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Restrictions of Riesz–Morrey potentials

David R. Adams and Jie Xiao

Abstract. This paper is devoted to exploiting the restrictions of Riesz–Morrey potentials on either unbounded or bounded domains in Euclidean spaces.

Introduction

This paper stems from an error by the present authors, an error discovered by X. Cabré, but suspected by others. It occurred first in [9] when we attempted to obtain an estimate for the Wolff potential associated with the Morrey capacities $C_\alpha(\cdot; L^{p,\lambda})$, to render a lower bound on these capacities in terms of a Hausdorff capacity of dimension $\lambda-\alpha p$ for $1<p<\lambda/\alpha$. Such an estimate now seems unlikely. Furthermore, this error has now percolated down through the next couple of articles by the present authors: [10], [11]. This paper, partially inspired by the fundamentals of the Riesz–Morrey potentials in Propositions 1.1–1.2, is our attempt to fix this error and its potential disruptive consequences.

A corrected version is given in Theorem 3.2(i) below—an estimate that now implies the (Riesz operator I_α-normalized) embedding with a constant $c_0>0$:

$$(*) \qquad \|I_\alpha f\|_{L^q_{\text{loc}}(\mu)} \le c_0 \|f\|_{L^{p,\lambda}}$$

for some $q>1$ and d-measure μ, i.e., the non-negative Radon measure on the $1\le N$-dimensional Euclidean space $\mathbb{R}^N$ enjoying

$$\sup_{(x,r)\in B\times(0,\infty)} r^{-d}\mu\big(\big\{y\in\mathbb{R}^N : |y-x|<r\big\}\big)<\infty$$

for some fixed ball $B\subseteq\mathbb{R}^N$ containing the support of μ. Here d plays the role of the dimension of the measure μ in the sense of Frostman; see [5, Theorem 5.1.12]. Of course, Lebesgue measures on hyper-planes of $\mathbb{R}^N$ are d-measures for d being

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dimension of the hyper-planes. Such estimates as $(*)$ now seem sparse, but, more interestingly, Theorems 3.1–3.2 (requiring $d>N-\alpha p$) and their by-products (Theorems 3.4–3.5 requesting $d>N-\alpha p$ again) are the best of this kind—as a matter of fact—[9, Remark 2.6(i)] ensures that if $1\le p<q<\infty$, $\alpha p<\lambda<N$ and

$$(**) \qquad \|I_\alpha f\|_{L^q(\mu)} \le c_0'\|f\|_{L^{p,\lambda}} \quad \text{for a constant } c_0'>0,$$

then

$$\begin{aligned}\|\mu\|_d := & \sup_{(x,r)\in\mathbb{R}^N\times(0,\infty)} r^{-d}\mu\big(\{y\in\mathbb{R}^N: |y-x|<r\}\big) \\ & < \infty \quad \text{with } d=p^{-1}q(\lambda-\alpha p)>\lambda-\alpha p.\end{aligned}$$

But, an application of Frostman's theorem (cf. [5, Theorem 5.1.12]) implies: if

$$(***) \qquad \begin{cases} d\in(\lambda-\alpha p, N-\alpha p]; \\ \lambda\in(\alpha p, N); \\ q\in\big(pd(\lambda-\alpha p)^{-1},\infty\big), \end{cases}$$

then there are a d-measure $\tilde{\mu}$ and a constant $c_1>0$ such that the d-dimensional Hausdorff capacity $\Lambda_d^{(\infty)}(E)$ of any set $E\subset\mathbb{R}^N$ is dominated by $c_1\tilde{\mu}(E)$—in particular—one has

$$\begin{aligned} r^d &= \Lambda_d^{(\infty)}\big(\{y\in\mathbb{R}^N: |y-x|<r\}\big) \\ &\le c_1\tilde{\mu}\big(\{y\in\mathbb{R}^N: |y-x|<r\}\big) \quad \forall (x,r)\in\mathbb{R}^N\times(0,\infty); \end{aligned}$$

furthermore, if

$$\|I_\alpha f\|_{L^q(\tilde{\mu})} \le c_2\|f\|_{L^{p,\lambda}} \quad \text{for a constant } c_2>0,$$

then [7, Theorem 5.3 (i)] is utilized to produce a constant $c_3>0$ such that for any $r\in(0,1)$,

$$\begin{aligned} r^d &= \Lambda_d^{(\infty)}\big(\{y\in\mathbb{R}^N: |y-x|<r\}\big) \\ &\le c_1\tilde{\mu}\big(\{y\in\mathbb{R}^N: |y-x|<r\}\big) \\ &\le c_1c_2^q\Big(C_\alpha\big(\{y\in\mathbb{R}^N: |y-x|<r\}; L^{p,\lambda}\big)\Big)^{p^{-1}q} \\ &\le c_1c_2^qc_3^{p^{-1}q}r^{p^{-1}q(\lambda-\alpha p)}, \end{aligned}$$

which, via letting $r\to 0$ and using $(***)$, yields a contradiction $\infty\le c_1c_2^qc_3^{p^{-1}q}$. Therefore, there is no embedding of type $(**)$ or $(*)$ for such a d-measure $\tilde{\mu}$ under $(***)$.

However, in this paper, we have still attempted to explore the limiting situations in a two-fold manner—first by relaxing the d-measure condition by other growth conditions (Theorem 2.4, as a consequence of the Wolff-type-estimate-based Theorem 2.1) or secondly by looking at the restriction of the Riesz–Morrey potential $I_\alpha f$ to a d-dimensional hyper-plane with the result that the Morrey norm is replaced by a mixed Morrey norm (Theorem 2.6).

1. Potentials

1.1. Riesz–Morrey potential I

For $(\alpha,p,\lambda)\in(0,N)\times[1,\infty)\times(0,N]$ and Ω—a domain in $\mathbb{R}^N$, we say that $f\colon \Omega\to\mathbb{R}$ is a Morrey function on Ω, denoted by $f\in L^{p,\lambda}(\Omega)$, provided

$$\|f\|_{L^{p,\lambda}(\Omega)}:=\sup_{x\in\Omega,\,0<r<\operatorname{diam}(\Omega)}\left(r^{\lambda-N}\int_{B(x,r)\cap\Omega}\big|f(y)\big|^p\,dy\right)^{\frac{1}{p}}<\infty,$$

where $\operatorname{diam}(\Omega)$ is the diameter of Ω, $B(x,r)$ is the open ball with center x and radius r, and the integral is taken with respect to the N-dimensional Lebesgue measure dy. In particular, we have $L^{p,N}(\Omega)=L^p(\Omega)$ and write $L^{p,\lambda}$ for $L^{p,\lambda}(\mathbb{R}^N)$.

Given $\alpha\in(0,N)$. A function g on $\mathbb{R}^N$ is said to be a Riesz–Morrey potential of order α provided

$$g(x)=I_\alpha f(x)=\int_{\mathbb{R}^N}f(y)|y-x|^{\alpha-N}\,dy\quad\text{for some } f\in L^{p,\lambda}.$$

Putting together all such potentials gives the so-called Riesz–Morrey space $I_\alpha L^{p,\lambda}$.

From now on, write 1_E and $\mathsf{X}\lesssim\mathsf{Y}$ respectively for the indicator of $E\subset\mathbb{R}^N$ and $\mathsf{X}\le c\mathsf{Y}$ with a constant $c>0$. Moreover, $\mathsf{X}\approx\mathsf{Y}$ means both $\mathsf{X}\lesssim\mathsf{Y}$ and $\mathsf{Y}\lesssim\mathsf{X}$.

Proposition 1.1. *Given $N\ge\lambda>\alpha p$ and $\infty>p\ge1$, let $\mathbb{B}^N$ be the unit open ball of $\mathbb{R}^N$ with ω_N being its volume. Then*

$$\|I_\alpha f\|_{L^q}\lesssim\|f\|_{L^{p,\lambda}}\ \forall 0\not\equiv f\in L^{p,\lambda}\quad\Longrightarrow\quad q=\frac{pN}{\lambda-\alpha p}.$$

On the other hand, if $f_0(x)=|x|^{-\frac{\lambda}{p}}1_{\mathbb{B}^N}(x)$ and $\lambda<N$ then $f_0\in L^{p,\lambda}$, but

$$\big(I_\alpha f_0(x)\big)1_{B(0,1/2)}(x)\approx|x|^{\alpha-\frac{\lambda}{p}}1_{B(0,1/2)}(x)\quad\textit{and}\quad\|I_\alpha f_0\|_{L^{\frac{pN}{\lambda-\alpha p}}}=\infty.$$

Proof. To check the first assertion, we assume that the last inequality is valid for $f_r(x)=f(rx)$, thereby calculating

$$\|f_r\|_{L^{p,\lambda}}=r^{-\frac{\lambda}{p}}\|f\|_{L^{p,\lambda}}\quad\text{and}\quad\|I_\alpha f_r\|_{L^q}=r^{-\alpha-\frac{N}{q}}\|I_\alpha f\|_{L^q}$$

and getting

$$\|I_\alpha f\|_{L^q}\lesssim r^{\alpha-\frac{\lambda}{p}+\frac{N}{q}}\|f\|_{L^{p,\lambda}}.$$

Consequently, if

$$\alpha-\frac{\lambda}{p}+\frac{N}{q}\neq 0$$

then $f\equiv 0$ follows from letting $r\to 0$ or $r\to\infty$, and so a contradiction occurs. Therefore,

$$q=\frac{pN}{\lambda-\alpha p}.$$

For the second assertion, given $(x,r)\in\mathbb{R}^N\times(0,\infty)$ we calculate

$$\begin{aligned}r^{\lambda-N}\int_{B(x,r)}\big|f_0(y)\big|^p\,dy&=r^{\lambda-N}\int_{B(x,r)\cap\mathbb{B}^N}|y|^{-\lambda}\,dy\\&=r^{\lambda-N}\int_{|z|<r,|x-z|<1}|x-z|^{-\lambda}\,dz,\end{aligned}$$

thereby establishing $f_0\in L^{p,\lambda}$ via handling two situations below.

Case 1—under $r\in[1,\infty)$ and $\lambda<N$ we have

$$r^{\lambda-N}\int_{B(x,r)}\big|f_0(y)\big|^p\,dy\le\int_{|x-z|<1}|x-z|^{-\lambda}\,dz\approx 1.$$

Case 2—under $r\in(0,1)$ and $\lambda<N$ we have

$$\begin{cases}|x|<2r\Rightarrow|x-z|<3r\\ \qquad\Rightarrow r^{\lambda-N}\displaystyle\int_{B(x,r)}\big|f_0(y)\big|^p\,dy\le r^{\lambda-N}\int_{|x-z|<3r}|x-z|^{-\lambda}\,dz\approx 1;\\ |x|\ge 2r\Longrightarrow|x-z|\ge r\Longrightarrow r^{\lambda-N}\displaystyle\int_{B(x,r)}\big|f_0(y)\big|^p\,dy\le r^{\lambda-N}\int_{|z|<r}r^{-\lambda}\,dz\approx 1.\end{cases}$$

At the same time, if $N\ge\lambda>\alpha p\ge\alpha$ and $|x|<1/2$ then

$$0<\alpha-\frac{\lambda}{p}<N\quad\text{and}\quad 2^{-1}|y|\le|x-y|\le 2|y|\ \forall y\in\mathbb{R}^N\setminus\mathbb{B}^N,$$

and hence it follows from [14, p. 132, (3)] that

$$\begin{aligned}
I_\alpha f_0(x) &= \int_{\mathbb{B}^N} |x-y|^{\alpha-N}|y|^{-\frac{\lambda}{p}}\,dy \\
&= \int_{\mathbb{R}^N} |x-y|^{\alpha-N}|y|^{-\frac{\lambda}{p}}\,dy - \int_{\mathbb{R}^N\setminus\mathbb{B}^N} |x-y|^{\alpha-N}|y|^{N-\frac{\lambda}{p}-N}\,dy \\
&\approx \left| |x|^{\alpha-\frac{\lambda}{p}} - \int_{\mathbb{R}^N\setminus\mathbb{B}^N} |x-y|^{\alpha-N}|y|^{-\frac{\lambda}{p}}\,dy \right| \\
&\approx \left| |x|^{\alpha-\frac{\lambda}{p}} - \int_{\mathbb{R}^N\setminus\mathbb{B}^N} |y|^{\alpha-N}|y|^{-\frac{\lambda}{p}}\,dy \right| \\
&\approx |x|^{\alpha-\frac{\lambda}{p}} \quad \forall |x|<1/2.
\end{aligned}$$

With the above estimate and $q=pN/(\lambda-\alpha p)$, we compute

$$\|I_\alpha f_0\|_{L^q}^q \geq \int_{|x|<1/2} \big(I_\alpha f_0(x)\big)^q\,dx \gtrsim \int_{|x|<1/2} |x|^{-N}\,dx = \infty. \qquad \square$$

1.2. Riesz–Morrey potential II

A further examination of Proposition 1.1 and its proof leads to the following result.

