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On systems of non-overlapping Haar polynomials

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Abstract. We prove that $\log n$ is an almost everywhere convergence Weyl multiplier for the orthonormal systems of non-overlapping Haar polynomials. Moreover, it is done for the general systems of martingale difference polynomials.

1. Introduction

Recall some definitions well-known in the theory of orthogonal series (see [4]).

Definition 1.1. Let $\Phi=\{\phi_n: n=1,2,\dots\}\subset L^2(0,1)$ be an orthonormal system. A sequence of positive numbers $\omega(n)\nearrow\infty$ is said to be an a.e. convergence Weyl multiplier (shortly C-multiplier) if every series

$$\sum_{n=1}^{\infty} a_n\phi_n(x),$$

with coefficients satisfying the condition $\sum_{n=1}^{\infty} a_n^2\omega(n)<\infty$ is a.e. convergent. If such series converges unconditionally a.e., then we say $\omega(n)$ is an a.e unconditional convergence Weyl multiplier (UC-multiplier) for Φ.

The Menshov-Rademacher classical theorem ([6], [10]) states that the sequence $\log^2 n$ is a C-multiplier for any orthonormal system. The sharpness of $\log^2 n$ in this theorem was proved by Menshov in the same paper [6]. That is any sequence $\omega(n)=o(\log^2 n)$ fails to be C-multiplier for some orthonormal system.

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The following inequality is the basic part in the proof of the Menshov-Rademacher theorem.

Theorem A. (Menshov-Rademacher, [6], [10], see also [4]) If $\{\phi_k\colon k=1,2,\dots,n\}\subset L^2(0,1)$ is an orthogonal system, then

$$\left\|\max_{1\le m\le n}\left|\sum_{k=1}^m \phi_k\right|\right\|_2 \le c\cdot\log n\left\|\sum_{k=1}^n\phi_k\right\|_2,$$

where $c>0$ is an absolute constant.

Similarly, the counterexample of Menshov is based on the following results.

Theorem B. (Menshov, [6]) For any natural number $n\in\mathbb{N}$ there exists an orthogonal system ϕ_k, $k=1,2,\dots,n$, such that

$$\left\|\max_{1\le m\le n}\left|\sum_{k=1}^m \phi_k\right|\right\|_2 \ge c\cdot\log n\left\|\sum_{k=1}^n\phi_k\right\|_2,$$

for an absolute constant $c>0$.

Let $\Phi=\{\phi_k(x),\ k=1,2,\dots\}\subset L^2(0,1)$ be an infinite orthogonal system of functions. Denote by $\mathcal{P}_n(\Phi)$ the family of all *monotonic* sequences of Φ-polynomials

$$p_k(x)=\sum_{j\in G_k} c_j\phi_j(x),\quad k=1,2,\dots,n,$$

where $G_1\subset G_2\subset\dots\subset G_n\subset\mathbb{N}$ and $\sum_{j\in G_n} c_j^2\neq 0$. Define

$$\mathcal{K}_n(\Phi)=\sup_{\{p_k\}\in\mathcal{P}_n(\Phi)}\frac{\|\max_{1\le m\le n}|p_m|\|_2}{\|p_n\|_2}.$$

From Theorem A it follows that $\mathcal{K}_n(\Phi)\le c\cdot\log n$ for every orthogonal system Φ, where c is an absolute constant. On the other hand, applying Theorem B, one can also construct an infinite orthogonal system with the lower bound $\mathcal{K}_n(\Phi)\ge c\cdot\log n$, $n=1,2,\dots$. Thus we conclude, in general, the logarithmic upper bound of $\mathcal{K}_n(\Phi)$ is optimal. We will see below that from results of Nikishin-Ulyanov [7] and Olevskii [8] it follows that $\mathcal{K}_n(\Phi)\gtrsim\sqrt{\log n}$ for any complete orthonormal system Φ.

