

# Stiefel-Whitney classes of curve covers

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**Abstract.** Let  $D$  be a Dedekind scheme with the characteristic of all residue fields not equal to 2. To every tame cover  $C \rightarrow D$  with only odd ramification we associate a second Stiefel-Whitney class in the second cohomology with mod 2 coefficients of a certain tame orbicurve  $[D]$  associated to  $D$ . This class is then related to the pull-back of the second Stiefel-Whitney class of the push-forward of the line bundle of half of the ramification divisor. This shows (indirectly) that our Stiefel-Whitney class is the pull-back of a sum of cohomology classes considered by Esnault, Kahn and Viehweg in ‘Coverings with odd ramification and Stiefel-Whitney classes’. Perhaps more importantly, in the case of a proper and smooth curve over an algebraically closed field, our Stiefel-Whitney class is shown to be the pull-back of an invariant considered by Serre in ‘Revêtements à ramification impaire et thêta-caractéristiques’, and in this case our arguments give a new proof of the main result of that article.

## 1. Introduction

Let  $C \rightarrow \text{Spec } k$  be a smooth and proper curve over an algebraically closed field with  $\text{char } k \neq 2$ . Mumford proved in [7] that if  $\mathcal{L}$  is a theta characteristic, i.e., a line bundle such that  $\mathcal{L} \otimes \mathcal{L} \simeq \Omega_{C/k}$ , then  $h^0(C, \mathcal{L} \otimes \mathcal{E})$  is constant mod 2 when  $\mathcal{E}$  varies in an algebraic family of orthogonal bundles. In [10], Serre found the following more precise version of this result:

$$(1) \quad h^0(C, \mathcal{L} \otimes \mathcal{E}) \equiv (m+1)h^0(C, \mathcal{L}) + h^0(C, \mathcal{L} \otimes \det \mathcal{E}) + w_2(\mathcal{E}) \pmod{2},$$

where the last term is the second Stiefel-Whitney class (or rather its image under the canonical isomorphism  $H^2(C, \mu_2) \rightarrow \mathbb{F}_2$ ) of the  $m$ -dimensional orthogonal bundle  $\mathcal{E}$ . (Serre only stated this result for Riemann surfaces, but the first of his arguments which uses the classification of orthogonal bundles generalises.)

One way in which orthogonal bundles on curves appear naturally is the following: let  $f: C \rightarrow D$  be an oddly and tamely branched cover of degree  $n$  of a smooth  $k$ -curve, let  $R$  be the ramification divisor and let  $E$  be the divisor such

that  $2E=R$ . Then duality theory for finite morphisms gives  $f_*\mathcal{O}(E)$  an orthogonal bundle structure (this will be explained in Section 2). The Hurwitz formula shows that if  $\mathcal{L}$  is a theta characteristic on  $D$ , then  $f^*\mathcal{L}\otimes\mathcal{O}(E)$  is a theta characteristic on  $C$ . Applying (1) to  $f_*\mathcal{O}(E)$  gives

$$(2) \quad h^0(C, \mathcal{L}\otimes\mathcal{O}(E)) \equiv (n+1)h^0(D, \mathcal{L}) + h^0(D, \mathcal{L}\otimes\det f_*\mathcal{O}(E)) + w_2(f_*\mathcal{O}(E)) \pmod{2}$$

This second formula grows considerably more interesting in light of the following: Serre defines a second invariant  $w_2^S(f) \in \mathbb{F}_2$  of such covers, and shows

$$(3) \quad w_2^S(f) = w_2(f_*\mathcal{O}(E)) + \omega(C/D) \quad \text{in } \mathbb{F}_2$$

where  $\omega(C/D)$  is the evaluation of a residue class character mod 8 on the sum of the branching indices. The point is that  $w_2^S(f)$  can be combinatorially defined, which means that the last term in (2) actually is easy to compute, which often is extremely useful.

Included in [loc. cit.] is an informal wish list, which slightly paraphrased reads as follows:

1. The given definition of  $w_2^S(f)$  is an element of  $\mathbb{F}_2$ , not a cohomology class. Remove this ad hoc-ness.

2. In an earlier paper [9], Serre had found a Stiefel-Whitney class  $w_2(j) \in H^2(K, \mu_2)$  for any separable field extension  $j: K \rightarrow L$ , and a formula which relates this class to the Stiefel-Whitney class of  $w_2(L/K)$ , where  $L/K$  denotes  $L$  considered as an orthogonal  $K$ -vector space by means of the trace pairing. Is there a common generalisation of this formula and Equation (3)?

3. The proof given in [10] for (3) uses (2), and then works by reduction to a special case. Is there a more direct proof of this result?

The second item on this list was picked up in [1] by Esnault, Kahn and Viehweg. They consider an arbitrary tame and oddly ramified cover  $f: C \rightarrow D$  of Dedekind schemes over  $\text{Spec } \mathbb{Z}[1/2]$ , define an invariant  $w_2(f)$  which is  $w_2(j)$  when  $C = \text{Spec } L$  and  $D = \text{Spec } K$ , and which satisfies the formula

$$w_2(f_*\mathcal{O}(E)) + \omega(X/Y) = w_2(f) + (2) \cup w_1(f_*\mathcal{O}(E)),$$

in general, where (2) denotes the image of  $2 \in H^0(D, \mathbb{G}_m)$  under the connecting homomorphism  $H^0(D, \mathbb{G}_m) \rightarrow H^1(D, \mu_2)$  of the Kummer sequence. Since (2)=0 for a curve over a separably closed field  $k$ , it follows from Serre's proof of (3) that  $w_2(f) = w_2^S(f)$  in the case of an oddly branched cover of a proper and smooth  $k$ -curve, and hence [loc. cit.] also gives an answer to the first item on the wish list, albeit an indirect one.

In the present paper we will consider a tamely ramified cover  $f: C \rightarrow D$  of Dedekind  $\mathbb{Z}[\frac{1}{2}]$ -schemes. Let  $[D]$  be the tame orbicurve associated to  $C \rightarrow D$ ; we will define a Stiefel-Whitney class  $w_2(C/D) \in H^2([D], \mu_2)$ . In Section 7 it will be directly shown that in case  $D$  is a complete and smooth  $k$ -curve, then  $w_2(C/D)$  agrees with  $w_2^S(f)$  under the canonical isomorphisms  $H^2(D, \mu_2) \rightarrow H^2([D], \mu_2)$  respectively  $H^2([D], \mu_2) \rightarrow \mathbb{F}_2$ , thereby ticking off the first item on the list. Finally, in Section 5 we will give a straight-forward proof of (3) with  $w_2^S(f)$  replaced by  $w_2(C/D)$  in the more general setting of an oddly ramified cover of Dedekind schemes. Specialised to the curve case, this is the third item on the wish list.