Proposition 1.2. *Let f be a non-negative $L^{p,\lambda}$ function with support in $\mathbb{B}^N$ and*

$$\begin{cases} \|f\|_{L^{p,\lambda}}>0; \\ f(x)=f\big(|x|\big) \quad \forall x\in\mathbb{R}^N; \\ f\big(|x|\big)>f\big(|y|\big) \quad \forall |x|<|y|. \end{cases}$$

(i) *If $N\geq\lambda>\alpha p$, $\infty>p\geq 1$ and $1<q<pN(\lambda-p\alpha)^{-1}$, then*

$$\int_{\mathbb{B}^N} \left(\frac{I_\alpha f(x)}{\|f\|_{L^{p,\lambda}}}\right)^q dx<\infty.$$

(ii) *If $N\geq\lambda>\alpha p$, $\infty>p\geq 1$ and $1<\gamma\leq q=pN(\lambda-p\alpha)^{-1}$, then*

$$\int_{\mathbb{B}^N} \left(\frac{I_\alpha f(x)}{\|f\|_{L^{p,\lambda}}}\right)^q \left(\ln\left(1+\frac{I_\alpha f(x)}{\|f\|_{L^{p,\lambda}}}\right)\right)^{-\gamma} dx<\infty.$$

(iii) *If $N\geq\lambda=\alpha p$, $\infty>p\geq 1$ and $\beta\in[0,1]$, then there is a constant $c_0>0$ such that*

$$\int_{\mathbb{B}^N} \exp\left(\frac{cI_\alpha f(x)}{\|f\|_{L^{p,\lambda}}}\right)^\beta dx<\infty \quad \forall c\in(0,c_0).$$

Proof. Using the hypothesis of f and its dilation f_r defined in Proposition 1.1, we get

$$\big(f(r)\big)^p \le \int_{|y|<1} \big(f(r|y|)\big)^p\,dy \le \|f_r\|^p_{L^{p,\lambda}} = r^{-\lambda}\|f\|^p_{L^{p,\lambda}} \quad \forall\ r>0,$$

whence controlling three cases (i), (ii) and (iii) below.

(i) If

$$N \ge \lambda > \alpha p \ge \alpha \quad \text{and} \quad 1<q<pN(\lambda-\alpha p)^{-1},$$

then a combination of the last inequality for $(f(r))^p$ and [14, p. 132, (3)] gives

$$\begin{aligned} I_\alpha f(x) &= \int_{\mathbb{B}^N} |x-y|^{\alpha-N} f\big(|y|\big)\,dy \\ &\le \|f\|_{L^{p,\lambda}} \int_{\mathbb{B}^N} |x-y|^{\alpha-N}|y|^{-\frac{\lambda}{p}}\,dy \lesssim \|f\|_{L^{p,\lambda}}|x|^{\alpha-\frac{\lambda}{p}} \quad \forall x\in\mathbb{B}^N, \end{aligned}$$

and hence

$$\int_{\mathbb{B}^N} \big(I_\alpha f(x)\big)^q\,dx \lesssim \|f\|^q_{L^{p,\lambda}} \int_{\mathbb{B}^N} |x|^{q(\alpha-\frac{\lambda}{p})}\,dx \lesssim \|f\|^q_{L^{p,\lambda}}.$$

This implies the desired estimate.

(ii) If

$$N \ge \lambda > \alpha p \quad \text{and} \quad 1<\gamma\le q = pN(\lambda-p\alpha)^{-1},$$

then the following function

$$s \longmapsto s\big(\ln(1+s)\big)^{-\frac{\gamma}{q}}$$

is increasing in $[0,\infty)$, and hence it follows that

$$\begin{aligned} \int_{\mathbb{B}^N} \left(\frac{I_\alpha f(x)}{\|f\|_{L^{p,\lambda}}}\right)^q \left(\ln\left(1+\frac{I_\alpha f(x)}{\|f\|_{L^{p,\lambda}}}\right)\right)^{-\gamma} dx &\lesssim \int_{\mathbb{B}^N} \big(|x|^{\alpha-\frac{\lambda}{p}}\big)^q\big(\ln\big(1+|x|^{\alpha-\frac{\lambda}{p}}\big)\big)^{-\gamma}\,dx \\ &\lesssim \int_0^1 t^{-1}\big(\ln\big(1+t^{-1}\big)\big)^{-\gamma}\,dt \\ &< \infty, \end{aligned}$$

whence deriving the desired inequality.

(iii) If

$$N \ge \lambda = \alpha p \ge \alpha \quad \text{and} \quad \beta\in[0,1],$$

then an application of the already-verified inequality

$$f\big(|y|\big) \le |y|^{-\lambda/p}\|f\|_{L^{p,\lambda}} \quad \forall |y|>0$$

implies

$$I_\alpha f(x) = \int_{\mathbb{B}^N} |x-y|^{\alpha-N} f\big(|y|\big)\,dy$$
$$\le \|f\|_{L^{p,\lambda}} \int_{\mathbb{B}^N} |x-y|^{\alpha-N}|y|^{-\alpha}\,dy \lesssim \|f\|_{L^{p,\lambda}} \ln\big(2|x|^{-1}\big) \quad \forall x\in\mathbb{B}^N,$$

and hence there is a constant $c_1>0$ such that

$$\int_{\mathbb{B}^N} \exp\bigg(\frac{cI_\alpha f(x)}{\|f\|_{L^{p,\lambda}}}\bigg)^\beta dx \lesssim \int_{\mathbb{B}^N} \exp\Big(\big(cc_1 \ln\big(2|x|^{-1}\big)\big)^\beta\Big)\,dx$$
$$\lesssim \int_0^1 \exp\Big((cc_1)^\beta\big(\ln t^{-1}\big)^\beta\Big)\,dt$$
$$\lesssim \int_0^\infty \exp\big((cc_1)^\beta s^\beta - s\big)\,ds.$$

Since $\beta\in[0,1]$, an appropriate choice of c ensures that the last integral is convergent, thereby producing the desired inequality. $\square$

2. Restrictions of Riesz–Morrey potentials on unbounded domains

2.1. Wolff type estimation

This will be one of the most effective tools for our approach to the restriction problem on unbounded domains; see e.g. [6], [13], [17], [3]. In what follows, $A_{p\in(1,\infty)}$ stands for the class of all non-negative functions w on $\mathbb{R}^N$ such that

$$\sup_{\text{coordinate cubes } Q} \bigg(\int_Q w(y)\frac{dy}{|Q|}\bigg)\bigg(\int_Q \big(w(y)\big)^{\frac{1}{1-p}}\frac{dy}{|Q|}\bigg)^{p-1} < \infty.$$

In the above and below, a coordinate cube always means a cube in $\mathbb{R}^N$ with edges parallel to the coordinate axes, and $|Q|$ stands for the Lebesgue measure of Q. As $p\to\infty$ we have $A_\infty = \cup_{p\in(1,\infty)} A_p$. Meanwhile, if $p\to 1$, then we get A_1—the class of all non-negative functions w on $\mathbb{R}^N$ such that

$$\sup_{\text{coordinate cubes } Q} \bigg(\int_Q w(y)\frac{dy}{|Q|}\bigg)\Big(\inf_{y\in Q} w(y)\Big)^{-1} < \infty.$$

It is well-known that $w\in A_1$ ensures $w\in\cap_{p\in(1,\infty)} A_p$. Moreover, $\mathcal{W}_{N-\lambda}$ $(0<\lambda<N)$ comprises all non-negative functions w on $\mathbb{R}^N$ obeying

$$\int_{\mathbb{R}^N} w\,d\Lambda^{(\infty)}_{N-\lambda} := \int_0^\infty \Lambda^{(\infty)}_{N-\lambda}\big(\big\{x\in\mathbb{R}^N : w(x)>t\big\}\big)\,dt \le 1$$

for which the symbol $\Lambda^{(\infty)}_{N-\lambda}$ expresses the $(N-\lambda)$-dimensional Hausdorff capacity, namely, $\Lambda^{(\infty)}_{N-\lambda}(E)=\inf\sum_j r_j^{N-\lambda}$ where the infimum is over all coverings $\cup_j B_j\supseteq E$ for which B_j is a ball with radius r_j. For simplicity, set

$$A_1^{(N-\lambda)}=A_1\cap\mathcal{W}_{N-\lambda}\quad\forall\lambda\in(0,N).$$

Theorem 2.1. *Let* $(\alpha,\lambda,p)\in(0,N)\times(0,N)\times(1,N/\alpha)$, $p'=p/(p-1)$, *and* μ *be a non-negative Radon measure on* $\mathbb{R}^N$. *Then*

(i)

$$\|I_\alpha\mu\|^{p'}_{L^{p',\lambda}}\approx\sup_{w\in A_1^{(N-\lambda)}}\int_{\mathbb{R}^N}W^{\mu,w,1}_{\alpha,p}(y)\,d\mu(y).$$

(ii)

$$\|I_\alpha\mu\|^{p'}_{H^{p',\lambda}}=\inf_{w\in\mathcal{W}_{N-\lambda}}\int_{\mathbb{R}^N}\big(I_\alpha\mu(y)\big)^{p'}\big(w(y)\big)^{1-p'}\,dy\approx\inf_{w\in A_1^{(N-\lambda)}}\int_{\mathbb{R}^N}W^{\mu,w,2}_{\alpha,p}(y)\,d\mu(y).$$

Here

$$W^{\mu,w,j}_{\alpha,p}(y)=\begin{cases}\int_0^\infty\big(t^{\alpha p-N}\mu(B(y,t))\big)^{\frac{1}{p-1}}\big(\int_{B(y,t)}w(z)\,dz\big)\frac{dt}{t^{N+1}} & as\ j=1;\\ \int_0^\infty\big(t^{\alpha p-N}\mu(B(y,t))\big)^{\frac{1}{p-1}}\big(\int_{B(y,t)}(w(z))^{\frac{1}{1-p}}\,dz\big)\frac{dt}{t^{N+1}} & as\ j=2.\end{cases}$$

Proof. (i) According to [8, Lemma 11], we have

$$\|I_\alpha\mu\|^{p'}_{L^{p',\lambda}}\approx\sup_{w\in A_1^{(N-\lambda)}}\int_{\mathbb{R}^N}\big(I_\alpha\mu(x)\big)^{p'}w(x)\,dx.$$

Note that

$$w\in A_1^{(N-\lambda)}\quad\Longrightarrow\quad w\in A_1\subset A_\infty.$$

So, we utilize [3, Theorem 3.2] to get

$$\int_{\mathbb{R}^N}\big(I_\alpha\mu(x)\big)^{p'}w(x)\,dx\approx\int_{\mathbb{R}^N}W^{\mu,w,1}_{\alpha,p}(y)\,d\mu(y)$$

which, plus taking the supremum over $A_1^{(N-\lambda)}$, implies the equivalence in (i).