In this paper we found the sharp rate of the growth of $\mathcal{K}_n(\sim\sqrt{\log n})$ for the generalized Haar systems. The classical Haar system case of the result is also new. The upper bound $\mathcal{K}_n\lesssim\sqrt{\log n}$ will be proved for the general systems of martingale type.

Given martingale $F_n\in L^2(0,1)$, $n=1,2,\dots$, defines an orthogonal system $f_n=F_{n+1}-F_n$ that is called a martingale difference. Consider the following general example of martingale difference. A partition is a family of pairwise disjoint measurable sets $\mathcal{A}=\{E_k\}$ such that $\mu(E_k)>0$, $\cup_k E_k=[0,1)$. A sequence of partitions $\mathcal{A}_n$, $n=1,2,\ldots$, is said to be a filtration if each $A\in\mathcal{A}_n$ is a union of some elements of $\mathcal{A}_{n+1}$. A martingale difference based on a filtration $\{\mathcal{A}_n\colon n=1,2,\ldots\}$ is a sequence of functions $f_n\in L^2(0,1)$, satisfying the conditions

(1) Every function f_n is constant on each $A\in\mathcal{A}_n$.

(2) We have $\int_A f_n=0$ for any $A\in\mathcal{A}_{n-1}$, $n\geq 2$.

We will call such a sequence of functions to be a discrete type martingale difference. Consider a filtration $\{\mathcal{A}_n\}$ such that 1) each $\mathcal{A}_n$ consists of n intervals of the form $[a,b)$, 2) the family $\mathcal{A}_{n+1}$ is obtained from $\mathcal{A}_n$ replacing a single interval $A\in\mathcal{A}_n$ by two disjoint intervals $A',A''(\in\mathcal{A}_{n+1})$ with $A=A'\cup A''$, and 3) $\max_{A\in\mathcal{A}_n}|A|\to 0$ as $n\to\infty$. A generalized Haar system is a L^2-normalized martingale difference generated by a such filtration. If in the replacements we additionally have $|A'|=|A''|$, then the filtration produces a rearranged classical Haar system. It is well-known that any generalized Haar system is complete.

Recall few standard notations. The relation $a\lesssim b$ ($a\gtrsim b$) will stand for the inequality $a\leq c\cdot b$ ($a\geq c\cdot b$), where $c>0$ is an absolute constant. Given two sequences of positive numbers $a_n,b_n>0$, we write $a_n\sim b_n$ if we have $c_1\cdot a_n\leq b_n\leq c_2\cdot a_n$, $n=1,2,\ldots$ for some constants $c_1,c_2>0$. Throughout the paper, the base of log is equal 2. The following theorems are the main results of the paper.

Theorem 1.2. If Φ is a martingale difference, then $\mathcal{K}_n(\Phi)\lesssim\sqrt{\log n}$.

Theorem 1.3. For any generalized Haar system $\mathcal{H}$ we have the relation

$$\mathcal{K}_n(\mathcal{H})\sim\sqrt{\log n}. \tag{1.1}$$

In the class of all martingale differences the upper bound in Theorem 1.2 is optimal that readily follows from Theorem 1.3. One can easily see that for the Rademacher system we have $\mathcal{K}_n\sim 1$. So relation (1.1) can not be extended for general martingale differences. Such estimates of $\mathcal{K}_n(\Phi)$ characterize Weyl multipliers of a given orthonormal system Φ. From Theorem 1.2 we deduce the following.

Corollary 1.4. If $\mathcal{F}=\{f_n\}$ is a martingale difference, then $\log n$ is a C-multiplier for any system of L^2-normalized *non-overlapping* $\mathcal{F}$-polynomials

$$p_n(x)=\sum_{j\in G_n}c_jf_j(x),\quad n=1,2,\ldots,$$

where $G_n\subset\mathbb{N}$ are finite and pairwise disjoint.

The following result is interesting and immediately follows from Corollary 1.4.

