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The explicit formulae for scaling limits in the ergodic decomposition of infinite Pickrell measures

Alexander I. Bufetov and Yanqi Qiu

Abstract. The main result of this paper, Theorem 1.5, gives explicit formulae for the kernels of the ergodic decomposition measures for infinite Pickrell measures on the space of infinite complex matrices. The kernels are obtained as the scaling limits of Christoffel-Uvarov deformations of Jacobi orthogonal polynomial ensembles.

1. Introduction

1.1. Pickrell measures

We start by recalling the definition of Pickrell measures [12]. Our presentation follows [6].

Let $s\in\mathbb{R}$ be a real parameter and let $n\in\mathbb{N}$, consider a measure $\mu_n^{(s)}$ on the space $\mathrm{Mat}(n,\mathbb{C})$ of $n\times n$-complex matrices:

$$d\mu_n^{(s)}(z)=\mathrm{const}_{n,s}\det\big(1+z^*z\big)^{-2n-s}\,dz, \tag{1}$$

where dz is the Lebesgue measure on $\mathrm{Mat}(n,\mathbb{C})$, and $\mathrm{const}_{n,s}$ is a normalization constant chosen in such a way (see, e.g., [6, Prop. 1.8]) that the push-forward of $\mu_{n+1}^{(s)}$ under the natural projection of cutting the northwest $n\times n$-corner of a $(n+1)\times(n+1)$-matrix is precisely $\mu_n^{(s)}$ (there is a subtlety, see [3], [6] and [7], in defining the push-forword for infinite measures $\mu_n^{(s)}$ when $s\le -1$). By the Kolmogorov Existence Theorem, one may define a measure $\mu^{(s)}$, called *Pickrell measure*, on the space $\mathrm{Mat}(\mathbb{N},\mathbb{C})$ of infinite complex matrices as the projective limit of the sequence $(\mu_n^{(s)})_{n=1}^{\infty}$. The measure $\mu^{(s)}$ is a probability measure if $s>-1$ and is an infinite measure if $s\le -1$. See also [9] and [10] for the study of similar measures.

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Let

$$U(\infty) = \bigcup_{n\in\mathbb{N}} U(n)$$

be the infinite unitary group. The group $U(\infty)\times U(\infty)$ acts on $\mathrm{Mat}(\mathbb{N},\mathbb{C})$ as follows:

$$T_{u_1,u_2}(z) = u_1 z u_2^*, \quad \text{for } (u_1,u_2)\in U(\infty)\times U(\infty),\ z\in \mathrm{Mat}(\mathbb{N},\mathbb{C}).$$

By definition, for all $s\in\mathbb{R}$, the measure $\mu^{(s)}$ are $U(\infty)\times U(\infty)$-invariant. One may study the ergodic decomposition of $\mu^{(s)}$. The probability case $(s>-1)$ was studied in [3] and the infinite measure case $(s\le -1)$ was studied in [6]. In this paper, we continue the study of the ergodic decomposition of the infinite Pickrell measures $\mu^{(s)}$.

Let us recall Pickrell's classification of $U(\infty)\times U(\infty)$-ergodic probability measures on $\mathrm{Mat}(\mathbb{N},\mathbb{C})$, see [11], [12] and [18]. Set

$$\Omega_P = \left\{\omega = (\gamma, x) : x = (x_1\ge x_2\ge \dots \ge x_i \ge \dots \ge 0),\ \sum_{i=1}^{\infty} x_i \le \gamma\right\}.$$

A probability measure η on $\mathrm{Mat}(\mathbb{N},\mathbb{C})$ is $U(\infty)\times U(\infty)$-ergodic if and only if there exists a unique $\omega{=}(\gamma,(x_n))\in\Omega_P$ such that for any $m\in\mathbb{N}$ and $\lambda_1,\dots,\lambda_m\in\mathbb{R}$, we have

$$\int_{\mathrm{Mat}(\mathbb{N},\mathbb{C})} \exp\left(i\sum_{j=1}^{m}\lambda_j \operatorname{Re} z_{jj}\right) d\eta(z) = \prod_{\ell=1}^{m} \frac{\exp(-4(\gamma - \sum_{k=1}^{\infty} x_k)\lambda_\ell^2)}{\prod_{k=1}^{\infty}(1+4x_k\lambda_\ell^2)}. \tag{2}$$

We denote by η_ω the ergodic probability measure on $\mathrm{Mat}(\mathbb{N},\mathbb{C})$ characterized by the formula (2).

If $s>-1$, then there exists a probability measure $\overline{\mu}^{(s)}$ on Ω_P, called the decomposition measure of $\mu^{(s)}$, such that the measure $\mu^{(s)}$ admits a unique ergodic decomposition

$$\mu^{(s)} = \int_{\Omega_P} \eta_\omega \, d\overline{\mu}^{(s)}(\omega). \tag{3}$$

Set

$$\Omega_P^0 := \left\{\omega = (\gamma, x)\in\Omega_P : x_i>0 \text{ for all } i, \text{ and } \gamma = \sum_{i=1}^{\infty} x_i\right\}.$$

We denote $\mathrm{Conf}((0,\infty))$ the space of configurations on $(0,\infty)$, i.e.,

$$\mathrm{Conf}\big((0,\infty)\big) := \big\{X\subset(0,\infty) \mid X \text{ is a discrete subset of } (0,\infty)\big\}.$$

Definition 1.1. Let $\mathbb{B}^{(s)}$ denote the push-forward of the measure $\mu^{(s)}$ under the following forgetting map:

$$\begin{aligned} \text{conf:} \quad \Omega_P^0 &\to \text{Conf}\big((0,\infty)\big) \\ \omega &\mapsto \{x_1, x_2, ..., x_i, ...\} \end{aligned}.$$

It has been proved (the case $s{=}0$ in [3], the general case $s{\in}\mathbb{R}$ in [6]), that

$$\overline{\mu}^{(s)}\big(\Omega_P \setminus \Omega_P^0\big) = 0,$$

and the map conf induces an isomorphism between probability spaces $(\Omega_P, \overline{\mu}^{(s)})$ and $(\text{Conf}((0,\infty), \mathbb{B}^{(s)})$. Moreover, the measure $\mathbb{B}^{(s)}$ is a determinantal probability measure on $\text{Conf}((0,\infty))$ induced by the correlation kernel (which is a kernel of an orthogonal projection):

$$J^{(s)}(x_1,x_2) := \frac{1}{x_1 x_2} \int_0^1 J_s\left(2\sqrt{\frac{t}{x_1}}\right) J_s\left(2\sqrt{\frac{t}{x_2}}\right) dt, \tag{4}$$

where J_s is the Bessel function of order s, see [1, p. 360, 9.1.10] for the definition of Bessel functions.

If $s{\le}{-1}$, the ergodic decomposition of the measure $\mu^{(s)}$ was described in [6]: there exists an *infinite measure* $\overline{\mu}^{(s)}$ on Ω_P, such that $\mu^{(s)}$ admits the same decomposition formula as (3). It was proved that the relation $\overline{\mu}^{(s)}(\Omega_P \setminus \Omega_P^0){=}0$ still holds. Moreover, the forgetting map conf defined in Definition 1.1 identifies the decomposition measure $\overline{\mu}^{(s)}$ with an *infinite determinantal measure* on $\text{Conf}((0,\infty))$. We denote this infinite determinantal measure on $\text{Conf}((0,\infty))$ by $\mathbb{B}^{(s)}$.

One way to describe $\mathbb{B}^{(s)}$ is as follows. Let $g{:}(0,\infty){\to}(0,\infty)$ be a positive function, the multiplicative functional Ψ_g on $\text{Conf}((0,\infty))$ is defined by the formula:

$$\Psi_g(X) = \prod_{x \in X} g(x) \quad \text{for any } X \in \text{Conf}\big((0,\infty)\big).$$

If the function g is appropriately chosen, one may get a *finite measure* $\Psi_g \mathbb{B}^{(s)}$ such that Ψ_g is $\mathbb{B}^{(s)}$-almost everywhere positive. Then the normalized probability measure

$$\mathbb{P}_g^{(s)} := \frac{\Psi_g \mathbb{B}^{(s)}}{\int_{\text{Conf}((0,\infty))} \Psi_g \, d\mathbb{B}^{(s)}} \tag{5}$$

is a determinantal probability measure on $\text{Conf}((0,\infty))$ induced by a kernel Π^g, see [4] [5]. For the theory of determinantal probability measures, we refer to [13], [14] and [15]. Here Π^g is the kernel of the orthogonal projection onto a certain subspace of $L^2(0,\infty)$. By definition of $\mathbb{P}_g^{(s)}$, we have

$$\mathbb{B}^{(s)} = \mathrm{Const} \cdot \Psi_g^{-1} \cdot \mathbb{P}_g^{(s)}. \tag{6}$$

If, in the above formula, g is explicitly given, then the explicit formula for the correlation kernel Π^g of $\mathbb{P}_g^{(s)}$ turns out to be non-trivial. Our aim in this paper is to give explicit formulae for the kernel Π^g with certain suitably chosen g.

Remark 1.2. Note that to describe $\mathbb{B}^{(s)}$, we only need to give one explicit function g such that the relation (6) holds and the kernel Π^g of $\mathbb{P}_g^{(s)}$ is explicitly calculated.

1.2. Formulation of the main result

Definition 1.3. Let $f_1, ..., f_n$ be complex-valued functions on an interval. Assume that all these functions are differentiable up to order $n-1$. Then the Wronskian $W(f_1, ..., f_n)$ of $f_1, ..., f_n$ is defined by the formula

$$W(f_1, ..., f_n)(t) = \det\big(f_i^{(j-1)}(t)\big)_{1 \le i,j \le n},$$

where $f_i^{(j-1)}$ is the $(j-1)$-th derivative of the function f_i.

Definition 1.4. For $\alpha > -1$, $v > 0$, we define

$$J_{\alpha,v}(t) \stackrel{\text{def}}{=} J_\alpha(t\sqrt{v}) \quad \text{and} \quad K_{\alpha,v}(t) \stackrel{\text{def}}{=} K_\alpha(t\sqrt{v}),$$

where J_α is the Bessel function of order α (cf. [1, 9.1.10]), and K_α is the modified Bessel function of second kind of order α (cf. [1, 9.6.24]).

Our main result is the following

Theorem 1.5. *Let $s \in \mathbb{R}$ and let $m \in \mathbb{N}$ be any natural number such that $s+m > -1$. Assume that $v_1, ..., v_m$ are distinct positive numbers. Take*

$$g(x) := \prod_{j=1}^{m} \frac{4}{4 + v_j x}. \tag{7}$$

Then the correlation kernel Π^g of the normalized determinantal probability measure $\mathbb{P}_g^{(s)}$ defined in (5) is given by the formula

$$\Pi^g(x, x') = \frac{1}{2} \cdot \frac{\mathscr{A}(1, 4/x)\mathscr{B}(1, 4/x') - \mathscr{A}(1, 4/x')\mathscr{B}(1, 4/x)}{\prod_{j=1}^m \sqrt{(v_j + 4/x)(v_j + 4/x')} \cdot [\mathscr{C}(1)]^2 \cdot (x' - x)}, \tag{8}$$

where $\mathscr{A}, \mathscr{B}, \mathscr{C}$ *are functions depending on parameters* $s, m, v_1, ..., v_m$, *given by formulae*:

$$\mathscr{A}(t,y) = W(K_{s+m,v_1}, ..., K_{s+m,v_m}, J_{s+m,y})(t), \quad \mathscr{B}(t,y) = \frac{\partial \mathscr{A}}{\partial t}(t,y) \quad \text{and}$$

$$\mathscr{C}(t) = W(K_{s+m,v_1}, ..., K_{s+m,v_m})(t).$$

Remark 1.6. Note that there is no restriction on the parameter s in Theorem 1.5. However, the case $s \le -1$ is of particular interests for us.

When $s > -1$ and $m \ge 1$, then by results of [6], the kernel Π^g in (8) is the kernel for the operator of orthogonal projection from $L_2(\mathbb{R}_+, \text{Leb})$ onto the subspace $\sqrt{g}\,\text{Range}(J^{(s)})$. Our result shows that the calculation of the correlation kernels of determinantal point processes deformed by multiplying multiplicative functional is in general difficult.