(ii) In accordance with [8, Theorem 7], we have

$$\|I_\alpha\mu\|^{p'}_{H^{p',\lambda}}\approx\inf_{w\in A_1^{(N-\lambda)}}\int_{\mathbb{R}^N}\big(I_\alpha\mu(y)\big)^{p'}\big(w(y)\big)^{1-p'}\,dy.$$

Since

$$w\in A_1^{(N-\lambda)}\quad\Longrightarrow\quad w^{\frac{1}{1-p}}\in A_\infty,$$

an application of [3, Theorem 3.2] yields

$$\int_{\mathbb{R}^N} \left(I_\alpha\mu(y)\right)^{p'} \left(w(y)\right)^{1-p'} dy \approx \int_{\mathbb{R}^N} W^{\mu,w,2}_{\alpha,p}(y)\, d\mu(y),$$

and then the estimate in (ii) after taking the infimum over $A_1^{(N-\lambda)}$. $\square$

Corollary 2.2. *Let* $(\alpha,\lambda,p)\in(0,N)\times(0,N)\times(1,N/\alpha)$, $p'=p/(p-1)$, *and* μ *be a non-negative Radon measure on* $\mathbb{R}^N$. *Then*

(i) [8, Theorem 21(i)]

$$\|I_\alpha\mu\|^{p'}_{L^{p',\lambda}} \lesssim \int_{\mathbb{R}^N}\left(\int_0^\infty \left(\frac{\mu(B(y,r))}{r^{\lambda+p(N-\lambda-\alpha)}}\right)^{\frac{1}{p-1}} \frac{dr}{r}\right) d\mu(y).$$

Meanwhile, when $0<\lambda<p'(N-\alpha)$ *and,* $supp(\mu)$ *(the support of* μ*) is contained in an origin-centered ball* $B(0,R)$, *one has*

$$\|I_\alpha\mu\|^{p'}_{L^{p',\lambda}} \gtrsim \int_{\mathbb{R}^N}\left(\int_{3R}^\infty \left(\frac{\mu(B(y,r))}{r^{\lambda+p(N-\lambda-\alpha)}}\right)^{\frac{1}{p-1}} \frac{dr}{r}\right) d\mu(y).$$

(ii) [8, Theorem 21(ii)]

$$\|I_\alpha\mu\|^{p'}_{H^{p',\lambda}} \approx \inf_{w\in A_1^{(N-\lambda)}} \int_{\mathbb{R}^N}\left(\int_0^\infty \left(\frac{t^{\alpha p}\mu(B(y,t))}{\int_{B(y,t)} w(z)\,dz}\right)^{\frac{1}{p-1}} \frac{dt}{t}\right) d\mu(y).$$

Proof. (i) Note that (cf. [8, p. 212])

$$\|w\|_{L^{\frac{N}{N-\lambda}}} \lesssim \int_{\mathbb{R}^N} w\, d\Lambda^{(\infty)}_{N-\lambda}.$$

So, using Hölder's inequality we get that for any ball $B(x,r)\subset\mathbb{R}^N$ and any $w\in A_1^{(N-\lambda)}$,

$$\int_{B(x,r)} w(y)\,dy \le \|w\|_{L^{\frac{N}{N-\lambda}}} r^\lambda \lesssim r^\lambda.$$

This in turn implies

$$W^{\mu,w,1}_{\alpha,p}(y) \lesssim \int_0^\infty \left(\frac{\mu(B(y,r))}{r^{\lambda+p(N-\lambda-\alpha)}}\right)^{\frac{1}{p-1}} \frac{dr}{r},$$

thereby deriving via Theorem 2.1(i)

$$\|I_\alpha\mu\|^{p'}_{L^{p',\lambda}} \lesssim \int_{\mathbb{R}^N}\left(\int_0^\infty \left(\frac{\mu(B(y,r))}{r^{\lambda+p(N-\lambda-\alpha)}}\right)^{\frac{1}{p-1}} \frac{dr}{r}\right) d\mu(y).$$

This inequality yields the upper bound estimate of [8, Theorem 21(i)].

However, as a replacement of the lower bound estimate of [8, Theorem 21(i)], we have the second part of (i). In order to verify this new estimation, we utilize [5, Lemma 3.3.1] to get

$$\begin{aligned}\|I_\alpha\mu\|^{p'}_{L^{p',\lambda}} &\gtrsim \sup_{(x,r)\in\mathbb{R}^N\times(0,\infty)} r^{\lambda-N}\int_{B(x,r)}\Big(\int_0^\infty \mu\big(B(y,t)\big)\,\frac{dt}{t}\Big)^{p'}dy\\ &\gtrsim \sup_{(x,r)\in\mathbb{R}^N\times(0,\infty)} r^{\lambda+(\alpha-N)p'}\Big(r^{-N}\int_{B(x,r)}\big(\mu\big(B(y,2r)\big)\big)^{p'}dy\Big)\\ &\gtrsim \sup_{(x,r)\in\mathbb{R}^N\times[2R,\infty)} r^{\lambda+(\alpha-N)p'}\Big(r^{-N}\int_{B(x,r)}\big(\mu\big(B(y,2r)\big)\big)^{p'}dy\Big).\end{aligned}$$

In the meantime, an application of Fubini's theorem gives that if $(x,r)\in\mathbb{R}^N\times[2R,\infty)$ then

$$\begin{aligned}&r^{-N}\int_{B(x,r)}\big(\mu\big(B(y,2r)\big)\big)^{p'}dy\\ &\quad= r^{-N}\int_{|x-y|<r}\big(\mu\big(B(y,2r)\big)\big)^{p'-1}\mu\big(B(y,2r)\big)\,dy\\ &\quad= r^{-N}\int_{|x-y|<r}\Big(\int_{|y-z|<2r}d\mu(z)\Big)^{p'-1}\Big(\int_{|y-\tilde z|<2r}d\mu(\tilde z)\Big)\,dy\\ &\quad\ge r^{-N}\int_{|\tilde z-x|<r/2}\Big(\int_{|z-\tilde z|<r}d\mu(z)\Big)^{p'-1}\Big(\int_{|x-y|<r,|y-z|<r/2,|y-\tilde z|<r/2}dy\Big)\,d\mu(\tilde z)\\ &\quad\gtrsim \int_{B(x,r/2)}\big(\mu\big(B(\tilde z,r)\big)\big)^{p'-1}d\mu(\tilde z)\\ &\quad\gtrsim \int_{B(x,R)}\big(\mu\big(B(\tilde z,r)\big)\big)^{p'-1}d\mu(\tilde z).\end{aligned}$$

This last estimate for $(x,r)\in\{0\}\times[2R,\infty)$, along with $\text{supp}(\mu)\subset B(0,R)$, derives that if $\lambda<(N-\alpha)p'$ then

$$\begin{aligned}\|I_\alpha\mu\|^{p'}_{L^{p',\lambda}} &\gtrsim \sup_{r\in[2R,\infty)}\sup_{x\in\mathbb{R}^N} r^{\lambda+(\alpha-N)p'}\Big(r^{-N}\int_{B(x,r)}\big(\mu\big(B(y,2r)\big)\big)^{p'}dy\Big)\\ &\gtrsim \sup_{r\in[2R,\infty)} r^{\lambda+(\alpha-N)p'}\int_{B(0,R)}\mu\big(B(\tilde z,r)\big)^{p'-1}d\mu(\tilde z)\end{aligned}$$

$$\approx \sup_{r\in[2R,\infty)} r^{\lambda+(\alpha-N)p'}\int_{\mathbb{R}^N}\mu\big(B(\tilde z,r)\big)^{\frac{1}{p-1}}\,d\mu(\tilde z)$$

$$\gtrsim \int_{\mathbb{R}^N}\int_{3R}^{\infty}\left(\frac{\mu(B(x,r))}{r^{\lambda+p(N-\lambda-\alpha)}}\right)^{\frac{1}{p-1}}\frac{dr}{r}\,d\mu(x).$$

Needless to say, the last inequality needs a verification. To do so, note that $\mathrm{supp}(\mu)\subset B(0,R)$ and

$$|x|<R\le r/3 \quad\text{and}\quad y\in B(x,r)\setminus B(x,2R) \quad\Longrightarrow\quad |y|\ge R$$

ensure

$$B(x,2R)\subset B(x,r)$$
$$\text{and}\quad \mu\big(B(x,r)\big)\le\mu\big(B(x,2R)\big)+\mu\big(B(x,r)\setminus B(x,2R)\big)=\mu\big(B(x,2R)\big).$$

Accordingly, if $\lambda<(\alpha-N)p'$ then

$$\int_{\mathbb{R}^N}\int_{3R}^{\infty}\left(\frac{\mu(B(x,r))}{r^{\lambda+p(N-\lambda-\alpha)}}\right)^{\frac{1}{p-1}}\frac{dr}{r}\,d\mu(x)$$

$$=\int_{B(0,R)}\int_{3R}^{\infty}\left(\frac{\mu(B(x,r))}{r^{\lambda+p(N-\lambda-\alpha)}}\right)^{\frac{1}{p-1}}\frac{dr}{r}\,d\mu(x)$$

$$=\int_{3R}^{\infty} r^{\lambda+(\alpha-N)p'}\left(\int_{B(0,R)}\big(\mu\big(B(x,r)\big)\big)^{\frac{1}{p-1}}\,d\mu(x)\right)\frac{dr}{r}$$

$$\le\int_{3R}^{\infty} r^{\lambda+(\alpha-N)p'}\left(\int_{B(0,R)}\big(\mu\big(B(x,2R)\big)\big)^{\frac{1}{p-1}}\,d\mu(x)\right)\frac{dr}{r}$$

$$\lesssim (2R)^{\lambda+(\alpha-N)p'}\int_{B(0,R)}\big(\mu\big(B(x,2R)\big)\big)^{p'-1}\,d\mu(x)$$

$$\lesssim \sup_{r\in[2R,\infty)} r^{\lambda+(\alpha-N)p'}\int_{\mathbb{R}^N}\big(\mu\big(B(x,r)\big)\big)^{\frac{1}{p-1}}\,d\mu(x).$$

(ii) Note that $w\in A_1^{(N-\lambda)}$ ensures $w\in A_p\ \forall\ p\in(1,\infty)$. So, the definition of A_p gives

$$r^{-N}\int_{B(z,r)}\big(w(x)\big)^{\frac{1}{1-p}}\,dx\lesssim\left(r^{-N}\int_{B(z,r)}w(x)\,dx\right)^{\frac{1}{1-p}}.$$

At the same time, a combination of $1<p<\infty$ and the Hölder inequality further derives

$$r^N\approx\int_{B(z,r)}dx\lesssim\left(\int_{B(z,r)}w(x)\,dx\right)^{\frac{1}{p}}\left(\int_{B(z,r)}\big(w(x)\big)^{\frac{1}{1-p}}\,dx\right)^{1-\frac{1}{p}},$$

whence yielding

$$r^{-N}\int_{B(z,r)}\big(w(x)\big)^{\frac{1}{1-p}}\,dx \gtrsim \left(r^{-N}\int_{B(z,r)} w(x)\,dx\right)^{\frac{1}{1-p}}.$$

Putting the foregoing estimates together, we obtain

$$r^{-N}\int_{B(z,r)}\big(w(x)\big)^{\frac{1}{1-p}}\,dx \approx \left(r^{-N}\int_{B(z,r)} w(x)\,dx\right)^{\frac{1}{1-p}},$$

thereby finding

$$W^{\mu,w,2}_{\alpha,p}(y) \approx \int_0^\infty \left(\frac{t^{\alpha p}\mu(B(y,t))}{\int_{B(y,t)} w(z)\,dz}\right)^{\frac{1}{p-1}}\frac{dt}{t}.$$

Therefore, Theorem 2.1(ii) is used to derive Corollary 2.2(ii), i.e., [8, Theorem 21(ii)]. □

Remark 2.3. Additionally, we would like to point out that the argument from line 5 of p. 220 to line 3 of p. 222 in [8] can only validate [8, Theorem 18] under $\lambda_0=\lambda_1=\lambda_\theta$—this has been examined in [15, Remark 1.6(iii)] through a Hausdorff-capacity-free treatment.

