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The primary process of a smoothing relation

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Chapter I

Auxiliaries

1. 1. Introduction

The object of this thesis is to study the primary process of a smoothing relation between two stationary stochastic processes of discrete as well as continuous parameters. As usually understood, the term smoothing indicates an operation of adjustment or rounding off. In the present study however, it is used in a somewhat different sense. When one desires to have certain data, it sometimes happens that the information is available in the shape of sums of the required data which may have been weighted uniformly or otherwise. For example, A. R. Prest [1] considers taxes as summed data of profits, and G. H. Orcutt mentions (in the discussion on a paper of D. G. Champernowne [1]) how water levels are summed data of the amounts of rainfall. The formation of such sums is commonly described as the construction of moving averages. Though we treat of similar sums in this thesis, still we refrain from using the term moving average in this connection, because in the theory of stochastic processes "moving average" has come to acquire a specific significance, standing as it does, for a linear relationship in which the process subjected to the summation is non-autocorrelated. For the sake of compactness, the term "smoothing" has been chosen to describe the same operation of forming weighted sums of the values of a process, irrespective of the nature of its autocorrelation.

In recent studies in Econometrics simple stochastic models have been considered with a view to explain economic fluctuations. Sometimes it happens that elimination gives rise to a stochastic difference equation for an economic variate such as the price of a commodity.

If the prices p_t at time t in the demand and supply functions are taken to satisfy the relations

$$X_t = \alpha p_t + \beta + u_t \qquad \text{(demand)}$$

and

$$X_t = h_2 p_{t-1} + k_2 + w_t \qquad \text{(supply)}$$

(cf. M. A. GIRSHICK and T. HAAVELMO [1]), then we get an equation of the form

$$p_t + a p_{t-1} = \xi_t.$$

Equations of a similar nature occur or are obtainable at times in connection with other economic models (e.g. TJ. KOOPMANS [1], L. HURWICZ [1], and G. H. ORCUTT and D. COCHRANE [1]. In this connection see also H. B. MANN and A. WALD [1]). When the mean value and the variance of the variable appearing in such an equation can be regarded as more or less constant, we may idealize the equation into a linear relation between two stationary processes. In the case when the parameter t ranges over integral values, the relation is a difference equation and is similar in form to an autoregression. If this relation is written as

$$\sum_{r=0}^{n} a_r X(t-r) = \xi(t),$$

it is called an autoregression if the coefficients (a_r) are suitably restricted and $\xi(t)$ is a non-autocorrelated process. Sometimes in econometric work $\xi(t)$ is termed the "residual" irrespective of the equation being an autoregression or not. Quite frequently the $\xi(t)$-process is autocorrelated for several reasons. For example, it might be that in the construction of the model all the relevant factors have not been taken into account, and hence the residual contains terms other than the purely random (or the shock) terms. Or again, as a consequence of the eliminations carried out, the residual appearing in the stochastic difference equation for the single variable is the sum of a number of residuals, which though purely random in themselves have their sums autocorrelated.

In the light of the foregoing considerations it appears that a study of a smoothing relation between two stationary processes of general nature is of particular interest.

In this thesis it is proposed to consider such a smoothing relation between two stationary stochastic processes. The process which is subjected to the operation of smoothing or summation is called the "primary process", and it is supposed that this process is unknown, except that we have an *a priori* knowledge that it is stationary in the wide sense. The process resulting from the summation is termed the "resulting process", and it is taken to be more or less known. Knowing the details of smoothing, our effort is directed to getting an insight into the primary process in terms of the resulting process.

To achieve our object we have to effect a filtering of the primary process as well as we can. This is done in the two-fold manner of inversion and linear estimation.

The problem of the inversion of a linear relationship between two stationary discrete parameter processes has been solved by H. WOLD [1] in the case when one of the processes involved is non-autocorrelated. In this connection it has been pointed out by WOLD that the position of the roots of the characteristic equation of the smoothing relation with respect to the unit circle is of fundamental importance for the representation and properties of the solution. In the case when the characteristic equation has a root on the unit circle, WOLD [2] has shown that it is possible to find a linear representation of the primary process by making use of the notion of summability.