A major reason behind the present work is our interest in whether there exists a generalisation of (3) to the case of wildly ramified covers of proper and smooth  $k$ -curves, where the tame part of the ramification is still assumed odd. Our definitions seem to be appropriate for such an investigation.

### 1.1. Terminology

All schemes and orbifolds are by definition  $\mathbb{Z}[\frac{1}{2}]$ -schemes or orbifolds.

## 2. Stiefel-Whitney classes of orthogonal vector bundles

Let  $p: [D] \rightarrow \text{Spec } \mathbb{Z}[\frac{1}{2}]$  be a tame Dedekind orbifold as defined in Appendix A. All sheaves on  $[D]$  will be taken with respect to the étale topology. An *orthogonal vector bundle of rank  $m$*  is a locally free sheaf  $\mathcal{E}$  of rank  $m$  which is equipped with a non-degenerate and symmetric pairing  $\mathcal{E} \otimes \mathcal{E} \rightarrow \mathcal{O}_X$ . If  $\mathcal{E}, \mathcal{F}$  are orthogonal vector bundles, let  $\text{Ortho}_{\mathcal{O}_{[D]}}(\mathcal{E}, \mathcal{F})$  be the sheaf where the sections are isomorphisms compatible with the pairings. Equip  $\mathbb{Z}[\frac{1}{2}]^m$  with the non-degenerate and symmetric pairing

$$(4) \quad \langle \cdot, \cdot \rangle: \mathbb{Z} \left[ \frac{1}{2} \right]^m \otimes \mathbb{Z} \left[ \frac{1}{2} \right]^m \rightarrow \mathbb{Z} \left[ \frac{1}{2} \right]$$

which in the standard basis is given by the identity matrix. Let  $O_m$  be the algebraic group defined by the functor from  $\mathbb{Z}[\frac{1}{2}]$ -algebras to groups

$$R \mapsto \{ M \in \text{End}_R(R^m) : \forall (x, y) \in R^m \times R^m \langle Mx, My \rangle = \langle x, y \rangle \}$$

Then

$$\mathcal{E} \mapsto \text{Ortho}_{\mathcal{O}_{[D]}} \left( p^* \mathbb{Z} \left[ \frac{1}{2} \right]^m, \mathcal{E} \right) =: O(\mathcal{E})$$

defines a category equivalence from the category of rank  $m$  orthogonal vector bundles to the category of  $O_m$ -torsors, with quasi-inverse given by  $\mathcal{P} \mapsto \mathcal{P} \times^{O_m} p^* \mathbb{Z}[\frac{1}{2}]^m$ .

There are exact sequences of sheaves of groups on  $[D]$

$$(5) \quad 1 \rightarrow \mathrm{SO}_m \rightarrow \mathrm{O}_m \xrightarrow{\det} \mu_2 \rightarrow 1,$$

$$(6) \quad 1 \rightarrow \mu_2 \rightarrow \mathrm{Pin}_m \rightarrow \mathrm{O}_m \rightarrow 1.$$

Using the interpretation of  $\mathrm{H}^1([D], \mathrm{O}_m)$  as equivalence classes of  $\mathrm{O}_m$ -torsors, define the first Stiefel-Whitney class  $w_1(\mathcal{E}) \in \mathrm{H}^1([D], \mu_2)$  as the class represented by the  $\mu_2$ -torsor  $\mathrm{O}(\mathcal{E}) \times^{\mathrm{O}_m} \mu_2$ . Similarly, define the second Stiefel-Whitney class  $w_2(\mathcal{E}) \in \mathrm{H}^2([D], \mu_2)$  to be the image of the class of  $\mathrm{O}(\mathcal{E})$  under the connecting map associated to the second sequence. For our purposes the formal properties of the connecting map will suffice. Nevertheless, it might be useful to bear in mind that if we interpret  $\mathrm{H}^2([D], \mu_2)$  as equivalence classes of  $\mu_2$ -banded gerbes, then  $w_2(\mathcal{E})$  is the class represented by the  $\mu_2$ -banded gerbe of local  $\mathrm{Pin}_m$ -structures of  $\mathcal{E}$ .

For a more thorough discussion of Stiefel-Whitney classes in algebraic geometry, cf., e.g., [1, §1].

### 3. Basic setup

Let  $D$  be a Dedekind scheme, and let  $f: C \rightarrow D$  be a connected tamely ramified cover of degree  $n$  with only odd ramification. For every  $c \in C$ , there is an inertia degree  $i := [k(c) : k(f(c))]$  (i.e., the degree of the residue field extension), and a ramification index  $e$  defined to be the positive integer such that  $\mathfrak{m}_{D, f(c)} \mathcal{O}_{C, c} = \mathfrak{m}_{C, c}^e$ . Tamely ramified with only odd ramification then means that  $e$  is coprime to  $2 \operatorname{char}(k(c))$ .

Let  $E$  be the divisor  $\sum_{c \in C} \frac{e_c - 1}{2} c$ , and let  $R := 2E$ . For every open subset  $U \subset D$ ,  $a \mapsto \operatorname{tr}_{K(C)/K(D)}(a)$  defines an  $\mathcal{O}_D(U)$ -linear map  $\operatorname{tr}_f(U): f_* \mathcal{O}(R)(U) \rightarrow \mathcal{O}_D(U)$ , and the pair  $(\mathcal{O}(R), \operatorname{tr}_f)$  is a dualising sheaf for  $f$ , cf., e.g., [4, §6.4]. This means in particular that the pairing

$$f_* \mathcal{H}om_{\mathcal{O}_C}(\mathcal{O}(E), \mathcal{O}(R)) \times f_* \mathcal{O}(E) \rightarrow f_* \mathcal{O}_R \xrightarrow{\operatorname{tr}_f} \mathcal{O}_D$$

is non-degenerate, and since  $f_* \mathcal{O}(E) = f_* \mathcal{H}om_{\mathcal{O}_C}(\mathcal{O}(E), \mathcal{O}(R))$  it follows that we have defined an orthogonal vector bundle.