2.2. Restriction under $1<p<\min\{q, N/\alpha\}$

Globally, we discover the following assertion.

Theorem 2.4. *Let*

$$\begin{cases}(\alpha,\lambda,p)\in(0,N)\times(0,N)\times\big(1,\min\{q,N/\alpha\}\big);\\ \|f\|_{H^{p,\lambda}} := \inf_{w\in\mathcal{W}_{N-\lambda}}\left(\int_{\mathbb{R}^N}\big|f(y)\big|^p\big(w(y)\big)^{1-p}\,dy\right)^{\frac{1}{p}},\end{cases}$$

and μ be a non-negative Radon measure on $\mathbb{R}^N$.

(i) *If*

$$\mu\big(B(x,r)\big) \lesssim \left(\int_r^\infty t^{\frac{\alpha p-N}{p-1}}\int_{B(x,t)}\big(w(y)\big)^{\frac{1}{1-p}}\,dy\,\frac{dt}{t^{N+1}}\right)^{\frac{q(1-p)}{p}} \quad \text{for some } w\in A_1^{(N-\lambda)},$$

then

$$\|I_\alpha f\|_{L^q(\mu)} \lesssim \|f\|_{L^{p,\lambda}}.$$

(ii)

$$\sup_{(x,r)\in\mathbb{R}^N\times(0,\infty)} \frac{\mu(B(x,r))}{\left(\sup_{w\in A_1^{(N-\lambda)}}\int_r^\infty t^{\frac{\alpha p-N}{p-1}}\int_{B(x,t)} w(y)\,dy\,\frac{dt}{t^{N+1}}\right)^{\frac{q(1-p)}{p}}}<\infty$$

$$\Longleftrightarrow \quad \|I_\alpha f\|_{L^q(\mu)}\lesssim\|f\|_{H^{p,\lambda}}.$$

Consequently, if

$$\|\mu\|_d<\infty \quad \text{for } 0<d=p^{-1}\big(\lambda-p(\alpha+\lambda-N)\big)q\le N,$$

then

$$\|I_\alpha f\|_{L^q(\mu)}\lesssim\|f\|_{H^{p,\lambda}}.$$

Proof. (i) If

$$\mu\big(B(x,r)\big)\lesssim\left(\int_r^\infty t^{\frac{\alpha p-N}{p-1}}\int_{B(x,t)}\big(w(y)\big)^{\frac{1}{1-p}}\,dy\,\frac{dt}{t^{N+1}}\right)^{-\frac{q(p-1)}{p}} \quad \text{for some } w\in A_1^{(N-\lambda)}$$

then by [3, Theorem 7.1] we find

$$\|I_\alpha f\|^p_{L^q(\mu)}\lesssim\int_{\mathbb{R}^N}\big|f(x)\big|^p w(x)\,dx$$

which, along with [8, Lemma 11], implies

$$\|I_\alpha f\|_{L^q(\mu)}\lesssim\|f\|_{L^{p,\lambda}}.$$

(ii) Since

$$\|f\|^p_{H^{p,\lambda}}\approx\inf_{w\in A_1^{(N-\lambda)}}\int_{\mathbb{R}^N}\big|f(x)\big|^p\big(w(x)\big)^{1-p}\,dx,$$

if

$$\|I_\alpha f\|_{L^q(\mu)}\lesssim\|f\|_{H^{p,\lambda}}$$

then

$$\|I_\alpha f\|^p_{L^q(\mu)}\lesssim\int_{\mathbb{R}^N}\big|f(x)\big|^p\big(w(x)\big)^{1-p}\,dx.$$

Thanks to

$$w\in A_1^{(N-\lambda)} \quad\Longrightarrow\quad w=\big(w^{1-p}\big)^{\frac{1}{1-p}}\in A_\infty,$$

it follows from [3, Theorem 7.1] that

$$\mu\big(B(x,r)\big)\lesssim\left(\int_r^\infty t^{\frac{\alpha p-N}{p-1}}\int_{B(x,t)} w(y)\,dy\,\frac{dt}{t^{N+1}}\right)^{-\frac{q(p-1)}{p}},$$

and so that

$$\mu\big(B(x,r)\big)\lesssim \inf_{w\in A_1^{(N-\lambda)}}\left(\int_r^\infty t^{\frac{\alpha p-N}{p-1}}\int_{B(x,t)}w(y)\,dy\,\frac{dt}{t^{N+1}}\right)^{-\frac{q(p-1)}{p}}.$$

Conversely, if the last inequality is true, then

$$\mu\big(B(x,r)\big)\lesssim \left(\int_r^\infty t^{\frac{\alpha p-N}{p-1}}\int_{B(x,t)}w(y)\,dy\,\frac{dt}{t^{N+1}}\right)^{-\frac{q(p-1)}{p}}\quad \forall w\in A_1^{(N-\lambda)}.$$

Again, applying [3, Theorem 7.1], we find

$$\|I_\alpha f\|^p_{L^q(\mu)}\lesssim\int_{\mathbb{R}^N}\big|f(y)\big|^p\big(w(y)\big)^{1-p}\,dy\quad \forall w\in A_1^{(N-\lambda)},$$

thereby reaching

$$\|I_\alpha f\|_{L^q(\mu)}\lesssim\|f\|_{H^{p,\lambda}}.$$

Furthermore, according to Corollary 2.2(ii) one has:

$$\begin{aligned} w\in A_1^{(N-\lambda)}\quad\text{and}\quad \frac{\lambda}{p'}<N-\alpha & \\ \Longrightarrow\quad & t^{-N}\int_{B(x,t)}w(y)\,dy\lesssim t^{\lambda-N}\\ \Longrightarrow\quad & r^{\frac{(\lambda-p(\alpha+\lambda-N))q}{p}}\lesssim\left(\int_r^\infty t^{\frac{\alpha p-N}{p-1}}\int_{B(x,t)}w(y)\,dy\,\frac{dt}{t^{N+1}}\right)^{-\frac{q(p-1)}{p}}. \end{aligned}$$

This, plus the above-verified equivalence, gives the desired implication:

$$\begin{aligned} &\|\mu\|_d<\infty \text{ under } 0<d=p^{-1}\big(\lambda-p(\alpha+\lambda-N)\big)q\le N\\ &\qquad\qquad\Longrightarrow\quad \|I_\alpha f\|_{L^q(\mu)}\lesssim\|f\|_{H^{p,\lambda}}. \qquad\square \end{aligned}$$

Remark 2.5. Here it is perhaps worth mentioning that if $w\in A_1^{(N-\lambda)}$ then

$$\left(\int_{B(x,r)}\big(w(y)\big)^{\frac{1}{1-p}}\,dy\right)^{1-\frac{1}{p}}\gtrsim r^{\frac{\lambda}{p}-N}$$

and hence

$$\begin{aligned} &\left(\int_r^\infty t^{\frac{\alpha p-N}{p-1}}\int_{B(x,t)}\big(w(y)\big)^{\frac{1}{1-p}}\,dy\,\frac{dt}{t^{N+1}}\right)^{\frac{q(1-p)}{p}}\\ &\qquad\lesssim\left(\int_r^\infty t^{\frac{\alpha p-N}{p-1}}\int_{B(x,r)}\big(w(y)\big)^{\frac{1}{1-p}}\,dy\,\frac{dt}{t^{N+1}}\right)^{\frac{q(1-p)}{p}}\\ &\qquad\lesssim r^{q(\lambda-\alpha p)/p}. \end{aligned}$$

2.3. Restriction on $\mathbb{R}^k$ with $1 \le k \le N-1$

Theorem 2.6. *Given*

$$\begin{cases} N \ge 2; \\ k \in \{1, ..., N-1\}; \\ 1 < p < \lambda\alpha^{-1}; \\ \lambda - \alpha p < d \le k < \lambda < N; \\ 1 < q < dp(\lambda - \alpha p)^{-1}. \end{cases}$$

Suppose f is a non-negative function with support in $\mathbb{R}^k \times \mathbb{B}^{N-k}$. Let μ be a non-negative Radon measure with support in $\mathbb{R}^k$ and $\|\mu\|_d < \infty$. Then

$$\|I_\alpha f\|_{L^q(\mu)} \lesssim \|f\|_{L^p(\mathbb{R}^k) L^{p,\lambda-k}(\mathbb{B}^{N-k})} := \Big\| \|f(\bar{y}, \hat{y})\|_{L^{p,\lambda-k}(\hat{y} \in \mathbb{B}^{N-k})} \Big\|_{L^p(\bar{y} \in \mathbb{R}^k)}.$$

Proof. Note that if $x \in \text{supp}(\mu) \subset \mathbb{R}^k$ then

$$x = (\bar{x}, 0) \in \mathbb{R}^k \times \mathbb{R}^{N-k},$$

and hence $y \in \mathbb{R}^N$ is written as $y = (\bar{y}, \hat{y}) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$.

Under

$$0 < \gamma = N - \lambda - \varepsilon < N - \lambda,$$

set

$$w(y) = w(\bar{y}, \hat{y}) = |\hat{y}|^{-\gamma} \quad \text{for } y = (\bar{y}, \hat{y}) \in \mathbb{R}^k \times \mathbb{R}^{N-k}.$$

Clearly, $w \in A_1$, and this weight satisfies

$$\int_{B(x=(\bar{x},0),r)} w(y)\, dy \approx \int_{|\hat{y}|<r} \int_{|\bar{x}-\bar{y}|<r} |\hat{y}|^{-\gamma}\, d\bar{y}\, d\hat{y} \approx r^{N-\gamma}.$$

If $\|\!|\mu|\!\|$ denotes the total variation of μ, then the Wolff potential in Theorem 2.1(ii) enjoys

$$\begin{aligned} W_{\alpha,p}^{\mu,w,2}(\bar{x}, 0) &\approx \int_0^\infty \left(\frac{t^{\alpha p} \mu(B(x,t))}{\int_{B(x,t)} w(y)\, dy} \right)^{\frac{1}{p-1}} \frac{dt}{t} \\ &\approx \left(\int_0^\delta + \int_\delta^\infty \right) \left(\frac{t^{\alpha p} \mu(B(x,t))}{t^{N-\gamma}} \right)^{\frac{1}{p-1}} \\ &\lesssim \|\mu\|_d^{\frac{1}{p-1}} \int_0^\delta t^{\frac{\alpha p + d - N + \gamma}{p-1}} \frac{dt}{t} + \|\!|\mu|\!\|^{\frac{1}{p-1}} \int_\delta^\infty t^{\frac{\alpha p - N + \gamma}{p-1}} \frac{dt}{t} \\ &\lesssim \|\mu\|_d^{\frac{1}{p-1}} \delta^{\frac{\alpha p + d + \gamma - N}{p-1}} + \|\!|\mu|\!\|^{\frac{1}{p-1}} \delta^{\frac{\alpha p + \gamma - N}{p-1}} \end{aligned}$$

under

$$N-\alpha p>\gamma>N-\alpha p-d \quad \Longleftrightarrow \quad \alpha p+d>\lambda+\varepsilon>\alpha p.$$