Note in the case where $s > -2$, $m = 1$ and $g(x) = (1 + vx/4)^{-1}$, by a simple computation, we can write more explicitly

$$\Pi^g(x, x') = \frac{1}{2} \cdot \frac{A(x)B(x') - A(x')B(x)}{\sqrt{(v+4/x)(v+4/x')} \cdot K_{s+1}(\sqrt{v})^2 \cdot (x' - x)},$$

where

$$A(x) = K_{s+1}(\sqrt{v}) J'_{s+1}(\sqrt{4/x}) \sqrt{4/x} - \sqrt{v} K'_{s+1}(\sqrt{v}) J_{s+1}(\sqrt{4/x}) \quad \text{and}$$

$$B(x) = K_{s+1}(\sqrt{v}) J''_{s+1}(\sqrt{4/x}) 4/x - v K''_{s+1}(\sqrt{v}) J_{s+1}(\sqrt{4/x}).$$

For reader's convenience, we also give an integral formula for Π^g in this case:

$$\Pi^g(x, x') = \frac{4}{xx'} \Bigg\{ N_s\left(\frac{4}{x}\right) N_s\left(\frac{4}{x'}\right) + \frac{1}{2} \left(\left(v + \frac{4}{x}\right) \left(v + \frac{4}{x'}\right) \right)^{-1/2} \int_0^1 F_s\left(\varkappa, \frac{4}{x}\right) F_s\left(\varkappa, \frac{4}{x'}\right) d\varkappa \Bigg\}, \tag{9}$$

where

$$F_s(\varkappa, t) = \sqrt{t} J'_{s+1}(\varkappa\sqrt{t}) - J_{s+1}(\varkappa\sqrt{t}) \frac{\sqrt{v} K'_{s+1}(\varkappa\sqrt{v})}{K_{s+1}(\varkappa\sqrt{v})}$$

and

$$N_s(t) = \begin{cases} (v^{s+1}\Gamma(-s-1)\Gamma(s+2))^{-1/2} t^{\frac{s+1}{2}} (v+t)^{-1/2} & \text{if } -2 < s < -1, \\ 0 & \text{if } s \ge -1. \end{cases}$$

The proof of formula (9) is straightforward. For details of the calculation, the reader is referred to our preprint [8, Prop. 4.22].

1.3. Organization of the paper

The paper is organized as follows. In Section 2, we give certain asymptotic formulae associated with Jacobi polynomials that will be useful for us. In Section 3, we prove that for appropriate functions g, the determinantal measures of type (5) have finite dimensional approximation. The finite dimensional approximations allow us to use the theory of orthogonal polynomial ensembles. Section 4 is the main section of this paper. We explain in Section 4.1 the strategy of the calculation of the correlation kernels as certain scaling limits. In Section 4.2, we obtain the scaling limits of Christoffel type deformation of Jacobi orthogonal polynomial ensembles, this section contains our main ideas in calculating the scaling limits. Finally, in Section 4.3, we adopt the ideas developed in Section 4.2 to obtain the scaling limits of Uvarov type deformation of Jacobi polynomial ensembles. Theorem 1.5 is derived as a corollary of the main result of Section 4.3.

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2. Some asymptotic formulae associated with Jacobi polynomials

2.1. Jacobi polynomials

For $\alpha, \beta > -1$, let $P_n^{(\alpha,\beta)}$ be the standard Jacobi polynomials (cf. [16, Section 4.1]), orthogonal on the interval $(-1,1)$ with weight $w_{\alpha,\beta}(t) = (1-t)^\alpha(1+t)^\beta$ and normalized by the condition

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}.$$

When $\alpha=s$, $\beta=0$, we will omit β in the notation.

2.2. Difference operators

Given a sequence $f=(f_n)_{n=0}^{\infty}$, we define the sequence of differences $\Delta(f)=(\Delta(f)_n)_{n=0}^{\infty}$ by the formula

$$\Delta(f)_n := f_{n+1}-f_n.$$

More generally, for $\ell\geq 1$ the sequence of $(\ell+1)$-order differences is defined by

$$\Delta^{\ell+1}(f) := \Delta\big(\Delta^{\ell}(f)\big).$$

By convention, we set $\Delta^0(f)_n=f_n$.

2.3. Asymptotics for differences of Jacobi polynomials

Let $P^{(\alpha,\beta)}=(P_n^{(\alpha,\beta)})_{n=0}^{\infty}$ be the sequence of Jacobi polynomials.

Lemma 2.1. *For $\ell\geq 0$ and $n\geq 1$, we have*

$$\begin{aligned}(n+1)\Delta^{\ell+1}\big(P^{(\alpha,\beta)}\big)_n(x)+\ell\Delta^{\ell}\big(P^{(\alpha,\beta)}\big)_{n+1}(x)+\ell(1-x)\Delta^{\ell-1}\big(P^{(\alpha+1,\beta)}\big)_{n+1}(x)\\ +\left(n+\frac{\alpha+\beta}{2}+1\right)(1-x)\Delta^{\ell}\big(P^{(\alpha+1,\beta)}\big)_n(x)=\alpha\Delta^{\ell}\big(P^{(\alpha,\beta)}\big)_n(x).\end{aligned} \tag{10}$$

Proof. If $\ell=0$, then the identity (10) is reduced to the known formula (cf. [16, 4.5.4]):

$$\begin{aligned}&\left(n+\frac{\alpha+\beta}{2}+1\right)(1-x)P_n^{(\alpha+1,\beta)}(x)\\ &\qquad=(n+1)\big(P_n^{(\alpha,\beta)}(x)-P_{n+1}^{(\alpha,\beta)}(x)\big)+\alpha P_n^{(\alpha,\beta)}(x).\end{aligned} \tag{11}$$

Now assume that the identity (10) holds for an integer ℓ and for *all* $n\geq 1$. In particular, we have

$$\begin{aligned}(n+2)\Delta^{\ell+1}\big(P^{(\alpha,\beta)}\big)_{n+1}(x)+\ell\Delta^{\ell}\big(P^{(\alpha,\beta)}\big)_{n+2}(x)\\ +\ell(1-x)\Delta^{\ell-1}\big(P^{(\alpha+1,\beta)}\big)_{n+2}(x)\\ +\left(n+\frac{\alpha+\beta}{2}+2\right)(1-x)\Delta^{\ell}\big(P^{(\alpha+1,\beta)}\big)_{n+1}(x)=\alpha\Delta^{\ell}\big(P^{(\alpha,\beta)}\big)_{n+1}(x).\end{aligned} \tag{12}$$

Taking (12)–(10), we get

$$(n+1)\Delta^{\ell+2}\big(P^{(\alpha,\beta)}\big)_n(x)+(\ell+1)\Delta^{\ell+1}\big(P^{(\alpha,\beta)}\big)_{n+1}(x)$$
$$+(\ell+1)(1-x)\Delta^{\ell}\big(P^{(\alpha+1,\beta)}\big)_{n+1}(x)$$
$$+\left(n+\frac{\alpha+\beta}{2}+1\right)(1-x)\Delta^{\ell+1}\big(P^{(\alpha+1,\beta)}\big)_n(x)=\alpha\Delta^{\ell+1}\big(P^{(\alpha,\beta)}\big)_n(x).$$

Thus the identity (10) holds for $\ell+1$ and for all $n\geq 1$. By induction, the identity (10) holds for all $\ell\geq 0$ and for all $n\geq 1$. $\square$

The classical Mehler-Heine theorem ([16, p. 192]) says that for $z\in\mathbb{C}\setminus\{0\}$,

$$\lim_{n\to\infty} n^{-\alpha}P_n^{(\alpha,\beta)}\left(1-\frac{z}{2n^2}\right)=2^{\alpha}z^{-\frac{\alpha}{2}}J_\alpha(\sqrt{z}). \tag{13}$$

This formula holds uniformly for z in a simply connected compact subset of $\mathbb{C}\setminus\{0\}$.

In what follows, let $(\varkappa_n)_{n=1}^{\infty}$ denote a sequence of natural numbers such that

$$\lim_{n\to\infty}\frac{\varkappa_n}{n}=\varkappa>0.$$

Proposition 2.2. *Let $x^{(n)}=1-\frac{z}{2n^2}$. Then for $\ell\geq 0$, we have*

$$\lim_{n\to\infty} n^{\ell-\alpha}\Delta^{\ell}\big(P^{(\alpha,\beta)}\big)_{\varkappa_n}\big(x^{(n)}\big)=2^{\alpha}z^{\frac{\ell-\alpha}{2}}J_\alpha^{(\ell)}(\varkappa\sqrt{z}), \tag{14}$$

where $J_\alpha^{(\ell)}$ is the ℓ-th derivative of the Bessel function J_α. The formula holds uniformly in $\varkappa$ and z as long as $\varkappa$ ranges in a compact subset of $(0,\infty)$ and z ranges in a compact simply connected subset of $\mathbb{C}\setminus\{0\}$.

Proof. If $\ell=0$, the identity (14) is reduced to the formula (13). Now assume that the identity (14) holds for $0,1,\dots,\ell$. By the relation (10), we have

$$\lim_{n\to\infty} n^{\ell+1-\alpha}\Delta^{\ell+1}\big(P^{(\alpha,\beta)}\big)_{\varkappa_n}\big(x^{(n)}\big)$$
$$=-\frac{\ell}{\varkappa}\cdot 2^{\alpha}z^{\frac{\ell-\alpha}{2}}J_\alpha^{(\ell)}(\varkappa\sqrt{z})-\frac{\ell}{\varkappa}\cdot\frac{z}{2}2^{\alpha+1}z^{\frac{\ell-1-(\alpha+1)}{2}}J_{\alpha+1}^{(\ell-1)}(\varkappa\sqrt{z})$$
$$-\frac{z}{2}2^{\alpha+1}z^{\frac{\ell-(\alpha+1)}{2}}J_{\alpha+1}^{(\ell)}(\varkappa\sqrt{z})+\frac{\alpha}{\varkappa}2^{\alpha}z^{\frac{\ell-\alpha}{2}}J_\alpha^{(\ell)}(\varkappa\sqrt{z})$$
$$=2^{\alpha}z^{\frac{\ell+1-\alpha}{2}}\left[-\ell\cdot\frac{J_\alpha^{(\ell)}(\varkappa\sqrt{z})}{\varkappa\sqrt{z}}-\ell\cdot\frac{J_{\alpha+1}^{(\ell-1)}(\varkappa\sqrt{z})}{\varkappa\sqrt{z}}-J_{\alpha+1}^{(\ell)}(\varkappa\sqrt{z})+\alpha\frac{J_\alpha^{(\ell)}(\varkappa\sqrt{z})}{\varkappa\sqrt{z}}\right]. \tag{15}$$

From the recurrence relation (cf. [1, 9.1.27])

$$J_\alpha'(z) = -J_{\alpha+1}(z) + \frac{\alpha}{z} J_\alpha(z), \tag{16}$$

one may derive, by induction on ℓ, that for all $\ell \geq 1$,

$$z\big[J_\alpha^{(\ell+1)}(z) + J_{\alpha+1}^{(\ell)}(z)\big] = (\alpha - \ell) J_\alpha^{(\ell)}(z) - \ell J_{\alpha+1}^{(\ell-1)}(z). \tag{17}$$

The identity (14) for $\ell+1$ follows from the identities (15) and (17). By induction on ℓ, Proposition 2.2 is completely proved. □

We will need the asymptotics for the derivative of the differences of Jacobi polynomials. The following formula will be useful for us (cf. [16, 4.21.7]):

$$\dot{P}_n^{(\alpha,\beta)}(t) = \frac{d}{dt}\{P_n^{(\alpha,\beta)}\}(t) = \frac{1}{2}(n+\alpha+\beta+1) P_{n-1}^{(\alpha+1,\beta+1)}(t). \tag{18}$$

Proposition 2.3. *Let $x^{(n)} = 1 - \frac{z}{2n^2}$. Then for $\ell \geq 0$, we have*

$$\lim_{n\to\infty} n^{-2+\ell-\alpha} \left[\frac{d}{dt} \Delta^\ell \big(P^{(\alpha,\beta)}\big)_{\varkappa_n}\right] \big(x^{(n)}\big) = 2^\alpha z^{\frac{-2+\ell-\alpha}{2}} \tilde{J}_{\alpha+1}^{(\ell)}(\varkappa\sqrt{z}),$$

where $\tilde{J}_{\alpha+1}(t) := t J_{\alpha+1}(t)$. The formula holds uniformly in $\varkappa$ and z as long as $\varkappa$ ranges in a compact subset of $(0,\infty)$ and z ranges in a compact simply connected subset of $\mathbb{C}\setminus\{0\}$.

