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The Hartogs extension theorem for holomorphic vector bundles and sprays

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Abstract. We give a detailed proof of Siu's theorem on extendibility of holomorphic vector bundles of rank larger than one, and prove a corresponding extension theorem for holomorphic sprays. We apply this result to study ellipticity properties of complements of compact subsets in Stein manifolds. In particular we show that the complement of a closed ball in $\mathbb{C}^n, n \geq 3$, is not subelliptic.

1. Introduction

The main purpose of the present article is to investigate ellipticity properties of complements of compact sets in Stein manifolds, and our main result is the following.

Theorem 1.1. *Let X be a Stein manifold with $\dim X \geq 3$ and let $K \subset X$ be a compact subset.*
  – If $X \setminus K$ is elliptic, then K has only finitely many accumulation points.
  – If $X \setminus K$ is subelliptic and smoothly bounded, then K is empty.

The notion of ellipticity for complex manifolds was introduced by Gromov in [12] where he proved an Oka principle for holomorphic sections of elliptic bundles, generalizing previous work of Grauert [13] for complex Lie groups. For ellipticity as well as the notion of subellipticity we refer to the book of Forstnerič [7]. A short overview can be found in this article in Section 7. The question which type of sets have Oka complements has been mentioned for example in [7, Problem 5.16.4] and Oka properties of ball complements have been investigated in [11]. In [7, Propo-

sitions 6.4.1, 6.4.3] (sub-)ellipticity properties for complements of subvarieties are studied.

The main ingredient in the proof of Theorem 1.1 is the following Hartogs type extension theorem for holomorphic vector bundles on Stein manifolds which was stated by Siu in [16]:

Theorem 1.2. *Let X be a Stein manifold and let $K \subset X$ be a holomorphically convex compact set with connected complement. Let $E \to X \setminus K$ be a holomorphic vector bundle. If $\dim X \geq 3$, then there is a finite set of points $P \subset K$ such that E extends to a holomorphic vector bundle on $X \setminus P$.*

For proving this result Siu suggested to use methods from Andreotti–Grauert theory. To our knowledge no proof of this statement exists in the literature, and so we carry it out in detail.

A proof of Theorem 1.2 in the case of line bundles was recently given by Fornæss–Sibony–Wold [3], and a description of the obstructions for the extension of roots of line bundles in dimension two was given by Ivashkovich [15].

It turns out that the extension of vector bundles across critical level sets of an exhaustion function poses some technical difficulties which may be of independent interest, see Theorem 1.3.

Theorem 1.3. *Let X be a complex manifold with $\dim X \geq 3$ and let $M \subset X$ be a closed totally real Lipschitz subset of X. Then for any holomorphic vector bundle $E \to X \setminus M$, there exists a discrete set of points $P \subset M$ such that E extends to a holomorphic vector bundle on $X \setminus P$.*

For the notion of a *totally real Lipschitz set*, see Definitions 2.5 and 2.7 on p. 4. In view of Theorem 1.3 the following corollary is an immediate consequence of the proof of Theorem 1.2:

Corollary 1.4. *Let X be a complex manifold and $M \subset X$ a closed, totally real Lipschitz set. If $X \setminus M$ is elliptic, then M is a discrete set of points.*

The paper is organized as follows. In Section 2 we present some results concerning the density of pseudoconcave points and the extension of mappings. Next, in Section 3, we show how to modify exhaustion functions near critical points. In Section 4, we consider extensions of sections and maps of fiber bundles, which—among other methods—are used to show the uniqueness of the extensions of bundles. In Section 5 we show how to extend bundles across non-critical level sets, and in Sec-

tion 6 we give the proofs of Theorems 1.2 and 1.3. Finally, in Section 7, we explain the application to Oka theory and prove Theorem 1.1.

2. Pseudoconcave points of compact sets and extension of mappings

Definition 2.1. By $\triangle^n := \{z = (z_1, ..., z_n) \in \mathbb{C}^n : |z_1| < 1, ..., |z_n| < 1\}$ we denote the open unit polydisc in $\mathbb{C}^n$.

For $0 \leq r < s$ define according to Siu [17, Chapter 2, Section 3]

$$G^n(r, s) := \big\{ (z_1, ..., z_n) \in (s \cdot \triangle)^n : |z_k| > r \text{ for some } k \in \{1, ..., n\} \big\}$$
$$= s \cdot \triangle^n \backslash r \cdot \overline{\triangle}^n.$$

For $q = 1, 2, ..., n-1$ we define a *standard q-Hartogs figure* $\mathfrak{h}^q$ by

$$\mathfrak{h}^q := \big(\triangle^q \times G^{n-q}(1/2, 1) \big) \cup \big(1/2 \cdot \triangle^q \times \triangle^{n-q} \big).$$

By a *q-Hartogs figure* we mean the biholomorphic image $\phi(\mathfrak{h}^q)$ of a standard q-Hartogs figure under an injective holomorphic map $\phi \colon \triangle^n \to X$ into a complex manifold X. We set $\tilde{\mathfrak{h}}^q := \phi(\triangle^n)$. If not stated explicitly otherwise, we always assume $n = \dim X$. When there is no danger of confusion, we will, by abuse of notation, write $\mathfrak{h}^q$ also for its biholomorphic image $\phi(\mathfrak{h}^q)$.

Definition 2.2. Let $K \subset \mathbb{C}^n$ be a compact set. We call a point $x \in K$ *strictly q-pseudoconcave* if for any open neighborhood U of x, there exists a q-Hartogs figure $\mathfrak{h}^q \subset U \backslash K$ with $x \in \tilde{\mathfrak{h}}^q$.

We will use the following notation: For any set $K \subset \mathbb{C}^k \times \mathbb{C}^m$ and $z_0 \in \mathbb{C}^k$, we denote

$$K_{z_0} := K \cap \big(\{z_0\} \times \mathbb{C}^m \big).$$

Note that if $x \in K_{z_0}$ is a strictly q-pseudoconcave point for K_{z_0} in $\mathbb{C}^m$, then x is strictly $(q+k)$-pseudoconcave for K, since q-Hartogs figures in $\{z_0\} \times \mathbb{C}^m$ can be fattened up slightly.

Proposition 2.3. *Let $K \subset \mathbb{C}^n$ be a compact set. Then the strictly q-pseudo-concave points in the Shilov boundary $\breve{S}(P(K))$ of the algebra $P(K)$ are dense in $\breve{S}(P(K))$ for $q = 1, 2, ..., n-1$.*

Remark 2.4. Let $S_q(K)$ denote the set of all strictly q-pseudoconcave points of the compact $K \subset \mathbb{C}^n$. Then the statement of Proposition 2.3 can be rephrased as

$$\overline{S_q(K) \cap \breve{S}(P(K))} = \breve{S}(P(K)) \quad \text{for } q = 1, 2, ..., n-1$$

Also note the following inclusions: $S_1(K) \subseteq S_2(K) \subseteq ... \subseteq S_{n-1}(K)$.

Proof. Note first that a global extreme point $x \in K$ is q-pseudoconcave for $0 < q < n$: Let $x \in K$ be such that $\|x\| = \sup_{z \in K} \|z\|$. After a unitary transformation we may assume that $x = e_1 = (1, 0, ..., 0)$. Then clearly $(\{e_1\} \times \{\mathbb{C}^{n-1}\}) \cap K = \{e_1\}$, so it is easy to construct q-Hartogs figures at x.