Recall that there exists an essentially unique pair  $(\tilde{C} \rightarrow D, \psi)$  where  $\tilde{C} \rightarrow D$  is a tame cover,  $\psi: \tilde{C} \times_D S_n \rightarrow \tilde{C}$  is an action of the symmetric group, and

$$\tilde{C} \times^{S_n} \{1, 2, \dots, n\} \rightarrow \tilde{C}/S_n$$

is isomorphic to  $C \rightarrow D$ . Using  $[\ ]$  to denote the stack quotient, there is a commutative diagram

$$(7) \quad \begin{array}{ccccc} U := f^{-1}V & \longrightarrow & [C] := [\tilde{C} \times^{S_n} \{1, 2, \dots, n\}] & \longrightarrow & C \\ \downarrow & & [f] \downarrow & & \downarrow f \\ V & \longrightarrow & [D] := [\tilde{C}/S_n] & \xrightarrow{\iota} & D \end{array}$$

of tame Dedekind orbifolds, where  $V \hookrightarrow D$  is the complement of the branch points of  $f$ , and  $[f]$  is étale. Hence the trace defines a non-degenerate and symmetric pairing

$$[f]_* \mathcal{O}_{[C]} \otimes [f]_* \mathcal{O}_{[C]} \rightarrow \mathcal{O}_{[D]},$$

i.e.,  $[f]_* \mathcal{O}_{[C]}$  is an orthogonal rank  $n$  vector bundle on  $[D]$ .

Now we are prepared to introduce the Stiefel-Whitney classes which the present paper is all about. Namely, for  $i=1, 2$ , let

$$w_i(E) := w_i(f_* \mathcal{O}(E)) \in H^i(D, \mu_2) \quad \text{and} \quad w_i(C/D) := w_i([f]_* \mathcal{O}_{[C]}) \in H^i([D], \mu_2).$$

We intend to compare  $\iota^* w_i(E)$  with  $w_i(C/D)$ . The next section will provide the computational tool for doing so.

#### 4. The second Stiefel-Whitney class

Once again  $[D]$  is a tame Dedekind orbifold. Let  $\bar{d}$  be a closed geometric point of  $[D]$ , and let  $\mathcal{U} \hookrightarrow [D]$  be the open substack complement of the residual gerbe  $\sigma$  of the point of  $[D]$  determined by  $\bar{d}$ . If  $\mathcal{E}$  is a bundle on  $[D]$ , we will write  $\mathcal{E}^{\mathcal{U}}$  for its restriction to  $\mathcal{U}$ .

**Proposition 1.** *Let  $\mathcal{E}, \mathcal{F}$  be orthogonal vector bundles of rank  $m \geq 2$  on  $[D]$ , and let  $\gamma: \mathcal{F}^{\mathcal{U}} \rightarrow \mathcal{E}^{\mathcal{U}}$  be an isomorphism. Choose an étale neighbourhood  $v: V \rightarrow [D]$  of  $\bar{d}$ , and isomorphisms  $\alpha: v^* \mathcal{E} \rightarrow \mathcal{O}_V^m$  respectively  $\beta: v^* \mathcal{F} \rightarrow \mathcal{O}_V^m$ . Let  $K_{\{\bar{d}\}}$  be the fraction field of  $\mathcal{O}_{[D], \bar{d}}$ , and  $A \in O_m(K_{\{\bar{d}\}})$  the matrix obtained from  $\alpha \gamma \beta^{-1}$ . Then*

$$w_2(\mathcal{E}) = \begin{cases} w_2(\mathcal{F}) & \text{if } A \text{ lifts to an element of } \text{Pin}_m(K_{\{\bar{d}\}}) \\ w_2(\mathcal{F}) + \delta(\mathcal{O}(\sigma)) & \text{else} \end{cases}$$

where  $\delta: H^1([D], \mathbb{G}_m) \rightarrow H^2([D], \mu_2)$  is the connecting homomorphism of the Kummer sequence.

*Proof.* Let  $V = \text{Spec } R$  be sufficiently small and note that  $\mathcal{F}$  is determined (up to isomorphism) by  $\mathcal{E}$  and  $\alpha\gamma\beta^{-1}$ . Every element of  $O_m(R)$  is a product of finitely many reflections. Moreover, all reflections are conjugate, and we may take (since we are working étale locally)  $O_m$  to be the group which preserves the bilinear form which in the standard basis on  $R^m$  is given by block diagonal matrix

$$\begin{pmatrix} H & & \\ & \ddots & \\ & & H \end{pmatrix} \quad \text{respectively} \quad \begin{pmatrix} 1 & & \\ & H & \\ & & \ddots \\ & & & H \end{pmatrix}$$

when  $m$  is even respectively odd, and where  $H := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ .

By induction on the number of reflections, and exploiting Proposition 11, it is enough to prove that if  $\alpha\gamma\beta^{-1} = \rho_e B$  where  $\rho_e$  is the reflection

$$\rho_e := \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & f^e \\ & & & f^{-e} & 0 \end{pmatrix} \in O_m(R),$$

then

- The induced matrix lifts to  $\text{Pin}_m(K_{\{\bar{d}\}})$  if and only if  $e$  is even.
- If  $\mathcal{F}'$  is the orthogonal vector bundle on  $[D]$  determined by  $\mathcal{E}$  and  $B$ , then

$$w_2(\mathcal{F}) = \begin{cases} w_2(\mathcal{F}') & \text{if } e \text{ is even} \\ w_2(\mathcal{F}') + \delta(\mathcal{O}(\sigma)) & \text{if } e \text{ is odd} \end{cases}$$

The first of these statements is proved as in the proof of Lemma 4 below. To prove the second, consider the diagram

$$\begin{array}{ccccc} & & \text{H } 1([D], O_m) & \longrightarrow & \text{H } 2([D], \mu_2) \\ & & \uparrow & & \parallel \\ & \mathcal{P} \mapsto \mathcal{P} \times^{O(\mathcal{F}')} \times^{O_m} O_m \times^{O(\mathcal{F}')} & & & \\ \text{H } 1([D], O(\mathcal{F}') \times^{O_m} \text{Pin}_m) & \longrightarrow & \text{H } 1([D], O(\mathcal{F}') \times^{O_m} O_m) & \longrightarrow & \text{H } 2([D], \mu_2) \\ \uparrow & & \uparrow & & \uparrow \\ \text{H } 1([D], \mathbb{G}_m) & \longrightarrow & \text{H } 1([D], \mathbb{G}_m) & \xrightarrow{\delta} & \text{H } 2([D], \mu_2) \end{array}$$

where the rows are parts of the cohomology sequences, and the vertical maps in the lower part of the diagram come from functoriality of cohomology with respect to the group homomorphism

$$\mathbb{G}_m \rightarrow \mathbb{O}_m \quad t \mapsto \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & t & & \\ & & & & t^{-1} & \end{pmatrix}$$