Proof. The relation (18) can be written as

$$2\frac{d}{dt}\Delta^0\big(P^{(\alpha,\beta)}\big)_n = (n+\alpha+\beta+1)\Delta^0\big(P^{(\alpha+1,\beta+1)}\big)_{n-1}.$$

From this formula, we may derive by induction that for all $\ell \geq 0$,

$$2\frac{d}{dt}\Delta^\ell\big(P^{(\alpha,\beta)}\big)_n = (n+\alpha+\beta+1)\Delta^\ell\big(P^{(\alpha+1,\beta+1)}\big)_{n-1} + \ell\cdot\Delta^{\ell-1}\big(P^{(\alpha+1,\beta+1)}\big)_n. \tag{19}$$

In view of Proposition 2.2 and the identity (19), we have

$$\begin{aligned}\lim_{n\to\infty} n^{-2+\ell-\alpha}\frac{d}{dt}\Delta^\ell\big(P^{(\alpha,\beta)}\big)_{\varkappa_n}\big(x^{(n)}\big) &= 2^\alpha z^{\frac{-2+\ell-\alpha}{2}}\big[\varkappa\sqrt{z} J_{\alpha+1}^{(\ell)}(\varkappa\sqrt{z}) + \ell J_{\alpha+1}^{(\ell-1)}(\varkappa\sqrt{z})\big] \\ &= 2^\alpha z^{\frac{-2+\ell-\alpha}{2}} \tilde{J}_{\alpha+1}^{(\ell)}(\varkappa\sqrt{z}).\end{aligned}$$

The last equality follows from Leibniz formula $(tJ_{\alpha+1}(t))^{(\ell)} = tJ_{\alpha+1}^{(\ell)}(t) + \ell J_{\alpha+1}^{(\ell-1)}(t)$. □

2.4. Asymptotics for differences of Jacobi's functions

Let $\alpha, \beta > -1$ and let $n \geq 0$ be an integer. The Jacobi's function of second kind $Q_n^{(\alpha,\beta)}$ is defined as follows. For $x \in \mathbb{C} \setminus [-1,1]$, set

$$Q_n^{(\alpha,\beta)}(x) := \frac{1}{2}(x-1)^{-\alpha}(x+1)^{-\beta} \int_{-1}^{1} (1-t)^{\alpha}(1+t)^{\beta} \frac{P_n^{(\alpha,\beta)}(t)}{x-t}\, dt. \tag{20}$$

Let $Q^{(\alpha,\beta)} = (Q_n^{(\alpha,\beta)})_{n=0}^{\infty}$ be the corresponding sequence.

If $\alpha = s, \beta = 0$, we denote $Q^{(s)} := (Q_n^{(s,0)})_{n=0}^{\infty}$.

Proposition 2.4. *Let $s > -1$ and $r_n = \frac{w}{2n^2}$. Then*

$$\lim_{n\to\infty} n^{-s} Q_{\varkappa_n}^{(s)}(1+r_n) = 2^s w^{-\frac{s}{2}} K_s(\varkappa\sqrt{w}),$$

where K_s is the modified Bessel function of the second kind of order s. Moreover, for any $\varepsilon > 0$, the convergence is uniform as long as $\varkappa \in [\varepsilon, 1]$ and w ranges in a bounded simply connected subset of $\mathbb{C} \setminus \{0\}$.

Proof. Let us prove Proposition 2.4 for $\varkappa_n = n$, the general case is similar. Define t_n by the formula

$$1 + r_n = \frac{1}{2}\left(t_n + \frac{1}{t_n}\right), \quad |t_n| < 1.$$

By definition, we have

$$\lim_{n\to\infty} n(1-t_n) = \sqrt{w}.$$

We now use the integral representation for the Jacobi's function (cf. [16, 4.82.4]):

$$\begin{aligned} Q_n^{(s)}(1+r_n) = \frac{1}{2}\left(\frac{4t_n}{1-t_n}\right)^s \int_{-\infty}^{\infty} \big((1+t_n)e^{\tau} + 1 - t_n\big)^{-s} \\ \times \left(1 + r_n + \left(2r_n + r_n^2\right)^{\frac{1}{2}} \cosh\tau\right)^{-n-1} d\tau. \end{aligned}$$

Taking $n \to \infty$ and using the integral representation for the modified Bessel function (cf. [1, 9.6.24]), we see that

$$\begin{aligned} \lim_{n\to\infty} n^{-s} Q_n^{(s)}(1+r_n) &= 2^{s-1} w^{-\frac{s}{2}} \int_{-\infty}^{\infty} e^{-s\tau - \sqrt{w}\cosh\tau}\, d\tau \\ &= 2^{s-1} w^{-\frac{s}{2}} \int_{-\infty}^{\infty} e^{-\sqrt{w}\cosh\tau} \cosh(s\tau)\, d\tau \\ &= 2^{s} w^{-\frac{s}{2}} \int_{0}^{\infty} e^{-\sqrt{w}\cosh\tau} \cosh(s\tau)\, d\tau = 2^s w^{-\frac{s}{2}} K_s(\sqrt{w}). \end{aligned}$$

□

Proposition 2.5. *Let $s>-1$ and $r_n=\frac{w}{2n^2}$. Then for all $\ell\geq 0$, we have*

$$\lim_{n\to\infty} n^{\ell-s}\Delta^\ell\big(Q^{(s)}\big)_{\varkappa_n}(1+r_n)=2^s w^{\frac{\ell-s}{2}}K_s^{(\ell)}(\varkappa\sqrt{w}), \tag{21}$$

where $K_s^{(\ell)}$ is the ℓ-th derivative of the modified Bessel function K_s. Moreover, for any $\varepsilon>0$, the convergence is uniform as long as $\varkappa\in[\varepsilon,1]$ and w ranges in a bounded simply connected subset of $\mathbb{C}\setminus\{0\}$.

Proof. It suffices to prove Proposition 2.5 in the case $\varkappa_n=n$. The general case can be easily deduced by using the uniform convergence.

From the identity (11) we obtain

$$(n+1)\Delta^1\big(Q^{(s)}\big)_n(x)+\left(n+\frac{s}{2}+1\right)(x-1)\Delta^0\big(Q^{(s+1)}\big)_n(x)=s\Delta^0\big(Q^{(s)}\big)_n(x).$$

By induction, for all $\ell\geq 0$, we have

$$\begin{aligned}(n+1)\Delta^{\ell+1}\big(Q^{(s)}\big)_n(x)+\ell\Delta^\ell\big(Q^{(s)}\big)_{n+1}(x)+\ell(x-1)\Delta^{\ell-1}\big(Q^{(s+1)}\big)_{n+1}(x)\\ +\left(n+\frac{s}{2}+1\right)(x-1)\Delta^\ell\big(Q^{(s+1)}\big)_n(x)=s\Delta^\ell\big(Q^{(s)}\big)_n(x).\end{aligned} \tag{22}$$

Using the formula ([1, 9.6.26])

$$K_s'(t)=-K_{s+1}(t)+\frac{s}{t}K_s(t), \tag{23}$$

we may derive that for $\ell\geq 1$,

$$t\big[K_s^{(\ell+1)}(t)+K_{s+1}^{(\ell)}(t)\big]=(s-\ell)K_s^{(\ell)}(t)-\ell K_{s+1}^{(\ell-1)}(t). \tag{24}$$

Proposition 2.4 says that the relation (21) holds for $\ell=0$. Now assume that the relation (21) holds for $0,1,\dots,\ell$. By the identity (22), we have

$$\begin{aligned}&\lim_{n\to\infty} n^{\ell+1-s}\Delta^{\ell+1}\big(Q^{(s)}\big)_n\big(1+r_j^{(n)}\big)\\ &=-2^s\ell w_j^{\frac{\ell-s}{2}}K_s^{(\ell)}(\sqrt{w_j})-\ell\frac{w_j}{2}2^{s+1}w_j^{\frac{\ell-s-2}{2}}K_{s+1}^{(\ell-1)}(\sqrt{w_j})\\ &\quad-\frac{w_j}{2}2^{s+1}w_j^{\frac{\ell-s-1}{2}}K_{s+1}^{(\ell)}(\sqrt{w_j})+s2^s w_j^{\frac{\ell-s}{2}}K_s^{(\ell)}(\sqrt{w_j})\\ &=2^s w_j^{\frac{\ell-s}{2}}\left[(s-\ell)K_s^{(\ell)}(\sqrt{w_j})-\ell K_{s+1}^{(\ell-1)}(\sqrt{w_j})-\sqrt{w_j}K_{s+1}^{(\ell)}(\sqrt{w_j})\right]\\ &=2^s w_j^{\frac{\ell+1-s}{2}}K_s^{(\ell+1)}(\sqrt{w_j}).\end{aligned}$$

This completes the proof. □

3. Bessel point processes and Pickrell measures

3.1. Radial parts of Pickrell measures and infinite Bessel point processes

Following Pickrell, we introduce a map $\mathfrak{rad}_n : \mathrm{Mat}(n, \mathbb{C}) \to \mathbb{R}_+^n$ by the formula

$$\mathfrak{rad}_n(z) = \big(\lambda_1(z), \dots, \lambda_n(z)\big),$$

where $(\lambda_1(z), \dots, \lambda_n(z))$ is the collection of the eigenvalues of the *positive matrix* $z^* z$ arranged in the order $\lambda_1(z) \leq \lambda_2(z) \leq \dots \leq \lambda_n(z)$.

When $s > -1$, the measure $\mu_n^{(s)}$ is a probability measure and the push-forward measure $(\mathfrak{rad}_n)_* \mu_n^{(s)}$ is well-defined. We call $(\mathfrak{rad}_n)_* \mu_n^{(s)}$ the *radial part* of $\mu_n^{(s)}$. Since finite-dimensional unitary groups are compact, when $s \leq -1$, although $\mu_n^{(s)}$ is an infinite measure, the measure $(\mathfrak{rad}_n)_* \mu_n^{(s)}$ is still well-defined once we have $n + s > 0$.

Denote $d\lambda$ the Lebesgue measure on $\mathbb{R}_+^n$, then $(\mathfrak{rad}_n)_* \mu_n^{(s)}$ is a measure on the subset $\{0 \leq \lambda_1 \leq \dots \leq \lambda_n\} \subset \mathbb{R}_+^n$ having the form

$$\mathrm{const}_{n,s} \cdot \prod_{1 \leq i < j \leq n} (\lambda_i - \lambda_j)^2 \cdot \prod_{i=1}^n \frac{1}{(1+\lambda_i)^{2n+s}} \cdot d\lambda. \tag{25}$$

The change of variables

$$u_i = \frac{\lambda_i - 1}{\lambda_i + 1},$$

transforms $(\mathfrak{rad}_n)_* \mu_n^{(s)}$ to a measure on $(-1, 1)^n$ given by the formula

$$\mathrm{const}_{n,s} \cdot \prod_{1 \leq i < j \leq n} (u_i - u_j)^2 \cdot \prod_{i=1}^n (1 - u_i)^s \, du_i. \tag{26}$$

For $s > -1$, the constants are chosen such that the measures (26) are probability measures. It is the Jacobi orthogonal polynomial ensemble (with parameters $\alpha = s$, $\beta = 0$) and hence represents a determinantal point process. The Heine-Mehler asymptotics of Jacobi polynomials imply that these determinantal point processes, when rescaled by

$$u_i = 1 - \frac{y_i}{2n^2} \quad \text{and} \quad y_i \in \big(0, 4n^2\big), \; i = 1, \dots, n, \tag{27}$$

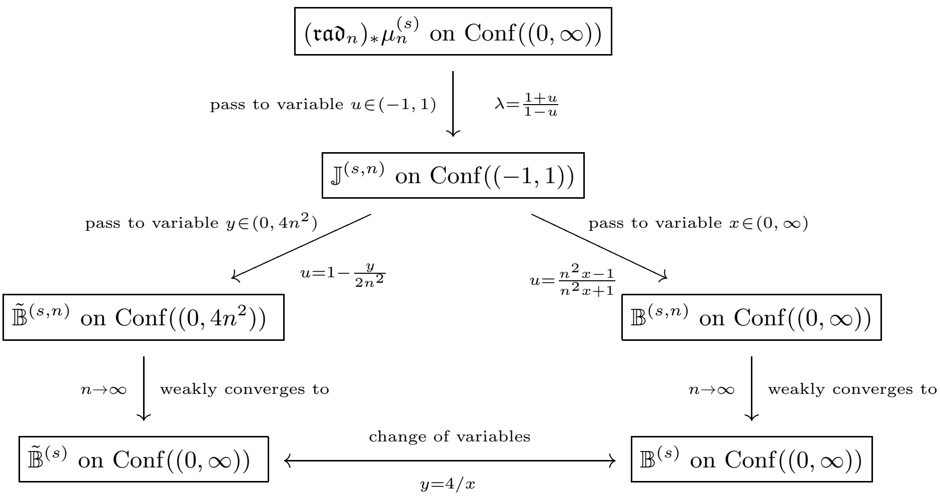
have as scaling limit the Bessel point process on $(0, \infty)$, see Tracy and Widom [17]. Using the same notation as in [6], we denote this point process by $\widetilde{\mathbb{B}}^{(s)}$. Recall that we denote $\mathrm{Conf}((0, \infty))$ the space of configurations on $(0, \infty)$. Hence $\widetilde{\mathbb{B}}^{(s)}$ is a probability measure on $\mathrm{Conf}((0, \infty))$.