We proceed to show that if $x \in \breve{S}(P(K))$ and if $\varepsilon > 0$, then there exists a point $x' \in K$ with $\|x' - x\| < \varepsilon$ and a holomorphic automorphism $\Lambda \in \mathrm{Aut}_{\mathrm{hol}}(\mathbb{C}^n)$ such that $\Lambda(x')$ is a global extreme point for the set $\Lambda(K)$.

The peak points for $P(K)$ are dense in $\breve{S}(P(K))$, so we may assume that the point x is a peak point for the algebra $P(K)$. Then x is not in the polynomially convex hull of the set $K' := K \setminus B_\varepsilon(x)$, where $B_\varepsilon(x)$ denotes the ball of radius ε with center x. Hence, for any $R > 0$ and any $\delta > 0$, by the Andersén–Lempert Theorem [1], [2], [4] and [5] there exists $\Lambda \in \mathrm{Aut}_{\mathrm{hol}}(\mathbb{C}^n)$ such that $\|\Lambda - \mathrm{id}\|_{K \setminus B_\varepsilon(x)} < \delta$ and such that $\|\Lambda(x)\| > R$. If R is chosen large and δ is chosen small, this proves the proposition. $\square$

We denote the coordinates on $\mathbb{C}^n$ by $z_1, ..., z_n$ with $z_j = x_j + iy_j$, and we let x denote the tuple $(x_1, ..., x_n)$ and y denote the tuple $(y_1, ..., y_n)$.

Definition 2.5. By a Lip-α-graph M at the origin we mean a set $M = \{y = \psi(x)\}$ with $\psi(0) = 0$ and ψ Lipschitz continuous with Lipschitz constant $\alpha > 0$, defined in a neighborhood of 0 in $\mathbb{R}_x^n$.

Corollary 2.6. *Let $M \subset \mathbb{C}^n$ be a compact subset of a closed Lip-α graph near the origin with $0 < \alpha < 1$. Then there exists a dense set of points $\Sigma \subset M$ such that for any point $x \in \Sigma$ and any open set U containing x, there exists an embedding $\phi: \triangle^n \to U$ containing x, with $\phi(\mathfrak{h}^1) \cap M = \varnothing$.*

Proof. In view of [18, Theorem 1.6.9, p. 55] we observe that a Lip-α graph M over $\mathbb{R}_x^n$ satisfies $M = \breve{S}(P(M))$. Then the statement follows from Proposition 2.3 with $K = M$ and $q = 1$. $\square$

Definition 2.7. Let X be a complex manifold. A subset $M \subseteq X$ is called a *totally real Lipschitz set* if for each point $p \in M$ there exists a holomorphic coordinate

neighborhood $z\colon U\to V$ containing p and a constant $0<\alpha<1$ such that $z(U\cap M)$ is contained in a Lip-α-graph of a map $\psi\colon \mathbb{R}^n_x\cap V\to\mathbb{R}^n_y\cap V$, where $z=x+iy$.

Corollary 2.8. *Let X be a complex manifold with $\dim X\geq 2$ and let $M\subset X$ be a closed totally real Lipschitz submanifold. Furthermore let Z be a complex manifold, let Y be a Stein manifold and let $f\colon (X\setminus M)\times Z\to Y$ be a holomorphic map. Then f extends holomorphically across $M\times Z$.*

Proof. The problem is local, so let W be an open connected subset of the origin, assume that M is a closed subset of W, and let $f\in\mathcal{O}((W\setminus M)\times Z)$. Since Y is Stein, it is enough to consider functions thanks to the existence of a proper holomorphic embedding of Y in some $\mathbb{C}^N$. Note that $W\setminus M$ is connected. By the previous corollary there exists a point $x\in M$ and an embedding $\phi\colon \triangle^n\to W$ with $x\in\phi(\triangle^n)$ and $\phi(\mathfrak{h}^1)\cap M=\varnothing$. Note that $\phi(\triangle^n)\setminus M$ is connected. Let $\tilde{f}$ be the extension of $f|_{\phi(\mathfrak{h}^1)\times Z}$ to $\phi(\triangle)\times Z$ according to Lemma 2.9. Now f and $\tilde{f}$ are both holomorphic on the connected open set $\phi(\triangle^n)\setminus M$ and they coincide on the open set $\phi(\mathfrak{h}^1)$. By the identity principle they coincide on $\phi(\triangle)\setminus M$, and then clearly $\tilde{f}$ extends f across $(M\cap\phi(\triangle^n))\times Z$.

Now let $\Omega\subset M$ be the largest open set (in the relative topology) such that f extends across $\Omega\times Z$. This is well defined since any extension across a point of M is unique if it exists, and it is non-empty by the argument above. Note that $M\setminus\Omega$ is either empty or it is a closed subset of a Lip-α-graph, and in the latter case we would get a contradiction to the assumption that Ω is maximal, since, due to Corollary 2.6, we could apply the above argument again. Hence, $M=\Omega$. $\square$

Lemma 2.9. *Let Z be a complex manifold and let $f\colon \mathfrak{h}^q\times Z\to\mathbb{C}$ be a holomorphic function, where $\mathfrak{h}^q$ stands for the standard q-Hartogs figure. Then f extends to a holomorphic function $\tilde{f}\colon \triangle^n\times Z\to\mathbb{C}$.*

Proof. Let (w,z) denote the coordinates on $\triangle^n\times Z$. By the classical Hartogs extension theorem, for each $z\in Z$ the map $f(\cdot,z)$ extends uniquely to a holomorphic function $\tilde{f}_z\colon \triangle^n\to\mathbb{C}$, and we let these extensions determine an extension $\tilde{f}$. It remains to show that the extension is actually holomorphic, and since holomorphicity is a local property, we can use local coordinates and assume that Z is a domain in $\mathbb{C}^m$. Using the classical Hartogs extension theorem for partial derivatives in the z-directions on $\mathfrak{h}^q\times\mathbb{C}^m$, it is easy to see that $\tilde{f}$ is holomorphic in each variable separately, hence $\tilde{f}$ is holomorphic. $\square$

If M is $\mathcal{C}^2$-smooth, it is easy to construct a 1-Hartogs figure directly; we denote by x resp. y the real resp. imaginary part of $z=x+iy\in\mathbb{C}^n$:
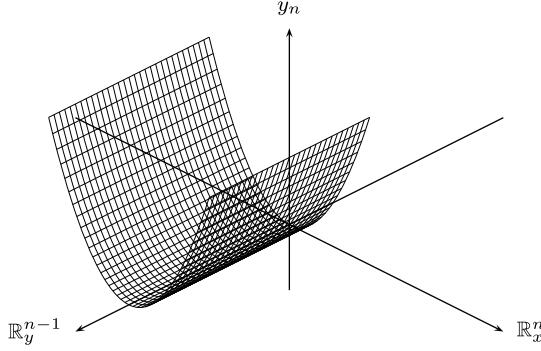
Figure 1. The lower bound $y_n = (C/2 - K) \cdot \|x\|^2$ for $\mathrm{Im}(f_C(x, \psi(x)))$ in Lemma 2.10.