Since the orthogonal vector bundle determined by the matrix  $\begin{pmatrix} 0 & f^e \\ f^{-e} & 0 \end{pmatrix}$  is the same as the one determined by the matrix  $\begin{pmatrix} f^e & 0 \\ 0 & f^{-e} \end{pmatrix}$ , it follows that  $\mathbb{O}_m(\mathcal{F})$  is isomorphic to image of  $\mathcal{I}som_{\mathcal{O}_{[D]}}(\mathcal{O}_{[D]}, \mathcal{O}(\sigma))$  under the composition of the maps in the middle vertical column of the Diagram. Hence it follows from [3, Chapter 4, Proposition 4.3.4] that  $w_2(\mathcal{F}) = w_2(\mathcal{F}') + \delta(\mathcal{O}(\sigma))$ .  $\square$

Apply the exact sequence (6) to  $\text{Spec } K_{\{\bar{d}\}}$ . Since  $H^1(\text{Spec } K_{\{\bar{d}\}}, \mu_2) \simeq \mathbb{Z}/2$ , we find that a matrix  $A \in \mathbb{O}_m(K_{\{\bar{d}\}})$  lifts to an element of  $\text{Pin}_m(K_{\{\bar{d}\}})$  if and only if the image of  $A$  under the group homomorphism  $\varkappa_m: \mathbb{O}_m(K_{\{\bar{d}\}}) \rightarrow H^1(\text{Spec } K_{\{\bar{d}\}}, \mu_2)$  is trivial.

**Corollary 2.** *Suppose that  $A = A_1 \times A_2 \times \dots \times A_s$  with  $A_i \in \mathbb{O}_{m_i}(K_{\{\bar{d}\}})$ . Then*

$$w_2(\mathcal{E}) = \begin{cases} w_2(\mathcal{F}) & \text{if the number of } A_i \text{ which do not lift to} \\ & \text{an element of } \text{Pin}_m(K_{\{\bar{d}\}}) \text{ is even} \\ w_2(\mathcal{F}) + \delta(\mathcal{O}(\sigma)) & \text{else} \end{cases}$$

*Proof.* There is a commutative diagram of sheaves of groups with exact rows

$$\begin{array}{ccccccc} 1 & \longrightarrow & \prod \mu_2 & \longrightarrow & \prod \text{Pin}_{m_i} & \longrightarrow & \prod \mathbb{O}_{m_i} \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mu_2 & \longrightarrow & \text{Pin}_m & \longrightarrow & \mathbb{O}_m \longrightarrow 1 \end{array}$$

This gives a commutative diagram of groups with exact rows

$$\begin{array}{ccccc}
 \prod \mathrm{H}^0(\mathrm{Spec} K_{\{\bar{c}\}}, \mathrm{Pin}_{m_i}) & \longrightarrow & \prod \mathrm{H}^0(\mathrm{Spec} K_{\{\bar{d}\}}, \mathrm{O}_{m_i}) & \xrightarrow{(\varkappa_1, \dots, \varkappa_s)} & \mathrm{H}^1(\mathrm{Spec} K_{\{\bar{d}\}}, \mu_2) \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathrm{H}^0(\mathrm{Spec} K_{\{\bar{d}\}}, \mathrm{Pin}_m) & \longrightarrow & \mathrm{H}^0(\mathrm{Spec} K_{\{\bar{d}\}}, \mathrm{O}_m) & \xrightarrow{\varkappa_m} & \mathrm{H}^1(\mathrm{Spec} K_{\{\bar{d}\}}, \mu_2)
 \end{array}$$

Hence  $\varkappa_m(A) = \sum \varkappa_{m_i}(A_i)$ , which gives the result.  $\square$

### 5. A local computation

We reuse the notation of Diagram (7). Also, let  $\tilde{\omega}: (\mathbb{Z}/8)^\times \rightarrow \{\pm 1\}$  be the (primitive) residue class character which is given by  $\tilde{\omega}(a) = 1$  if and only if  $a \equiv \pm 1 \pmod{8}$  (incidentally, this is the character which cuts out  $\mathbb{Q}(\sqrt{2})$  from the cyclotomic field of 8-th roots of unity), and let  $\omega: \mathbb{Z} \rightarrow \{\pm 1\}$  be the corresponding extended residue class character. Define the  $D$ -divisor

$$\omega(C/D) = \sum_{d \in D} n_d d$$

where

$$n_d = \begin{cases} 1 & \text{if } \omega(\prod_{\{c \in C: f(c)=d\}} e_c^{i_c}) = \prod \omega(e_c)^{i_c} = -1 \\ 0 & \text{else.} \end{cases}$$

Finally, let  $\delta\omega(C/D)$  be the image of  $\omega(C/D)$  under

$$\mathrm{Div} D \rightarrow \mathrm{Pic} D \rightarrow \mathrm{H}^2(D, \mu_2)$$

where the second homomorphism comes from the Kummer sequence. Our main result is:

**Theorem 3.**  $\iota^*(w_2(E) + \delta\omega(C/D)) = w_2(C/D)$

*Proof.* There are natural sheaf morphisms

$$\begin{array}{ccc}
 \iota^* f_* \mathcal{O}_C & \longrightarrow & \iota^* f_* \mathcal{O}(E) \\
 \downarrow & & \\
 [f]_* \mathcal{O}_{[C]} & & 
 \end{array}
 \tag{8}$$



which restrict to isomorphisms of orthogonal bundles on  $V$ . Note that we can construct a finite sequence of orthogonal bundles on  $[D]$

$$[f]_*\mathcal{O}_{[C]} =: \mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_a := \iota^* f_*\mathcal{O}(E)$$

such that  $\mathcal{E}_i$  differs from  $\mathcal{E}_{i+1}$  only at a single residual gerbe. Hence it is enough to consider the case when  $[f]_*\mathcal{O}_{[C]}$  differs from  $\iota^* f_*\mathcal{O}(E)$  only at a single residual gerbe. We are going to use Corollary 2, i.e., show that the matrix which results from comparing  $[f]_*\mathcal{O}_{[C]}$  with  $\iota^* f_*\mathcal{O}(E)$  by the procedure of Proposition 1, lifts to an element of  $\text{Pin}_n(K_{\{\bar{d}\}})$  if and only if  $n_d=0$ .