Taking

$$\delta=\left(\frac{\|\mu\|}{\|\mu\|_d}\right)^{\frac{1}{d}},$$

we find

$$W^{\mu,w,2}_{\alpha,p}(\bar{x},0)\lesssim \|\mu\|^{\frac{1}{p-1}}\left(\frac{\|\mu\|}{\|\mu\|_d}\right)^{\frac{\alpha p-N+\gamma}{d(p-1)}}.$$

In the meantime, we utilize Fubini's theorem, Hölder's inequality and [3, Theorem 3.2] to make the following estimation:

$$\begin{aligned}
\int_{\mathbb{R}^N} I_\alpha f\,d\mu &= \int_{\mathbb{R}^N} f(y) I_\alpha\mu(y)\,dy \\
&= \int_{\mathbb{R}^N} f(y)\big(w(y)\big)^{\frac{1}{p}} I_\alpha\mu(y)\big(w(y)\big)^{-\frac{1}{p}}\,dy \\
&\le \left(\int_{\mathbb{R}^N}\big(f(y)\big)^p w(y)\,dy\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}^N}\big(I_\alpha\mu(y)\big)^{p'}\big(w(y)\big)^{-\frac{1}{p-1}}\,dy\right)^{\frac{1}{p'}} \\
&\approx \left(\int_{\mathbb{R}^N}\big(f(y)\big)^p w(y)\,dy\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}^N} W^{\mu,w,2}_{\alpha,p}(y)\,d\mu(y)\right)^{\frac{1}{p'}} \\
&=: \|f\|_{L^p(w)}\left(\int_{\mathbb{R}^N} W^{\mu,w,2}_{\alpha,p}(y)\,d\mu(y)\right)^{\frac{1}{p'}}.
\end{aligned}$$

Next, for any $\sigma>0$ let

$$E_\sigma=\left\{z\in\mathbb{R}^N:\, I_\alpha f(z)>\sigma\right\} \quad \text{and} \quad \mu_{E_\sigma}=\mu\lfloor E_\sigma.$$

Using the observation on $\operatorname{supp}(\mu)\subset\mathbb{R}^k$ made at the beginning of the proof, we obtain that if the support of f is contained in $\mathbb{R}^k\times\mathbb{B}^{N-k}$ then

$$\begin{aligned}
\sigma\mu(E_\sigma) &\le \int_{E_\sigma} I_\alpha f\,d\mu \\
&= \int_{\mathbb{R}^N} f(y) I_\alpha\mu_{E_\sigma}(y)\,dy \\
&\lesssim \|f\|_{L^p(w)}\left(\int_{\mathbb{R}^N} W^{\mu_{E_\sigma},w,2}_{\alpha,p}\,d\mu_{E_\sigma}\right)^{\frac{1}{p'}}
\end{aligned}$$

$$\lesssim \|f\|_{L^p(w)} \left(\left(\mu(E_\sigma)\right)^{\frac{1}{p-1}} \left(\frac{\mu(E_\sigma)}{\|\mu\|_d} \right)^{\frac{\alpha p-N+\gamma}{d(p-1)}} \mu(E_\sigma) \right)^{\frac{1}{p'}}$$

$$\lesssim \|f\|_{L^p(w)} \|\mu\|_d^{\frac{N-\alpha p-\gamma}{dp'(p-1)}} \left(\left(\mu(E_\sigma)\right)^{\frac{\alpha p-N+d+\gamma}{d(p-1)}+1} \right)^{\frac{1}{p'}},$$

and hence

$$\mu(E_\sigma) \lesssim \|\mu\|_d \left(\frac{\|f\|_{L^p(w)}}{\sigma} \right)^{\frac{dp}{N-\alpha p-\gamma}}.$$

Furthermore, an application of the layer cake representation (cf. [14, Theorem 1.13]) yields

$$\begin{aligned} \|I_\alpha f\|_{L^q(\mu)}^q &= \left(\int_0^\eta + \int_\eta^\infty \right) \mu\left(\left\{ y \in \mathbb{R}^N : I_\alpha f(y) > t \right\} \right) dt \\ &\lesssim \|\mu\| \eta^q + \|\mu\|_d \int_\eta^\infty \left(\frac{\|f\|_{L^p(w)}}{\sigma} \right)^{\frac{dp}{N-\alpha p-\gamma}} d\sigma^q \\ &\lesssim \|\mu\| \eta^q + \|\mu\|_d \|f\|_{L^p(w)}^{\frac{dp}{N-\alpha p-\gamma}} \int_\eta^\infty t^{q-\frac{dp}{N-\alpha p-\gamma}} \frac{dt}{t} \\ &\approx \|\mu\| \eta^q + \|\mu\|_d \|f\|_{L^p(w)}^{\frac{dp}{N-\alpha p-\gamma}} \eta^{q-\frac{dp}{N-\alpha p-\gamma}} \end{aligned}$$

under

$$\eta > 0, \quad q < \frac{dp}{N-\alpha p-\gamma} = \frac{dp}{\lambda-\alpha p+\varepsilon} < \frac{dp}{\lambda-\alpha p} \quad \text{and} \quad \alpha p \le \lambda < N.$$

Upon taking

$$\eta = \|f\|_{L^p(w)} \left(\frac{\|\mu\|_d}{\|\mu\|} \right)^{\frac{N-\alpha p-\gamma}{dp}},$$

we get

$$\|I_\alpha f\|_{L^q(\mu)}^q \lesssim \|f\|_{L^p(w)}^q \|\mu\| \left(\frac{\|\mu\|_d}{\|\mu\|} \right)^{\frac{q(N-\alpha p-\gamma)}{dp}}.$$

It remains to control $\|f\|_{L^p(w)}$ from above. To do so, writing

$$\left(g(\hat{y})\right)^p = \int_{\mathbb{R}^k} \left(f(\bar{y},\hat{y})\right)^p d\bar{y} \quad \forall \hat{y} \in \mathbb{B}^{N-k},$$

we find

$$\|f\|_{L^p(w)}^p = \int_{\mathbb{R}^k} \int_{\mathbb{B}^{N-k}} \left(f(\bar{y},\hat{y})\right)^p |\hat{y}|^{-\gamma} \, d\bar{y} d\hat{y}.$$

Now, if $k<\lambda<N$ and $E\subset\mathbb{R}^{N-k}$ is covered by a sequence of $(N-k)$-dimensional balls $\widehat{B}_j$ with radius r_j then

$$\begin{aligned}\int_E \big(g(\hat{y})\big)^p\,d\hat{y} &\le \sum_j \int_{\widehat{B}_j}\big(g(\hat{y})\big)^p\,d\hat{y}\\ &=\sum_j r_j^{N-\lambda}\left(r_j^{\lambda-N}\int_{\widehat{B}_j}\big(g(\hat{y})\big)^p\,d\hat{y}\right)\\ &\lesssim \|g\|^p_{L^{p,\lambda-k}(\mathbb{B}^{N-k})}\sum_j r_j^{N-\lambda}.\end{aligned}$$

Via taking the infimum in the last inequality over all such coverings $\cup_j\widehat{B}_j\supset E$, we obtain

$$\int_E \big(g(\hat{y})\big)^p\,d\hat{y}\lesssim \|g\|^p_{L^{p,\lambda-k}(\mathbb{B}^{N-k})}\Lambda^{(\infty)}_{N-\lambda}(E).$$

Upon choosing

$$E=\big\{\hat{y}\in\mathbb{B}^{N-k}:\,|\hat{y}|^{-\gamma}>t\big\}$$

and integrating the last inequality over $t\in[0,\infty)$, we get that if

$$k<\lambda<N\quad\text{and}\quad 0<\gamma=N-\lambda-\varepsilon<N-\lambda$$

then

$$\begin{aligned}\|f\|^p_{L^p(w)} &\lesssim \left(\int_{\mathbb{R}^k}\|f(\bar{y},\cdot)\|^p_{L^{p,\lambda-k}(\mathbb{B}^{N-k})}\,d\bar{y}\right)\int_0^\infty \Lambda^{(\infty)}_{N-\lambda}\big(\{\hat{y}\in\mathbb{B}^{N-k}:\,|\hat{y}|^{-\gamma}>t\}\big)\,dt\\ &\lesssim \left(\int_{\mathbb{R}^k}\|f(\bar{y},\cdot)\|^p_{L^{p,\lambda-k}(\mathbb{B}^{N-k})}\,d\bar{y}\right)\\ &\quad\times\left(1+\int_1^\infty \Lambda^{(\infty)}_{N-\lambda}\big(\{\hat{y}\in\mathbb{B}^{N-k}:\,|\hat{y}|<t^{-\frac{1}{\gamma}}\}\big)\,dt\right)\\ &\lesssim \left(\int_{\mathbb{R}^k}\|f(\bar{y},\cdot)\|^p_{L^{p,\lambda-k}(\mathbb{B}^{N-k})}\,d\bar{y}\right)\left(1+\int_1^\infty t^{-\frac{N-\lambda}{\gamma}}\,dt\right)\\ &\lesssim \int_{\mathbb{R}^k}\|f(\bar{y},\cdot)\|^p_{L^{p,\lambda-k}(\mathbb{B}^{N-k})}\,d\bar{y}.\end{aligned}$$

This actually implies the desired estimate for $\|f\|_{L^p(w)}$. $\square$

3. Restrictions of Riesz–Morrey potentials on bounded domains

3.1. Restriction under $q\in\{1,\infty\}$

The following $L^1(\mu)$ and L^∞ estimates for $I_\alpha f$ with $f\in L^{p,\lambda}$ on a bounded domain Ω of $\mathbb{R}^N$ form two supporting points to discover $L^q(\mu)$-estimate.

Theorem 3.1. *Suppose μ and f are a non-negative Radon measure and a non-negative $L^{p,\lambda}$ function with support in a bounded domain $\Omega\subset\mathbb{R}^N$ respectively.*

(i) *If*

$$1\le p<\infty,\quad 0<N-\alpha p<d\le N \quad \textit{and} \quad \|\mu\|_d<\infty,$$

then

$$\int_\Omega |I_\alpha f|\,d\mu\le c_1(p,N,d,\alpha)\omega_N^{1-\frac{1}{p}}\big(\mathrm{diam}(\Omega)\big)^{d+\alpha-\frac{\lambda}{p}}\|\mu\|_d\|f\|_{L^{p,\lambda}},$$

where

$$c_1(p,N,d,\alpha)=\begin{cases}\frac{d}{d+\alpha-N} & \textit{as } p=1;\\ (\frac{N-\alpha p}{pd-N+\alpha p})^{\frac{N-\alpha p}{pd}}(\frac{2(N-\alpha)pd}{(N-\alpha p)(d-N+\alpha p)}) & \textit{as } p>1.\end{cases}$$

(ii) *If*

$$1\le p<\infty \quad \textit{and} \quad 0<\lambda<\alpha p,$$

then

$$\sup_{x\in\Omega} I_\alpha f(x)\le c_\infty(p,N,\lambda,\alpha)\omega_N^{1-\frac{1}{p}}\big(\mathrm{diam}(\Omega)\big)^{\alpha-\frac{\lambda}{p}}\|f\|_{L^{p,\lambda}},$$

where

$$c_\infty(p,N,\lambda,\alpha)=2^{N-\alpha}\left(\frac{p(N+\alpha)-(\lambda+\alpha)}{p\alpha-\lambda}\right).$$