Lemma 2.10. *Let $M = \{y = \psi(x)\} \subset \mathbb{C}^n$ be a $\mathcal{C}^2$-smooth graph at the origin with $d\psi(0) = 0$. Then for any open set U containing the origin, there exists an embedding $\phi \colon \triangle^n \to U$ containing the origin, with $\phi(\mathfrak{h}^1) \cap M = \varnothing$.*

Proof. For $C > 0$ we let $F_C \colon \mathbb{C}^n \to \mathbb{C}^n$ be the map

$$F_C(z) = \left(z_1, ..., z_{n-1}, f_C(z) \right) := \left(z_1, ..., z_{n-1}, z_n + Ci \cdot \sum_{j=1}^n z_j^2 \right).$$

Since F_C is injective near the origin, there exists a constant $K > 0$ such that $\|\psi(x)\| < K\|x\|^2$, and by staying close enough to the origin we may assume that $\|\psi(x)\| \leq 1/2 \cdot \|x\|$. Choosing $C > 2K$ we see that

$$\mathrm{Im}\left(f_C\left(x, \psi(x) \right) \right) = C \sum_{j=1}^n \left(x_j^2 - \psi_j(x)^2 \right) + \psi_n(x)$$

$$\geq C \sum_{j=1}^n 1/2 \cdot x_j^2 - K\|x\|^2,$$

Therefore we obtain $\mathrm{Im}(f_C(x, \psi(x))) \geq (C/2 - K) \cdot \|x\|^2$. Using this it is easy to construct a Hartogs figure, see also Figure 1. $\square$

3. Modification of a strictly plurisubharmonic function near a Morse critical point

In this section we show how to modify a strictly plurisubharmonic function near a nice Morse critical point, in order to extend sections and bundles across

critical points. This is very similar to what is done in Oka–Grauert theory, and the reader can compare the following proposition with Lemma 3.10.1 in [7]. We feel however that the proof we give here is simpler. It is motivated by ideas in [3].

Definition 3.1. We denote the super-level sets of an exhaustion function ρ of the complex manifold X by $X^c:=\{x\in X : \rho(x)>c\}$, the sub-level sets by $X_c:=\{x\in X : \rho(x)<c\}$, and its level sets by $\Gamma_c:=\{x\in X : \rho(x)=c\}$.

Definition 3.2. Let $\rho\colon X\to\mathbb{R}$ be a Morse exhaustion function of the complex manifold X. A critical point $x_0\in X$ of ρ is called *nice* (see [14, Definition 2.4]) if there exist local holomorphic coordinates $z=(z_1,...,z_n)=(x_1+iy_n,...,x_n+iy_n)$ such that ρ is of the form

$$(1) \qquad \rho(z)=\rho(x_0)+\sum_{j=1}^{m}\left(x_j^2+\mu_j y_j^2\right)+\sum_{j=m+1}^{n}\left(x_j^2-\mu_j y_j^2\right),$$

with $0\leq\mu_j\leq1$ for $j=1,...,n$ and $\mu_j<1$ for $j=m+1,...,n$.

For a compact set $L\subset X$ the Morse exhaustion function is called *nice* on $X\setminus L$ if the set of critical points of ρ in $X\setminus L$ is discrete and if each of them is nice, with not more than one critical point in each level set.

Proposition 3.3. *Let $\rho\colon X\to\mathbb{R}$ be a strictly plurisubharmonic exhaustion function, let $x\in X$ be a nice critical point with $\rho(x)=c$ and let some $c'>c$ be such that there are no critical values in the interval $(c,c']$. Choose coordinates z in which ρ assumes the form as in (1) for $\|z\|<1$. We can assume that all μ_j for $j=1,...,m$ in (1) are positive (for if not, we may move the corresponding x_j^2 terms to the second sum and renumber). Then for every $\varepsilon>0$, there exist $\varepsilon_1<\varepsilon_2<\varepsilon$ and $s>0$, and there exists a strictly plurisubharmonic function $\tilde\rho$ on X such that the following hold:*
  (i) *in the coordinate z, on $\mathbb{B}_{\varepsilon_1}(0)$, we have that*

$$(2) \qquad \tilde\rho(z)=c+\sum_{j=1}^{m}\left(x_j^2+\mu_j y_j^2\right)+\sum_{j=m+1}^{n}x_j^2,$$

with $0<\mu_j\leq1$ for $j=1,...,m$,
  (ii) $\{x\in X:\tilde\rho(x)=c\}\setminus\mathbb{B}_{\varepsilon_1}\subset\{x\in X : \rho(x)<c\}$,
  (iii) $\tilde\rho(x)=\rho(x)+s$ *for $x\in X\setminus\mathbb{B}_{\varepsilon_2}(0)$, and*
  (iv) *there are no critical values for $\tilde\rho$ in the interval $(c,c'+s]$.*

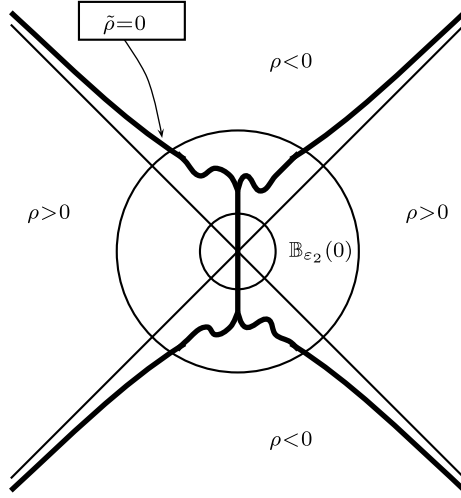See also Figure 2 for an illustration of the properties (i)–(iv) of Proposition 3.3.

Figure 2. Modification of ρ to $\tilde{\rho}$ in Proposition 3.3, assuming for simplicity that the form (2) is reached at this step.

Proof. For simplicity we assume that $c=0$, and hence we can locally represent the function ρ as

$$(3) \qquad \rho(z) = \sum_{j=1}^{m} \left(x_j^2 + \mu_j y_j^2\right) + \sum_{j=m+1}^{n} \left(x_j^2 - \mu_j y_j^2\right).$$

Fix a cutoff function $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$ with $\chi \equiv 1$ on $[-1/2, 1/2]$, $\operatorname{supp} \chi \subset (-1, 1)$, and $0 \leq \chi \leq 1$. Furthermore, for every $0 < \varepsilon < 1$, and every $\delta \in \{0\}^m \times [0, 1)^{n-m}$, we set

$$(4) \qquad \psi_{\varepsilon, \delta}(z) := \chi\left(\frac{\|z\|^2}{\varepsilon^2}\right) \cdot \sum_{j=m+1}^{n} \delta_j y_j^2$$

The function $\tilde{\rho}$ shall be defined as a sum

$$(5) \qquad \tilde{\rho}(z) := \rho(z) + \sum_{k=1}^{\ell} \psi_{\varepsilon_k, \delta(k)}(z) + s \cdot \left(1 - \chi\left(\frac{\|z\|^2}{\varepsilon_{\ell+1}^2}\right)\right).$$

The following lemma provides the crucial step in the proof.