Let  $c \in C$  map to  $d$ , and let  $\bar{c}$  be a geometric point of  $C$  which maps to  $c$ . Let  $e$  (respectively  $i$ ) be the ramification index (respectively the inertia degree) at  $c$ , let  $t$  be the least common multiple of the ramification indices at the preimages of  $f(c)$ , and put  $u=t/e$ . After strictly localising in the lower row and replacing  $[\tilde{C}/G]$  with a suitable étale cover, the lower right corner of Diagram (7) becomes

$$\begin{array}{ccc} & & \prod_{j=1}^i R[Y]/(Y^e - \pi) \\ & & \uparrow \\ R[Z]/(Z^t - \pi) & \longleftarrow & R \end{array}$$

where  $R := \mathcal{O}_{D, \bar{d}}$  and  $\pi$  is a uniformiser of  $R$ . The relevant part of Diagram (8) becomes

$$(9) \quad \begin{array}{ccc} R[Z]/(Z^t - \pi) \otimes_R R[Y]/(Y^e - \pi) & \longrightarrow & R[Z]/(Z^t - \pi) \otimes_R W \\ \downarrow & & \\ \prod_{j=1}^e R[Z]/(Z^t - \pi) & & \end{array}$$

where  $W$  is the free  $R$ -submodule of  $R[Y, Y^{-1}]/(Y^e - \pi)$  generated by

$$(1, Y, Y^{-1}, \dots, Y^{(e-1)/2}, Y^{-(e-1)/2}).$$

The trace pairing for  $W$  in terms of the indicated basis is the standard split pairing. Hence for the rest of this section we find it convenient to use the same convention as in the proof of Proposition 1, i.e., let  $O_m$  (respectively  $\text{Pin}_m$ ) denote the orthogonal group (respectively Pin group) with respect to the standard split pairing, which once again is permissible since we are working étale locally.

Next, choose

$$(1, Y, Y^{e-1}, \dots, Y^{(e-1)/2}, Y^{(e+1)/2}).$$

as  $R[Z]/(Z^t - \pi)$ -basis for the upper left corner of Diagram (9). The restriction of the horizontal map of this diagram is given by the diagonal matrix

$$N = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & Z^t & & & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & Z^t \end{pmatrix},$$

and the restriction of the vertical map is given by

$$K := \begin{pmatrix} 1 & \rho Z & \rho^{e-1} Z^{e-1} & \dots & \rho^{(e+1)/2} Z^{(e+1)/2} \\ 1 & \rho^2 Z & \rho^{2(e-1)} Z^{e-1} & \dots & \rho^{(e+1)/2} Z^{(e+1)/2} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \rho^e Z & \rho^{e(e-1)} Z^{e-1} & \dots & \rho^{e(e+1)/2} Z^{(e+1)/2} \end{pmatrix}$$

where  $\rho$  is a primitive  $e$ -th root of unity. Finally note that the transposed matrix  $K(1)^T$  gives an orthogonal isomorphism from  $(R[Z]/(Z^t - \pi))^e$  equipped with the trace pairing, to  $(R[Z]/(Z^t - \pi))^e$  equipped with the standard split pairing in terms of its standard basis.

Composing, we get the block diagonal matrix

$$P_e := K(1)^T K N^{-1} = \begin{pmatrix} 1 & & & & & \\ & 0 & Z^{-u} & & & \\ & Z^u & 0 & & & \\ & & & \ddots & & \\ & & & & 0 & Z^{-\frac{e-1}{2}u} \\ & & & & Z^{\frac{e-1}{2}u} & 0 \end{pmatrix}$$

with coefficients in the fraction field of  $R[Z]/(Z^t - \pi)$ . Hence the matrix  $A$  as in Proposition 1 which results from the comparison of  $[f]_* \mathcal{O}_{[C]}$  with  $\iota^* f_* \mathcal{O}(E)$  is

$$\prod_{\{c \in C: f(c)=d\}} P_{e_c}^{i_c}.$$

The result then follows from Corollary 2 and Corollary 5 below.  $\square$

It only remains to state and prove the Corollary referred to in the proof of the Theorem. First a lemma:

**Lemma 4.** *Let  $L$  be the fraction field of a strictly henselian ring with  $\rho$  as uniformiser. Equip  $L^2$  with the standard split pairing. Then the orthogonal transformation given by the matrix  $\begin{pmatrix} 0 & \rho^{-a} \\ \rho^a & 0 \end{pmatrix}$  ( $a \in \mathbb{Z}$ ) lifts to an element of  $\text{Pin}_2(L)$  if and only if  $a$  is even.*

*Proof.* Note that the matrix describes the reflection in the hyperplane orthogonal to  $-e_1 + \rho^a e_2$ , and that the quadratic form takes  $-e_1 + \rho^a e_2$  to  $-2\rho^a$ . Since a vector of  $L^2$  on which the quadratic form is  $-1$ , when considered as an element of the Clifford algebra, belongs to  $\text{Pin}_2(L^2)$  and maps to the reflection in the hyperplane orthogonal to the vector (cf., e.g., [2, Proposition 20.28]), the result follows.  $\square$

Now let  $L$  be the fraction field of  $R[Z]/(Z^t - \pi)$  and note that  $u$  is an odd number. Then:

**Corollary 5.**  *$P_e$  is in the image of  $\text{Pin}_e(L) \rightarrow \text{O}_s(L)$  if and only if  $e \equiv 1, -1 \pmod{8}$ .*

## 6. Mod 2 cohomology of orbicurves

In order to connect with Serre's invariant, we will from now on consider the geometric case when  $[D]$  is a smooth  $k$ -orbicurve, where  $k$  is an algebraically closed field with  $p := \text{char } k \neq 2$ . We begin by deducing the mod 2 cohomology of  $[D]$  from the computation of the cohomology with  $\mathbb{G}_m$ -coefficients in [8].