Corollary 1.5. The sequence $\log n$ is a C-multiplier for any rearrangement of a generalized Haar system.

Corollary 1.6. Let $\{p_n\}$ be a sequence of L^2-normalized non-overlapping polynomials with respect to a martingale difference. If $\omega(n)/\log n$ is increasing and

$$\sum_{n=1}^{\infty}\frac{1}{n\omega(n)}<\infty, \tag{1.2}$$

then $\omega(n)$ is an UC-multiplier for $\{p_n\}$.

The only prior result in this context is due to Ulyanov (see [11], [12] or [4], ch. 2 Theorem 17). It states that condition (1.2) is a necessary and sufficient for $\omega(n)$ to be an UC-multiplier for the Haar classical system. The optimality of $\log n$ in Corollary 1.5 as well as condition (1.2) in Corollary 1.6 both follows just from this result of Ulyanov.

We prove Theorem 1.2 using a good-λ inequality due to Chang-Wilson-Wolff [1], which is an extension of classical Azuma-Hoeffding and Bernstein inequalities for martingales. See also [2], where the same method has been first applied in the study of maximal functions of Mikhlin-Hörmander multipliers.

Remark. Recall that an orthonormal system Φ is said to be a convergence system if $\omega(n)\equiv 1$ is a C-multiplier for Φ. It was proved by Komlós-Révész [5] that if an orthonormal system $\Phi=\{\phi_n\}\subset L^2(0,1)$ satisfies $\|\phi_n\|_4\le M$, $n=1,2,\ldots$, and we have

$$\int_0^1 \phi_{n_1}\phi_{n_2}\phi_{n_3}\phi_{n_4}=0 \tag{1.3}$$

for any choice of different indexes n_1,n_2,n_3,n_4, then Φ is a convergence system. One can check that systems of non-overlapping martingale difference polynomials satisfy (1.3). Thus, with the extra condition $\|p_n\|_4\le M$ in Corollary 1.4 we can claim that $\{p_n\}$ is a convergence system.

Question. Is the additional condition $\|p_n\|_p\le M$ in Corollary 1.4, with a fixed $2<p<4$, is sufficient for $\{p_n\}$ to be a convergence system?

2. Measure-preserving transformations

A mapping $\tau:[0,1)\to[0,1)$ is said to be measure-preserving (MP) transformation if $|\tau^{-1}(A)|=|A|$ for any Lebesgue measurable set $A\subset[0,1)$. A set in $[0,1)$ is said to be simple, if it is a finite union of intervals (of the form $[\alpha,\beta)$). Let a be a simple set. One can easily check, that the function

$$\xi_a(x)=\frac{|[0,x)\cap a|}{|a|}$$

defines a one to one mapping from a to $[0,1)$, such that $|\xi_a(E)|=|E|/|a|$ for any Lebesgue measurable set $E\subset a$. Given integer $n\geq 1$ the mapping $\eta_n(x)=\{nx\}$ defines an MP-transformation of $[0,1)$. Observe that if a is a simple set, then for any integer $n\geq 1$ the mapping

$$u_{a,n}(x)=\begin{cases}((\xi_a)^{-1}\circ\eta_n\circ\xi_a)(x) & \text{if } x\in a,\\ x & \text{if } x\in[0,1)\setminus a,\end{cases}$$

determines an MP-transformation of $[0,1)$ that maps the set a to itself. Moreover, for any functions $f,g\in L^2(0,1)$ we have

$$\lim_{n\to\infty}\int_a f(u_{a,n}(x))g(x)\,dx=\int_a f(x)\,dx\cdot\int_a g(x)\,dx \tag{2.1}$$

that is a well-known standard argument. Let $\mathcal{A}=\{a_j\}$ be a partition such that each element a_j is simple. Given integer $n\geq 1$ we consider the MP-transformation

