is absolutely continuous with respect to the n-dimensional Lebesgue measure. Indeed, let f^{-1} map a set N of the n-dimensional Lebesgue measure 0 to a set $O=f^{-1}(N)$. If χ_N and χ_O stand for the indicators of N and O respectively, then $k\chi_O, k\chi_N\in Q_{0<\alpha<2^{-1}}(\mathbb{R}^n)$ for any $k\in\mathbb{N}$, but $k\chi_N=0$ in $Q_{0<\alpha<2^{-1}}(\mathbb{R}^n)$, and hence from the first $\triangleright$-hypothesis in Theorem 1.2 it follows that $k\chi_O=0$ in $Q_{0<\alpha<2^{-1}}(\mathbb{R}^n)$ and so $O=f^{-1}(N)$ is of the n-dimensional Lebesgue measure 0.

(ii) In accordance with [9, Theorem 3] (cf. [1, Theorem] & [3, Theorem 3.1] for some generalizations), we have that if the first requirement on $\mathbf{C}_f$ & $\mathbf{C}_{f^{-1}}$ in Theorem 1.2 is replaced by the condition that f^{-1} is absolutely continuous and the second requirement on f is kept the same then the boundedness of $\mathbf{C}_f$ on $BMO(\mathbb{R}^n)$ derives that f is a quasiconformal self-map of $\mathbb{R}^n$. Accordingly, this $BMO(\mathbb{R}^n)$-result can be naturally strengthened via Theorem 1.2 thanks to $Q_{0<\alpha<2^{-1}}(\mathbb{R}^n)\subset BMO(\mathbb{R}^n)$.

In addition, while focusing on the planar situation of Theorem 1.1 and observing that the Jacobian determinant of any quasiconformal self-map of $\mathbb{R}^{n\geq 2}$ is an A_∞-weight (cf. [4, Theorem 15.32]) we readily discover

Theorem 1.4. [8, *Theorem* 1.3: $n=2\,\&\,E=\varnothing$] *Let* $f:\mathbb{R}^2\to\mathbb{R}^2$ *be quasiconformal. If* J_f *is an* A_1*-weight on* $\mathbb{R}^2$, *i.e.*, $J_f\in A_1(\mathbb{R}^2;\varnothing)$, *then* $\mathbf{C}_f$ *is bounded on* $Q_{0<\alpha<1}(\mathbb{R}^2)$.

On the basis of the planar cases of Theorem 1.2 and Remark 1.3(ii), a partial converse to Theorem 1.4 (under $2^{-1}<\alpha<1$) is naturally given by

Theorem 1.5. *Let $f:\mathbb{R}^2\to\mathbb{R}^2$ be a homeomorphism. If $\mathbf{C}_f$ is bounded on $Q_{2^{-1}<\alpha<1}(\mathbb{R}^2)$, then f is quasiconformal.*

Remark 1.6. Let $n\geq 2$. Recall that if a homeomorphism of $\mathbb{R}^n$ preserves either the Sobolev space $W^{1,n}(\mathbb{R}^n)$ or the Triebel-Lizorkin space $\dot{F}^s_{n/s,q}(\mathbb{R}^n)$ with $s\in(0,1)$ & $q\in[1,\infty)$, it must be quasiconformal. But any homeomorphism preserving the Besov space $\dot{B}^s_{n/s,q}(\mathbb{R}^n)$ with $s\in(0,1)$ & $q\in[1,\infty)\setminus\{n/s\}$ or $s\in(0,1)$ & $q=n/s$ must be bi-Lipschitz or quasiconformal; see also [6], [7] and the references therein. By Reimann's paper [9], a homeomorphism of $\mathbb{R}^n$ preserving the John-Nirenberg space $BMO(\mathbb{R}^n)$ and satisfying the assumptions of Theorem 1.2 must be quasiconformal.

The rest of this paper is organized as follows: §2 is employed to prove Theorem 1.2 in terms of Lemmas 2.1–2.2 & 2.4 & 2.6 as well as Corollaries 2.3 & 2.5 producing a suitable $Q_\alpha(\mathbb{R}^n)$-function. More precisely, we borrow some of Reimann's ideas from [9] to prove Theorem 1.2, namely, prove that

$$\sup_{y\in\mathbb{R}^n\ \&\ |y|=1}\left|\left(Df^{-1}(x)\right)y\right|^n\lesssim J_{f^{-1}}(x)$$

holds for almost all $x\in\mathbb{R}^n$, where Df^{-1} and $J_{f^{-1}}$ are the formal derivative and Jacobian determinant of f^{-1} (cf. [4, Chapters 14-15]) – equivalently – we show that the maximal eigenvalue λ_1 of $Df^{-1}(x)$ is bounded by the minimal eigenvalue λ_n of $Df^{-1}(x)$ – in fact – by comparing the norms of suitable scalings of some special $Q_\alpha(\mathbb{R}^n)$-functions $u_\star$ (cf. Corollary 2.5 & Lemma 2.6) and their compositions with f, we can obtain the desired inequality $\lambda_1\lesssim\lambda_n$. §3 is designed to demonstrate Theorem 1.5 through a $Q_\alpha(\mathbb{R}^n)$-capacity estimate given in Lemma 3.1 and a technique for reducing the space dimension shown in Lemma 2.1.

Notation In the above and below, $X\lesssim Y$ stands for $X\leq\varkappa Y$ with a constant $\varkappa>0$.

2. Validation of Theorem 1.2

In order to prove the validity of Theorem 1.2, we need four lemmas and two corollaries.

Lemma 2.1. *Let $(\alpha,n,m)\in\mathbb{R}\times\mathbb{N}\times\mathbb{N}$ and $u:\mathbb{R}^n\to\mathbb{R}$. Then $u\in Q_\alpha(\mathbb{R}^n)$ if and only if $\mathbb{R}^n\times\mathbb{R}^m\ni(x,y)\mapsto U(x,y)=u(x)$ belongs to $Q_\alpha(\mathbb{R}^{n+m})$.*

Proof. This follows immediately from [2, Theorem 2.6] and its demonstration. □

Lemma 2.2. *Let* $(\alpha,n)\in\big[0,\min\{1,2^{-1}n\}\big)\times\mathbb{N}$. *Then* $x\mapsto\ln|x|$ *is in* $Q_\alpha(\mathbb{R}^n)$.

Proof. For any Euclidean ball $B=B(x_0,r)$ with centre $x_0\in\mathbb{R}^n$ and radius $r\in(0,\infty)$ and a measurable function u on $\mathbb{R}^n$ let

$$\Phi_\alpha(u,B)=r^{2\alpha-n}\int_B\int_B\frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}}\,dx\,dy.$$

So, it suffices to verify that if $u_{\ln}(x)=\ln|x|$ then $\Phi_\alpha(u_{\ln},B)\lesssim 1$.