**Proposition 6.** *Let  $[D]$  be a smooth  $k$ -orbicurve where all inertia groups are prime to  $2p$ , and let  $\gamma: [D] \rightarrow D$  be the morphism from  $[D]$  to its moduli curve. Then  $\gamma^* := \text{H}^i(\gamma): \text{H}^i(D, \mu_2) \rightarrow \text{H}^i([D], \mu_2)$  is an isomorphism of groups if  $i=0, 1, 2$ . In particular, if  $[D]$  is proper, then  $\text{H}^2([D], \mu_2) = \{\pm 1\}$ .*

*Proof.* Case  $i=0$ . The result follows from [8, Corollary 4.15], since  $\text{H}^0([D], \mu_2) = {}_2\Gamma([D], \mathbb{G}_m)$  and  $\text{H}^0(D, \mu_2) = {}_2\Gamma(D, \mathbb{G}_m)$ , where  ${}_2A$  for an abelian group  $A$  means its kernel under multiplication by 2.

Case  $i=1$ . The Kummer sequences for  $D$  and  $[D]$  give a commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \Gamma(D, \mathbb{G}_m)/2\Gamma(D, \mathbb{G}_m) & \longrightarrow & \text{H}^1(D, \mu_2) & \longrightarrow & {}_2\text{Pic } D \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \Gamma([D], \mathbb{G}_m)/2\Gamma([D], \mathbb{G}_m) & \longrightarrow & \text{H}^1([D], \mu_2) & \longrightarrow & {}_2\text{Pic}[D] \longrightarrow 0
 \end{array}$$

with exact rows. Now [8, Corollary 4.15] means that the left and right vertical homomorphisms are bijective (the latter since taking 2-torsion is a left exact functor). Hence the result.

Case  $i=2$ . The Kummer sequences on  $D$  and  $[D]$ , combined with [8, Corollary 4.15] yield a commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & \text{Pic } D & \longrightarrow & \text{Pic } D & \longrightarrow & \text{H}^2(D, \mu_2) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \text{Pic } [D] & \longrightarrow & \text{Pic } [D] & \longrightarrow & \text{H}^2([D], \mu_2) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \bigoplus_l \mathbb{Z}/d_l \mathbb{Z} & \longrightarrow & \bigoplus_l \mathbb{Z}/d_l \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 
 \end{array}$$

where the rows and the two first columns are exact. (Here  $d_l$  is the order of an inertia group, and the sum runs over the points of  $[D]$  with non-trivial inertia.) The snake lemma then shows that  $\text{H}^2(D, \mu_2) \rightarrow \text{H}^2([D], \mu_2)$  is an isomorphism.  $\square$

This, in the notation of Section 3, immediately gives the following result:

**Corollary 7.**  $w_1(C/D) = \iota^* w_1(E)$

*Proof.* The Proposition says that there is an unramified double cover  $\tilde{D} \rightarrow D$  such that

$$[D] \times_D \tilde{D} \simeq \mathcal{O}([f]_* \mathcal{O}_{[C]}) \times^{\text{O}_n} \mu_2.$$

It is then enough to see that  $\tilde{D} \rightarrow D$  represents  $w_1(E)$ . But this follows since the morphism of sheaves  $\mathcal{O}_C \rightarrow \mathcal{O}(E)$  gives rise to an isomorphism of  $\mu_{2,V}$ -torsors

$$\mathcal{O}([f]_* \mathcal{O}_{[C]})^V \times^{\text{O}_n} \mu_2 \rightarrow \mathcal{O}(f_* \mathcal{O}(E))^V \times^{\text{O}_n} \mu_2,$$

where  $V$  is as in Diagram (7), and  $V$  as an upper index means restriction.  $\square$

## 7. Combinatorial nature of $w_2(C/D)$

Identify  $[f]: [C] \rightarrow [D]$  with the sheaf it represents on  $[D]$ . Then

$$\mathcal{I}som_{[D]}([C], \{1, 2, \dots, n\}_{[D]})$$

is a right  $S_n$ -torsor if we equip it with the action  $(\alpha, \beta) \mapsto \beta^{-1}\alpha$ . Note that

$$\mathcal{I}som_{[D]}([C], \{1, 2, \dots, n\}_{[D]}) \rightarrow \mathcal{O}rtho_{\mathcal{O}_{[D]}}(\mathcal{O}_{[D]}^n, [f]_*\mathcal{O}_{[C]}), \quad \alpha \mapsto \alpha^\#$$

gives a morphism of sheaves which is compatible with the homomorphism  $S_n \rightarrow O_n$  which comes from the standard embedding of  $S_n$  in  $O_n$ . Let  $\tilde{S}_n$  be the inverse image of  $S_n$  under the homomorphism  $\text{Pin}_n \rightarrow O_n$ . It follows that  $w_2(C/D)$  is the image of the  $S_n$ -torsor  $\mathcal{I}som_{[D]}([C], \{1, 2, \dots, n\}_{[D]})$  under the second coboundary associated to the exact sequence of sheaves of groups

$$1 \rightarrow \mu_2 \rightarrow \tilde{S}_n \rightarrow S_n \rightarrow 1.$$

Now, in the notation of Section 3,  $\mathcal{I}som_{[D]}([C], \{1, 2, \dots, n\}_{[D]})$  is represented by  $[D] \times_D \tilde{C}$ . Hence:

**Proposition 8.**  *$w_2(C/D)$  is trivial if and only if there exists an  $\tilde{S}_n$ -cover  $\tilde{C} \rightarrow D$  such that  $\tilde{C} \rightarrow \tilde{C} \times^{\tilde{S}_n} S_n$  is unramified, and  $\tilde{C} \times^{\tilde{S}_n} S_n \rightarrow D$  is  $D$ -equivalent as  $S_n$ -cover to  $\tilde{C} \rightarrow D$ .*

It follows that our  $w_2(C/D)$  agrees with Serre's  $w_2(G, \pi)$  defined in [10, §3] (and which we denoted  $w_2^S(f)$  in the Introduction). As Serre observed, it means that we may calculate  $w_2(C/D)$  in the following combinatorial fashion: order a fibre of  $f: C \rightarrow D$  such that the monodromy action on this fibre is given by  $\varphi: \pi_1(V) \rightarrow G \subset S_n$ . Lift  $\varphi$  to a group homomorphism  $\tilde{\varphi}: \pi_1(V) \rightarrow \tilde{G}$ , where  $\tilde{G}$  is the inverse image of  $G$  under  $\tilde{S}_n \rightarrow S_n$ . (This is always possible if  $V$  is non-proper, since then  $\pi_1(V)$  is free.) If  $d$  belongs to the complement of  $V$ , let  $I_d \subset \pi_1(V)$  be the inertia group at  $d$ , as usual well-defined up to conjugation. Let  $\varepsilon_d(\tilde{\varphi}) = 0$  if  $\tilde{\varphi}(I_d)$  is of odd order, and  $\varepsilon_d(\tilde{\varphi}) = 1$  if not. Then (if  $D$  is complete and under the identification  $H^2(D, \mu_2) = \mathbb{Z}/2$ )

$$w_2(C/D) = \sum_{d \in D \setminus V} \varepsilon_d(\tilde{\varphi}),$$

which is what we referred to as the combinatorial nature of  $w_2(C/D)$  in the Introduction.