For $s \le -1$, the scaling limit of (26) under the scaling regime (27) is an *infinite determinantal measure* $\tilde{\mathbb{B}}^{(s)}$ on $\mathrm{Conf}((0,\infty))$, see [6] [7].

Fix $s \in \mathbb{R}$. Let us denote by $\mathbb{J}^{(s,n)}$ the determinantal measure on $\mathrm{Conf}((-1,1))$ (finite or infinite) corresponding to the measure (26). The change of variable $u = 1 - \frac{y}{2n^2}$ transforms $\mathbb{J}^{(s,n)}$ to a determinantal measure on $\mathrm{Conf}((0,4n^2))$ (which will be viewed as a subset of $\mathrm{Conf}((0,\infty))$). We denote this measure on $\mathrm{Conf}((0,4n^2))$ by $\tilde{\mathbb{B}}^{(s,n)}$.

Here we recall that a change of variables $x_1 = \rho(y_1)$, $x_2 = \rho(y_2)$ transforms a kernel $K(x_1,x_2)$ to a kernel of the form $\sqrt{\rho'(y_1)\rho'(y_2)}K(\rho(y_1),\rho(y_2))$.

The relations between the determinantal measures we are going to deal with are demonstrated in the following diagram:



Remark 3.1. For obtaining the relation between $\tilde{\mathbb{B}}^{(s)}$ and $\mathbb{B}^{(s)}$, we used the relation:

$$u = \frac{n^2x-1}{n^2x+1} = 1 - \frac{4/x}{2n^2+2/x} \sim 1 - \frac{4/x}{2n^2}.$$

3.2. Christoffel-Uvarov deformations and the scaling limits

Let $(w_1,\dots,w_m)$ and $(v_1,\dots,v_m)$ be two fixed m-tuples of *distinct* positive numbers. Define functions $F_n, G_n : (-1,1) \to \mathbb{R}_+$ by the formulae:

$$F_n(u) = \prod_{i=1}^{m} \frac{(1 - \frac{w_i}{2n^2} - u)^2}{(1-u)^2} \quad \text{and} \quad G_n(u) = \prod_{i=1}^{m} \frac{1-u}{1 + \frac{v_i}{2n^2} - u}.$$

Let $f_n(x)$, $g_n(x)$ denote the functions given by the formulae

$$f_n(x)=F_n\left(\frac{n^2x-1}{n^2x+1}\right) \quad \text{and} \quad g_n(x)=G_n\left(\frac{n^2x-1}{n^2x+1}\right). \tag{28}$$

By direct computations, the sequence of functions $(f_n)_{n=1}^\infty$ and the sequence of functions $(g_n)_{n=1}^\infty$ admits pointwise limit f and g respectively, given by the formulae:

$$f(x)=\prod_{i=1}^m\left(1-\frac{w_i}{4}x\right)^2 \quad \text{and} \quad g(x)=\prod_{i=1}^m \frac{4}{4+v_ix}. \tag{29}$$

Let $(h_n)_{n=1}^\infty$ be either the sequence $(f_n)_{n=1}^\infty$ or the sequence $(g_n)_{n=1}^\infty$, and let $h:(0,\infty)\to[0,1]$ denote f or g accordingly. There exists a constant $M>0$ such that

(a) for any (finite or infinite) sequence of positive real numbers $(x_i)_{i=1}^N$, we have

$$\prod_{i=1}^N h_n(x_i)\le M\cdot\prod_{i=1}^N h(x_i). \tag{30}$$

(b) for any sequence $\{(x_i^{(n)})_{1\le i\le n}\}_{n=1}^\infty$ satisfying $x_i^{(n)}\ge 0$,

$$\lim_{n\to\infty} x_i^{(n)}=x_i \quad \text{and} \quad \lim_{n\to\infty}\sum_{i=1}^n x_i^{(n)}=\sum_{i=1}^\infty x_i<\infty,$$

we have

$$\lim_{n\to\infty}\prod_{i=1}^n h_n\big(x_i^{(n)}\big)=\prod_{i=1}^\infty h(x_i). \tag{31}$$

Let H_n denote the function F_n or G_n. On the cube $(-1,1)^n$, the probability measure

$$\mathrm{const}_{n,s}\cdot\prod_{1\le i<j\le n}(u_i-u_j)^2\prod_{i=1}^n(1-u_i)^sH_n(u_i)\,du_i \tag{32}$$

gives a determinantal point process. The change of variable $y=\frac{n^2x-1}{n^2x+1}\in\mathbb{R}$, transforms the point process (32) to

$$\mathbb{P}_{h_n}^{(s,n)}:=\frac{\Psi_{h_n}\mathbb{B}^{(s,n)}}{\int_{\mathrm{Conf}((0,+\infty))}\Psi_{h_n}\,d\mathbb{B}^{(s,n)}}, \tag{33}$$

where $\mathbb{B}^{(s,n)}$ is the point process $(\mathfrak{rad}_n)_*\mu_n^{(s)}$ after the change of variable $y=\frac{n^2x-1}{n^2x+1}$ and h_n is the function f_n or g_n on $(0,\infty)$ defined by the formulae (28).

We shall need the following elementary lemma.

Lemma 3.2. *Let* (Ω, m) *be a measure space,* m *is a* σ*-finite measure. Given two sequences of positive functions* $(\varphi_n)_{n=1}^{\infty}$ *and* $(\Phi_n)_{n=1}^{\infty}$ *in* $L^1(\Omega, m)$ *satisfying*

(i) *for any* $n\in\mathbb{N}$, $\varphi_n \leq \Phi_n$.

(ii) $\lim_{n\to\infty} \varphi_n = \varphi$, *a.e. and* $\lim_{n\to\infty} \Phi_n = \Phi$, *a.e.*

(iii) $\lim_{n\to\infty} \int \Phi_n \, dm = \int \Phi \, dm < \infty$.

Then

$$\lim_{n\to\infty} \int \varphi_n \, dm = \int \varphi \, dm.$$

Proposition 3.3. *Assume that we are in one of the following situations:*

I. $h_n = f_n$, $h = f$ *and* $s - 2m > -1$;

II. $h_n = g_n, h = g$ *and* $s + m > -1$.

Then the determinantal probability measure in (33) *converges weakly in the space of finite Radon measures on* $\mathrm{Conf}((0,\infty))$ *to the probability measure:*

$$\mathbb{P}_h^{(s)} := \frac{\Psi_h \mathbb{B}^{(s)}}{\int_{\mathrm{Conf}((0,\infty))} \Psi_h \, d\mathbb{B}^{(s)}}. \tag{34}$$

Proof. In this proof, we will use without explanation the notation in [6, p. 25]. It is proved in [6] that the measure $\mu^{(s)}$ is supported on the subset $\mathrm{Mat}_{\mathrm{reg}}(\mathbb{N}, \mathbb{C})$ for any $s \in \mathbb{R}$. By the properties (30) and (31) for the sequence (h_n), for any $z \in \mathrm{Mat}_{\mathrm{reg}}(\mathbb{N}, \mathbb{C})$, we have

$$\lim_{n\to\infty} \Psi_{h_n}\big(\mathfrak{r}^{(n)}(z)\big) = \Psi_h\big(\mathfrak{r}^{\infty}(z)\big) \quad \text{and} \quad \Psi_{h_n}\big(\mathfrak{r}^{(n)}(z)\big) \leq M \cdot \Psi_h\big(\mathfrak{r}^{(n)}(z)\big).$$

Now take any bounded and continuous function φ on $\mathrm{Conf}((0,\infty))$, we have

$$\int \phi(X) \, d\mathbb{P}_{h_n}^{(s,n)}(X) = \frac{\int_{\mathrm{Mat}_{\mathrm{reg}}(\mathbb{N},\mathbb{C})} \varphi(\mathfrak{r}^{(n)}(z)) \Psi_{h_n}(\mathfrak{r}^{(n)}(z)) \, d\mu^{(s)}(z)}{\int_{\mathrm{Mat}_{\mathrm{reg}}(\mathbb{N},\mathbb{C})} \Psi_{h_n}(\mathfrak{r}^{(n)}(z)) \, d\mu^{(s)}(z)}.$$

By Lemma 3.2, to prove Proposition 3.3, it suffices to show that

$$\lim_{n\to\infty} \int_{\mathrm{Mat}(\mathbb{N},\mathbb{C})} \Psi_h\big(\mathfrak{r}^{(n)}(z)\big) \, d\mu^{(s)}(z) = \int_{\mathrm{Mat}(\mathbb{N},\mathbb{C})} \Psi_h\big(\mathfrak{r}^{(\infty)}(z)\big) \, d\mu^{(s)}(z). \tag{35}$$

If $s > -1$, the measure $\mu^{(s)}$ is a probability measure, by dominated convergence theorem, the equality (35) holds.

If $s \leq -1$, the measure $\mu^{(s)}$ is infinite. The radial part of $\mu^{(s)}$ is an infinite determinantal process described in Section 5.2 in [6]. By using the asymptotic formulae

$$f(x) \sim 1 - \sum_i \frac{w_i}{2} x \quad \text{and} \quad g(x) \sim 1 - \sum_i \frac{v_i}{4} x, \quad \text{as } x \to 0^+,$$

we can check that the conditions of Proposition 3.6 in [6] are satisfied. For instance, let us check the following condition

$$\lim_{n\to\infty} \mathrm{tr}\sqrt{1-h}\Pi^{(s,n)}\sqrt{1-h} = \mathrm{tr}\sqrt{1-h}\Pi^{(s)}\sqrt{1-h}, \tag{36}$$

where $\Pi^{(s,n)}$ is the orthogonal projection onto the subspace $L^{(s+2n_s,n-n_s)}$ described in Section 5.2.1 in [6]. Combining the estimates given in Proposition 5.11 and Proposition 5.13 in [6], the integrands $(1-h(x))\Pi^{(s,n)}(x,x)$ appeared in the calculation of

$$\mathrm{tr}\sqrt{1-h}\Pi^{(s,n)}\sqrt{1-h}$$

are uniformly integrable, hence by the Heine-Mehler classical asymptotics, the equality (36) indeed holds. Now by Corollary 3.7 in [6], we have

$$\frac{\Psi_h \mathbb{B}^{(s,n)}}{\int_{\mathrm{Conf}((0,\infty))} \Psi_h \, d\mathbb{B}^{(s,n)}} \longrightarrow \frac{\Psi_h \mathbb{B}^{(s)}}{\int_{\mathrm{Conf}((0,\infty))} \Psi_h \, d\mathbb{B}^{(s)}}.$$

It follows that

$$\begin{aligned} &\lim_{n\to\infty} \frac{\int_{\mathrm{Mat}(\mathbb{N},\mathbb{C})} \varphi(\mathfrak{r}^{(n)}(z))\Psi_h(\mathfrak{r}^{(n)}(z))\, d\mu^{(s)}(z)}{\int_{\mathrm{Mat}(\mathbb{N},\mathbb{C})} \Psi_h(\mathfrak{r}^{(n)}(z))\, d\mu^{(s)}(z)} \\ &\qquad = \frac{\int_{\mathrm{Mat}(\mathbb{N},\mathbb{C})} \varphi(\mathfrak{r}^{(\infty)}(z))\Psi_h(\mathfrak{r}^{(\infty)}(z))\, d\mu^{(s)}(z)}{\int_{\mathrm{Mat}(\mathbb{N},\mathbb{C})} \Psi_h(\mathfrak{r}^{(\infty)}(z))\, d\mu^{(s)}(z)}. \end{aligned} \tag{37}$$

Moreover, by Lemma 1.14 in [6], there exists a positive bounded continuous function ϕ such that

$$\lim_{n\to\infty} \int_{\mathrm{Mat}(\mathbb{N},\mathbb{C})} \phi\big(\mathfrak{r}^{(n)}(z)\big)\, d\mu^{(s)}(z) = \int_{\mathrm{Mat}(\mathbb{N},\mathbb{C})} \phi\big(\mathfrak{r}^{(\infty)}(z)\big)\, d\mu^{(s)}(z).$$

Again by Lemma 3.2, we have

$$\begin{aligned} &\lim_{n\to\infty} \int_{\mathrm{Mat}(\mathbb{N},\mathbb{C})} \phi\big(\mathfrak{r}^{(n)}(z)\big)\Psi_h\big(\mathfrak{r}^{(n)}(z)\big)\, d\mu^{(s)}(z) \\ &\qquad = \int_{\mathrm{Mat}(\mathbb{N},\mathbb{C})} \phi\big(\mathfrak{r}^{(\infty)}(z)\big)\Psi_h\big(\mathfrak{r}^{(\infty)}(z)\big)\, d\mu^{(s)}(z). \end{aligned} \tag{38}$$

Finally, by taking $\varphi = \phi$, the relation (35) now follows from the relations (37) and (38). $\square$

4. Scaling limits of Christoffel-Uvarov deformations of Jacobi orthogonal polynomial ensembles

In this section, we calculate the kernels of the determinantal probability measures $\mathbb{P}_h^{(s)}$ in Proposition 3.3.