- *Case* $|x_0|>2r$. Note that there is $\theta\in(0,1)$ obeying

$$\begin{aligned}x,y\in B&\Longrightarrow r<|x|,|y|\le 3r\\&\Longrightarrow\big|\ln|x|-\ln|y|\big|=\frac{\big||x|-|y|\big|}{(1-\theta)|x|+\theta|y|}\le\frac{|x-y|}{r}.\end{aligned}$$

So

$$\begin{aligned}\Phi_\alpha(u_{\ln},B)&=r^{2\alpha-n-2}\int_B\int_B|x-y|^{2-n-2\alpha}\,dx\,dy\\&\le r^{2\alpha-n-2}\int_B\int_{B(x,2r)}|x-y|^{2-n-2\alpha}\,dy\,dx\\&\lesssim r^{2\alpha-2}\int_0^r t^{1-2\alpha}\,dt\\&\lesssim 1,\end{aligned}$$

as desired.

- *Case* $|x_0|\le 2r$. Since $B(x_0,r)\subseteq B(0,3r)$ – the origin-centered ball with radius $3r$, we only need to estimate $\Phi_\alpha(u_{\ln},B)$ for $B=B(0,r)$.

Firstly, write

$$\begin{cases}\Phi_\alpha(u_{\ln},B)=I_1+I_2+I_3;\\ I_1=r^{2\alpha-n}\int_B\int_{B(x,2^{-1}|x|)}\frac{|\ln|x|-\ln|y||^2}{|x-y|^{n+2\alpha}}\,dy\,dx\\ I_2=r^{2\alpha-n}\int_B\int_{B\setminus B(x,4|x|)}\frac{|\ln|x|-\ln|y||^2}{|x-y|^{n+2\alpha}}\,dy\,dx;\\ I_3=r^{2\alpha-n}\int_B\int_{B(x,4|x|)\setminus B(x,2^{-1}|x|)}\frac{|\ln|x|-\ln|y||^2}{|x-y|^{n+2\alpha}}\,dy\,dx.\end{cases}$$

Since

$$|x-y|\le 2^{-1}|x|\Longrightarrow|\ln|x|-\ln|y||\le 2|x-y||x|^{-1},$$

one has

$$I_1\lesssim r^{2\alpha-n}\int_B|x|^{-2}\int_{B(x,2^{-1}|x|)}|x-y|^{2-n-2\alpha}\,dy\,dx\lesssim 1.$$

Secondly, write

$$\int_{B\setminus B(x,4|x|)} \frac{|\ln|x|-\ln|y||^2}{|x-y|^{n+2\alpha}}\,dy \le \sum_{j\ge 3}\int_{B(x,2^j|x|)\setminus B(x,2^{j-1}|x|)} \frac{|\ln|x|-\ln|y||^2}{|x-y|^{n+2\alpha}}\,dy.$$

Observe that if $j-2\in\mathbb{N}$ then

$$2^{j-1}|x|\le|x-y|\le 2^j|x| \Longrightarrow 2^{j-2}|x|\le|y|\le 2^{j+1}|x|$$
$$\Longrightarrow \int_{B(x,\ 2^j|x|)\setminus B(x,\ 2^{j-1}|x|)} \frac{|\ln|x|-\ln|y||^2}{|x-y|^{n+2\alpha}}\,dy \lesssim \frac{2^{j(2-2\alpha)}}{|x|^{2\alpha}}.$$

Thus

$$I_2\lesssim r^{2\alpha-n}\int_B |x|^{-2\alpha}\sum_{j=3}^{\infty}(...)\,dy\lesssim r^{2\alpha-n}\int_B |x|^{-2\alpha}\,dx\lesssim 1.$$

Thirdly, note that

$$y\in B(x,\ 4|x|)\setminus B(x,\ 2^{-1}|x|)\Longrightarrow |y|\le 5|x|.$$

So

$$\begin{aligned} I_3 &\lesssim r^{2\alpha-n}\int_B\int_{B(x,\ 4|x|)\setminus B(x,\ 2^{-1}|x|)} \frac{|\ln|x|-\ln|y||^2}{|x-y|^{n+2\alpha}}\,dy\,dx\\ &\lesssim r^{2\alpha-n}\int_B |x|^{-(n+2\alpha)}\int_{B(0,\ 5|x|)}\left(\ln\frac{|x|}{|y|}\right)^2 dy\,dx\\ &\lesssim r^{2\alpha-n}\int_B |x|^{-(n+2\alpha)}\sum_{i=1}^{\infty}(2^{-i}5|x|)^n i^2\,dx\\ &\lesssim r^{2\alpha-n}\int_B |x|^{-2\alpha}\,dx\\ &\lesssim 1. \quad \square \end{aligned}$$

Corollary 2.3. *Let* $(n-1,c)\in\mathbb{N}\times\mathbb{R}$. *Then*
(i)

$$x=(x_1,x_2,...,x_n)\longmapsto \max\left\{c,\ln(x_1^{-2})\right\}$$

is in $Q_{0\le\alpha<2^{-1}}(\mathbb{R}^n)$.
(ii)

$$x=(x_1,x_2,...,x_n)\longmapsto \max\left\{c,\ln(x_1^2+x_2^2)^{-1}\right\}$$

is in $Q_{0\le\alpha<1}(\mathbb{R}^n)$.

Proof. This follows from

$$\max\{u,v\}=2^{-1}\big(u+v+|u-v|\big)=u+\max\{v-u,0\},$$

the basic fact that $Q_\alpha(\mathbb{R}^n)$ is a linear space with

$$w\in Q_\alpha(\mathbb{R}^n)\Longrightarrow |w|\in Q_\alpha(\mathbb{R}^n),$$

and Lemmas 2.1–2.2. $\square$

Lemma 2.4. *Let* $(\alpha,n-1)\in(0,1)\times\mathbb{N}$. *If*

$$\begin{cases}|||u|||_{Q_\alpha}=\|u\|_{Q_\alpha(\mathbb{R}^n)}+\sup_{(x_0,r)\in\mathbb{R}^n\times[1,\infty)}\Big(r^{2\alpha-n}\int_{B(x_0,r)}|u(x)|^2\,dx\Big)^{2^{-1}}<\infty;\\ \|g\|_{\infty,Lip}=\|g\|_{L^\infty(\mathbb{R})}+\sup_{z_1,z_2\in\mathbb{R},\,z_1\neq z_2}|g(z_1)-g(z_2)||z_1-z_2|^{-1}<\infty,\end{cases}$$

then $\mathbb{R}^n\times\mathbb{R}\ni(x,z)\mapsto u(x)g(z)$ *belongs to* $Q_\alpha(\mathbb{R}^n\times\mathbb{R})$.