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## Appendix A. Cohomology of tame Dedekind orbifolds

The present appendix intends to generalise a few well-known facts from the étale cohomology theory of Dedekind schemes (cf., e.g., [5]), to tame Dedekind orbifolds. Throughout we take a completely utilitarian approach: the results are only proved in the generality needed, some arguments which translate word-for-word from the scheme case are omitted. No new ideas are necessary; the only purpose is to make it easier for the reader to check that everything works as expected.

Define a *tame Dedekind orbifold*  $[D]$  to be a Deligne-Mumford stack which is isomorphic to a stack quotient  $[C/G]$ , where  $G$  is a finite group, and  $C$  is a Dedekind scheme with an action  $G \times C \rightarrow C$  such that the inertia group of the generic point is trivial, and the *wild* ramification groups of each closed point is trivial as well. Note that  $[D] \rightarrow C/G =: D$  is the coarse moduli space.

Let  $[D]$  be a tame Dedekind orbifold, let  $i: Z \hookrightarrow [D]$  be a closed substack, and  $j: U \hookrightarrow [D]$  its open complement. Then there is a category equivalence

$$\mathrm{Ab}([D]) \rightarrow \mathbf{T}([D]), \quad \mathcal{F} \mapsto (i^* \mathcal{F}, j^* \mathcal{F}, i^* \eta_{(j^*, j_*)}(\mathcal{F})),$$

where  $\mathrm{Ab}([D])$  is the category of abelian sheaves on the (small) étale site of  $[D]$ ,  $\mathbf{T}([D])$  is the mapping cylinder category of the triple  $(\mathrm{Ab}(Z), \mathrm{Ab}(U), i^* j_*)$  (i.e., the category where the objects are diagrams  $\varphi: \mathcal{F}_1 \rightarrow i^* j_* \mathcal{F}_2$  where  $\mathcal{F}_1$  is an object of  $\mathrm{Ab}(Z)$ ,  $\mathcal{F}_2$  is an object of  $\mathrm{Ab}(U)$ ), and  $\eta_{(j^*, j_*)}: \mathrm{id}_{\mathrm{Ab}([D])} \rightarrow j^* j_*$  is the counit.

From this we may deduce the existence of a right adjoint  $i^!$  of  $i_*$ , a left adjoint  $j_!$  of  $j^*$ , their usual descriptions in terms of objects of  $\mathbf{T}([D])$ , and their usual properties (cf., e.g., [6, Chapter 2, §3]). In particular,  $i^!$  is left exact and preserves injectives.

For a point  $\sigma \in [D]$ , let  $i: \tilde{\sigma} \rightarrow [D]$  be its corresponding residual gerbe, and let  $j: U \rightarrow [D]$  be the open complement of the residual gerbe. Define the relative cohomology groups as  $H_\sigma^m([D], \mathcal{F}) := \mathrm{Ext}_{[D]}^m(i_* \mathbb{Z}, \mathcal{F})$ . There is a short exact sequence

$$0 \rightarrow j_! j^* \mathcal{F} \rightarrow \mathcal{F} \rightarrow i_* i^* \mathcal{F} \rightarrow 0$$

of abelian sheaves on  $[D]$ , which gives a long exact sequence of abelian groups

$$\dots \rightarrow H_\sigma^i([D], \mathcal{F}) \rightarrow H^i([D], \mathcal{F}) \rightarrow H^i(U, \mathcal{F}) \rightarrow H_\sigma^{i+1}([D], \mathcal{F}) \rightarrow \dots$$

Write  $d$  for the image of  $\sigma$  in  $D$ , let  $[D]_{\{\sigma\}} := \tilde{\mathcal{O}}_{D,d} \otimes_D [D]$  where  $\tilde{\mathcal{O}}_{D,d}$  is the henselisation of  $\mathcal{O}_{D,d}$  with respect to  $d$ , and let  $p$  be the projection to the second factor. The usual excision result for Dedekind schemes (cf., e.g., [5, p. 535]) generalises to the following excision result for tame Dedekind orbifolds:

**Proposition 9.** *For any abelian sheaf  $\mathcal{F}$  on  $[D]$ , the natural homomorphism*

$$H_{\sigma}^m([D]_{\{\sigma\}}, p^* \mathcal{F}) \rightarrow H_{\sigma}^m([D], \mathcal{F})$$

*is an isomorphism.*

Excision will be useful, because we can compute the first three cohomology groups of the multiplicative group on  $[D]_{\{\sigma\}}$ :

**Proposition 10.**

$$H_{\sigma}^m([D]_{\{\sigma\}}, \mathbb{G}_m) = H^{m-1}(\sigma, \mathbb{Z}) = \begin{cases} 0 & \text{if } m=0 \\ \mathbb{Z} & \text{if } m=1 \\ 0 & \text{if } m=2 \end{cases}$$