4.1. Strategy

We are going to deal with two cases.

(i) Case I: for any integer $m \geq 1$ and real number $s > 2m-1$, we obtain the correlation kernel for the determinantal probability measure

$$\frac{\Psi_f \mathbb{B}^{(s)}}{\int_{\mathrm{Conf}((0,\infty))} \Psi_f \, d\mathbb{B}^{(s)}}.$$

Recall that f is the function defined by the formula

$$f(x) = \prod_{k=1}^{m} \left(1 - \frac{w_k}{4} x\right)^2.$$

Note that in this case, we have $s > -1$ and the measure $\mathbb{B}^{(s)}$ is obtained by Borodin and Olshanski. Our result in this case provides an example for calculating the correlation kernels for determinantal measures deformed by multiplying a multiplicative functional.

(ii) Case II: for any $s \in \mathbb{R}$, we choose any integer $m \in \mathbb{N}$ such that $s + m > -1$ and we obtain the correlation kernel for the determinantal probability measure

$$\frac{\Psi_g \mathbb{B}^{(s)}}{\int_{\mathrm{Conf}((0,\infty))} \Psi_g \, d\mathbb{B}^{(s)}}$$

for a function g given by

$$g(x) = \prod_{k=1}^{m} \frac{4}{4 + v_k x}.$$

We emphasize that in this case, there is no restriction on the parameter $s \in \mathbb{R}$. If $s > -1$, we get similar result as in Case I. However, the case where $s \leq -1$ is of particular interests for us. Our main result, Theorem 1.5, is derived as a corollary of the main result in this case.

Let us briefly explain our strategy in our main case, the Case II. The analysis of Case I is similar.

By the diagram above Remark 3.1, the change of variables $y=4/x$ transforms the determinantal probability measure

$$\frac{\Psi_g \mathbb{B}^{(s)}}{\int_{\mathrm{Conf}((0,\infty))} \Psi_g \, d\mathbb{B}^{(s)}}$$

to the determinantal probability measure

$$\frac{\Psi_{\tilde{g}} \tilde{\mathbb{B}}^{(s)}}{\int_{\mathrm{Conf}((0,\infty))} \Psi_{\tilde{g}} \, d\tilde{\mathbb{B}}^{(s)}}, \tag{39}$$

where $\tilde{g}$ is a function defined by $\tilde{g}(y)=g(4/y)$. For analyzing the measure (39), we need to consider the determinantal probability measure

$$\frac{\Psi_{G_n} \mathbb{J}^{(s,n)}}{\int_{\mathrm{Conf}((0,4n^2))} \Psi_{G_n} \, d\mathbb{J}^{(s,n)}}. \tag{40}$$

The change of variable $u=1-\frac{y}{2n^2}$ transforms (40) to the following probability measure

$$\frac{\Psi_{\tilde{g}_n} \tilde{\mathbb{B}}^{(s,n)}}{\int_{\mathrm{Conf}((0,\infty))} \Psi_{\tilde{g}_n} \, d\tilde{\mathbb{B}}^{(s,n)}}, \tag{41}$$

where $\tilde{g}_n : (0, 4n^2) \to (0,1)$ is given by the formula

$$\tilde{g}_n(y) := G_n\left(1 - \frac{y}{2n^2}\right) = \prod_{k=1}^{m} \frac{y}{v_k + y}.$$

By Remark 3.1, $\tilde{g}_n(y)$ is approximately $g_n(4/y)$. Slightly modifying the proof of Proposition 3.3, one may prove that

$$\frac{\Psi_{\tilde{g}_n} \tilde{\mathbb{B}}^{(s,n)}}{\int_{\mathrm{Conf}((0,\infty))} \Psi_{\tilde{g}_n} \, d\tilde{\mathbb{B}}^{(s,n)}} \xrightarrow{n\to\infty} \frac{\Psi_{\tilde{g}} \tilde{\mathbb{B}}^{(s)}}{\int_{\mathrm{Conf}((0,\infty))} \Psi_{\tilde{g}} \, d\tilde{\mathbb{B}}^{(s)}}.$$

We will show by direct computation that the correlation kernel of the determinantal probability measure

$$\frac{\Psi_{\tilde{g}_n} \tilde{\mathbb{B}}^{(s,n)}}{\int_{\mathrm{Conf}((0,\infty))} \Psi_{\tilde{g}_n} \, d\tilde{\mathbb{B}}^{(s,n)}},$$

converges pointwise and uniformly on compact sets to a limit kernel. This limit kernel is then automatically the correlation kernel of

$$\frac{\Psi_{\tilde{g}} \tilde{\mathbb{B}}^{(s)}}{\int_{\mathrm{Conf}((0,\infty))} \Psi_{\tilde{g}} \, d\tilde{\mathbb{B}}^{(s)}}.$$

4.2. Case I

Recall that in Case I, we have fixed $s\in\mathbb{R}$, $m\in\mathbb{N}$ such that $s>2m-1$ and have fixed an m-tuple of distinct positive numbers $(w_1,\dots,w_m)$. Note that in Case I, we must have $s>-1$.

As explained in Section 4.1, we need to consider the determinantal probability measure

$$\frac{\Psi_{F_n}\mathbb{J}^{(s,n)}}{\int_{\mathrm{Conf}((0,4n^2))}\Psi_{F_n}\,d\mathbb{J}^{(s,n)}}. \tag{42}$$

The above determinantal measure turns out to be given by the orthogonal polynomial ensemble

$$\mathrm{const}_{n,s}\prod_{1\le i<j\le n}(u_i-u_j)^2\prod_{i=1}^{n}\left[\prod_{k=1}^{m}\left(1-\frac{w_k}{2n^2}-u_i\right)^2\right](1-u_i)^{s-2m}\,du_i. \tag{43}$$

The ensemble (43) is a Christoffel deformation of the following Jacobi polynomial ensemble (which is a probability measure since $s-2m>-1$):

$$\mathrm{const}_{n,s}\prod_{1\le i<j\le n}(u_i-u_j)^2\prod_{i=1}^{n}(1-u_i)^{s-2m}\,du_i.$$

Let $\omega_n:(-1,1)\to\mathbb{R}_+$ be the weight:

$$\omega_n(u):=\prod_{k=1}^{m}\left(1-\frac{w_k}{2n^2}-u\right)^2(1-u)^{s-2m},$$

and let $(\pi_j)_{j\ge 0}$ denote the system of monic orthogonal polynomials associated with the weights ω_n (we omit in the notation the dependence of π_j on n, s and $w_1,\dots,w_m$). Let $K_n(u_1,u_2)$ denote the associated n-th Christoffel-Darboux kernel given by the formula:

$$K_n(u_1,u_2)=\sqrt{\omega_n(u_1)\omega_n(u_2)}\cdot\sum_{j=0}^{n-1}\frac{\pi_j(u_1)\cdot\pi_j(u_2)}{h_j},$$

where

$$h_j=\int_{-1}^{1}\pi_j(u)^2\omega_n(u)\,du. \tag{44}$$

By classical results on orthogonal polynomial ensembles, we know that $K_n(u_1,u_2)$ is the correlation kernel for (43) and hence for (42).

Our aim of this section is to establish the scaling limit of $K_n(u_1,u_2)$ in the regime:

(45) $$u_i = 1 - \frac{y_i}{2n^2} \quad \text{and} \quad y_i > 0, \ i = 1, 2.$$

We show by direct computation the existence of the following limit kernel:

(46) $$K_\infty(y_1,y_2) := \lim_{n\to\infty} \frac{1}{2n^2} K_n\left(1-\frac{y_1}{2n^2}, 1-\frac{y_2}{2n^2}\right).$$

The limit kernel $K_\infty(y_1,y_2)$ is then automatically the correlation kernel for the determinantal probability measure

$$\frac{\Psi_{\tilde{f}} \tilde{\mathbb{B}}^{(s)}}{\int_{\mathrm{Conf}((0,\infty))} \Psi_{\tilde{f}}\, d\tilde{\mathbb{B}}^{(s)}}.$$

4.2.1. Explicit formulae for $K_n(u_1,u_2)$

Let $k_j^{(s-2m)}$ denote the leading coefficient of the Jacobi polynomial $P_j^{(s-2m)}$ and denote

$$h_j^{(s-2m)} := \int \left[P_j^{(s-2m)}(u)\right]^2 (1-u)^{s-2m}\, du.$$

Let

$$\xi_i^{(n)} = 1 - \frac{w_i}{2n^2}, \quad 1 \le i \le m.$$

Then the *monic* polynomial π_j is given by the Christoffel formula ([16, Thm 2.5.]):

$$\pi_j(u) = \frac{1}{\prod_{i=1}^m (\xi_i^{(n)} - u)^2} \cdot \frac{D_j(u)}{k_{j+2m}^{(s-2m)} \cdot \delta_j},$$

where

$$D_j(u) = \begin{vmatrix} P_j^{(s-2m)}(\xi_1^{(n)}) & P_{j+1}^{(s-2m)}(\xi_1^{(n)}) & \dots & P_{j+2m}^{(s-2m)}(\xi_1^{(n)}) \\ \vdots & \vdots & & \vdots \\ P_j^{(s-2m)}(\xi_m^{(n)}) & P_{j+1}^{(s-2m)}(\xi_m^{(n)}) & \dots & P_{j+2m}^{(s-2m)}(\xi_m^{(n)}) \\ \dot{P}_j^{(s-2m)}(\xi_1^{(n)}) & \dot{P}_{j+1}^{(s-2m)}(\xi_1^{(n)}) & \dots & \dot{P}_{j+2m}^{(s-2m)}(\xi_1^{(n)}) \\ \vdots & \vdots & & \vdots \\ \dot{P}_j^{(s-2m)}(\xi_m^{(n)}) & \dot{P}_{j+1}^{(s-2m)}(\xi_m^{(n)}) & \dots & \dot{P}_{j+2m}^{(s-2m)}(\xi_m^{(n)}) \\ P_j^{(s-2m)}(u) & P_{j+1}^{(s-2m)}(u) & \dots & P_{j+2m}^{(s-2m)}(u) \end{vmatrix};$$

and $\dot{P}_j^{(s-2m)}$ denotes the derivative of the polynomial $P_j^{(s-2m)}$; the quantity δ_j is the coefficient of $P_{j+2m}^{(s-2m)}(u)$ in the expansion of the determinant $D_j(u)$ with respect to the last row.