Lemma 3.4. *Fix the function χ as above and let $0 < \mu < 1$ and $\mu' > 0$. Then there exists $0 < \delta_0 < 1$ such that if ρ is a function of the form (3) on $\mathbb{B}_\varepsilon^n$ with $\mu_j \geq \mu'$ for $j = 1, ..., m$, and $\mu_j \leq \mu$ for $j = m+1, ..., n$, and if $\psi_{\varepsilon, \delta}$ as above satisfies $0 < \delta_j \leq \mu_j$ and $\delta_j < \delta_0$ for $j = m+1, ..., n$, then $\rho_{\varepsilon, \delta} = \rho + \psi_{\varepsilon, \delta}$ is a strictly plurisubharmonic function on $\mathbb{B}_\varepsilon^n$, with all critical points contained in $\{\rho_{\varepsilon, \delta} \leq 0\}$.*

Before we prove the lemma, we show how it is used by completing the proof of the above proposition:

Let $\mu:=\max_{m+1\leq j\leq n}\{\mu_j\}$, $\mu':=\min_{1\leq j\leq m}\{\mu_j\}$, and let δ_0 be as in the lemma, depending on μ and μ', and fix an $l\in\mathbb{N}$ and δ_j for $j=m+1,...,n$ such that $\mu_j=l\delta_j$ with $\delta_j<\delta_0$. Set $\varepsilon_k=(1/2)^k$ for $k=1,...,l+1$, and set $\delta(k)=(\delta_{m+1},...,\delta_n)$ for $k=1,...,l$. The function $\tilde{\rho}$ defined by (5) will now satisfy all the desired properties for sufficiently small s. $\quad\square$

Proof of Lemma 3.4. By computing the Levi form of $\psi_{\varepsilon,\delta}$, we see that there exists a constant M, independent of ε and δ, such that $\mathcal{L}_z(\psi_{\varepsilon,\delta};w)\geq -M\delta\|w\|^2$, where $\delta=\max_{m<j\leq n}\{\delta_j\}$, hence

$$\mathcal{L}_z(\rho+\psi_{\varepsilon,\delta};w)\geq\frac{1}{2}\left(\sum_{j=1}^m(1+\mu_j-M\delta)|w_j|^2+\sum_{j=m+1}^n(1-\mu_j-M\delta)|w_j|^2\right).$$

So if $\delta_0<\frac{1-\mu}{M}$ we see that $\rho_{\varepsilon,\delta}$ is strictly plurisubharmonic.

Next we compute the gradient

$$\begin{aligned}\nabla\rho_{\varepsilon,\delta}(z) &= \nabla\rho(z)+\nabla\psi_{\varepsilon,\delta}(z)\\
&= 2(x_1,...,x_n,\mu_1y_1,...,\mu_my_m,-\mu_{m+1}y_{m+1},...,-\mu_ny_n)\\
&\quad+\chi\left(\frac{\|z\|^2}{\varepsilon^2}\right)\cdot 2(0,...,0,\delta_{m+1}y_{m+1},...,\delta_ny_n)\\
&\quad+\left(\frac{2}{\varepsilon^2}\chi'\left(\frac{\|z\|^2}{\varepsilon^2}\right)\cdot\sum_{j=m+1}^n\delta_jy_j^2\right)\cdot(x_1,...,x_n,y_1,...,y_n).\end{aligned}$$

Now fix z and assume that $\nabla\rho_{\varepsilon,\delta}(z)=0$. If $y_{m+1}=y_{m+2}=...=y_n=0$, then $\nabla\rho_{\varepsilon,\delta}(z)=\nabla\rho(z)$, hence $x_1=x_2=...=x_n=y_1=y_2=...=y_m=0$.

Next we assume that $x\neq 0$ and observe that this requires

$$\frac{2}{\varepsilon^2}\chi'\left(\frac{\|z\|^2}{\varepsilon^2}\right)\cdot\sum_{j=1}^n\delta_jy_j^2=-2.$$

Now there is at least one $j\in\{m+1,...,n\}$ such that $y_j\neq 0$. The y_j-component in the sum of the first two lines above is $2y_j(\chi(\frac{\|z\|^2}{\varepsilon^2})\delta_j-\mu_j)$. But the expression in brackets can never be 1, so the y_j-component of $\nabla\rho_{\varepsilon,\delta}(z)$ is not zero. Finally, assume that $y_k\neq 0$ for $1\leq k\leq m$. Note that $\sum_{j=m+1}^n\delta_jy_j^2\leq\delta_0\|z\|^2<\delta_0\varepsilon^2$, so the y_k-component of

the third term in the sum above is less than $2\|\chi'\|\delta_0|y_k|$ in absolute value. Therefore, as long as δ_0 is chosen small enough so that $2\|\chi'\|\delta_0 < \mu'$, the gradient is non-zero.

We conclude that $x_1 = x_2 = \ldots = x_n = y_1 = \ldots = y_m = 0$, and $\rho_{\varepsilon,\delta}(z) \leq 0$. $\quad\square$

4. The Hartogs extension theorem for holomorphic fiber bundles

Theorem 4.1. *Let X be a Stein manifold, $\dim X \geq 2$, let $K \subset K' \subset X$ be compact sets with K holomorphically convex and $X \setminus K'$ connected, and let $Q \subset X \setminus K$ be a closed discrete set of points. Let $\pi \colon E \to X \setminus (K \cup Q)$ be a holomorphic fiber bundle with Stein fiber, and let Y be a Stein manifold. Then the following hold:*

(i) Any holomorphic map $f \colon \pi^{-1}(X \setminus (K' \cup Q)) \to Y$ extends uniquely to a holomorphic map $\tilde{f} \colon E \to Y$, and

(ii) any holomorphic section $s \colon X \setminus (K' \cup Q) \to E$ extends uniquely to a section $\tilde{s} \colon X \setminus (K \cup Q) \to E$.

Proof. We first argue that we can assume that $Q = \varnothing$. For this purpose we let ρ be a strictly plurisubharmonic exhaustion function of X, and we pick $c > 0$ such that $K' \subset \{\rho < c\}$ and $\{\rho = c\} \cap Q = \varnothing$. Choose $c' < c$ such that $K' \cup (Q \cap \{\rho < c\}) \subset \{\rho < c'\}$. Then $K \cup (Q \cap \{\rho < c\})$ is holomorphically convex in $\{\rho < c\}$, so we can let this be our new set K, and K' be the set $\{\rho \leq c'\}$, and we replace X by $\{\rho < c\}$.

Let $\rho \colon X \to \mathbb{R}$ be a $\mathcal{C}^\infty$-smooth non-negative plurisubharmonic exhaustion function such that $K = \{\rho = 0\}$, $\rho > 0$ on $X \setminus K$, and ρ is strictly plurisubharmonic on $X \setminus K$ (see e.g [18, Theorem 1.3.8]). By the Morse lemma and [14, Lemma 2.5] we can assume that ρ is a nice exhaustion function, i.e. near each critical point $x_0 \in X \setminus K$ the exhaustion function ρ is of the form (1) in suitable local coordinates. We will give the proof of (i); the proof of (ii) is almost identical.

Choose a non-critical value $c > 0$ such that $K' \subset X_c$, and define

$$(6) \qquad s = \inf\big\{0 \leq t \leq c : f \text{ extends to } \pi^{-1}(X^t)\big\}.$$

By Corollary 4.3 below we obtain that $s < c$. Assume to get a contradiction that $s > 0$. By Corollary 4.3 or Lemma 4.4 f would extend further. $\quad\square$

We now give the lemmas used in the above proof.