$$u_{\mathcal{A},n}(x)=\sum_j u_{a_j,n}(x)\cdot\mathbf{1}_{a_j}(x)$$

that maps every a_j to itself. This is an MP-transformation of $[0,1)$ that maps each set a_j to itself and from (2.1) it follows that

$$\lim_{n\to\infty}\int_0^1 f(u_{\mathcal{A},n}(x))g(x)\,dx=\sum_j\int_{a_j} f(x)\,dx\cdot\int_{a_j} g(x)\,dx \tag{2.2}$$

for any functions $f,g\in L^2(0,1)$. An MP-transformation τ is said to be simple if $\tau^{-1}(a)$ is simple set whenever a is simple. Obviously all above described MP-transformations are simple.

We say that a function system $\{\tilde{f}_n\}$ is a transformation of another system $\{f_n\}$ if for every choice of numbers $m_k\in\mathbb{N}$ and $\lambda_k\in\mathbb{R}$ it holds the equality

$$|\{f_{m_k}(x)>\lambda_k,\ k=1,2,...,n\}|=|\{\tilde{f}_{m_k}(x)>\lambda_k,\ k=1,2,...,n\}|.$$

For example, this relation occurs when $\tilde{f}_k(x)=f_k(\tau(x))$ for some MP-transformation τ.

The following lemma is an extension of a lemma of Olevksii [8] (see also [4], ch. 10, Lemma 1) proving the same for the classical Haar system.

Lemma 2.1. Let $\Phi=\{\phi_k(x)\}$ be a complete orthonormal system and $\mathcal{F}=\{f_n\}$ be a martingale difference. Then for any sequence of numbers $\varepsilon_k>0$ there exists a transformation $\widetilde{\mathcal{F}}=\{\tilde{f}_n\}$ of the system $\mathcal{F}$ and a sequence of non-overlapping Φ-polynomials p_k such that

$$\|\tilde{f}_k-p_k\|_2<\varepsilon_k, \quad k=1,2,\dots. \tag{2.3}$$

Proof. Given general martingale difference f_n and numbers $\varepsilon_n>0$ one can find a discrete type martingale difference g_n such that $\|g_n-f_n\|_2<\varepsilon_n$. Moreover, we can also suppose that g_n is based on a filtration consisting of intervals. So without loss of generality we can assume that f_n is itself a such martingale difference. We shall realize the constructions of sequences $\tilde{f}_k$ and p_k by induction. First, we take $\tilde{f}_1=f_1$. Approximation of f_1 by a Φ-polynomial p_1 gives (2.3) for $k=1$ that is the base of induction. Then suppose that we have already defined $\tilde{f}_k$, p_k, $k=1,2,\dots,l$, satisfying the condition (2.3) such that $\tilde{f}_k(x)=f_k(\tau_l(x))$, $k=1,2,\dots,l$, where τ_l is a simple MP-transformation (maps a simple set to a simple set). Let $\mathcal{A}=\{a_j\}$ be the partition of $[0,1)$ that is formed by the maximal sets, where each function $\tilde{f}_k$, $k=1,2,\dots,l$ is constant. Clearly each a_j is a simple set. Since $u_{\mathcal{A},n}$ maps each a_j to itself, $\tau_{l+1}=\tau_l\circ u_{\mathcal{A},n}$ determines a simple MP-transformation so that $f_k(\tau_{l+1}(x))=f_k(\tau_l(x))=\tilde{f}_k(x)$, $k=1,2,\dots,l$, and

$$\int_{\alpha_i} f_{l+1}(\tau_l(x))\,dx=0, \quad i=1,2,\dots. \tag{2.4}$$

From (2.2) and (2.4) it follows that

$$\begin{aligned}\lim_{n\to\infty}\int_0^1 f_{l+1}(\tau_{l+1}(x))\phi_i(x)\,dx &= \lim_{n\to\infty}\int_0^1 f_{l+1}(\tau_l\circ u_{\alpha,n})(x)\phi_i(x)\,dx \\ &= \sum_i \int_{\alpha_i} f_{l+1}(\tau_l(x))\,dx \int_{\alpha_i}\phi_j(x)\,dx\,dx=0\end{aligned} \tag{2.5}$$