Proof. (i) If $N/\alpha>p=1$ and $N\ge d>N-\alpha$, then according to [5, Lemma 3.1.1] and Fubini's theorem we have

$$\begin{aligned}\|I_\alpha f\|_{L^1(\mu)}&=\int_\Omega\left(\int_\Omega\frac{d\mu(x)}{|x-y|^{N-\alpha}}\right)f(y)\,dy\\ &\le\int_\Omega\left(\int_{B(y,\mathrm{diam}(\Omega))}\frac{d\mu(x)}{|x-y|^{N-\alpha}}\right)f(y)\,dy\\ &\le\int_\Omega\Bigg((N-\alpha)\int_0^{\mathrm{diam}(\Omega)}\left(\frac{\mu(B(y,r))}{r^{N-\alpha}}\right)\frac{dr}{r}\\ &\quad+\big(\mathrm{diam}(\Omega)\big)^{\alpha-N}\mu\big(B\big(y,\mathrm{diam}(\Omega)\big)\big)\Bigg)f(y)\,dy\end{aligned}$$

$$\leq \int_\Omega \|\mu\|_d\Big((N-\alpha)\int_0^{\mathrm{diam}(\Omega)} r^{d+\alpha-N}\,\frac{dr}{r}+\big(\mathrm{diam}(\Omega)\big)^{d+\alpha-N}\Big) f(y)\,dy$$
$$\leq \Big(\frac{d}{d+\alpha-N}\Big)\big(\mathrm{diam}(\Omega)\big)^{d+\alpha-\lambda}\|\mu\|_d\|f\|_{L^{1,\lambda}}.$$

Furthermore, if $N/\alpha>p>1$ and $N\geq d>N-\alpha p$, then Hölder's inequality and [1, Theorem 2] are utilized to deduce

$$\begin{aligned}\|I_\alpha f\|_{L^1(\mu)} &\leq \|\mu\|^{1-\frac{N-\alpha p}{dp}}\|I_\alpha f\|_{L^{\frac{dp}{N-\alpha p}}(\mu)}\\ &\lesssim \|\mu\|^{1-\frac{N-\alpha p}{dp}}\|\mu\|_d^{\frac{N-\alpha p}{dp}}\|f\|_{L^p}\\ &\lesssim \|\mu\|^{1-\frac{N-\alpha p}{dp}}\big(\mathrm{diam}(\Omega)\big)^{\frac{N-\lambda}{p}}\|\mu\|_d^{\frac{N-\alpha p}{dp}}\|f\|_{L^{p,\lambda}}\\ &\lesssim \big(\mathrm{diam}(\Omega)\big)^{d+\alpha-\frac{\lambda}{p}}\|\mu\|_d\|f\|_{L^{p,\lambda}}.\end{aligned}$$

Nevertheless, the problem is that the last estimate produces only a rough constant. In order to overcome such a problem, we employ [16, p. 55, (1.89)] to achieve the weak-type inequality:

$$\mu\big(\{y\in\Omega:\ I_\alpha f(y)>t\}\big)\leq \|\mu\|_d t^{-\frac{pd}{N-\alpha p}}\bigg(\frac{(N-\alpha)pd\omega_N^{1-\frac{1}{p}}\|f\|_{L^p}}{(N-\alpha p)(d-N+\alpha p)}\bigg)^{\frac{pd}{N-\alpha p}}\quad \forall t>0.$$

This, along with the above-mentioned layer cake representation, the definition of $\|f\|_{L^{p,\lambda}}$ and the condition

$$d>N-\alpha p>(N-\alpha p)/p\quad\text{and}\quad a>0,$$

derives

$$\begin{aligned}\|I_\alpha f\|_{L^1(\mu)} &= \int_0^\infty \mu\big(\{y\in\Omega:\ I_\alpha f(y)>t\}\big)\,dt\\ &= \Big(\int_0^a+\int_a^\infty\Big)\mu\big(\{y\in\Omega:\ I_\alpha f(y)>t\}\big)\,dt\\ &\leq \|\mu\|a+\|\mu\|_d\bigg(\frac{(N-\alpha)pd\omega_N^{1-\frac{1}{p}}\|f\|_{L^p}}{(N-\alpha p)(d-N+\alpha p)}\bigg)^{\frac{pd}{N-\alpha p}}\int_a^\infty t^{-\frac{pd}{N-\alpha p}}\,dt\\ &\leq \|\mu\|a+\Big(\frac{N-\alpha p}{pd-N+\alpha p}\Big)\|\mu\|_d a^{\frac{N-\alpha p-pd}{N-\alpha p}}\bigg(\frac{(N-\alpha)pd\omega_N^{1-\frac{1}{p}}}{(N-\alpha p)(d-N+\alpha p)}\bigg)^{\frac{pd}{N-\alpha p}}\\ &\quad\times\|f\|_{L^{p,\lambda}}^{\frac{pd}{N-\alpha p}}\big(\mathrm{diam}(\Omega)\big)^{\frac{d(N-\lambda)}{N-\alpha p}}.\end{aligned}$$

Upon choosing

$$a=\left(\left(\frac{N-\alpha p}{pd-N+\alpha p}\right)\frac{\|\mu\|_d}{\|\mu\|}\right)^{\frac{N-\alpha p}{pd}}\left(\frac{(N-\alpha)pd\omega_N^{1-\frac{1}{p}}}{(N-\alpha p)(d-N+\alpha p)}\right)\big(\mathrm{diam}(\Omega)\big)^{\frac{N-\lambda}{p}}\|f\|_{L^{p,\lambda}}$$

and using

$$\|\mu\|\le\big(\mathrm{diam}(\Omega)\big)^d\|\mu\|_d,$$

we obtain

$$\begin{aligned}\|I_\alpha f\|_{L^1(\mu)}&\le 2a\|\mu\|\\&\le\left(\frac{N-\alpha p}{pd-N+\alpha p}\right)^{\frac{N-\alpha p}{pd}}\left(\frac{2(N-\alpha)pd\omega_N^{1-\frac{1}{p}}}{(N-\alpha p)(d-N+\alpha p)}\right)\\&\quad\times\big(\mathrm{diam}(\Omega)\big)^{d+\alpha-\frac{\lambda}{p}}\|\mu\|_d\|f\|_{L^{p,\lambda}}.\end{aligned}$$

(ii) Although the argument for [18, Lemma 2] can be modified to establish an inequality for $\sup_{x\in\Omega} I_\alpha f(x)$ with a precise constant, we here offer a different approach. According to [5, Lemma 3.1.1] we have that if $x\in\Omega$ then $\Omega\subseteq B(x,\mathrm{diam}(\Omega))$ and hence

$$\begin{aligned}I_\alpha f(x)&\le\int_{|x-y|<\mathrm{diam}(\Omega)} f(y)|y-x|^{\alpha-N}\,dy\\&\le\int_0^{\mathrm{diam}(\Omega)}(N-\alpha)\left(\int_{B(x,t)}f(y)\,dy\right)t^{\alpha-N}\,\frac{dt}{t}\\&\quad+\big(\mathrm{diam}(\Omega)\big)^{\alpha-N}\int_{B(x,\mathrm{diam}(\Omega))}f(y)\,dy\\&=:(N-\alpha)\mathsf{X}+\mathsf{Y}.\end{aligned}$$

For $(N-\alpha)\mathsf{X}$, using the condition $f\in L^{p,\lambda}$ with support in Ω and the Hölder inequality we get

$$\begin{aligned}&\int_0^{2^{-1}\mathrm{diam}(\Omega)}\left(\int_{B(x,t)}f(y)\,dy\right)t^{\alpha-N}\,\frac{dt}{t}\\&\qquad\le\omega_N^{1-\frac{1}{p}}\|f\|_{L^{p,\lambda}}\int_0^{2^{-1}\mathrm{diam}(\Omega)}t^{\alpha-\frac{\lambda}{p}}\,\frac{dt}{t}\\&\qquad=\omega_N^{1-\frac{1}{p}}2^{\frac{\lambda}{p}-\alpha}\left(\frac{p}{\alpha p-\lambda}\right)\big(\mathrm{diam}(\Omega)\big)^{\alpha-\frac{\lambda}{p}}\|f\|_{L^{p,\lambda}}\end{aligned}$$

and

$$\int_{2^{-1}\mathrm{diam}(\Omega)}^{\mathrm{diam}(\Omega)} \Big(\int_{B(x,t)} f(y)\,dy \Big) t^{\alpha-N} \frac{dt}{t}$$
$$\le \|f\|_{L^{p,\lambda}} \big(\mathrm{diam}(\Omega)\big)^{N-\frac{\lambda}{p}} \omega_N^{1-\frac{1}{p}} \int_{2^{-1}\mathrm{diam}(\Omega)}^{\mathrm{diam}(\Omega)} t^{\alpha-N} \frac{dt}{t}$$
$$= \big(\mathrm{diam}(\Omega)\big)^{\alpha-\frac{\lambda}{p}} \omega_N^{1-\frac{1}{p}} \left(\frac{2^{N-\alpha}}{N-\alpha}\right) \|f\|_{L^{p,\lambda}}.$$

Putting the above two estimates together yields

$$(N-\alpha)\mathsf{X} \le 2^{N-\alpha}\left(\frac{pN-\lambda}{p\alpha-\lambda}\right)\omega_N^{1-\frac{1}{p}}\big(\mathrm{diam}(\Omega)\big)^{\alpha-\frac{\lambda}{p}}\|f\|_{L^{p,\lambda}}.$$

For Y, we still utilize the Hölder inequality and the assumption on $f\in L^{p,\lambda}$ to obtain

$$\mathsf{Y} = \big(\mathrm{diam}(\Omega)\big)^{\alpha-N}\int_{B(x,\mathrm{diam}(\Omega))} f(y)\,dy \le \omega_N^{1-\frac{1}{p}}\big(\mathrm{diam}(\Omega)\big)^{\alpha-\frac{\lambda}{p}}\|f\|_{L^{p,\lambda}}.$$

Now, the inequalities for $(N-\alpha)\mathsf{X}$ and Y are placed together to derive

$$I_\alpha f(x) \le 2^{N-\alpha}\left(\frac{p(N+\alpha)-(\lambda+\alpha)}{p\alpha-\lambda}\right)\omega_N^{1-\frac{1}{p}}\big(\mathrm{diam}(\Omega)\big)^{\alpha-\frac{\lambda}{p}}\|f\|_{L^{p,\lambda}}. \qquad \square$$

3.2. Restriction under $q\in(1,\infty)$

Theorem 3.2. *Suppose μ and f are a non-negative Radon measure and a non-negative function with support in a bounded domain $\Omega\subset\mathbb{R}^N$ respectively. Let*

$$\begin{cases} 1\le p<\infty; \\ 0\le N-\alpha p<d\le N; \\ \|\mu\|_d<\infty; \\ 0<\|f\|_{L^{p,\lambda}}<\infty. \end{cases}$$

(i) *If*

$$N\ge\lambda>\alpha p \quad \textit{and} \quad 1<q<\frac{p(d+\lambda-N)}{\lambda-\alpha p},$$

then

$$\int_\Omega \left(\frac{I_\alpha f}{\|f\|_{L^{p,\lambda}}}\right)^q d\mu \lesssim \|\mu\|_d.$$

(ii) *If*

$$N\ge\lambda>\alpha p \quad \textit{and} \quad 1<\gamma\le q=\frac{p(d+\lambda-N)}{\lambda-p\alpha},$$

then

$$\int_\Omega \left(\frac{I_\alpha f}{\|f\|_{L^{p,\lambda}}}\right)^q \left(\ln\left(1+\frac{I_\alpha f}{\|f\|_{L^{p,\lambda}}}\right)\right)^{-\gamma} d\mu \lesssim \|\mu\|_d.$$

(iii) *If*

$$N \ge \lambda = \alpha p \quad \textit{and} \quad 0 \le \beta \le 1,$$

then there is a constant $\varkappa_0 > 0$ *such that*

$$\int_\Omega \exp\left(\frac{\varkappa I_\alpha f}{\|f\|_{L^{p,\lambda}}}\right)^\beta d\mu \lesssim \|\mu\|_d \quad \forall \varkappa \in (0, \varkappa_0).$$