Lemma 4.2. *Let $c' > 0$ be a non-critical value for ρ, and assume that $g \colon \pi^{-1}(X^{c'}) \to Y$ is a holomorphic map. Then there exists $c'' < c'$ and an extension of g to $\pi^{-1}(X^{c''})$.*

Proof. For any point $x \in \Gamma_{c'}$ there exist local holomorphic coordinates such that $\Gamma_{c'}$ is strictly convex near x. Hence, by compactness, there exist Hartogs figures $(\mathfrak{h}_j^1, \tilde{\mathfrak{h}}_j^1)$, $j = 1, ..., m$, in $X^{c'}$ such that for all j

(7) $\qquad \tilde{\mathfrak{h}}_j^1 \cap \widetilde{X}^{c'}$ is connected for all small $\mathcal{C}^2$-perturbations $\widetilde{X}^{c'}$ of $X^{c'}$.

and such that $\bigcup_j \tilde{\mathfrak{h}}_j^1$ covers $\Gamma_{c'}$. Choose finitely many compact sets $C_j \subset \Gamma_{c'} \cap \tilde{\mathfrak{h}}_j^1$ such that $\bigcup_j C_j$ covers $\Gamma_{c'}$. Since Y is Stein, we can think of each $g|_{\pi^{-1}(\mathfrak{h}_j^1)}$ as a holomorphic map to $\mathbb{C}^k$ and so a local extension reduces to extension of each component. Hence Lemma 2.9 gives an extension to each $\pi^{-1}(\tilde{\mathfrak{h}}_j^1)$. It is intuitively clear that we get a jointly well defined extension across $\Gamma_{c'}$. The following is a precise argument.

We proceed by induction on l and assume that we have found a small $\mathcal{C}^2$-perturbation ρ_l of ρ defining a small perturbation $X_l^{c'}$ of $X^{c'}$, with $X^{c'} \cup C_1 \cup ... \cup C_l \subset X_l^{c'}$ and an extension of f to $X_l^{c'}$.

By (7) we get a well defined extension of f to $X_l^{c'} \cup \tilde{\mathfrak{h}}_{l+1}^1$. Let $\chi \in \mathcal{C}_0^\infty(X)$, $\chi \geq 0$, $\mathrm{supp}\,\chi \subset \tilde{\mathfrak{h}}_{l+1}^1$ and $\chi \equiv 1$ near C_{l+1}. Then $\rho_{l,\varepsilon} := \rho_l + \varepsilon \cdot \chi$ converges to ρ_l in $\mathcal{C}^2$-norm, and defines ρ_{l+1} and $X_{l+1}^{c'}$ for $\varepsilon > 0$ small enough.

After m steps we have an extension to a full neighborhood of $\Gamma_{c'}$. $\quad\square$

Corollary 4.3. *Let $c' > 0$ be a non-critical value for ρ, and $g \colon \pi^{-1}(X^{c'}) \to Y$ be a holomorphic map. Let $0 \leq c'' < c'$ and assume that there are no critical values in the interval (c'', c'). Then g extends to a holomorphic map $\tilde{g} \colon \pi^{-1}(X^{c''}) \to Y$.*

Proof. We let $s := \inf\{c'' < t < c' : g$ extends to $\pi^{-1}(X^t)\}$. Then $c'' \leq s < c'$ and g extends to some $\tilde{g} \colon \pi^{-1}(X^s) \to Y$. If $c'' < s$ we would get a contradiction since $\tilde{g}$ would extend across Γ_s. $\quad\square$

Next we consider the critical case.

Lemma 4.4. *Let $c' > 0$ be a critical value for ρ and $g \colon \pi^{-1}(X^{c'}) \to Y$ be any holomorphic map. Then there exists a $c'' \in [0, c')$ such that g extends to $\pi^{-1}(X^{c''})$.*

Proof. If c' corresponds to a local minimum, it is easy to extend using a Hartogs figure and Lemma 2.9. Otherwise we choose $c > c'$ such that there are no critical values in the interval (c', c). Let $\tilde{\rho}$ be a function as in Proposition 3.3 below. Then, by Corollary 4.3 with the function $\tilde{\rho}$, the map g extends to $\pi^{-1}(\{\tilde{\rho} > c'\})$, and, by

Lemma 2.8, g extends further across the critical point. So there exists some $c'' < c'$ such that the extension is defined on $\pi^{-1}(X^{c''})$. $\square$

Remark 4.5. Note that an isomorphism between two holomorphic vector bundles of rank r is a section of an associated holomorphic fiber bundle with fibers $\mathrm{Gl}_r(\mathbb{C})$. Hence part (ii) of Theorem 4.1 can be applied for extension of isomorphisms of bundles.

Before we give two corollaries, we state the definition of the extension of a bundle.

Definition 4.6. Let X be a complex manifold, let $U_1 \subset U_2 \subset X$ be open sets and let $E_j \to U_j$ be holomorphic vector bundles for $j=1,2$. We say that E_2 is an *extension* of E_1 if $E_2|_{U_1}$ is isomorphic to E_1.

Corollary 4.7. *Let X be a Stein manifold with a nice exhaustion function ρ. Let $c_3 < c_2 < c_1$, let Q_j be closed discrete subsets of X^{c_j} for $j=1,2,3$, and let $E_1 \to X^{c_1} \setminus Q_1$ be a holomorphic vector bundle, and assume that E_j is an extension of E_1 to $X^{c_j} \setminus Q_j$ for $j=2,3$. Moreover, assume for each $j=1,2,3$ that Q_j is the singularity set of the vector bundle $E_j \to X^{c_j} \setminus Q_j$, i.e. E_j cannot be extended as a holomorphic vector bundle through any point in Q_j. Then $Q_1 \subset Q_2 \subset Q_3$, $Q_1 = Q_2 \cap X^{c_1} = Q_3 \cap X^{c_1}$, $Q_2 = Q_3 \cap X^{c_2}$ and E_2 and E_3 are isomorphic over $X^{c_2} \setminus Q_2$.*

Proof. By definition of an extension we have that $Q_j \cap X^{c_1} \subset Q_1$ for $j=2,3$, and by transitivity, the three bundles are isomorphic over $X^{c_1} \setminus Q_1$. Now let $x \in X^{c_1} \setminus Q_j$ for $j=2$ or $j=3$. The isomorphism implies that E_1 is trivial in a punctured neighborhood of x, hence $x \notin Q_1$. Hence $Q_1 \subset Q_2$ and $Q_1 \subset Q_3$. It follows that $Q_1 = Q_2 \cap X^{c_1} = Q_3 \cap X_1^c$. Using a similar argument, we conclude that $Q_2 \subset Q_3$ and $Q_2 = Q_3 \cap X^{c_2}$.