for any $i=1,2,\dots$. We will chose n bigger enough and define $\tilde{f}_{l+1}(x)=f_{l+1}(\tau_{l+1}(x))$. Let c_i be the Fourier coefficients of the function $\tilde{f}_{l+1}$ in system Φ. Suppose that each polynomial p_k, $k=1,2,\dots,l$, is a linear combination of functions ϕ_j, $j=1,2,\dots,m$. From (2.5) it follows that for a bigger enough n we have $\sum_{i=1}^m c_i^2<\varepsilon_{l+1}^2/4$. Then we can chose an integer $r>m$ such that $\sum_{i=r+1}^\infty c_i^2<\varepsilon_{l+1}^2/4$. Define

$$p_{l+1}(x)=\sum_{i=m+1}^{r} c_i\phi_i(x).$$

Since Φ is a complete system, one can easily check that (2.3) is satisfied for $k=l+1$ that finalizes the induction and so the proof of lemma. $\square$

3. Proof of Theorem 1.2

We will first prove the theorem for the classical Haar system. Let h_n be the L^2-normalized classical Haar system. For a given function $f\in L^1(0,1)$ let $\sum_{k=1}^\infty a_k h_k$ be the Fourier-Haar series of f. Recall the maximal and the square functions operators defined by

$$\mathcal{M}f(x)=\sup_{n\geq 1}\left|\sum_{k=1}^n a_k h_k(x)\right|, \quad Sf(x)=\left(\sum_{k=1}^\infty a_k^2 h_k^2(x)\right)^{1/2}.$$

It is well known the boundedness of both operators on L^p, $1<p<\infty$ (see for example [4], ch. 3, Theorems 4 and 9). A key point in the proof of Theorem 1.2 is the following good-λ inequality due to Chang-Wilson-Wolff (see [1], Corollary 3.1):

$$\begin{aligned}(3.1)\quad &|\{x\in[0,1): \mathcal{M}f(x)>\lambda,\ Sf(x)<\varepsilon\lambda\}|\\ &\lesssim \exp\left(-\frac{c}{\varepsilon^2}\right)|\{\mathcal{M}f(x)>\lambda/2\}|,\quad \lambda>0,\ 0<\varepsilon<1.\end{aligned}$$

So let p_k, $k=1,2,\dots,n$, be a monotonic sequence of Haar polynomials. We have $|g(x)|\leq\mathcal{M}g(x)$ a.e. for any function $g\in L^1$, as well as $Sp_k(x)\leq Sp_n(x)$, $k=1,2,\dots,n$. Thus, applying inequality (3.1) with $\varepsilon_n=(c/\ln n)^{1/2}$, we obtain

$$\begin{aligned}(3.2)\quad &|\{|p_k(x)|>\lambda,\ Sp_n(x)\leq\varepsilon_n\lambda\}|\\ &\lesssim \exp\left(-\frac{c}{\varepsilon_n^2}\right)|\{\mathcal{M}p_k(x)>\lambda/2\}|.\end{aligned}$$

For $p^*(x)=\max_{1\leq m\leq n}|p_m(x)|$ we obviously have

$$\begin{aligned}\{p^*(x)>\lambda\}\subset\{&p^*(x)>\lambda,\ Sp_n(x)\leq\varepsilon_n\lambda\}\\ &\cup\{Sp_n(x)>\varepsilon_n\lambda\}=A(\lambda)\cup B(\lambda),\end{aligned}$$

and thus

$$\|p^*\|_2^2\leq 2\int_0^\infty \lambda|A(\lambda)|\,d\lambda+2\int_0^\infty\lambda|B(\lambda)|\,d\lambda.$$