Proof. For any

$$(x_0,z_0,\rho,r,k+2)\in\mathbb{R}^n\times\mathbb{R}\times(0,\infty)\times(0,\infty)\times\mathbb{N},$$

set

$$\begin{cases}C(x_0,z_0,\rho)=\big\{(x,z)\in\mathbb{R}^n\times\mathbb{R}:\ |(x-x_0,z-z_0)|\le\rho\big\};\\ A(k,x_0,z_0,r)=C(x_0,z_0,2^{-k}r)\setminus C(x_0,z_0,2^{-k-1}r);\\ a_{k,r}(x_0,z_0)=u_{A(k,x_0,z_0,r)}g(z_0).\end{cases}$$

Here and henceforth, for a given set $E\subset\mathbb{R}^{m\ge1}$ with the m-dimensional Lebesgue measure $|E|>0$, the symbol

$$u_E=\fint_E u(x)\,dx=|E|^{-1}\int_E u(x)\,dx$$

stands for the average of u over E. We make the following claim

$$\begin{aligned}&\Psi_\alpha\big(ug,C(x_0,z_0,r)\big)\\ &:=\sum_{k\ge-1}2^{2k\alpha}\fint_{C(x_0,z_0,r)}\fint_{A(k,x,z,r)}|u(\tilde{x})g(\tilde{z})-a_{k,r}(x,z)|^2\,d\tilde{z}\,d\tilde{x}\,dz\,dx\\ &\lesssim\big(\|g\|_{\infty,Lip}|||u|||_{Q_\alpha}\big)^2.\end{aligned}$$

Assume that the last estimation holds for the moment. Then an application of the basic fact that

$$\begin{cases}C(x,z,2r)=\bigcup_{k\ge-1}A(k,x,z,r);\\ A(k,x,z,r)\cap A(l,x,z,r)=\varnothing\quad\forall\ k\neq l;\\ \big((x,z),(y,w)\big)\in C(x_0,z_0,r)\times C(x_0,z_0,r)\Longrightarrow(y,w)\in C(x,z,2r)\subset C(x_0,z_0,3r),\end{cases}$$

the Hölder inequality and Lemma 2.1 gives

$$
\begin{aligned}
&r^{2\alpha-n-1}\int_{C(x_0,z_0,r)}\int_{C(x_0,z_0,r)}\frac{|u(x)g(z)-u(y)g(w)|^2}{|(x,z)-(y,w)|^{n+1+2\alpha}}\,dx\,dz\,dy\,dw\\
&\quad\lesssim r^{2\alpha}\fint_{C(x_0,z_0,r)}\int_{C(x,z,2r)}\frac{|u(x)g(z)-u(y)g(w)|^2}{|(x,z)-(y,w)|^{n+1+2\alpha}}\,dy\,dw\,dx\,dz\\
&\quad\lesssim \fint_{C(x_0,z_0,r)}\sum_{k\ge -1}\frac{2^{2k\alpha}}{(2^{-k}r)^{n+1}}\int_{A(k,x,z,r)}\frac{dy\,dw\,dx\,dz}{|u(x)g(z)-u(y)g(w)|^{-2}}\\
&\quad\lesssim \fint_{C(x_0,z_0,r)}\sum_{k\ge -1}\fint_{A(k,x,z,r)}\frac{2^{2k\alpha}dy\,dw\,dx\,dz}{\big|(a_{k,r}(x,z)-u(y)g(w))+(u(x)g(z)-a_{k,r}(x,z))\big|^{-2}}\\
&\quad\lesssim \Psi_\alpha\big(ug,C(x_0,z_0,r)\big)+\|g\|^2_{\infty,Lip}\sum_{k\ge -1}2^{2k\alpha}\fint_{C(x_0,z_0,r)}|u(x)-u_{A(k,x,z,r)}|^2\,dx\,dz\\
&\quad\lesssim \Psi_\alpha\big(ug,C(x_0,z_0,r)\big)\\
&\qquad+\|g\|^2_{\infty,Lip}\sum_{k\ge -1}2^{2k\alpha}\fint_{C(x_0,z_0,r)}\fint_{A(k,x,z,r)}|u(x)-u(y)|^2\,dy\,dw\,dx\,dz\\
&\quad\lesssim \Psi_\alpha\big(ug,C(x_0,z_0,r)\big)\\
&\qquad+\|g\|^2_{\infty,Lip}\sum_{k\ge -1}\int_{C(x_0,z_0,r)}\int_{A(k,x,z,r)}\frac{|u(x)-u(y)|^2\,dy\,dw\,dx\,dz}{r^{n+1-2\alpha}|(x-y,z-w)|^{1+n+2\alpha}}\\
&\quad\lesssim \Psi_\alpha\big(ug,C(x_0,z_0,r)\big)\\
&\qquad+\|g\|^2_{\infty,Lip}\int_{C(x_0,z_0,3r)}\int_{C(x,z,2r)\subset C(x_0,z_0,3r)}\frac{|u(x)-u(y)|^2\,dy\,dw\,dx\,dz}{r^{n+1-2\alpha}|(x-y,z-w)|^{1+n+2\alpha}}\\
&\quad\lesssim \Psi_\alpha\big(ug,C(x_0,z_0,r)\big)+\big(\|g\|_{\infty,Lip}\|\|u\|\|_{Q_\alpha}\big)^2.
\end{aligned}
$$

This, plus the foregoing claim, yields

$$
\begin{aligned}
&\|ug\|^2_{Q_\alpha(\mathbb{R}^{n+1})}\\
&\quad=\sup_{(x_0,z_0,r)\in\mathbb{R}^n\times\mathbb{R}\times(0,\infty)}\int_{C(x_0,z_0,r)}\int_{C(x_0,z_0,r)}\frac{|u(x)g(z)-u(y)g(w)|^2}{|(x,z)-(y,w)|^{n+1+2\alpha}}\,\frac{dx\,dz\,dy\,dw}{r^{n+1-2\alpha}}\\
&\quad\lesssim \sup_{(x_0,z_0,r)\in\mathbb{R}^n\times\mathbb{R}\times(0,\infty)}\Psi_\alpha\big(ug,C(x_0,z_0,r)\big)+\big(\|g\|_{\infty,Lip}\|\|u\|\|_{Q_\alpha}\big)^2\\
&\quad\lesssim \big(\|g\|_{\infty,Lip}\|\|u\|\|_{Q_\alpha}\big)^2,
\end{aligned}
$$

Now, it remains to verify the above claim.