Finally, we observe that by Theorem 4.1 the isomorphism between E_2 and E_3 on $X^{c_1} \setminus Q_1$ extends to an isomorphism on $X^{c_2} \setminus Q_2$. $\square$

Corollary 4.8. *Let X be a Stein manifold with a nice exhaustion function ρ. Let $c_j \in \mathbb{R}$, $c_j < c_{j-1}$, for $j=1,2,3,\ldots$. Let $E_1 \to X^{c_1} \setminus Q_1$ be holomorphic vector bundle, and assume that for every $j=1,2,3,\ldots$ there exist holomorphic extensions $E_j \to X^{c_j} \setminus Q_j$, where Q_j is the singularity set of E_j. Then E_1 extends to $X^s \setminus Q$ as a holomorphic vector bundle, where $s = \inf\{c_j\}$ and $Q = \bigcup_{j=1}^{\infty} Q_k$.*

Proof. The previous corollary allows us to define the inductive limit of all the extensions, removing unnecessary singular points. $\square$

Proposition 4.9. *Let $K \subset \triangle \times r \cdot \overline{\triangle}^{n-1}$, $0 < r < 1$, $n \geq 3$, be a closed set with $\triangle^n \setminus K$ connected, and with K_{z_1} polynomially convex (possibly empty for some z_1) for all $z_1 \in \triangle$. Let $\pi \colon E \to \triangle^n \setminus K$ be a holomorphic fiber bundle with Stein fiber, and let Y be a Stein manifold. If we denote by $\mathfrak{D}_r^{n-1}$ the domain $\triangle \times G^{n-1}(r, 1) = \triangle \times (\triangle^{n-1} \setminus r \cdot \overline{\triangle}^{n-1})$, then the following hold*

(i) *Any holomorphic map $f \colon \pi^{-1}(\mathfrak{D}_r^{n-1}) \to Y$ extends uniquely to a holomorphic map $\tilde{f} \colon E \to Y$, and*

(ii) *any holomorphic section $s \colon \mathfrak{D}_r^{n-1} \to E$ extends uniquely to a section $\tilde{s} \colon \triangle^n \setminus K \to E$.*

Proof. Theorem 4.1 implies that f and s have well defined extensions to the slices $\triangle_{z_1}^n \setminus K_{z_1}$ for all $z_1 \in \triangle$. This gives us extensions $\tilde{f}$ and $\tilde{s}$ of f and s respectively, but we need to show that they are holomorphic in the z_1-direction.

In order to show this, say for s, we let Ω be the largest open subset of $\triangle^n \setminus K$ such that s is holomorphic on Ω, and write $\widetilde{K} = \triangle^n \setminus \Omega$. Assume by contradiction that $\widetilde{K} \neq K$. Then there exists $z_0 \in \triangle$ such that K_{z_0} is a proper subset of $\widetilde{K}_{z_0}$. Since K_{z_0} is polynomially convex we see that K_{z_0} cannot contain $\breve{S}(P(\widetilde{K}_{z_0}))$, hence there is a point $x \in \breve{S}(P(\widetilde{K}_{z_0})) \setminus K_{z_0}$. It follows then from Proposition 2.3 that there is a strongly pseudoconcave point $x' \in \widetilde{K}_{z_0} \setminus K_{z_0}$, and hence there exists an arbitrarily small Hartogs figure $(\mathfrak{h}^1, \tilde{\mathfrak{h}}^1)$ of dimension $(n-1)$ with $\mathfrak{h}^1 \subset \triangle_{z_0}^n \setminus \widetilde{K}_{z_0}$ and $x' \in \tilde{\mathfrak{h}}_{z_0}^1$. For some small $\varepsilon > 0$ we have that $\triangle_\varepsilon(z_0) \times \mathfrak{h}^1 \subset \triangle^n \setminus \widetilde{K}$, and this gives us a Hartogs figure on which s, by local triviality of the fiber bundle π, extends to some open neighborhood of x' in $\triangle^n$. Contradiction.

The case of a holomorphic map follows the same argument. $\square$

5. Extending vector bundles across smooth level sets of a strictly plurisubharmonic exhaustion function

In this section we will assume throughout that X is a complex manifold with $\dim X = n \geq 3$, and we let ρ be a nice exhaustion function.

Proposition 5.1. *Let c be a non-critical value of ρ, $Q \subset X^c$ be a closed (in X^c) discrete set of points and let $E \to X^c \setminus Q$ be a holomorphic vector bundle whose singular set is Q. Then Q is finite near Γ_c and there exist $c' < c$ and a finite set of points $P \subset X^{c'}$, such that E extends to $X^{c'} \setminus (Q \cup P)$.*

The first step for proving the proposition will be the following lemma:

Lemma 5.2. *Under the assumptions of Proposition 5.1 the set Q is finite near Γ_c and there exists a finite set of points $P \subset \Gamma_c$ such that for any point $x \in \Gamma_c \setminus P$, there exists an open neighborhood U of x in X with the property that E is a holomorphically trivial vector bundle on $U \cap X_c$. Moreover, there exists an open neighborhood V of P such that E extends as a coherent analytic sheaf $\mathcal{S}$ on $X^c \cup V$.*

Proof. Let $x \in \Gamma_c$. Then we can assume that Γ_c is strictly convex in suitable local holomorphic coordinates, and hence it is easy to construct a holomorphic embedding $\phi \colon \triangle^n \to X$ with $\phi(0) = x$ and a set W of the form

$$\tag{8} \left(\triangle_\varepsilon(z^*) \times \triangle^{n-1} \right) \cup \left(\triangle \times G^{n-1}(r,1) \right)$$

for some $z^* \in \triangle$, $\varepsilon \in (0,1)$, $r \in (0,1)$ and $\triangle_\varepsilon(z^*) = \{z : |z - z^*| < \varepsilon\}$ with $\varphi(W) \subset X^c \setminus Q$, where Q is the singularity set of E.

Indeed, due to the strict convexity of Γ_c, we can first choose an embedding $\varphi \colon \triangle^n \to X$ such that $\varphi(0) = x$ and $\varphi((\{0\} \times \triangle^{n-1}) \cup (\triangle \times b\triangle^{n-1})) \subset X^c \cup \{x\}$. Then, taking into account the fact that Q is countable, we can choose $z^* \in \triangle$ and $t \in (0,1)$ such that $\varphi((\{z^*\} \times \triangle_t^{n-1}) \cup (\triangle \times b\triangle_t^{n-1})) \subset X^c \setminus Q$ and hence, by closedness of Q in X^c, there exists a small enough neighborhood of the set $(\{z^*\} \times \triangle_t^{n-1}) \cup (\triangle \times b\triangle_t^{n-1})$ in $\triangle^n$ which will give, after a homothety $\triangle_t^{n-1} \to \triangle^{n-1}$, the open set W in (8) with the required properties.

Then by Siu, p. 225 in [17], the vector bundle E extends to a coherent analytic sheaf $\mathcal{S}$ on $\phi(\triangle^n)$, which is a holomorphic vector bundle $\widetilde{E} \to \phi(\triangle^n) \setminus \widetilde{Q}$ where $\widetilde{Q}$ is a finite set of points. In view of Remark 4.5 above we can conclude by Proposition 4.9, applied to the fiber bundle with fibres $\mathrm{Gl}_r(\mathbb{C})$, that any isomorphism $\Psi \colon E|_G \to \widetilde{E}|_G$, where $G := \phi(\triangle \times G^{n-1}(r,1))$, extends to a vector bundle isomorphism

$$\widetilde{\Psi} \colon E|_{\phi(\triangle^n) \setminus (X_c \cup Q \cup \widetilde{Q})} \to \widetilde{E}|_{\phi(\triangle^n) \setminus (X_c \cup Q \cup \widetilde{Q})}.$$