Lemma 4.1. *For any $j \geq 0$, we have*

$$h_j := \int_{-1}^{1} \pi_j(u)^2 \omega_n(u)\, du = \frac{h_j^{(s-2m)}}{k_j^{(s-2m)} k_{j+2m}^{(s-2m)}} \cdot \frac{\delta_{j+1}}{\delta_j}.$$

Proof. By orthogonality, for any $\ell \geq 1$, we have

$$\int_{-1}^{1} P_{j+\ell}^{(s-2m)}(u) \pi_j(u)(1-u)^{s-2m}\, du = 0.$$

Note that

$$D_j(u) = \delta_{j+1} P_j^{(s-2m)}(u) + \text{ linear combination of } P_{j+1}^{(s-2m)}(u), \dots, P_{j+2m}^{(s-2m)}(u),$$

whence

$$\begin{aligned} h_j &= \frac{1}{k_{j+2m}^{(s-2m)} \delta_j} \int D_j(u) \pi_j(u)(1-u)^{s-2m}\, du \\ &= \frac{1}{k_{j+2m}^{(s-2m)} \delta_j} \int \delta_{j+1} P_j^{(s-2m)}(u) \pi_j(u)(1-u)^{s-2m}\, du \\ &= \frac{\delta_{j+1}}{k_{j+2m}^{(s-2m)} \delta_j} \int \left\{ P_j^{(s-2m)}(u) \right\}^2 \frac{1}{k_j^{(s-2m)}} (1-u)^{s-2m}\, du \\ &= \frac{h_j^{(s-2m)}}{k_j^{(s-2m)} k_{j+2m}^{(s-2m)}} \cdot \frac{\delta_{j+1}}{\delta_j}. \quad \square \end{aligned}$$

By the Christoffel-Darboux formula (cf. [16, Thm 3.2.2]), we have:

$$K_n(u_1, u_2) = \frac{\sqrt{\omega_n(u_1)\omega_n(u_2)}}{h_{n-1}} \cdot \frac{\pi_n(u_1)\pi_{n-1}(u_2) - \pi_n(u_2)\pi_{n-1}(u_1)}{u_1 - u_2}.$$

For simplifying the notation, we denote

$$u_i^{(n)} := 1 - \frac{y_i}{2n^2}, \quad i = 1, 2.$$

Then the change of variables (45) transforms the kernel $K_n(u_1, u_2)$ to the kernel:

$$\widetilde{K}_n(y_1, y_2) = \frac{1}{2n^2} K_n\big(u_1^{(n)}, u_2^{(n)}\big) = \frac{(y_1 y_2)^{\frac{s-2m}{2}}}{\left|\prod_{i=1}^{m} (y_1 - w_i)(y_2 - w_i)\right|} \cdot S_n(y_1, y_2), \tag{47}$$

where

$$(48)\qquad S_n(y_1,y_2)=\frac{(2n^2)^{4m-s}}{\frac{h_{n-1}^{(s-2m)}k_{n+2m}^{(s-2m)}}{k_{n-1}^{(s-2m)}}\delta_n^2}\cdot\frac{D_n(u_1^{(n)})D_{n-1}(u_2^{(n)})-D_n(u_2^{(n)})D_{n-1}(u_1^{(n)})}{y_2-y_1}.$$

4.2.2. Scaling limits

We divide the calculation of $K_\infty(y_1,y_2)$ into five steps.

Step 1: Asymptotic of $\frac{h_{n-1}^{(s-2m)}k_{n+2m}^{(s-2m)}}{k_{n-1}^{(s-2m)}}$.

The following formulae are well-known [16, p. 63, p. 68]:

$$(49)\qquad k_j^{(s-2m)}=\frac{1}{2^j\cdot j!}\frac{\Gamma(2j+s-2m+1)}{\Gamma(j+s-2m+1)}\quad\text{and}\quad h_j^{(s-2m)}=\frac{2^{s-2m+1}}{2j+s-2m+1}.$$

For all $a\in\mathbb{R}$, we have $\lim_{n\to\infty}\frac{\Gamma(n+a)}{n^a\Gamma(n)}=1$. This implies that if $p\in\mathbb{Z}$, then

$$(50)\qquad \lim_{n\to\infty}\frac{k_{\varkappa_n+p}^{(s-2m)}}{k_{\varkappa_n}^{(s-2m)}}=2^p.$$

Thus we get the following

Lemma 4.2. *We have*

$$\lim_{n\to\infty} n\cdot\frac{h_{n-1}^{(s-2m)}k_{n+2m}^{(s-2m)}}{k_{n-1}^{(s-2m)}}=2^{s+1}.$$

Step 2: Asymptotic of δ_n.

Recall that we defined in Definition 1.4 and Proposition 2.3 that

$$J_{\alpha,v}(t):=J_\alpha(t\sqrt{v})\quad\text{and}\quad \tilde{J}_\alpha(t):=tJ_\alpha(t).$$

So we also define

$$\tilde{J}_{\alpha,v}(t):=\tilde{J}_\alpha(t\sqrt{v}).$$

Recall that we denote by $(\varkappa_n)_{n=1}^\infty$ a sequence in $\mathbb{N}$, such that

$$\lim_{n\to\infty}\varkappa_n/n=\varkappa>0.$$

Lemma 4.3. *Let* $\alpha = 6m^2 - 2sm - 3m$. *Then we have*

$$\lim_{n\to\infty} n^{\alpha}\delta_{\varkappa_n} = \frac{2^{2m(s-2m)}}{(w_1 \dots w_m)^{1+s-2m}} C(\varkappa),$$

where $C(\varkappa) = W(J_{s-2m,w_1}, \dots, J_{s-2m,w_m}, \tilde{J}_{s-2m+1,w_1}, \dots \tilde{J}_{s-2m+1,w_m})(\varkappa)$. *Moreover, the above convergence is uniform as long as* $\varkappa$ *is in a compact subset of* $(0,\infty)$.

Proof. By the multi-linearity of the determinant with respect to the columns, the determinant $\delta_{\varkappa_n}$ equals to the following determinant:

$$\begin{vmatrix} \Delta^0(P^{(s-2m)})_{\varkappa_n}(\xi_1^{(n)}) & \Delta^1(P^{(s-2m)})_{\varkappa_n}(\xi_1^{(n)}) & \dots & \Delta^{2m-1}(P^{(s-2m)})_{\varkappa_n}(\xi_1^{(n)}) \\ \vdots & \vdots & & \vdots \\ \Delta^0(P^{(s-2m)})_{\varkappa_n}(\xi_m^{(n)}) & \Delta^1(P^{(s-2m)})_{\varkappa_n}(\xi_m^{(n)}) & \dots & \Delta^{2m-1}(P^{(s-2m)})_{\varkappa_n}(\xi_m^{(n)}) \\ \dot{\Delta}^0(P^{(s-2m)})_{\varkappa_n}(\xi_1^{(n)}) & \dot{\Delta}^1(P^{(s-2m)})_{\varkappa_n}(\xi_1^{(n)}) & \dots & \dot{\Delta}^{2m-1}(P^{(s-2m)})_{\varkappa_n}(\xi_1^{(n)}) \\ \vdots & \vdots & & \vdots \\ \dot{\Delta}^0(P^{(s-2m)})_{\varkappa_n}(\xi_m^{(n)}) & \dot{\Delta}^1(P^{(s-2m)})_{\varkappa_n}(\xi_m^{(n)}) & \dots & \dot{\Delta}^{2m-1}(P^{(s-2m)})_{\varkappa_n}(\xi_m^{(n)}) \end{vmatrix}.$$

Multiplying the matrix used in the above determinant on the right by the diagonal matrix

$$\mathrm{diag}\big(n^{-s+2m}, n^{1-s+2m}, \dots, n^{4m-1-s}\big)$$

and on the left by the diagonal matrix

$$\mathrm{diag}(\underbrace{1,\dots,1}_{m \text{ terms}}, \underbrace{n^{-2},\dots,n^{-2}}_{m \text{ terms}})$$

and taking determinant, we obtain that $n^{\alpha}\delta_{\varkappa_n}$ equals to the following determinant:

$$\begin{vmatrix} n^{-s'}\Delta^0(P^{(s')})_{\varkappa_n}(\xi_1^{(n)}) & n^{1-s'}\Delta^1(P^{(s')})_{\varkappa_n}(\xi_1^{(n)}) & \dots & n^{2m-1-s'}\Delta^{2m-1}(P^{(s')})_{\varkappa_n}(\xi_1^{(n)}) \\ \vdots & \vdots & & \vdots \\ n^{-s'}\Delta^0(P^{(s')})_{\varkappa_n}(\xi_m^{(n)}) & n^{1-s'}\Delta^1(P^{(s')})_{\varkappa_n}(\xi_m^{(n)}) & \dots & n^{2m-1-s'}\Delta^{2m-1}(P^{(s')})_{\varkappa_n}(\xi_m^{(n)}) \\ n^{-2-s'}\dot{\Delta}^0(P^{(s')})_{\varkappa_n}(\xi_1^{(n)}) & n^{-1-s'}\dot{\Delta}^1(P^{(s')})_{\varkappa_n}(\xi_1^{(n)}) & \dots & n^{2m-3-s'}\dot{\Delta}^{2m-1}(P^{(s')})_{\varkappa_n}(\xi_1^{(n)}) \\ \vdots & \vdots & & \vdots \\ n^{-2-s'}\dot{\Delta}^0(P^{(s')})_{\varkappa_n}(\xi_m^{(n)}) & n^{-1-s'}\dot{\Delta}^1(P^{(s')})_{\varkappa_n}(\xi_m^{(n)}) & \dots & n^{2m-3-s'}\dot{\Delta}^{2m-1}(P^{(s')})_{\varkappa_n}(\xi_m^{(n)}) \end{vmatrix},$$

where $s' = s - 2m$. Applying Propositions 2.2 and 2.3, we obtain the desired formula. The last statement follows from the uniform convergences in Propositions 2.2, 2.3. □

Step 3: Asymptotic of $D_n(1-\frac{y}{2n^2})$.

Lemma 4.4. *Let $\beta=6m^2+m-2ms-s$. Then we have*

$$\lim_{n\to\infty} n^\beta D_{\varkappa_n}\left(1-\frac{y}{2n^2}\right)=\frac{2^{(2m+1)(s-2m)}}{(w_1...w_m)^{1+s-2m}}y^{-\frac{s-2m}{2}}\cdot A(\varkappa,y),$$

where

$$A(\varkappa,y)=W(J_{s-2m,w_1},...,J_{s-2m,w_m},\tilde{J}_{s-2m+1,w_1},...,\tilde{J}_{s-2m+1,w_m},J_{s-2m,y})(\varkappa).$$

Moreover, the convergence is uniform as long as $\varkappa$ is in a compact subset of $(0,\infty)$.

Proof. The proof is similar to that of Lemma 4.3. □

Step 4: Asymptotic of $D_n(u_1^{(n)})D_{n-1}(u_2^{(n)})-D_n(u_2^{(n)})D_{n-1}(u_1^{(n)})$.

The following elementary identity will be useful for us:

$$D_n(u_1^{(n)})D_{n-1}(u_2^{(n)})-D_n(u_2^{(n)})\cdot D_{n-1}(u_2^{(n)})$$
$$=\begin{vmatrix} D_{n-1}(u_2^{(n)}) & D_n(u_2^{(n)})-D_{n-1}(u_2^{(n)}) \\ D_{n-1}(u_1^{(n)}) & D_n(u_1^{(n)})-D_{n-1}(u_1^{(n)})\end{vmatrix}.$$

By Step 3, the asymptotics of $D_n(u_2^{(n)})$ and $D_n(u_2^{(n)})$ are already known, thus we need to obtain the asymptotics of $D_n(u_1^{(n)})-D_{n-1}(u_1^{(n)})$ and $D_n(u_2^{(n)})-D_{n-1}(u_2^{(n)})$.