From (3.2) it follows that

$$\begin{aligned}\int_0^\infty\lambda|A(\lambda)|\,d\lambda&\leq\sum_{m=1}^n\int_0^\infty\lambda|\{|p_m|>\lambda,\ Sp_n\leq\varepsilon_n\lambda\}|\,d\lambda\\ &\leq\exp\left(-\frac{c}{\varepsilon_n^2}\right)\sum_{m=1}^n\int_0^\infty\lambda|\{\mathcal{M}p_m>\lambda/2\}|\,d\lambda\\ &\lesssim\frac{1}{n}\sum_{m=1}^n\|\mathcal{M}p_m\|_2^2\end{aligned}$$

$$\lesssim \frac{1}{n}\sum_{m=1}^{n}\|p_m\|_2^2$$
$$\le \|p_n\|_2^2.$$

Combining this and

$$2\int_0^\infty \lambda|B(\lambda)|\,d\lambda = \varepsilon_n^{-2}\|Sp_n\|_2^2 \lesssim \log n\cdot\|p_n\|_2^2,$$

we get

$$\|p^*\|_2 = \left\|\max_{1\le m\le n}|p_m(x)|\right\|_2 \lesssim \sqrt{\log n}\cdot\|p_n\|_2$$

that proves the theorem for the Haar system. Clearly we will have the same bound also for any transformation of the Haar system. To proceed the general case we suppose that $\mathcal{F}=\{f_n\}$ is an arbitrary martingale difference and let

$$F_k = \sum_{j\in G_k} c_j f_j, \quad k=1,2,...,n,$$

be an arbitrary monotonic sequence of $\mathcal{F}$-polynomials. Apply Lemma 2.1, choosing Φ to be the Haar classical system and $\varepsilon_j=\varepsilon$ for $j\in G_n$. So we get (2.3) for non-overlapping Haar polynomials p_k. Denote $\widetilde{F}_k=\sum_{j\in G_k} c_j\tilde{f}_j$. Obviously,

$$P_k = \sum_{j\in G_k} c_j p_j, \quad k=1,2,...,n,$$

forms a monotonic sequence of Haar polynomials. For a small enough ε we will have

$$\|\widetilde{F}_k - P_k\|_2 \le \left(\sum_{j\in G_n} c_j^2\right)^{1/2}\left(\sum_{j\in G_n}\varepsilon_j^2\right)^{1/2}$$
$$= \varepsilon\sqrt{\#(G_n)}\left(\sum_{j\in G_n} c_j^2\right)^{1/2} \le \frac{\|P_n\|_2}{n}.$$

Therefore, taking into account that the theorem is true for the Haar system, we get

$$\left\|\max_{1\le m\le n}|F_k|\right\|_2 = \left\|\max_{1\le m\le n}\left|\widetilde{F}_k\right|\right\|_2 \le \left\|\max_{1\le m\le n}|P_k|\right\|_2 + \|P_n\|_2$$
$$\lesssim \sqrt{\log n}\cdot\|P_n\|_2 \lesssim \sqrt{\log n}\cdot\|F_n\|_2.$$

This completes the proof of theorem.

4. Proof of Theorem 1.3

The upper bound $\mathcal{K}_n(\mathcal{H})\lesssim\sqrt{\log n}$ follows from Theorem 1.2. The lower bound

$$\mathcal{K}_n(\mathcal{H})\gtrsim\sqrt{\log n} \tag{4.1}$$

for the classical Haar system follows from the Nikishin-Ulyanov [7] inequality

$$\left\|\sup_{1\le m\le n}\left|\sum_{k=1}^{m}a_k\chi_{\sigma(k)}\right|\right\|_2\gtrsim\sqrt{\log n}\cdot\left(\sum_{k=1}^{n}a_k^2\right)^{1/2},$$