In particular, the set $Q \cap \varphi(\triangle^n) \subset \widetilde{Q}$ is finite. By the compactness of Γ_c we can find a finite covering Γ_c by the sets $\varphi_j(\triangle^n)$, $j = 1, 2, ..., m$, described above. Therefore, the set P, defined as the union of the corresponding sets $\widetilde{Q}_j \cap \Gamma_c$ (where $\widetilde{Q}_j$ is the described set $\widetilde{Q}$ which corresponds to $\varphi_j(\triangle^n)$), as well as the set Q near Γ_c are finite. For $x \in \Gamma_c \setminus P$, we can take an embedding $\varphi \colon \triangle^n \to X$ with $x \in \varphi(\triangle^n)$ and such a small image that $\varphi(\triangle^n) \cap Q = \varnothing$. This shows that for $U = \varphi(\triangle^n)$ the bundle E is trivial on $U \cap X_c$. The existence of the required neighborhood V of P and the extension of the vector bundle $E \to X^c \setminus Q$ to a coherent sheaf $\mathcal{S}$ on $X^c \cup V$ follows from the above argument applied to a covering of P by finitely many polydiscs $\varphi_j(\triangle^n)$, $j = 1, ..., m$, and then extending E to a coherent sheaf $\mathcal{S}_j$ for each $j = 1, ..., m$. $\quad \square$

Proof of Proposition 5.1. We now let $\chi \in \mathcal{C}_0^\infty(X)$ be such that $\chi \geq 0$ and $\chi(p) > 0$ for all $p \in P$ provided by the preceding lemma. If $\varepsilon > 0$ is small enough, then the function $\rho_\varepsilon := \rho + \varepsilon \chi$ is strongly plurisubharmonic without critical points near $\Gamma_{c,\varepsilon} := \{x \in X : \rho_\varepsilon(x) = c\}$, and by the last lemma we have an extension of E across $\Gamma_{c,\varepsilon}$ which we may assume to be a trivial bundle near any point of $\Gamma_{c,\varepsilon}$. To avoid too many indices we now drop the subscript ε, assume that E was already locally trivial near Γ_c, and proceed with the extension.

Let $U_i, i = 1, ..., l$, be an open cover of Γ_c such that $E|_{X^c \cap U_i}$ is trivial for every i. Choose a finite number of Hartogs figures $(\mathfrak{h}_j^1, \tilde{\mathfrak{h}}_j^1)$ such that for each j one has $\mathfrak{h}_j^1 \subset X^c$, $\tilde{\mathfrak{h}}_j^1 \subset U_i$ for some i, $\tilde{\mathfrak{h}}_j^1 \cap X^c$ is connected, and such that $\bigcup_j \tilde{\mathfrak{h}}_j^1$ covers Γ_c. It follows from Proposition 4.9 and Remark 4.5 that if $\widetilde{X}^c$ is a sufficiently small $\mathcal{C}^2$-perturbation of X^c with $X^c \subset \widetilde{X}^c$, and if $\widetilde{E}$ is an extension of E to $\widetilde{X}^c$, then $\widetilde{E}$ is holomorphically trivial on $\tilde{\mathfrak{h}}_j^1 \cap \widetilde{X}^c$ for each j. Now, an argument as at the end of the proof of Lemma 3.4 shows that E extends across Γ_c. $\square$

Corollary 5.3. *Let X be a Stein manifold with a nice exhaustion function ρ, let $Q \subset X^c$ be a closed discrete set for a non-critical value c, and let $E \to X^c \setminus Q$ be a holomorphic vector bundle such that Q is its singularity set. Then for each $c' < c$ with no critical values in $(c', c]$ there exists a discrete subset $Q' \subset X^{c'}$ and an extension of E to $X^{c'} \setminus Q'$ with Q' being singularity set of this extension.*

Proof. Let $s := \inf\{c'' < c : E$ extends to a vector bundle on $X^{c''} \setminus Q''$, where $Q'' \subset X^{c''}$ is a closed discrete set$\}$. By Corollary 4.8 we have that E extends to $X^s \setminus Q^s$ for some closed discrete set $Q^s \subset X^s$ and so $s = c'$. $\square$

6. Proof of Theorems 1.2 and 1.3

Proof of Theorem 1.3. By Corollary 2.8, transition maps extend across totally real Lipschitz sets. Hence it is enough to show that there is a discrete set of points $P \subset M$ such that for any point $x_0 \in M \setminus P$, there is an open neighborhood U of x_0 with the property that E is holomorphically trivial on $U \setminus M$.

Let $\Omega \subset M$ be the set consisting of all points $x \in M$ that have a neighborhood $U \subset X$ such that $E|_{U \setminus M}$ is trivial. We will first show that $\Omega \neq \varnothing$, and then that every point $x \in b\Omega$ is isolated in M.

By Corollary 2.6, there exists an embedding $\phi : \triangle^n \to X$ with $\phi(\triangle^n) \cap M \neq \varnothing$, and such that $\phi(\mathfrak{h}^1) \subset X \setminus M$.

By choosing a possibly smaller Hartogs figure, we may assume that E is a vector bundle on $\phi(\mathfrak{h}^1)$. Now E extends to a coherent analytic sheaf $\mathcal{S}$ on $\phi(\triangle^n)$, and $\mathcal{S}$

is locally free outside a finite set of points. Arguing as in the proof of Lemma 5.2, we see that E may be trivialized outside a locally finite set of points in M.

Next, $\widetilde{M} := M \setminus \Omega$ is itself a totally real Lipschitz set, and E extends to a vector bundle on $X \setminus \widetilde{M}$. Unless $\widetilde{M}$ is a discrete set of points, the whole argument can be repeated to show that E is a trivial vector bundle near a relatively open subset of $\widetilde{M}$, contradicting the maximality of Ω. $\square$

Remark 6.1. We believe that the same method of proof for Theorem 1.3 works also for a $(n-2)$-complete manifold X instead of a Stein manifold X, i.e. when X has a smooth exhaustion function whose Levi form has a least 3 positive eigenvalues in every point.

Proof of Theorem 1.2. We choose a nice nonnegative plurisubharmonic exhaustion function ρ of X, i.e. as in Definition 3.2 with $L = \varnothing$. Then, by the compactness of K, the vector bundle E is defined on a superlevel set $X^a \subset X \setminus K$ of ρ for some regular value $a > 0$. Let

$$c := \inf\big\{ b < a : E \text{ extends to } X^b \setminus Q_b, \text{ where } Q_b \text{ is a discrete closed subset of } X^b$$
$$\text{which is the singularity set of } E \text{ in } X^b \big\}.$$

It follows from Corollary 4.8 that E extends to $X^c \setminus Q_c$ where Q_c is also is a discrete closed subset of X^c which is the singularity set of E in X^c. By Corollary 5.3, c cannot be a regular value of ρ. If the critical point is not a local minimum, we can consider the function $\tilde\rho$ constructed in Proposition 3.3 and conclude that E extends to $\{\tilde\rho > c\} \setminus \widetilde{Q}_c$. By Lemma 2.10 and an argument as in the Lemma 5.2 the set $\widetilde{Q}_c$ is locally finite near $M := \{\tilde\rho = 0\} \cap \mathbb{B}_{\varepsilon_1}$. Finally, E extends across M by Theorem 1.3. If the critical point is a local minimum, it is easy to find a Hartogs figure, and by the same argument as before, the singularity set cannot accumulate there. We conclude that $c = 0$ and that the singularity set is finite. $\square$

7. Oka theory

In this section we will use our extension result to prove the following theorem.

Theorem 7.1. *Let X be a Stein manifold with an exhaustion function ρ, let $K \subset X$ be a compact set, and let $E_j \to X \setminus K$ be a holomorphic vector bundle with sprays $s_j : E \to X \setminus K$ for $j = 1, ..., m$ finite. Assume further that the family of sprays is dominating on a super-level set $X^c \supset K$.*

1. *If $m=1$, then K has only finitely many accumulation points which lie in X_c.*
2. *If $m\geq1$ and K is smoothly bounded, then K is actually the empty set.*

The statement of Theorem 7.1 contains Theorem 1.1.