First of all, we have

$$
\fint_{A(k,x,x,r)}|u(\tilde{x})g(\tilde{z})-a_{k,r}(x,z)|^2\,d\tilde{x}\,d\tilde{z}
$$

$$\lesssim \fint_{A(k,x,z,r)} |u(\tilde{x})-u_{A(k,x,z,r)}|^2|g(\tilde{z})|^2\,d\tilde{x}\,d\tilde{z}+\fint_{A(k,x,z,r)} \frac{|g(\tilde{z})-g(z)|^2}{|u_{A(k,x,z,r)}|^{-2}}\,d\tilde{x}\,d\tilde{z}$$
$$\lesssim \|g\|^2_{\infty,Lip}\left(\fint_{A(k,x,z,r)} |u(\tilde{x})-u_{A(k,x,z,r)}|^2\,d\tilde{x}\,d\tilde{z}+\min\{2^{-k}r,1\}^2|u_{A(k,x,z,r)}|^2\right),$$

thereby finding that if

$$I(u,\alpha)=\sum_{k\ge -1} 2^{2k\alpha}\min\{2^{-k}r,1\}^2\fint_{C(x_0,z_0,r)} |u_{A(k,x,z,r)}|^2\,dx\,dz$$

then an application of the triangle inequality, the Hölder inequality and Lemma 2.1 derives

$$\begin{aligned}
&\Psi_\alpha\big(ug, C(x_0,z_0,r)\big)\\
&\lesssim \|g\|^2_{\infty,Lip}\Bigg(\sum_{k\ge -1} 2^{2k\alpha}\fint_{C(x_0,z_0,r)}\fint_{A(k,x,z,r)} |u(\tilde{x})-u_{A(k,x,z,r)}|^2\,d\tilde{x}\,d\tilde{z}\,dx\,dz\\
&\quad +I(u,\alpha)\Bigg)\\
&\lesssim \|g\|^2_{\infty,Lip}\Bigg(\sum_{k\ge -1} 2^{2k\alpha}\fint_{C(x_0,z_0,r)}\fint_{A(k,x,z,r)} \frac{d\tilde{x}\,d\tilde{z}\,dx\,dz}{\big(|u(x)-u_{A(k,x,z,r)}|^2+|u(\tilde{x})-u(x)|^2\big)^{-1}}\\
&\quad +I(u,\alpha)\Bigg)\\
&\lesssim \|g\|^2_{\infty,Lip}\Bigg(|||u|||^2_{Q_\alpha}\\
&\quad +\fint_{C(x_0,z_0,r)}\sum_{k\ge -1}\fint_{A(k,x,z,r)} \frac{2^{2k\alpha}|u(\tilde{x})-u(x)|^2\,d\tilde{x}\,d\tilde{z}\,dx\,dz}{|(\tilde{x},\tilde{z})-(x,z)|^{1+n+2\alpha}(2^{-k}r)^{-n-1-2\alpha}}\\
&\quad +I(u,\alpha)\Bigg)\\
&\lesssim \|g\|^2_{\infty,Lip}\Bigg(|||u|||^2_{Q_\alpha}\\
&\quad +\int_{C(x_0,z_0,r)}\sum_{k\ge -1}\int_{A(k,x,z,r)} \frac{|u(\tilde{x})-u(x)|^2\,d\tilde{x}\,d\tilde{z}\,dx\,dz}{|(\tilde{x},\tilde{z})-(x,z)|^{1+n+2\alpha}r^{1+n-2\alpha}}+I(u,\alpha)\Bigg)\\
&\lesssim \|g\|^2_{\infty,Lip}\Bigg(|||u|||^2_{Q_\alpha}
\end{aligned}$$

$$+\int_{C(x_0,z_0,3r)}\int_{C(x,z,2r)\subset C(x_0,z_0,3r)}\frac{|u(\tilde{x})-u(x)|^2\,d\tilde{x}\,d\tilde{z}\,dx\,dz}{|(\tilde{x},\tilde{z})-(x,z)|^{1+n+2\alpha}r^{1+n-2\alpha}}+I(u,\alpha)\Bigg)$$
$$\lesssim \|g\|^2_{\infty,Lip}\Big(|||u|||^2_{Q_\alpha}+I(u,\alpha)\Big).$$

Next, we handle $I(u,\alpha)$ according to the following two cases.

- *Case* $r<2$. By the hypothesis on u and the inclusion

$$Q_\alpha(\mathbb{R}^n)\subseteq BMO(\mathbb{R}^n)$$

we obtain that if $k+2\in\mathbb{N}$ then Lemma 2.1 yields

$$\begin{aligned}|u_{A(k,x,z,r)}| &\lesssim (2^{-k}r)^{-n-1}\left|\int_{C(x,z,2^{-k}r)}u(y)\,dy\,dw-\int_{C(x,z,2^{-k-1}r)}u(y)\,dy\,dw\right|\\ &\lesssim |u_{C(x,z,2^{-k}r)}|+|u_{C(x,z,2^{-k-1}r)}|\\ &\lesssim |u_{C(x,z,2)}|+|u_{C(x,z,2)}-u_{C(x,z,2^{-k}r)}|+|u_{C(x,z,1)}|\\ &\quad +|u_{C(x,z,1)}-u_{C(x,z,2^{-k-1}r)}|\\ &\lesssim \Big((|u|^2)_{B(x,2)}\Big)^{2^{-1}}+\Big((|u|^2)_{B(x,1)}\Big)^{2^{-1}}+\left(k+1+\ln\frac{4}{r}\right)\|u\|_{Q_\alpha(\mathbb{R}^n)}\\ &\lesssim \left(k+2+\ln\frac{4}{r}\right)|||u|||_{Q_\alpha}\end{aligned}$$

and hence

$$I(u,\alpha)\lesssim |||u|||^2_{Q_\alpha}\sum_{k\geq -1}2^{2k\alpha-2k}r^2\left(k+2+\ln\frac{4}{r}\right)^2\lesssim |||u|||^2_{Q_\alpha}.$$