Proof. (i)–(ii) According to [2, Theorem 5.1], under the conditions on p, α, d, μ we have the following weak-type estimate:

$$\mu\big(\{y \in B(x,r) \cap \Omega:\ I_\alpha f(y) > t\}\big)$$
$$\lesssim \|\mu\|_d \big(\|f\|_{L^{p,\lambda}} t^{-1}\big)^{\frac{(d+\lambda-N)p}{\lambda-\alpha p}} r^{N-\lambda} \quad \forall (x,r,t) \in \mathbb{R}^N \times (0,\infty) \times (0,\infty),$$

where

$$N - \alpha p < d \le N \quad \text{and} \quad \alpha p < \lambda \le N \quad \Longrightarrow \quad d + \lambda - N > 0.$$

Selecting $x \in \Omega$ and $r = \mathrm{diam}(\Omega)$ in the above estimate, we find the following weak type inequality:

$$\mu\big(\{y \in \Omega:\ I_\alpha f(y) > t\}\big)$$
$$\lesssim \|\mu\|_d \big(\|f\|_{L^{p,\lambda}} t^{-1}\big)^{\frac{(d+\lambda-N)p}{\lambda-\alpha p}} \big(\mathrm{diam}(\Omega)\big)^{N-\lambda} \quad \text{under } 0 \le N - d < \alpha p < \lambda \le N.$$

If $q < p(d+\lambda-N)/(\lambda - p\alpha)$, then an application of the last inequality gives that if $a > 0$ then

$$\begin{aligned}
\int_\Omega \left(\frac{I_\alpha f}{\|f\|_{L^{p,\lambda}}}\right)^q d\mu &= \int_0^\infty \mu\left(\left\{y \in \Omega:\ \frac{I_\alpha f(y)}{\|f\|_{L^{p,\lambda}}} > t\right\}\right) dt^q \\
&\lesssim \int_0^a \mu\left(\left\{y \in \Omega:\ \frac{I_\alpha f(y)}{\|f\|_{L^{p,\lambda}}} > t\right\}\right) dt^q \\
&\quad + \|\mu\|_d \big(\mathrm{diam}(\Omega)\big)^{N-\lambda} \int_a^\infty t^{q - \frac{(d+\lambda-N)p}{\lambda-\alpha p}} \frac{dt}{t} \\
&\lesssim |\!|\!|\mu|\!|\!| a^q + \|\mu\|_d \big(\mathrm{diam}(\Omega)\big)^{N-\lambda} a^{q - \frac{(d+\lambda-N)p}{\lambda-\alpha p}}.
\end{aligned}$$

Via minimizing the last summation, we take

$$a = \Big(|\!|\!|\mu|\!|\!|^{-1} \|\mu\|_d \big(\mathrm{diam}(\Omega)\big)^{N-\lambda}\Big)^{\frac{\lambda-\alpha p}{p(d+\lambda-N)}}$$

and then employ

$$\|\mu\| \le \big(\mathrm{diam}(\Omega)\big)^d \|\mu\|_d \quad \text{and} \quad \frac{q}{p} < \frac{d+\lambda-N}{\lambda-\alpha p}$$

to gain

$$\int_\Omega \left(\frac{I_\alpha f}{\|f\|_{L^{p,\lambda}}}\right)^q d\mu \lesssim \|\mu\| a^q \lesssim \big(\mathrm{diam}(\Omega)\big)^{d-p^{-1}q(\lambda-\alpha p)}\|\mu\|_d.$$

If $q=p(d+\lambda-N)/(\lambda-p\alpha)$ and $1<\gamma\le q$, then using the basic fact that

$$t \longmapsto \phi(t)=t^q\big(\ln(1+t)\big)^{-\gamma}$$

increases on $[0,\infty)$, the foregoing layer cake representation and the weak type inequality for μ, one gets

$$\begin{aligned}
&\int_\Omega \left(\frac{I_\alpha f}{\|f\|_{L^{p,\lambda}}}\right)^q \left(\ln\left(1+\frac{I_\alpha f}{\|f\|_{L^{p,\lambda}}}\right)\right)^{-\gamma} d\mu \\
&\qquad = \int_0^\infty \mu\left(\left\{y\in\Omega:\ \frac{I_\alpha f(y)}{\|f\|_{L^{p,\lambda}}}>t\right\}\right) d\phi(t)\\
&\qquad \le \int_0^1 \|\mu\|\, d\phi(t) + \int_1^\infty \mu\left(\left\{y\in\Omega:\ \frac{I_\alpha f(y)}{\|f\|_{L^{p,\lambda}}}>t\right\}\right) d\phi(t)\\
&\qquad \lesssim \|\mu\| + \|\mu\|_d^{\frac{(d+\lambda-N)p}{\lambda-\alpha p}}\big(\mathrm{diam}(\Omega)\big)^{N-\lambda}\int_1^\infty t^{-\frac{(d+\lambda-N)p}{\lambda-\alpha p}}\, d\phi(t)\\
&\qquad \lesssim \|\mu\| + \|\mu\|_d^{\frac{(d+\lambda-N)p}{\lambda-\alpha p}}\big(\mathrm{diam}(\Omega)\big)^{N-\lambda}\int_1^\infty \phi(t)t^{-\frac{(d+\lambda-N)p}{\lambda-\alpha p}}\,\frac{dt}{t}\\
&\qquad \lesssim \|\mu\| + \|\mu\|_d^{\frac{(d+\lambda-N)p}{\lambda-\alpha p}}\big(\mathrm{diam}(\Omega)\big)^{N-\lambda}\int_1^\infty \big(\ln(1+t)\big)^{-\gamma}\,\frac{dt}{t}\\
&\qquad \lesssim \|\mu\|_d\Big(\big(\mathrm{diam}(\Omega)\big)^d + \|\mu\|_d^{\frac{(d+\lambda-N)p}{\lambda-\alpha p}-1}\big(\mathrm{diam}(\Omega)\big)^{N-\lambda}\Big).
\end{aligned}$$

(iii) First of all, given $N\ge\lambda=\alpha p$ and for a sufficiently large natural number q let us choose three positive constants: $\varepsilon;\alpha_1;\alpha_2$ such that

$$\begin{cases}
\varepsilon=\dfrac{d+\alpha p-N}{pq}<d;\\
\alpha_1=\dfrac{N-d}{p}+\varepsilon<N;\\
\alpha_2=\alpha+\varepsilon<N;\\
N-\alpha=\dfrac{N-\alpha_1}{q}+\dfrac{(q-1)(N-\alpha_2)}{q}.
\end{cases}$$

An application of the Hölder inequality yields

$$
\begin{aligned}
I_\alpha f(x) &= \int_\Omega \frac{f(y)^{\frac{1}{q}} f(y)^{\frac{q-1}{q}}}{|y-x|^{\frac{N-\alpha_1}{q}} |y-x|^{\frac{(q-1)(N-\alpha_2)}{q}}}\,dy \\
&\le \left(\int_\Omega \frac{f(y)}{|y-x|^{N-\alpha_1}}\,dy\right)^{\frac{1}{q}} \left(\int_\Omega \frac{f(y)}{|y-x|^{N-\alpha_2}}\,dy\right)^{\frac{q-1}{q}}.
\end{aligned}
$$

This, along with Theorem 3.1(i) for $I_{\alpha_1} f$ and Theorem 3.1(ii) for $I_{\alpha_2} f$, implies

$$
\begin{aligned}
\|I_\alpha f\|^q_{L^q(\mu)} &\le \|I_{\alpha_1} f\|_{L^1(\mu)} \Big(\sup_{x\in\Omega} I_{\alpha_2} f(x)\Big)^{q-1} \\
&\le c_1(p,N,d,\alpha_1)\big(c_\infty(p,N,\lambda,\alpha_2)\big)^{q-1} \omega_N^{q(1-\frac{1}{p})} \\
&\quad \times \big(\mathrm{diam}(\Omega)\big)^{d+\alpha_1-\frac{1}{p}+(\alpha_2-\frac{\lambda}{p})(q-1)} \|\mu\|_d \|f\|^q_{L^{p,\lambda}} \\
&= c_1(p,N,d,\alpha_1)\big(c_\infty(p,N,\lambda,\alpha_2)\big)^{q-1} \omega_N^{q(1-\frac{1}{p})} \big(\mathrm{diam}(\Omega)\big)^d \|\mu\|_d \|f\|^q_{L^{p,\lambda}}.
\end{aligned}
$$

Next, if $p=1$, then a simple calculation with $\lambda=p\alpha=\alpha$ gives

$$
\begin{cases}
c_1(p,N,d,\alpha_1) \le \varepsilon^{-1} d; \\
c_\infty(p,N,\lambda,\alpha_2) \le \varepsilon^{-1} 2^N (N-\alpha),
\end{cases}
$$

and hence

$$
c_1(p,N,d,\alpha_1)\big(c_\infty(p,N,\lambda,\alpha_2)\big)^{q-1} \le \left(\frac{d2^N}{N-\alpha}\right)\left(\frac{p(N-\alpha)q}{d+\alpha-N}\right)^q.
$$

Moreover, if $p>1$, then a slightly long (but elementary) estimation with

$$
\lambda = p\alpha, \quad N-\alpha p < d \quad \text{and} \quad q/(q-1) \le 2
$$

yields

$$
\begin{cases}
c_1(p,N,d,\alpha_1) \le 2\varepsilon^{-1}\Big((p-1)^{-\frac{d+N-\alpha p}{pd}} + (p-1)^{-\frac{1}{2p}}\Big); \\
c_\infty(p,N,\lambda,\alpha_2) \le \varepsilon^{-1} N 2^{N-\alpha+1},
\end{cases}
$$

and hence

$$
\begin{aligned}
&c_1(p,N,d,\alpha_1)\big(c_\infty(p,N,\lambda,\alpha_2)\big)^{q-1} \\
&\qquad\qquad \le N^{-1} 2^{1-N+\alpha}\Big((p-1)^{-\frac{d+N-\alpha p}{pd}} + (p-1)^{-\frac{1}{2p}}\Big)\left(\frac{2^{N-\alpha+1} Npq}{d+\alpha p-N}\right)^q.
\end{aligned}
$$

Putting the previous two cases together deduces

$$c_1(p,N,d,\alpha_1)\big(c_\infty(p,N,\lambda,\alpha_2)\big)^{q-1}\le \mathsf{c}(\tilde{\mathsf{c}}q)^q,$$

where

$$\begin{cases}\mathsf{c}:=\begin{cases}\frac{d2^N}{N-\alpha} & \text{as } p=1;\\ N^{-1}2^{1-N+\alpha}((p-1)^{-\frac{d+N-\alpha p}{pd}}+(p-1)^{-\frac{1}{2p}}) & \text{as } p>1,\end{cases}\\ \tilde{\mathsf{c}}:=\begin{cases}(\frac{d2^N}{N-\alpha})(\frac{p(N-\alpha)}{d+\alpha-N}) & \text{as } p=1;\\ \frac{2^{N-\alpha+1}Np}{d+\alpha p-N} & \text{as } p>1.\end{cases}\end{cases}$$

Consequently,

$$\|I_\alpha f\|^q_{L^q(\mu)}\le \mathsf{c}\big(\mathrm{diam}(\Omega)\big)^d\big(\omega_N^{1-\frac{1}{p}}\tilde{\mathsf{c}}q\big)^q\|\mu\|_d\|f\|^q_{L^{p,\lambda}}.$$