Before proving this theorem we first give some background on Oka theory.

Gromov [12] introduced 1989 in his seminal paper the notion of an elliptic manifold and proved an Oka principle for holomorphic sections of elliptic bundles, generalizing previous work of Grauert [13] for complex Lie groups. The (basic) Oka principle of a complex manifold X says that every continuous map $Y\to X$ from a Stein space Y is homotopic to a holomorphic map.

In Oka theory exist many interesting classes of complex manifolds with weaker properties than ellipticity. However, all the known inclusion relations between these classes are yet not known to be proper inclusions. Following the book of Forstnerič [7, Chapter 5] we want to mention in particular the following classes and then prove that at least one of these inclusions has to be proper.

Definition 7.2. A *spray* on a complex manifold X is a triple (E,π,s) consisting of a holomorphic vector bundle $\pi\colon E\to X$ and a holomorphic map $s\colon E\to X$ such that for each point $x\in X$ we have $s(0_x)=x$ where 0_x denotes the zero in the fiber over x.

The spray (E,π,s) is said to be *dominating* if for every point $x\in X$ we have

$$\mathrm{d}_{0_x}s(E_x)=\mathrm{T}_xX$$

A complex manifold is called *elliptic* if it admits a dominating spray.

In this definition we adapted the convention used e.g. in the textbook [7] identifying the fiber E_x over x with its tangent space in 0_x.

A weaker notion, subellipticity, was introduced later by Forstnerič [6] where he proved the Oka principle for subelliptic manifolds:

Definition 7.3. A finite family of sprays (E_j,π_j,s_j), $j=1,...,m$, on X is called *dominating* if for every point $x\in X$ we have

$$(9) \qquad \mathrm{d}_{0_x}s_1(E_{1,x})+...+\mathrm{d}_{0_x}s_m(E_{m,x})=\mathrm{T}_xX$$

A complex manifold X is called *subelliptic* if it admits a finite dominating family of sprays.

It is immediately clear from the definition that an elliptic manifold is subelliptic. However, the Oka principle holds under even weaker conditions. Forstnerič [8], [9] and [10] showed that the following condition (CAP) is equivalent for a manifold to satisfy the Oka principle (and versions of the Oka principle with interpolation and approximation), hence justifying the name Oka manifold:

Definition 7.4. A complex manifold X is said to satisfy the *convex approximation property* (CAP) if on any compact convex set $K \subset \mathbb{C}^n$, $n \in \mathbb{N}$, every holomorphic map $f \colon K \to X$ can be approximated, uniformly on K, by entire holomorphic maps $\mathbb{C}^n \to X$. A manifold satisfying CAP is called an *Oka manifold*. If this approximation property holds only for $n \leq N$ for some $N \in \mathbb{N}$, then X is said to satisfy CAP_N.

A subelliptic manifold is always Oka. Whether an Oka manifold is elliptic or subelliptic, is on the other hand not known—this implication holds however under the extra assumption that it is Stein.

We want to mention also these weaker properties:

Definition 7.5. A complex manifold X of dimension n is called *dominable* if there exists a point $x_0 \in X$ and a holomorphic map $f \colon \mathbb{C}^n \to X$ with $f(0) = x_0$ and rank $d_0 f = n$.

Definition 7.6. A complex manifold X of dimension n is called *strongly dominable* if for every point $x_0 \in X$ there exists a holomorphic map $f \colon \mathbb{C}^n \to X$ with $f(0) = x_0$ and rank $d_0 f = n$.

Summarizing the previous, the following inclusions are known [7, Corollary 5.15.4]:

$$\text{elliptic} \subseteq \text{subelliptic} \subseteq \text{Oka} \subseteq \text{strongly dominable} \subseteq \text{dominable}$$

There are several known candidates to prove that one of these inclusions is proper. Theorem 7.1 gives a class of examples of complex manifolds which are not subelliptic, but strongly dominable.

Corollary 7.7. *Let K be the closure of a smoothly bounded non-empty open subset of $\mathbb{C}^n$, $n \geq 3$. Then $\mathbb{C}^n \setminus K$ is not subelliptic, but dominable. If K is holomorphically convex, then $\mathbb{C}^n \setminus K$ is strongly dominable.*

Proof. Since $X:=\mathbb{C}^n$ is a Stein manifold, it is a direct consequence of Theorem 7.1 that $X\setminus K$ is not subelliptic. However for all $x\in\mathbb{C}^n\setminus\widehat{K}$ there exists a Fatou–Bieberbach domain $\Omega\subseteq\mathbb{C}^n\setminus\widehat{K}, \Omega\cong\mathbb{C}^n, x\in\Omega$, therefore $\mathbb{C}^n\setminus K$ is dominable resp. strongly dominable. $\square$

Whether or not these examples are Oka manifolds, remains an open question. A partial result in this direction has been obtained in [11] where they prove that $\mathbb{C}^n\setminus\overline{\mathbb{B}^n}$ satisfies CAP_k for $k<n$.

Proof of Theorem 7.1. In the subelliptic case (which covers also the elliptic case) we are given a finite family of sprays (E_j, π_j, s_j) for $j=1,...,m$ on $X\setminus K$, which is dominating on X^c. By Theorem 1.2 we can extend it to a family $(\widetilde{E}_j, \widetilde{\pi}_j, \tilde{s}_j)$ of sprays on $X\setminus P$, $\tilde{s}_j\colon\widetilde{E}_j\to X$, where P is a finite set of points. Now $(\widetilde{E}_j, \widetilde{\pi}_j, \tilde{s}_j)$ is dominating outside an analytic subset of $X\setminus P$, hence a closed discrete set of points P' in $X\setminus P$. We denote by $P'':=P\cup P'$. Consider a point $p\in bK\setminus P''$, i.e. a boundary point of K where the family of the extended sprays exists and is dominating. By continuity of the derivative and lower semicontinuity of its rank, the family of sprays is then also dominating in a open neighborhood $V\subset X\setminus P''$ of p.

1. If $m=1$, every sequence in $V\setminus K$ converging to p will be such that the fibres above all but finitely many points of this sequence hit p. For if not, s_1 could not be dominating in p by the continuity of the derivative. Therefore $K\subseteq P''$, since we have just shown that there cannot exist a dominating spray $\tilde{s}_1\colon E_1|_{X\setminus K}\to X\setminus K$ if $K\nsubseteq P''$.

2. If $m\geq1$, assume for a contradiction that $K\neq\varnothing$ is smoothly bounded. For each spray $\tilde{s}_j$ we denote by $A_{p,j}:=\{q\in V : \tilde{s}_j(E_{j,q})\ni p\}=\pi_j((\tilde{s}_j|_V)^{-1}(p))$ the set of points whose fiber hits p. By the domination property in p there exists at least one spray $\tilde{s}_j$ such that $A_{p,j}$ has a tangent in p which is transversal to the boundary of K. In analogue to the situation $m=1$ we can choose a sequence in $A_{p,j}\setminus K$ converging to p such that the fibres above the points of this sequence hit p under $\tilde{s}_j$. Again we can conclude $K\subseteq P''$. Now if the boundary of K is actually smooth and consists of only finitely many accumulation points, K is empty. $\square$

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