- *Case* $r\geq 2$. An application of the hypothesis on u, the Hölder inequality and the Fubini theorem gives that if $k+2\in\mathbb{N}$ then

$$\begin{aligned}&\fint_{C(x_0,z_0,r)}|u_{A(k,x,z,r)}|^2\,dx\,dz\\ &\quad\lesssim \fint_{C(x_0,z_0,r)}\Big(|u|_{C(x,z,2^{-k}r)}\Big)^2\,dx\,dz\\ &\quad\lesssim \fint_{C(x_0,z_0,r)}\fint_{C(x,z,2^{-k}r)}|u(y)|^2\,dy\,dw\,dx\,dz\\ &\quad\lesssim \fint_{C(x_0,z_0,r)}\fint_{C(0,0,2^{-k}r)}|u(x+z)|^2\,dx\,dz\,dy\,dw\\ &\quad\lesssim r^{-2\alpha}|||u|||^2_{Q_\alpha}\end{aligned}$$

and hence

$$I(u,\alpha)\lesssim |||u|||_{Q_\alpha}^2\left(\sum_{k\geq \ln r}2^{2k\alpha-2k}r^{2-2\alpha}+\sum_{-1\leq k\leq \ln r}2^{2k\alpha}r^{-2\alpha}\right)\lesssim |||u|||_{Q_\alpha}^2.$$

Finally, upon putting the previous two cases together, we achieve the desired estimation

$$\Psi_\alpha\big(ug,C(x_0,z_0,r)\big)\lesssim \|g\|_{\infty,Lip}^2\big(|||u|||_{Q_\alpha}^2+I(u,\alpha)\big)\lesssim \big(\|g\|_{\infty,Lip}|||u|||_{Q_\alpha}\big)^2. \quad \square$$

Corollary 2.5. *For* $n-1\in\mathbb{N}$ *let*

$$\phi(t)=\begin{cases}0 & as\ t\in(-\infty,-2];\\ 1-|1+t| & as\ t\in[-2,0];\\ 1-|1-t| & as\ t\in[0,2];\\ 0 & as\ t\in[2,\infty),\end{cases}$$

and

$$\psi(t)=\begin{cases}1 & as\ |t|\leq 1;\\ 2-|t| & as\ 1\leq |t|\leq 2;\\ 0 & as\ |t|\geq 2.\end{cases}$$

If

$$u_\star(x_1,...,x_n)=\begin{cases}\max\big\{0,\ln(x_1^{-2})\big\}\phi(x_2) & for\ n=2;\\ \Big(\max\big\{0,\ln(x_1^{-2})\big\}\Big)\psi(x_2)...\psi(x_{n-1})\phi(x_n) & for\ n\geq 3,\end{cases}$$

then $u_\star\in Q_{0<\alpha<2^{-1}}(\mathbb{R}^n)$.

Proof. Note that

$$\|\phi\|_{\infty,Lip}+\|\psi\|_{\infty,Lip}<\infty$$

holds and (via Corollary 2.3(i))

$$u(x_1,...,x_n)=\max\big\{0,\ln(x_1^{-2})\big\}\quad\text{enjoys}\quad |||u|||_{Q_{0<\alpha<2^{-1}}}<\infty.$$

So, the assertion $u_\star\in Q_{0<\alpha<2^{-1}}(\mathbb{R}^n)$ follows from Lemma 2.4. $\square$

Lemma 2.6. *For $n-1\in\mathbb{N}$ let $\mathbf{a}=(a_1,...,a_n)$ be with $0<a_1\le a_2\le...\le a_n=1$. Given $r>0$ set*

$$\begin{cases}(u_\star)_r(x)=u_\star(r^{-1}x);\\ P_{\mathbf{a},r}=\{x=(x_1,...,x_n)\in\mathbb{R}^n:\ |x_1|\le a_1r,...,|x_n|\le a_nr\};\\ (u_\star)_{\mathbf{a},r}=\frac{(u_\star)_r\chi_{P_{\mathbf{a},r}}}{|P_{\mathbf{a},r}|}=\frac{(u_\star)_r\chi_{P_{\mathbf{a},r}}}{(2r)^n a_1\dots a_n};\\ c_{\mathbf{a}}=\int_{\mathbb{R}^n}|(u_\star)_{\mathbf{a},r}(x)|\,dx=⨍_{P_{\mathbf{a},r}}|(u_\star)_r(x)|\,dx=⨍_{P_{\mathbf{a},1}}|u_\star(x)|\,dx.\end{cases}$$

If $h\in L^1(\mathbb{R}^n)$, then there exists a subsequence $\{r_j\}$ converging to 0 such that for any rational point $\mathbf{a}\in\mathbb{R}^n$ one has that

$$\begin{cases}(u_\star)_{\mathbf{a},r_j}*h(y)=\int_{\mathbb{R}^n}(u_\star)_{\mathbf{a},r_j}(z)h(y-z)\,dz\to 0;\\ |(u_\star)_{\mathbf{a},r_j}|*h(y)=\int_{\mathbb{R}^n}|(u_\star)_{\mathbf{a},r_j}(z)|h(y-z)\,dz\to c_{\mathbf{a}}h(y),\end{cases}$$

holds for almost all $y\in\mathbb{R}^n$.

Proof. The argument is similar to the proof of [9, Lemma 8]. □

Proof of Theorem 1.2. We are about to use Reimann's procedure in [9]. Rather than showing that f is quasiconformal, we prove that f^{-1} (the inverse of f) is quasiconformal. It suffices to verify that

$$\sup_{y\in\partial B(0,1)}\left|\big(Df^{-1}(x)\big)y\right|^n\lesssim J_{f^{-1}}(x)$$

holds for almost all $x\in\mathbb{R}^n$ where Df^{-1} and $J_{f^{-1}}$ are the formal derivative and Jacobian determinant of f^{-1} (cf. [4, p.250]). Since f^{-1} is absolutely continuous with respect to the n-dimensional Lebesgue measure, one has

$$J_{f^{-1}}(x)=\lim_{r\to 0}\frac{|f^{-1}(B(x,r))|}{|B(x,r)|}$$

almost everywhere and $J_{f^{-1}}\in L^1_{\text{loc}}(\mathbb{R}^n)$ where the absolute values right after $\lim_{r\to 0}$ stand for the n-dimensional Lebesgue measures of the sets $f^{-1}\big(B(x,r)\big)$ and $B(x,r)$ respectively. Also our hypothesis implies that f^{-1} is (totally) differentiable almost everywhere, and $J_{f^{-1}}>0$ holds almost everywhere. We may assume $J_{f^{-1}}(0)>0$ and $h=\chi_{B(0,1)}\,J_{f^{-1}}$ in Lemma 2.6. Up to some rotation, translation and scaling which preserve the $Q_\alpha(\mathbb{R}^n)$-norm, we may also assume

$$\begin{cases}f^{-1}(0)=0;\\ Df^{-1}(0)=\operatorname{diag}\{\lambda_1,...,\lambda_n\};\\ \lambda_1\ge...\ge\lambda_n=1.\end{cases}$$

and so are required to verify

$$(\sharp)\quad \lambda_1^n \lesssim \lambda_1 \ldots \lambda_n.$$

Given any sufficiently small $\varepsilon>0$, we choose

$$\mathbf{a}_m = (a_{m1}, \ldots, a_{mn})$$

rationally such that

$$0 < a_{m1} \le a_{m2} \le \ldots \le a_{mn} = 1 \quad \& \quad \sum_{k=1}^{n} |a_{mk}\lambda_k - 1| < \varepsilon.$$