*Proof.* Let  $K_{\{d\}}$  be the fraction field of  $\tilde{\mathcal{O}}_{D,d}$ , and let  $\tilde{j}: \text{Spec } K_{\{d\}} \rightarrow [D]_{\{\sigma\}}$  be the generic point. There is a Weil divisor exact sequence of abelian sheaves on  $[D]_{\{\sigma\}}$

$$(10) \quad 0 \rightarrow \mathbb{G}_m \rightarrow \tilde{j}_* \mathbb{G}_m \rightarrow i_* \mathbb{Z} \rightarrow 0,$$

where  $i: \tilde{\sigma} \rightarrow [D]_{\{\sigma\}}$ . From the spectral sequence

$$R^p i^! R^q \tilde{j}_* \mathbb{G}_m \implies R^{p+q}(i^! \tilde{j}_*) \mathbb{G}_m$$

and the fact that  $i^! \tilde{j}_* = 0$ , we conclude that  $R^q i^! (\tilde{j}_* \mathbb{G}_m) = 0$  for every  $q$ . Also,

$$R^p i^! (i_* \mathbb{Z}) = R^p (i^! i_*) \mathbb{Z} = R^p (\text{id}) \mathbb{Z} = \begin{cases} \mathbb{Z} & \text{if } p=0 \\ 0 & \text{else} \end{cases}$$

where the first equality holds since  $i_*$  is exact and preserves injectives. From the exact sequence (10), we get

$$(11) \quad R^p i^! \mathbb{G}_m = \begin{cases} \mathbb{Z} & \text{if } p=1 \\ 0 & \text{else} \end{cases}$$

Since the functor  $\mathcal{F} \mapsto \text{Hom}_{[D]}(i_* \mathbb{Z}, \mathcal{F})$  is equal to the composition of functors  $\mathcal{F} \mapsto i^! \mathcal{F} \mapsto \text{Hom}_{\sigma}(\mathbb{Z}, i^! \mathcal{F})$  and the first of these functors preserves injectives, there is a spectral sequence

$$H^p(\tilde{\sigma}, R^q i^! \mathbb{G}_m) \implies H_{\sigma}^{p+q}([D]_{\{\sigma\}}, \mathbb{G}_m).$$

From (11), we see that the spectral sequence collapses, which gives the first equality in the Proposition.

It remains to compute  $H^m(\tilde{\sigma}, \mathbb{Z})$ , which is the part of the argument which differs from the Dedekind scheme case. There exists a finite group  $G_d$ , and a local artinian  $k(d)$ -algebra  $(B, \mathfrak{m})$  such that  $\tilde{\sigma} = [\text{Spec } B/G_d]$ . From the Lyndon-Serre spectral sequence

$$H^p(G_d, H^q(\text{Spec } B, \mathbb{Z})) \implies H^{p+q}(\tilde{\sigma}, \mathbb{Z})$$

we get

$$H^0(\tilde{\sigma}, \mathbb{Z}) = H^0(G_d, H^0(\text{Spec } B, \mathbb{Z})) = H^0(G_d, \mathbb{Z}) = \mathbb{Z}$$

and an exact sequence

$$H^1(G_d, H^0(\text{Spec } B, \mathbb{Z})) \rightarrow H^1(\tilde{\sigma}, \mathbb{Z}) \rightarrow H^0(G_d, H^1(\text{Spec } B, \mathbb{Z})).$$

Since  $H^1(G_d, \mathbb{Z}) = 0$  and  $H^1(\text{Spec } B, \mathbb{Z}) = H^1(B/\mathfrak{m}, \mathbb{Z}) = H^1(\text{Gal}(B/\mathfrak{m}), \mathbb{Z}) = 0$  (since  $\text{Gal}(B/\mathfrak{m})$  is a profinite group), we get the result.  $\square$

Let us return to the cohomology of  $[D]$ . The Kummer sequence gives the exact rows, and the relative cohomology sequence yields the exact columns of the commutative diagram

$$(12) \quad \begin{array}{ccccccc} H_\sigma^1([D], \mathbb{G}_m) & \longrightarrow & H_\sigma^1([D], \mathbb{G}_m) & \longrightarrow & H_\sigma^2([D], \mu_2) & \longrightarrow & H_\sigma^2([D], \mathbb{G}_m) \\ & & \downarrow & & \downarrow & & \\ & & H^1([D], \mathbb{G}_m) & \xrightarrow{\delta} & H^2([D], \mu_2) & & \\ & & & & \downarrow & & \\ & & & & H^2(U, \mu_2) & & \end{array}$$

Excision (i.e., Proposition 9) gives that  $H_\sigma^i([D], \mathcal{F}) = H_\sigma^i([D]_{\{\sigma\}}, \mathcal{F})$  for any sheaf of abelian groups  $\mathcal{F}$  on  $[D]$ , and hence Proposition 10 shows that

$$(13) \quad H_\sigma^1([D], \mathbb{G}_m) = \mathbb{Z} \text{ and } H_\sigma^2([D], \mathbb{G}_m) = 0.$$

We are now able to prove the result we want:

**Proposition 11.** *The kernel of  $H^2([D], \mu_2) \rightarrow H^2(U, \mu_2)$  is generated by  $\delta(\mathcal{O}(\tilde{\sigma}))$ , and is either trivial or of order two.*



*Proof.* According to (13), the upper row of Diagram (12) is

$$\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0.$$

Hence the kernel of  $H^2([D], \mu_2) \rightarrow H^2(V, \mu_2)$  is either trivial or of order two.

It then suffices to prove that the left vertical map of Diagram (12) maps the positive generator of  $H_\sigma^1([D], \mathbb{G}_m)$  to  $\mathcal{O}(\sigma)$ . Let  $K$  be the function field of  $D$ , and let  $g: \text{Spec } K \rightarrow [D]$  be the generic point. There is a Weil divisor exact sequence on  $[D]$

$$0 \rightarrow \mathbb{G}_m \rightarrow g_*\mathbb{G}_m \rightarrow \bigoplus_{d \in D^0} i(d)_*\mathbb{Z} \rightarrow 0,$$

where  $i(d): \sigma(d) \rightarrow [D]$  are the closed immersions of the residual gerbes. Since (10) is the pull-back of this sequence to  $[D]_{\{\sigma\}}$ , we get a commutative diagram

$$\begin{array}{ccc} H_\sigma^0([D]_{\{\sigma\}}, i_*\mathbb{Z}) & \longrightarrow & H_\sigma^1([D]_{\{\sigma\}}, \mathbb{G}_m) \\ \downarrow & & \downarrow \\ H^0([D], \bigoplus i(d)_*\mathbb{Z}) & \longrightarrow & H^1([D], \mathbb{G}_m). \end{array}$$

Since  $H_\sigma^k([D]_{\{\sigma\}}, \tilde{j}_*\mathbb{G}_m) = 0$  for all  $k$  (argue as in the proof of Proposition 10), the upper horizontal morphism must be an isomorphism. This gives the desired conclusion.  $\square$

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