Definition 4.5. Set $\boldsymbol{\theta}_j^{(n)}(u)$ to be the column vector:

$$\left(P_j^{(s-2m)}(\xi_1^{(n)}),...,P_j^{(s-2m)}(\xi_m^{(n)}),\dot{P}_j^{(s-2m)}(\xi_1^{(n)}),...,\dot{P}_j^{(s-2m)}(\xi_m^{(n)}),P_j^{(s-2m)}(u)\right)^T.$$

We have

$$\begin{aligned} D_n(u_i^{(n)}) &= \det\left[\boldsymbol{\theta}_n^{(n)}(u_i^{(n)})...\boldsymbol{\theta}_{n+2m}^{(n)}(u_i^{(n)})\right], \\ D_{n-1}(u_i^{(n)}) &= \det\left[\boldsymbol{\theta}_{n-1}^{(n)}(u_i^{(n)})...\boldsymbol{\theta}_{n+2m-1}^{(n)}(u_i^{(n)})\right] \\ &= \det\left[\boldsymbol{\theta}_n^{(n)}(u_i^{(n)})...\boldsymbol{\theta}_{n+2m-1}^{(n)}(u_i^{(n)}) \quad \boldsymbol{\theta}_{n-1}^{(n)}(u_i^{(n)})\right], \end{aligned}$$

whence

$$D_n(u_i^{(n)})-D_{n-1}(u_i^{(n)})$$
$$=\det\left[\boldsymbol{\theta}_n^{(n)}(x_i^{(n)})...\boldsymbol{\theta}_{n+2m-1}^{(n)}(x_i^{(n)}) \quad \boldsymbol{\theta}_{n+2m}^{(n)}(x_i^{(n)})-\boldsymbol{\theta}_{n-1}^{(n)}(x_i^{(n)})\right]. \tag{51}$$

Lemma 4.6. *Let* $\gamma = 1+6m^2+m-2ms-s$. *Then we have*

$$\lim_{n\to\infty} n^{\gamma}\big(D_{\varkappa_n}(u_i^{(n)}) - D_{\varkappa_n-1}(u_i^{(n)})\big) = \frac{2^{(2m+1)(s-2m)}}{(w_1\dots w_m)^{1+s-2m}} y_i^{-\frac{s-2m}{2}} \cdot B(\varkappa, y_i),$$

where $B(\varkappa, y_i) = |\boldsymbol{\eta}_{y_i}(\varkappa)\ \boldsymbol{\eta}'_{y_i}(\varkappa)\ \dots\ \boldsymbol{\eta}_{y_i}^{(2m-1)}(\varkappa)\ \boldsymbol{\eta}_{y_i}^{(2m+1)}(\varkappa)|$ *and* $\boldsymbol{\eta}_{y_i}(\varkappa)$ *is the column vector*

$$\big(J_s(\varkappa\sqrt{w_1}), \dots, J_{s-2m}(\varkappa\sqrt{w_m}), \tilde{J}_{s-2m+1}(\varkappa\sqrt{w_1}), \dots,$$
$$\tilde{J}_{s-2m+1}(\varkappa\sqrt{w_1}), J_{s-2m}(\varkappa\sqrt{y_i})\big)^T.$$

Proof. The proof is similar to that of Lemma 4.3, we emphasize that in the proof we used the elementary equality

$$\Delta^{2m+1}\big(P^{(s-2m)}\big)_{\varkappa_n-1} = P^{(s-2m)}_{\varkappa_n+2m} + (-1)^{2m+1} P^{(s-2m)}_{\varkappa_n-1}$$
$$+ \text{ linear combination of } P^{(s-2m)}_{\varkappa_n}, P^{(s-2m)}_{\varkappa_n+1}, \dots, P^{(s-2m)}_{\varkappa_n+2m-1}.$$

□

Step 5: Calculation of the limit kernel $K_\infty(y_1, y_2)$.

Theorem 4.7. *The limit kernel* $K_\infty(y_1, y_2)$ *in* (46) *exists and is given by*

$$K_\infty(y_1, y_2) = \frac{A(1, y_1)B(1, y_2) - A(1, y_2)B(1, y_1)}{2|\prod_{i=1}^m (y_1 - w_i)(y_2 - w_i)| \cdot [C(1)]^2 \cdot (y_1 - y_2)}.$$

Proof. Recall that we defined in Lemmas 4.3, 4.4 and 4.6 that

$$\alpha = 6m^2 - 2sm - 3m, \quad \beta = 6m^2 + m - 2ms - s \quad \text{and} \quad \gamma = 1 + 6m^2 + m - 2ms - s.$$

Note that

$$\beta + \gamma - 2\alpha - 1 = 8m - 2s,$$

whence

$$S_n(y_1, y_2) = \frac{2^{4m-s}}{n \cdot \frac{h^{(s-2m)}_{n-1} k^{(s-2m)}_{n+2m}}{k^{(s-2m)}_{n-1}} [n^{\alpha}\delta_n]^2} \cdot \frac{\begin{vmatrix} n^{\beta} D_{n-1}(u_2^{(n)}) & n^{\gamma}[D_n(u_2^{(n)}) - D_{n-1}(u_2^{(n)})] \\ n^{\beta} D_{n-1}(u_1^{(n)}) & n^{\gamma}[D_n(u_1^{(n)}) - D_{n-1}(u_1^{(n)})] \end{vmatrix}}{y_2 - y_1}.$$

By Lemmas 4.2–4.4 and 4.6, we obtain that

$$\lim_{n\to\infty} S_n(y_1, y_2) = \frac{1}{2[C(1)]^2} \frac{\begin{vmatrix} y_2^{-\frac{s-2m}{2}} A(1, y_2) & y_2^{-\frac{s-2m}{2}} B(1, y_2) \\ y_1^{-\frac{s-2m}{2}} A(1, y_1) & y_1^{-\frac{s-2m}{2}} B(1, y_1) \end{vmatrix}}{y_2 - y_1}$$

$$=\frac{(y_1y_2)^{-\frac{s-2m}{2}}}{2[C(1)]^2}\frac{A(1,y_1)B(1,y_2)-A(1,y_2)B(1,y_1)}{y_1-y_2}.$$

Substituting the above formula into (47), we get the desired result. □

4.3. Case II

Recall that in Case II, we have fixed $s\in\mathbb{R}, m\in\mathbb{N}$ such that $s+m>-1$ and have fixed an m-tuple of *distinct* positive numbers $(v_1,...,v_m)$.

Similar to Case I, we shall consider, in this case, the determinantal probability measure

$$\frac{\Psi_{G_n}\mathbb{J}^{(s,n)}}{\int_{\mathrm{Conf}((0,4n^2))}\Psi_{G_n}\,d\mathbb{J}^{(s,n)}}. \tag{52}$$

The measure (52) is given by the orthogonal polynomial ensemble

$$\mathrm{const}_{n,s}\prod_{1\le i<j\le n}(u_i-u_j)^2\prod_{i=1}^{n}\frac{(1-u_i)^{s+m}}{\prod_{k=1}^{m}(1+\frac{v_k}{2n^2}-u_i)}\,du_i. \tag{53}$$

The ensemble (53) is an Uvarov deformation of the following Jacobi polynomial ensemble:

$$\mathrm{const}_{n,s}\prod_{1\le i<j\le n}(u_i-u_j)^2\prod_{i=1}^{n}(1-u_i)^{s+m}du_i.$$

Let $\Omega_n:(-1,1)\to\mathbb{R}_+$ be the weight given by the formula

$$\Omega_n(u):=\frac{(1-u)^{s+m}}{\prod_{k=1}^{m}(1+\frac{v_k}{2n^2}-u)}.$$

Let $(p_j)_{j\ge0}$ denote a system of orthogonal polynomials (not necessarily monic, the choice of the system will be specified later) associated with the weight Ω_n. Note that as in Case I, here we omit in the notation the dependence of p_j on n, s and $v_1,...,v_m$. The correlation kernel for (52) and for (53) is the n-th Christoffel-Darboux kernel of the weight Ω_n, given by the formula:

$$\mathscr{K}_n(u_1,u_2):=\sqrt{\Omega_n(u_1)\Omega_n(u_2)}\cdot\sum_{j=0}^{n-1}\frac{p_j(u_1)\cdot p_j(u_2)}{e_j},$$

where

$$e_j=\int_{-1}^{1}p_j(u)^2\Omega_n(u)\,du. \tag{54}$$

We will give explicit formula for the following limit kernel:

$$\mathscr{K}_\infty(y_1,y_2):=\lim_{n\to\infty}\frac{1}{2n^2}\mathscr{K}_n\left(1-\frac{y_1}{2n^2},1-\frac{y_2}{2n^2}\right). \tag{55}$$

The limit kernel $\mathscr{K}_\infty(y_1,y_2)$ is then automatically the correlation kernel for the determinantal probability measure

$$\frac{\Psi_{\tilde g}\tilde{\mathbb{B}}^{(s)}}{\int_{\mathrm{Conf}((0,\infty))}\Psi_{\tilde g}\,d\tilde{\mathbb{B}}^{(s)}}.$$

4.3.1. Explicit formulae for $\mathscr{K}_n(u_1,u_2)$

Set

$$r_i^{(n)}=1+\frac{v_i}{2n^2},\quad i=1,...m.$$

Recall that we set $u_i^{(n)}:=1-\frac{y_i}{2n^2}, i=1,2$. Recall also that the sequence of Jacobi functions of second kind $Q^{(s+m)}=(Q_n^{(s+2m,0)})_{n=0}^\infty$ is given by the formula (20).

The Christoffel-Uvarov formula (cf., e.g., [2, Lemma 2.5]) implies that we may take p_j for $j\geq m$ to be the following polynomial:

$$p_j(u):=\begin{vmatrix} Q_{j-m}^{(s+m)}(r_1^{(n)}) & \dots & Q_j^{(s+m)}(r_1^{(n)})\\ \vdots & & \vdots\\ Q_{j-m}^{(s+m)}(r_m^{(n)}) & \dots & Q_j^{(s+m)}(r_m^{(n)})\\ P_{j-m}^{(s+m)}(u) & \dots & P_j^{(s+m)}(u)\end{vmatrix}.$$

If $0\leq j<m$, we may take p_j to be the j-th monic orthogonal polynomial with respect to the weight Ω_n (the explicit form of p_j for $0\leq j<m$ is not necessary for us).

Let d_j denote the coefficient of $P_j^{(s+m)}(u)$ in the determinant form of the polynomial p_j, i.e.,

$$d_j:=\begin{vmatrix} Q_{j-m}^{(s+m)}(r_1^{(n)}) & \dots & Q_{j-1}^{(s+m)}(r_1^{(n)})\\ \vdots & & \vdots\\ Q_{j-m}^{(s+m)}(r_m^{(n)}) & \dots & Q_{j-1}^{(s+m)}(r_m^{(n)})\end{vmatrix}.$$

Denote by l_j the leading coefficient of p_j. In particular, if $j\geq m$, then

$$l_j=d_jk_j^{(s+m)},\quad \left(\text{recall that } k_j^{(s+m)} \text{ is the leading coefficient of } P_j^{(s+m)}\right).$$

Lemma 4.8. *For any $j \geq m$, we have*

$$e_j := \int_{-1}^{1} p_j(u)^2 \Omega_n(u)\,du = \frac{d_j d_{j+1} k_j^{(s+m)} h_{j-m}^{(s+m)}}{k_{j-m}^{(s+m)}}.$$

Proof. The proof is similar to the proof of Lemma 4.1. □

By the Christoffel-Darboux formula (cf. [16, Thm 3.2.2]), we have:

$$\mathscr{K}_n(u_1, u_2) = \frac{\sqrt{\Omega_n(u_1)\Omega_n(u_2)}}{e_{n-1}} \cdot \frac{p_n(u_1)p_{n-1}(u_2) - p_n(u_2)p_{n-1}(u_1)}{u_1 - u_2}.$$

The change of variables (45) transforms the kernel $\mathscr{K}_n(u_1, u_2)$ to the kernel:

$$\widetilde{\mathscr{K}}_n(y_1, y_2) = \frac{1}{2n^2} \mathscr{K}_n\left(u_1^{(n)}, u_2^{(n)}\right) = \frac{(y_1 y_2)^{\frac{s+m}{2}}}{\prod_{k=1}^{m} (v_k + y_1)^{\frac{1}{2}} (v_k + y_2)^{\frac{1}{2}}} \Sigma_n(y_1, y_2), \tag{56}$$

where

$$\Sigma_n(y_1, y_2) = \frac{(2n^2)^{-s}}{\frac{h_{n-1-m}^{(s+m)} k_n^{(s+m)}}{k_{n-1-m}^{(s+m)}} \cdot d_n^2} \cdot \frac{p_n(u_1^{(n)}) p_{n-1}(u_2^{(n)}) - p_n(u_2^{(n)}) p_{n-1}(u_1^{(n)})}{y_2 - y_1}. \tag{57}$$

4.3.2. Scaling limits

Now we investigate the scaling limits.

Step 1: Asymptotic of $\frac{h_{n-1-m}^{(s+m)} k_n^{(s+m)}}{k_{n-1-m}^{(s+m)}}$.

By changing $s-2m$ to $s+m$ in the formulae (49) and (50), we immediately get the following

Lemma 4.9. *We have*

$$\lim_{n \to \infty} n \cdot \frac{h_{n-1-m}^{(s+m)} k_n^{(s+m)}}{k_{n-1-m}^{(s+m)}} = 2^{s+2m+1}.$$

Step 2: Asymptotic of d_n.