valid for appropriate coefficients a_k and permutation σ of the numbers $\{1,2,...,n\}$. We will have the same estimate (4.1) also for any transformation of the classical Haar system. Then we apply Olevskii lemma ([4], ch. 10, Lemma 1), that is the case of Lemma 2.1 when $\mathcal{F}$ coincides with the classical Haar system. So we get a transformed Haar system $\{\tilde{h}_n\}$ and a sequence of non-overlapping Φ-polynomials p_k such that

$$\|\tilde{h}_k-p_k\|_2<\varepsilon_k,\quad k=1,2,....$$

Since ε_k's here can be arbitrarily small, one can conclude $\mathcal{K}_n(\Phi)\ge\mathcal{K}_n(\mathcal{H})$. Combining this and (4.1) we get the following.

Proposition 4.1. If Φ is a complete orthonormal system, then $\mathcal{K}_n(\Phi)\gtrsim\sqrt{\log n}$.

Since any generalized Haar system is complete, the lower bound (4.1) immediately follows from Proposition 4.1.

5. Proof of corollaries

Lemma 5.1. ([3], Theorem 5.3.2) Let $\{\phi_n(x)\}$ be an orthonormal system and $\omega(n)\nearrow\infty$ be a sequence of positive numbers. If an increasing sequence of indexes n_k satisfy the bound $\omega(n_k)\ge k$, then the condition $\sum_{k=1}^{\infty}a_k^2\omega(k)<\infty$ implies a.e. convergence of sums $\sum_{j=1}^{n_k}a_j\phi_j(x)$ as $k\to\infty$.

Proof of Corollary 1.4. Consider the series

$$\sum_{k=1}^{\infty}a_kp_k(x)$$

with coefficients satisfying the condition $\sum_{k=1}^{\infty}a_k^2\log k<\infty$ and denote $S_n=\sum_{k=1}^{n}p_k$. Since $\omega(n)=\log n$ satisfies the condition $\omega(2^k)\ge k$, from Lemma 5.1 we have a.e. convergence of subsequences $S_{2^k}(x)$. So we just need to show that

$$\delta_k(x)=\max_{2^k<n\le 2^{k+1}}|S_n(x)-S_{2^k}(x)|\longrightarrow 0\quad\text{a.e. as } k\longrightarrow\infty. \tag{5.1}$$

We have

$$\|\delta_k\|_2 \le \mathcal{K}_{2^k}(\mathcal{F}) \left(\sum_{j=2^k+1}^{2^{k+1}} a_j^2 \right)^{1/2} \lesssim \sqrt{k} \left(\sum_{j=2^k+1}^{2^{k+1}} a_j^2 \right)^{1/2}.$$

So we get

$$\sum_{k=1}^{\infty} \|\delta_k\|_2^2 \le \sum_{k=1}^{\infty} k \sum_{j=2^k+1}^{2^{k+1}} a_j^2 \le \sum_{j=1}^{\infty} a_j^2 \log j < \infty,$$

which implies (5.1). □

To prove the next corollary we will need another lemma.

Lemma 5.2. ([13], [9]) Let $u(n)$ be a C-multiplier for any rearrangement of the orthonormal system $\Phi=\{\phi_n(x)\}$. If an increasing sequence of positive numbers $\delta(k)$ satisfies the condition

$$\sum_{k=1}^{\infty} \frac{1}{\delta(k) k \log k} < \infty, \tag{5.2}$$

then $\delta(n)u(n)$ turns to be a UC-multiplier for Φ.

Proof of Corollary 1.6. According to Corollary 1.4 $u(n)=\log n$ is a C-multiplier for the systems of non-overlapping MD-polynomials and their rearrangements. By the hypothesis of Corollary 1.6 the sequence $\delta(n)=\omega(n)/\log n$ is increasing and satisfies (5.2). Thus, the combination of Corollary 1.4 and Lemma 5.2 completes the proof. □

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