Let

$$\begin{cases} P_r = \{z=(z_1,\ldots,z_n)\in\mathbb{R}^n :\ |z_1|,\ldots,|z_n|\le r\}; \\ P_{\mathbf{a}_m,r} = \{z=(z_1,\ldots,z_n)\in\mathbb{R}^n :\ |z_1|\le a_{m1}r,\ldots,|z_n|\le a_{mn}r\}. \end{cases}$$

Upon using Lemma 2.6 with $\mathbf{a}=\mathbf{a}_m$, we write

$$c_{\mathbf{a}_m} = \fint_{P_{\mathbf{a}_m,1}} |u_\star(x)|\,dx.$$

By the definition of $u_\star$ as in Corollary 2.5 with $\mathbf{a}=\mathbf{a}_m$ we have

$$(\dagger)\quad c_{\mathbf{a}_m} \gtrsim -\ln a_{m1}.$$

Indeed, if $n=2$, then

$$0 < a_{m1} \le 1 = a_{m2}$$

derives

$$\begin{aligned} \fint_{P_{\mathbf{a}_m,1}} |u_\star(x)|\,dx &= (4a_{m1}a_{m2})^{-1} \int_{-a_{m1}}^{a_{m1}} \int_{-a_{m2}}^{a_{m2}} \max\{0, \ln(x_1^{-2})\} |\phi(x_2)|\,dx_1\,dx_2 \\ &\gtrsim (a_{m1}a_{m2})^{-1} \int_0^{a_{m2}} \left(\int_0^{a_{m1}} \ln(x_1^{-2})\,dx_1 \right) x_2\,dx_2 \\ &\gtrsim -\ln a_{m1}. \end{aligned}$$

Furthermore, if $n\ge 3$, then a similar argument, along with

$$\psi(t) = 1 \quad \forall\ |t| \le 1,$$

will also ensure $(\dagger)$.

In this way, for a sufficiently small $r<\delta_1$ we have that $f^{-1}(P_{\mathbf{a}_m,r})$ contains

$$R = \{z=(z_1,\ldots,z_n)\in\mathbb{R}^n :\ |z_1|,\ldots,|z_n| \le r(1-\varepsilon)\}$$

and is contained in

$$S=\big\{z=(z_1,...,z_n)\in\mathbb{R}^n:\ |z_1|,...,|z_n|\le r(1+\varepsilon)\big\}.$$

In fact, this can be obtained by the differentiability of f^{-1} & f at 0, and

$$Df^{-1}(0)=\mathrm{diag}\{\lambda_1,...,\lambda_n\}\quad \&\quad Df(0)=\mathrm{diag}\{\lambda_1^{-1},...,\lambda_n^{-1}\}.$$

By virtue of the assumption on f and the function $u_\star$ constructed in Corollary 2.5, we have

$$(\ddagger)\quad \|\mathbf{C}_f u_\star\|_{Q_{-\frac{n}{2}}(\mathbb{R}^n)}\lesssim\|\mathbf{C}_f u_\star\|_{Q_{0<\alpha<2^{-1}}(\mathbb{R}^n)}\lesssim\|(u_\star)_r\|_{Q_{0<\alpha<2^{-1}}(\mathbb{R}^n)}\\ \lesssim\|u_\star\|_{Q_{0<\alpha<2^{-1}}(\mathbb{R}^n)}\lesssim 1.$$

Since

$$Q_{0<\alpha<2^{-1}}(\mathbb{R}^n)\subset BMO(\mathbb{R}^n)=Q_{-\frac{n}{2}}(\mathbb{R}^n)$$

we are required to control

$$\|\mathbf{C}_f u_\star\|_{BMO(\mathbb{R}^n)}=\|\mathbf{C}_f u_\star\|_{Q_{-\frac{n}{2}}(\mathbb{R}^n)}$$

via

$$\|\mathbf{C}_f u_\star\|_{BMO(\mathbb{R}^n)}\gtrsim⨍_{f^{-1}(P_{\mathbf{a}_m,r})}\left|\mathbf{C}_f u_\star(x)-⨍_{f^{-1}(P_{\mathbf{a}_m,r})}\mathbf{C}_f u_\star(y)\,dy\right|\,dx.$$

Note that if

$$h(x)=\begin{cases}J_f(x) & \text{for } x\in B(0,1);\\ 0 & \text{for } x\in\mathbb{R}^n\setminus B(0,1),\end{cases}$$

then

$$⨍_{f^{-1}(P_{\mathbf{a}_m,r})}\mathbf{C}_f u_\star(x)\,dx=\frac{|P_{\mathbf{a}_m,r}|}{|f^{-1}(P_{\mathbf{a}_m,r})|}⨍_{P_{\mathbf{a}_m,r}}(u_\star)_r(z)J_{f^{-1}}(z)\,dz\\ =\frac{|P_{\mathbf{a}_m,r}|}{|f^{-1}(P_{\mathbf{a}_m,r})|}(u_\star)_{\mathbf{a}_m,r}*h(0).$$

So, upon applying Lemma 2.6, we obtain a constant $\delta_2\in(0,\delta_1)$ and a sequence $r_j<\delta_2$ such that

$$\left|⨍_{f^{-1}(P_{\mathbf{a}_m,r})}\big(\mathbf{C}_f(u_\star)_{r_j}\big)(x)\,dx\right|\le\varepsilon\quad\forall\quad\mathbf{a}_m.$$

Accordingly,

$$\|\mathbf{C}_f u_\star\|_{BMO(\mathbb{R}^n)} \ge \fint_{f^{-1}(P_{\mathbf{a}_m,r})} |\mathbf{C}_f u_\star(x)|\, dx - \varepsilon \quad \forall \quad r\in(0,\infty).$$