Recall that in Definition 1.4, we define $K_{\alpha,v}(t) = K_\alpha(t\sqrt{v})$ and $J_{\alpha,v}(t) = J_\alpha(t\sqrt{v})$. Recall also by $(\varkappa_n)_{n=1}^{\infty}$, we denote a sequence in $\mathbb{N}$, such that $\lim_{n\to\infty} \varkappa_n / n = \varkappa > 0$.

Lemma 4.10. *Let $\alpha'=-\frac{m^2+m}{2}-ms$. Then we have*

$$\lim_{n\to\infty} n^{\alpha'} d_{\varkappa_n} = 2^{m(s+m)}(v_1 \ldots v_m)^{-\frac{s+m}{2}}\mathscr{C}(\varkappa),$$

where

$$\mathscr{C}(\varkappa) = W(K_{s+m,v_1},\ldots,K_{s+m,v_m})(\varkappa).$$

Proof. For $\ell \geq 1$, we have

(58)
$$\Delta^\ell\big(Q^{(s+m)}\big)_n = Q^{(s+m)}_{n+\ell} + (-1)^\ell Q^{(s+m)}_n + \text{linear combination of } Q^{(s+m)}_{n+1},\ldots,Q^{(s+m)}_{n+\ell-1}.$$

Hence for $k_n \geq m$, by linearity of determinant with respect to columns, we get

$$d_{\varkappa_n} = \begin{vmatrix} \Delta^0(Q^{(s+m)})_{\varkappa_n-m}(r^{(n)}_1) & \ldots & \Delta^{m-1}(Q^{(s+m)})_{\varkappa_n-m}(r^{(n)}_1) \\ \vdots & & \vdots \\ \Delta^0(Q^{(s+m)})_{\varkappa_n-m}(r^{(n)}_m) & \ldots & \Delta^{m-1}(Q^{(s+m)})_{\varkappa_n-m}(r^{(n)}_m) \end{vmatrix}.$$

Lemma 4.10 is completely proved by applying the same arguments as in the proof of Lemma 4.3 and by applying Propositions 2.4 and 2.5. □

Step 3: Asymptotic of $p_{\varkappa_n}(u^{(n)}_i)$.

Lemma 4.11. *Let $\beta'=-(m+1)(s+\frac{m}{2})$. Then we have*

$$\lim_{n\to\infty} n^{\beta'} p_{\varkappa_n}\big(u^{(n)}_i\big) = 2^{(m+1)(s+m)}(v_1\ldots v_m)^{-\frac{s+m}{2}} y_i^{-\frac{s+m}{2}}\mathscr{A}(\varkappa,y_i),$$

where

$$\mathscr{A}(\varkappa,y_i) = W(K_{s+m,v_1},\ldots,K_{s+m,v_m},J_{s+m,y_i})(\varkappa).$$

Proof. Similar to (58), for $\ell\geq 1$, we have

(59)
$$\Delta^\ell\big(P^{(s+m)}\big)_n = P^{(s+m)}_{n+\ell} + (-1)^\ell P^{(s+m)}_n + \text{linear combination of } P^{(s+m)}_{n+1},\ldots,P^{(s+m)}_{n+\ell-1},$$

where the coefficients in the linear combination of (58) and (59) coincide. Hence for $\varkappa_n \geq m$, we have

$$p_{\varkappa_n}\big(u^{(n)}_i\big) = \begin{vmatrix} \Delta^0(Q^{(s+m)})_{\varkappa_n-m}(r^{(n)}_1) & \ldots & \Delta^{m}(Q^{(s+m)})_{\varkappa_n-m}(r^{(n)}_1) \\ \vdots & & \vdots \\ \Delta^0(Q^{(s+m)})_{\varkappa_n-m}(r^{(n)}_m) & \ldots & \Delta^{m}(Q^{(s+m)})_{\varkappa_n-m}(r^{(n)}_m) \\ \Delta^0(P^{(s+m)})_{\varkappa_n-m}(u^{(n)}_i) & \ldots & \Delta^{m}(P^{(s+m)})_{\varkappa_n-m}(u^{(n)}_i) \end{vmatrix}.$$

Lemma 4.11 is completely proved by applying the same arguments as in the proof of Lemma 4.3 and by applying Propositions 2.2, 2.3, 2.4 and 2.5. $\square$

Step 4: Asymptotic of $p_n(u_1^{(n)})p_{n-1}(u_2^{(n)}) - p_n(u_2^{(n)})p_{n-1}(u_1^{(n)})$.

By similar argument as in studying $D_n(u_1^{(n)})D_{n-1}(u_2^{(n)}) - D_n(u_2^{(n)})D_{n-1}(u_1^{(n)})$, here we shall consider the asymptotic of $p_n(u_i^{(n)}) - p_{n-1}(u_i^{(n)})$. More generally, we may consider $p_{\varkappa_n}(u_i^{(n)}) - p_{\varkappa_n - 1}(u_i^{(n)})$.

Lemma 4.12. *Let* $\gamma' = -(m+1)(s+\frac{m}{2})+1$. *Then we have*

$$\lim_{n\to\infty} n^{\gamma'}\left[p_{\varkappa_n}\big(u_i^{(n)}\big) - p_{\varkappa_n-1}\big(u_i^{(n)}\big)\right] = 2^{(m+1)(s+m)}(v_1 \dots v_m)^{-\frac{s+m}{2}} y_i^{-\frac{s+m}{2}} \mathscr{B}(\varkappa, y_i),$$

where

$$\mathscr{B}(\varkappa, y_i) = \left|\boldsymbol{\phi}_{y_i}(\varkappa), \boldsymbol{\phi}'_{y_i}(\varkappa), \dots, \boldsymbol{\phi}_{y_i}^{(m-1)}(\varkappa), \boldsymbol{\phi}_{y_i}^{(m+1)}(\varkappa)\right|,$$

and $\boldsymbol{\phi}_{y_i}(\varkappa)$ *is the column vector* $(K_{s+m,v_1}(\varkappa), \dots K_{s+m,v_m}(\varkappa), J_{s+m,y_i}(\varkappa))^T$.

Remark 4.13. Note that $\mathscr{B}(\varkappa, y_i) = \frac{\partial}{\partial \varkappa}\mathscr{A}(\varkappa, y_i)$.

Proof of Lemma 4.12. To simplify notation, we prove Lemma 4.12 in the case $\varkappa_n = n$, the proof in the general case is similar. Denote $s'' = s+m$. Set $\beta_j^{(n)}(u)$ to be the column vector

$$\beta_j^{(n)}(u) = \big(Q_j^{(s'')}\big(r_1^{(n)}\big), \dots, Q_j^{(s'')}\big(r_m^{(n)}\big), P_j^{(s'')}(u)\big)^T.$$

Then for $i = 1, 2$,

$$\begin{aligned} p_n\big(u_i^{(n)}\big) &= \left|\beta_{n-m}^{(n)}(u_i^{(n)}) \quad \dots \quad \beta_{n-1}^{(n)}(u_i^{(n)}) \quad \beta_n^{(n)}(u_i^{(n)})\right|; \\ p_{n-1}\big(u_i^{(n)}\big) &= \left|\beta_{n-1-m}^{(n)}(u_i^{(n)}) \quad \dots \quad \beta_{n-2}^{(n)}(u_i^{(n)}) \quad \beta_{n-1}^{(n)}(u_i^{(n)})\right| \\ &= (-1)^m \left|\beta_{n-m}^{(n)}(u_i^{(n)}) \dots \beta_{n-1}^{(n)}(u_i^{(n)})\ \beta_{n-1-m}^{(n)}(u_i^{(n)})\right|. \end{aligned}$$

Hence

$$\begin{aligned} &p_n\big(u_i^{(n)}\big) - p_{n-1}\big(u_i^{(n)}\big) \\ &\quad = \left|\beta_{n-m}^{(n)}(u_i^{(n)}) \dots \beta_{n-1}^{(n)}(u_i^{(n)})\ \beta_n^{(n)}(u_i^{(n)}) + (-1)^{m+1}\beta_{n-1-m}^{(n)}(u_i^{(n)})\right| \end{aligned}$$

$$= \begin{vmatrix} \Delta^0(Q^{(s'')})_{n-m}(r_1^{(n)}) & \dots & \Delta^{m-1}(Q^{(s'')})_{n-m}(r_1^{(n)}) & \Delta^{m+1}(Q^{(s'')})_{n-m-1}(r_1^{(n)}) \\ \vdots & & \vdots & \vdots \\ \Delta^0(Q^{(s'')})_{n-m}(r_m^{(n)}) & \dots & \Delta^{m-1}(Q^{(s'')})_{n-m}(r_m^{(n)}) & \Delta^{m+1}(Q^{(s'')})_{n-m-1}(r_m^{(n)}) \\ \Delta^0(P^{(s'')})_{n-m}(u_i^{(n)}) & \dots & \Delta^{m-1}(P^{(s'')})_{n-m}(u_i^{(n)}) & \Delta^{m+1}(P^{(s'')})_{n-m-1}(u_i^{(n)}) \end{vmatrix}.$$

We can finish the proof of Lemma 4.12 by using Propositions 2.2, 2.3, 2.4 and 2.5. ☐

Step 5: Calculation of the limit kernel $\mathscr{K}_\infty(y_1,y_2)$.

Theorem 4.14. *The limit kernel $\mathscr{K}_\infty(y_1,y_2)$ in (55) exists and is given by*

$$\mathscr{K}_\infty(y_1,y_2) = \frac{\mathscr{A}(1,y_1)\mathscr{B}(1,y_2)-\mathscr{A}(1,y_2)\mathscr{B}(1,y_1)}{2\prod_{i=1}^m \sqrt{(v_i+y_1)(v_i+y_2)}\cdot[\mathscr{C}(1)]^2\cdot(y_1-y_2)}.$$

Proof. For the numbers α',β',γ' defined in Lemmas 4.10, 4.11 and 4.12, we have

$$\beta'+\gamma'-2\alpha'-1=-2s,$$

whence

$$\Sigma_n(y_1,y_2) = \frac{2^{-s}}{n\cdot \frac{h_{n-1-m}^{(s+m)}k_n^{(s+m)}}{k_{n-1-m}^{(s+m)}}\cdot[n^{\alpha'}d_n]^2}\cdot \frac{\begin{vmatrix} n^{\beta'}p_n(u_2^{(n)}) & n^{\gamma'}[p_n(u_2^{(n)})-p_{n-1}(u_2^{(n)})] \\ n^{\beta'}p_n(u_1^{(n)}) & n^{\gamma'}[p_n(u_1^{(n)})-p_{n-1}(u_1^{(n)})] \end{vmatrix}}{y_2-y_1}.$$

By Lemmas 4.9, 4.10, 4.11 and 4.12, we obtain that

$$\lim_{n\to\infty}\Sigma_n(y_1,y_2) = \frac{(y_1y_2)^{-\frac{s+m}{2}}}{2[\mathscr{C}(1)]^2}\cdot\frac{\mathscr{A}(1,y_1)\mathscr{B}(1,y_2)-\mathscr{A}(1,y_2)\mathscr{B}(1,y_1)}{y_1-y_2}.$$

Combining this with (56), we get the desired result. ☐

Proof of Theorem 1.5. By the diagram in Section 3.1, we know that the correlation kernel Π^g is obtained by the change of variables $y_1=4/x, y_2=4/x'$ from the kernel $\mathscr{K}_\infty(y_1,y_2)$, i.e.,

$$\Pi^g(x,x') = \frac{4}{xx'}\mathscr{K}_\infty(4/x,4/x').$$

Direct computation shows that this is the kernel announced in Theorem 1.5. ☐

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Alexander I. Bufetov
Aix-Marseille Université, Centrale
Marseille, CNRS, I2M, UMR7373
39 Rue F. Joliot Curie
FR-13453, Marseille Cedex 13
France
and
Steklov Institute of Mathematics
Moscow
Russia
and
Institute for Information Transmission
Problems
Moscow
Russia
and
National Research University Higher
School of Economics
Moscow
Russia
bufetov@mi.ras.ru

Yanqi Qiu
Aix-Marseille Université, Centrale
Marseille, CNRS, I2M, UMR7373
39 Rue F. Joliot Curie
FR-13453, Marseille Cedex 13
France
yqi.qiu@gmail.com

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