Finally, let us verify the desired inequality. Of course, it is enough to handle the situation under $\beta\in(0,1]$. Assuming $\varkappa>0$ and using the Hölder inequality with $0<\beta\le 1$ we obtain

$$\begin{aligned}\int_\Omega \exp\left(\frac{\varkappa I_\alpha f}{\|f\|_{L^{p,\lambda}}}\right)^\beta d\mu &= \sum_{j=0}^\infty \frac{1}{j!}\left(\frac{\varkappa\|I_\alpha f\|_{L^{j\beta}(\mu)}}{\|f\|_{L^{p,\lambda}}}\right)^{j\beta}\\ &\le |||\mu|||+|||\mu|||^{1-\beta}\left(\sum_{j=1}^{q-1}+\sum_{j=q}^\infty\right)\frac{1}{j!}\left(\int_\Omega\left(\frac{\varkappa I_\alpha f}{\|f\|_{L^{p,\lambda}}}\right)^j d\mu\right)^\beta.\end{aligned}$$

If $1\le j<q$, then a combination of the Hölder inequality and the estimate for $\|I_\alpha f\|^q_{L^q(\mu)}$ gives

$$\begin{aligned}|||\mu|||^{1-\beta}\left(\int_\Omega\left(\frac{\varkappa I_\alpha f}{\|f\|_{L^{p,\lambda}}}\right)^j d\mu\right)^\beta &\le \varkappa^{j\beta}\left(\int_\Omega\left(\frac{I_\alpha f}{\|f\|_{L^{p,\lambda}}}\right)^q d\mu\right)^{\frac{j\beta}{q}}|||\mu|||^{1-\frac{j\beta}{q}}\\ &\le \big(\varkappa \mathsf{c}^{\frac{1}{q}}\big(\omega_N^{1-\frac{1}{p}}\tilde{\mathsf{c}}q\big)\big)^{j\beta}\big(\mathrm{diam}(\Omega)\big)^d\|\mu\|_d.\end{aligned}$$

Moreover, if $j\ge q$, then the estimate for $\|I_\alpha f\|^q_{L^q(\mu)}$ also applies to $\|I_\alpha f\|^j_{L^j(\mu)}$, and hence for

$$\varkappa<e^{-\frac{1}{\beta}}\big(\omega_N^{1-\frac{1}{p}}\tilde{\mathsf{c}}\big)^{-1}=:\varkappa_0$$

one has

$$\|\mu\|^{1-\beta}\sum_{j=q}^{\infty}\frac{1}{j!}\left(\int_\Omega\left(\frac{\varkappa I_\alpha f}{\|f\|_{L^{p,\lambda}}}\right)^j d\mu\right)^\beta$$

$$\lesssim \|\mu\|^{1-\beta}\left(\big(\mathrm{diam}(\Omega)\big)^d\|\mu\|_d\right)^\beta\sum_{j=q}^{\infty}\frac{(j(\omega_N^{1-\frac{1}{p}}\tilde{\mathfrak{c}}\varkappa)^\beta)^j}{j!}$$

$$\lesssim \big(\mathrm{diam}(\Omega)\big)^d\|\mu\|_d.$$

Accordingly, the previous treatment ensures

$$\int_\Omega \exp\left(\frac{\varkappa I_\alpha f}{\|f\|_{L^{p,\lambda}}}\right)^\beta d\mu \lesssim \big(\mathrm{diam}(\Omega)\big)^d\|\mu\|_d\ \forall\ \varkappa\in(0,\varkappa_0). \qquad \square$$

Remark 3.3. Theorem 3.2 leads to the following question: if

$$\alpha p<\lambda<N \quad \text{and} \quad \frac{p(d+\lambda-N)}{\lambda-\alpha p}<q\le\frac{pd}{\lambda-\alpha p},$$

are there embeddings of type $(*)$ or $(**)$ (stated in the introduction of this paper) for a d-measure μ? The conjecture now is that there are none.

3.3. Two consequences

As a straightforward application of Theorem 3.2, the forthcoming two theorems characterize the restrictions of a k-th order Sobolev–Morrey function $f\in C_0^\infty(\mathbb{R}^N)$ with $|\nabla^k f|\in L^{p,\lambda}$ being compactly supported in a bounded domain Ω and a Riesz–Morrey potential on $\mathbb{B}^k\subset\mathbb{R}^k\subset\mathbb{R}^N$ respectively. In the above and below, $C_0^\infty(\mathbb{R}^N)$ consists of all infinitely differentiable functions with compact support in $\mathbb{R}^N$ and

$$\nabla^k f=\begin{cases}(-\Delta)^{\frac{k}{2}}f & \text{as } k \text{ is even;}\\ \nabla(-\Delta)^{\frac{k-1}{2}}f & \text{as } k \text{ is odd.}\end{cases}$$

Theorem 3.4. *Given $f\in C_0^\infty(\mathbb{R}^N)$, $N\ge 2$ and $k\in\{1,2,...,N-1\}$, suppose μ and $|\nabla^k f|$ are a non-negative Radon measure and a non-negative function with support in a bounded domain $\Omega\subset\mathbb{R}^N$ respectively. Let*

$$\begin{cases}1\le p<\infty;\\ 0\le N-kp<d\le N;\\ \|\mu\|_d<\infty;\\ 0<\big\||\nabla^k f|\big\|_{L^{p,\lambda}}<\infty.\end{cases}$$

(i) *If*

$$N\ge\lambda>kp\quad\textit{and}\quad 1<q<\frac{p(d+\lambda-N)}{\lambda-kp},$$

then

$$\int_\Omega\left(\frac{|f|}{\|\nabla^k f\|_{L^{p,\lambda}}}\right)^q d\mu\lesssim\|\mu\|_d.$$

(ii) *If*

$$N\ge\lambda>kp\quad\textit{and}\quad 1<\gamma\le q=\frac{p(d+\lambda-N)}{\lambda-kp},$$

then

$$\int_\Omega\left(\frac{|f|}{\|\nabla^k f\|_{L^{p,\lambda}}}\right)^q\left(\ln\left(1+\frac{|f|}{\|\nabla^k f\|_{L^{p,\lambda}}}\right)\right)^{-\gamma} d\mu\lesssim\|\mu\|_d.$$

(iii) *If*

$$N\ge\lambda=kp\quad\textit{and}\quad 0\le\beta\le 1,$$

then there is a constant $c_0>0$ such that

$$\int_\Omega\exp\left(\frac{c|f|}{\|\nabla^k f\|_{L^{p,\lambda}}}\right)^\beta d\mu\lesssim\|\mu\|_d\quad\forall c\in(0,c_0).$$

Proof. Utilizing the following representation (cf. [4, Lemma 2] for general k and [12, Lemma 7.14] for $k=1$) for $f\in C_0^\infty(\mathbb{R}^N)$:

$$f(x)=\varkappa_{k,N}\begin{cases}\int_{\mathbb{R}^N}|x-y|^{k-N}\nabla^k f(y)\,dy & \text{for } k=\text{even};\\ \int_{\mathbb{R}^N}|x-y|^{k-N-1}(x-y)\cdot\nabla^k f(y)\,dy & \text{for } k=\text{odd};\end{cases}$$

with a constant $\varkappa_{k,N}$ depending only on k and N, plus Theorem 3.2, we achieve

$$\left|f(x)\right|\lesssim\left(I_k\left|\nabla^k f\right|\right)(x)$$

whence reaching the desired restriction results. □

Theorem 3.5. *Given $N\ge2$ and $k\in\{1,2,...,N-1\}$, suppose μ and f are a non-negative Radon measure and a non-negative function with support in $\mathbb{B}^k$ and $\mathbb{B}^k\times\mathbb{B}^{N-k}$ respectively. Let*

$$\begin{cases}1\le p<\infty;\\ N<\alpha+k<\lambda+k;\\ 0\le N-\alpha p<d\le k;\\ \|\mu\|_d<\infty;\\ 0<\|f\|_{L^{p,\lambda+k-N}(\mathbb{B}^k)L^p(\mathbb{B}^{N-k})}:=\Big\|\big\|f(\bar y,\hat y)\big\|_{L^p(\hat y\in\mathbb{B}^{N-k})}\Big\|_{L^{p,\lambda+k-N}(\bar y\in\mathbb{B}^k)}<\infty.\end{cases}$$

(i) *If*

$$k\geq k+\lambda-N>(\alpha+k-N)p \quad \text{and} \quad 1<q<\frac{p(d+\lambda-k)}{\lambda-p\alpha+(N-k)(p-1)},$$

then

$$\int_{\mathbb{B}^k}\left(\frac{I_\alpha f(\bar{x},0)}{\|f\|_{L^{p,\lambda+k-N}(\mathbb{B}^k)L^p(\mathbb{B}^{N-k})}}\right)^q d\mu(\bar{x},0)\lesssim\|\mu\|_d$$

(ii) *If*

$$k\geq k+\lambda-N>(\alpha+k-N)p \quad \text{and} \quad 1<\gamma\leq q=\frac{p(d+\lambda-k)}{\lambda-p\alpha+(N-k)(p-1)},$$

then

$$\int_{\mathbb{B}^k}\left(\frac{I_\alpha f(\bar{x},0)}{\|f\|_{L^{p,\lambda+k-N}(\mathbb{B}^k)L^p(\mathbb{B}^{N-k})}}\right)^q\left(\ln\left(1+\frac{I_\alpha f(\bar{x},0)}{\|f\|_{L^{p,\lambda+k-N}(\mathbb{B}^k)L^p(\mathbb{B}^{N-k})}}\right)\right)^{-\gamma} d\mu\lesssim\|\mu\|_d.$$

(iii) *If*

$$k\geq\lambda+k-N=(\alpha+k-N)p \quad \text{and} \quad 0\leq\beta\leq1,$$

then there is a constant $c_0>0$ *such that*

$$\int_{\mathbb{B}^k}\exp\left(\frac{cI_\alpha f(\bar{x},0)}{\|f\|_{L^{p,\lambda+k-N}(\mathbb{B}^k)L^p(\mathbb{B}^{N-k})}}\right)^\beta d\mu(\bar{x},0)\lesssim\|\mu\|_d \quad \forall\ c\in(0,c_0).$$

Proof. Writing

$$\begin{cases} x=(\bar{x},\hat{x})\in\mathbb{R}^k\times\mathbb{R}^{N-k};\\ y=(\bar{y},\hat{y})\in\mathbb{R}^k\times\mathbb{R}^{N-k};\\ g(\bar{y})=\left(\int_{\mathbb{B}^{N-k}}\big(f(\bar{y},\hat{y})\big)^p\,d\hat{y}\right)^{\frac{1}{p}}=\big\|f(\bar{y},\cdot)\big\|_{L^p(\mathbb{B}^{N-k})}, \end{cases}$$

we use the Hölder inequality to get

$$\begin{aligned} I_\alpha f(\bar{x},0)&=\int_{\mathbb{B}^{N-k}}\int_{\mathbb{B}^k}\left(|\bar{x}-\bar{y}|^2+|\hat{y}|^2\right)^{\frac{\alpha-N}{2}}f(\bar{y},\hat{y})\,d\bar{y}\,d\hat{y}\\ &\leq\int_{\mathbb{B}^k}|\bar{x}-\bar{y}|^{\alpha-N}\left(\int_{\mathbb{B}^{N-k}}f(\bar{y},\hat{y})\,d\hat{y}\right)d\bar{y}\\ &\lesssim\int_{\mathbb{B}^k}|\bar{x}-\bar{y}|^{\alpha-N}g(\bar{y})\,d\bar{y}\\ &\lesssim I_{\alpha+k-N}g(\bar{x}), \end{aligned}$$

whence reaching the desired estimates through Theorem 3.2 with $f;N;\alpha;\lambda$ being replaced by $g;k;\alpha+k-N;\lambda+k-N$ respectively. $\square$

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David R. Adams
Department of Mathematics
University of Kentucky
Lexington
KY US-40506-0027
U.S.A.
dave.adams@uky.edu

Jie Xiao
Department of Mathematics and Statistics
Memorial University
St. John's
NL CA-A1C 5S7
Canada
jxiao@mun.ca

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