Similarly, we have

$$\begin{aligned}\fint_{f^{-1}(P_{\mathbf{a}_m,r})} |\mathbf{C}_f u_\star(x)|\, dx &= \left(\frac{|P_{\mathbf{a}_m,r}|}{|f^{-1}(P_{\mathbf{a}_m,r})|}\right)\fint_{P_{\mathbf{a}_m,r}} |(u_\star)_r(z)| J_{f^{-1}}(z)\, dz\\ &= \left(\frac{|P_{\mathbf{a}_m,r}|}{|f^{-1}(P_{\mathbf{a}_m,r})|}\right) |(u_\star)_{\mathbf{a}_m,r}| * h(0),\end{aligned}$$

thereby using Lemma 2.6 to discover

$$\liminf_{r_j\to 0} \fint_{f^{-1}(P_{\mathbf{a}_m,r_j})} \left|\left(\mathbf{C}_f (u_\star)_{\mathbf{a}_m,r_j}\right)(x)\right| dx = \left(\liminf_{r_j\to 0} \frac{|P_{\mathbf{a}_m,r_j}|}{|f^{-1}(P_{\mathbf{a}_m,r_j})|}\right) c_{\mathbf{a}_m} h(0).$$

For $r_j<\delta_1$, we utilize

$$1-\varepsilon \le a_{mk}\lambda_k \le 1+\varepsilon \quad \forall \quad k\in\{1,...,n\},$$

to deduce

$$\left(\frac{|P_{\mathbf{a}_m,r_j}|}{|f^{-1}(P_{\mathbf{a}_m,r_j})|}\right) h(0) \ge (1+\varepsilon)^{-n}(a_{m1}...a_{mn})(\lambda_1...\lambda_n) \ge \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^n,$$

whence

$$\liminf_{r_j\to 0} \fint_{f^{-1}(P_{\mathbf{a},r})} \left|\left(\mathbf{C}_f (u_\star)_{r_j}\right)(x)\right| dx \ge \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^n c_{\mathbf{a}_m},$$

which in turn implies

$$\|\mathbf{C}_f u_\star\|_{BMO(\mathbb{R}^n)} \ge \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^n c_{\mathbf{a}_m} - \varepsilon.$$

Upon combining this with (†)–(‡), we achieve a constant $\varkappa>0$ (independent of $\mathbf{a}_m$) such that

$$-\ln a_{m1} \le \varkappa \quad \& \quad a_{m1} \ge e^{-\varkappa}.$$

Consequently, we gain

$$1=\lambda_n \le \lambda_{n-1} \le ... \le \lambda_1 \le 2e^{\varkappa},$$

thereby reaching (♯). □

3. Validation of Theorem 1.5

In order to prove Theorem 1.5, we need the concept of a $Q_\alpha(\mathbb{R}^n)$-capacity. For $(\alpha, n)\in(-\infty,1)\times\mathbb{N}$ and any pair of disjoint continua $E, F\subset\mathbb{R}^n$, let

$$\mathrm{Cap}_{Q_\alpha(\mathbb{R}^n)}(E,F)=\inf\left\{\|u\|^2_{Q_\alpha(\mathbb{R}^n)}:\ u\in\Delta_\alpha(E,F)\right\}$$

be the $Q_\alpha(\mathbb{R}^n)$-capacity of the pair (E,F), where $\Delta_\alpha(E,F)$ is the class of all continuous functions $u\in Q_\alpha(\mathbb{R}^n)$ enjoying

$$\begin{cases} 0\le u\le 1 & \text{on } \mathbb{R}^n;\\ u=0 & \text{on } E;\\ u=1 & \text{on } F.\end{cases}$$

Obviously, if $\widetilde{E}\,\&\,\widetilde{F}$ are disjoint continua satisfying $E\subseteq\widetilde{E}\,\&\,F\subseteq\widetilde{F}$, then

$$\mathrm{Cap}_{Q_\alpha(\mathbb{R}^n)}(E,F)\le\mathrm{Cap}_{Q_\alpha(\mathbb{R}^n)}(\widetilde{E},\widetilde{F}).$$

Moreover, we have

Lemma 3.1. *Given a constant $\delta\in(0,\infty)$ let $n=1$ & $\alpha\in(0,2^{-1}]$ or $n=2$ & $\alpha\in(2^{-1},1)$. If $E\,\&\,F$ are disjoint continua in $\mathbb{R}^n$ such that their diameters* $\operatorname{diam} E$ & $\operatorname{diam} F$ *and Euclidean distance* $\operatorname{dist}(E,F)$ *obey*

$$\min\{\operatorname{diam} E,\ \operatorname{diam} F\}\ge\delta\operatorname{dist}(E,F)>0,$$

then

$$\mathrm{Cap}_{Q_\alpha(\mathbb{R}^n)}(E,F)\gtrsim 1.$$

Proof. Without loss of generality we may assume

$$\operatorname{diam} E=\operatorname{diam} F\ge\delta\operatorname{dist}(E,F).$$

If

$$x_0\in E\quad\&\quad r=(2+\delta^{-1})\operatorname{diam} E,$$

then

$$E,F\subseteq B(x_0,r).$$

Thanks to either $n=1$ & $\alpha\in(0,2^{-1}]$ or $n=2$ & $\alpha\in(2^{-1},1)$, we may assume

$$\begin{cases} u\in\Delta_\alpha(E,F);\\ u_{B(x_0,r)}\ge 2^{-1};\\ 0<\varepsilon\le 1-n+2\alpha.\end{cases}$$

For every $x\in E$ and $\rho>0$ we utilize

$$\Phi_\alpha\big(u,B(x,\rho)\big)=\rho^{2\alpha-n}\int_{B(x,\rho)}\int_{B(x,\rho)}\frac{|u(z)-u(w)|^2}{|z-w|^{n+2\alpha}}\,dz\,dw \gtrsim ⨍_{B(x,\rho)}⨍_{B(x,\rho)}|u(z)-u(w)|\,dz\,dw$$

to estimate

$$\begin{aligned}2^{-1}&\le|u(x)-u_{B(x_0,r)}|\\ &\le\sum_{i=-1}^{\infty}|u_{B(x,2^{-i}r)}-u_{B(x,2^{-i-1}r)}|+|u_{B(x,2r)}-u_{B(x_0,r)}|\\ &\lesssim\sum_{i=-1}^{\infty}\left(⨍_{B(x,2^{-i}r)}⨍_{B(x,2^{-i}r)}|u(z)-u(w)|^2\,dz\,dw\right)^{2^{-1}}\\ &\lesssim\sum_{i=-1}^{\infty}\big(\Phi_\alpha(u,B(x,2^{-i}r))\big)^{2^{-1}}\\ &\lesssim\sum_{i=-1}^{\infty}(2^{-i}r)^{\frac{\varepsilon}{2}}\sup_{t\le2r}t^{-\frac{\varepsilon}{2}}[\Phi_\alpha(u,B(x,t))]^{2^{-1}}\\ &\lesssim r^{\frac{\varepsilon}{2}}\sup_{t\le2r}t^{-\frac{\varepsilon}{2}}\big(\Phi_\alpha(u,B(x,t))\big)^{2^{-1}}.\end{aligned}$$

Accordingly, for each $x\in E$ there exists a $t_x\in(0,2r]$ such that

$$\begin{cases}1\lesssim r^\varepsilon t_x^{-\varepsilon}\Phi_\alpha(u,B(x,t_x));\\ t_x^{n-2\alpha+\varepsilon}\lesssim r^\varepsilon\int_{B(x,t_x)}\int_{B(x,t_x)}\frac{|u(z)-u(w)|^2}{|z-w|^{n+2\alpha}}\,dz\,dw.\end{cases}$$

By the Vitali covering lemma, we can find points $x_i\in E$ and radii $r_i>0$ as above such that ball $B(x_i,t_i)$ are mutually disjoint and $E\subseteq\bigcup_i B(x_i,5t_i)$. Hence,

$$\operatorname{diam}E\lesssim\sum_{i=-1}^{\infty}t_i\lesssim r^{\frac{\varepsilon}{n-2\alpha+\varepsilon}}\sum_{i=-1}^{\infty}\left(\int_{B(x_i,t_i)}\int_{B(x_i,t_i)}\frac{|u(z)-u(w)|^2}{|z-w|^{n+2\alpha}}\,dz\,dw\right)^{\frac{1}{n-2\alpha+\varepsilon}}.$$

Upon noticing $1/(n-2\alpha+\varepsilon)\ge1$, we obtain

$$\begin{aligned}\frac{r}{2+\delta^{-1}}&\lesssim r^{\frac{\varepsilon}{n-2\alpha+\varepsilon}}\left(\sum_{i=-1}^{\infty}\int_{B(x_i,t_i)}\int_{B(x_i,t_i)}\frac{|u(z)-u(w)|^2}{|z-w|^{n+2\alpha}}\,dz\,dw\right)^{\frac{1}{n-2\alpha+\varepsilon}}\\ &\lesssim r^{\frac{\varepsilon}{n-2\alpha+\varepsilon}}\left(\int_{B(x_0,4r)}\int_{B(x_0,4r)}\frac{|u(z)-u(w)|^2}{|z-w|^{n+2\alpha}}\,dz\,dw\right)^{\frac{1}{n-2\alpha+\varepsilon}},\end{aligned}$$

whence

$$\Phi_\alpha(u, B(x_0, 4r)) \gtrsim 1,$$

which yields

$$\mathrm{Cap}_{Q_\alpha(\mathbb{R}^n)}(E, F) \gtrsim 1. \quad \square$$

Proof of Theorem 1.5. By the metric characterization of a quasiconformal mapping (cf. [7]), it is enough to validate that if

$$\begin{cases} \ell(f,r)=\inf\left\{|f(x)-f(x_0)|:\ |x-x_0|\geq r\right\}; \\ L(f,r)=\sup\left\{|f(x)-f(x_0)|:\ |x-x_0|\leq r\right\}; \\ (x_0,r)\in\mathbb{R}^2\times(0,\infty), \end{cases}$$

then

$$L(f,r) \leq c(f)\,\ell(f,r),$$

where $c(f)$ is a positive constant depending on f.

To this end, if

$$v(y) = \begin{cases} 1 & \text{as } |y-x_0|\leq\ell(f,r); \\ \frac{\ln L(f,r)-\ln|y-x_0|}{\ln L(f,r)-\ln\ell(f,r)} & \text{as } \ell(f,r)\leq|y-x_0|\leq L(f,r); \\ 0 & \text{as } |y-x_0|\geq L(f,r), \end{cases}$$

then

$$|\nabla v(y)| = \begin{cases} 0 & \text{as } |y-x_0|\leq\ell(f,r); \\ \frac{|y-x_0|^{-1}}{\ln L(f,r)-\ln\ell(f,r)} & \text{as } \ell(f,r)\leq|y-x_0|\leq L(f,r); \\ 0 & \text{as } |y-x_0|\geq L(f,r), \end{cases}$$

and hence

$$\begin{aligned} \|v\|^2_{W^{1,2}(\mathbb{R}^2)} &= \int_{\mathbb{R}^2} |\nabla v(y)|^2 dy \\ &= \left(\ln\frac{L(f,r)}{\ell(f,r)}\right)^{-2} \int_{l\leq|y-x_0|\leq L} \frac{dy}{|y-x_0|^2} \\ &\lesssim \left(\ln\frac{L(f,r)}{\ell(f,r)}\right)^{-1}. \end{aligned}$$

This last estimation, along with [10, Theorem 4.1] under $n=2$ & $\alpha<1$, implies

$$\|v\|_{Q_{2^{-1}<\alpha<1}(\mathbb{R}^2)} \lesssim \|v\|_{W^{1,2}(\mathbb{R}^2)} \lesssim \left(\ln\frac{L(f,r)}{\ell(f,r)}\right)^{-2^{-1}}.$$

Let

$$E=f^{-1}\Big(B\big(f(x_0),\ell\big)\Big),$$

i.e., the preimage of $B\big(f(x_0),\ell\big)$ under f. Then E is connected and enjoys

$$E\subseteq B(x_0,r)\quad \&\quad \operatorname{diam} E\geq r.$$

Moreover, observe that as the connected preimage of $\mathbb{R}^2\setminus B\big(f(x_0),L\big)$ under f,

$$f^{-1}\Big(\mathbb{R}^2\setminus B\big(f(x_0),L\big)\Big)$$

joins

$$\overline{B}(x_0,r)=\{x\in\mathbb{R}^2:|x-x_0|\leq r\}\quad \&\quad \mathbb{R}^2\setminus B(x_0,2r).$$

So we can find a connected continuum F such that it is contained in

$$f^{-1}(\mathbb{R}^2\setminus B\big(f(x_0),L\big))$$

and joins $\overline{B}(x_0,r)$ and $\mathbb{R}^2\setminus B(x_0,2r)$, and consequently we may assume

$$F\subseteq\overline{B}(x_0,2r)\setminus B(x_0,r).$$

Obviously, we have

$$\operatorname{diam} F\geq r\quad \&\quad 0<\operatorname{dist}(E,F)\leq 5r\leq 10\min\{\operatorname{diam} E,\,\operatorname{diam} F\}.$$

Upon applying Lemma 3.1 under $n=2$ & $2^{-1}<\alpha<1$ we discover

$$\operatorname{Cap}_{Q_\alpha(\mathbb{R}^2)}(E,F)\gtrsim 1,$$

thereby arriving at the required inequality

$$\ln\frac{L(f,r)}{\ell(f,r)}\lesssim 1.\quad\square$$

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