In the present study we shall be concerned with the inversion problem for a smoothing relation between two stationary processes, neither of which is assumed to be non-autocorrelated. Both the discrete and the continuous parameter cases will be considered. It is of interest to mention here that, in the discrete parameter case, the roots of the characteristic equation play a similar rôle as in the cases treated by WOLD. Accordingly, our results in the discrete case will be closely connected with the results previously obtained by WOLD. In particular, when there are roots on the unit circle, we shall have recourse to methods of summability in norm.

In the continuous parameter case, the roots of the characteristic equation of modulus unity will be replaced by the set of real zeros of the Fourier transform of the weight function connected with the smoothing relation. It will be shown that the process obtained by inversion can, in many important cases, be represented as the limit in the mean of a sequence of processes, each of which is a smoothing of the resulting process.

The condition of the identity between the entire closed Hilbert spaces generated by the two processes occurring in the given smoothing relation will be met with very frequently in the sequel. This is but natural, since we are throughout dealing with inversion by means of linear methods, thus trying to find linear representations of one of the processes concerned in terms of the other.

When the available information of the resulting process is incomplete (e.g. only the past values are known), we have to construct an estimate of the primary process at the time instant t or at a later instant in terms of the known values. We are thus led on to the problem of estimation and prediction. In this connection we shall see how our problem is related to the filtering and prediction of N. WIENER [1]. Also in the case of the discrete parameter we shall see that the spectral method can be used to solve the estimation of R. FRISCH [1] in the case of a moving average.

When we desire to form linear, unbiassed and minimum variance estimates of the mean values of the primary process, with a knowledge of the values of the other process, it is seen that this problem is essentially the same as forming such estimates of the mean value of the resulting process in terms of its own values, a topic which has been studied by U. GRENANDER [1].

In connection with a smoothing relation between two stationary processes, we examine the extent to which the resulting process shares the nature of the primary process in respect of metric transitivity, Markoff nature, and stochastic periodic terms. This has been considered in chapter IV.

Derived processes and non-stationary processes of bounded norm are touched upon towards the end.

The method of analysis employed in this thesis is the spectral representation of a wide sense stationary process by H. CRAMÉR [1] and the related theory of the Hilbert space of a process developed by K. KARHUNEN [1].

The following is an outline of the thesis. Chapter I contains an introduction, notation, and the auxiliary notions required for the analysis that is to follow in the subsequent chapters. Chapters II and III deal with inversion in the discrete and the continuous parameter cases respectively. In chapter IV we compare the primary and the resulting processes. Estimation and prediction form the subject matter of chapter V, and in the last chapter we consider some generalizations.

1.2. Notation

The following explains in a general way the notation used in the thesis.

t is a real variable, usually spoken of as time, and ranging over integral values in the case of the discrete parameter and all real values in the continuous parameter case,

$X(t)$: the primary process,

$\xi(t)$: the resulting process,

$Z_X(\lambda)$, $\sigma_X(\lambda)$, M_X, $R_X(t)$ stand respectively for the random spectral function in the spectral representation of $X(t)$, the spectrum of $X(t)$, the mean value of the X-process, and the covariance function of the $X(t)$-process. (Here λ is a real variable ranging over values in W, where W stands for the range $(-\pi, \pi)$ in the discrete parameter case and $(-\infty, \infty)$ in the continuous case). A similar notation denotes the corresponding quantities connected with the ξ-process. However, in the preliminary stage, when we consider only one stationary process, it is needless to show by a suffix the process to which the expressions refer, and as such, the suffix is dropped.

E stands for the operation of taking expectation with respect to the probability measure put upon the space of random functions.

L stands for the smoothing operation, and L^{-1} for the inverse operation, while $L_2(X; -\infty, t)$ stands for the closed Hilbert space of the linear manifold of random variables constituting the process $X(t)$ up to the time instant t. Also we write shortly $L_2(X)$ for $L_2(X; -\infty, \infty)$.

$\{a_r\}$: the sequence of weights used in the smoothing in the discrete case.

$f(u)$: a bounded function of the real variable u such that

$$F(\lambda) = \int_{-\infty}^{\infty} e^{-iu\lambda} f(u)\, du$$

exists and is bounded for all λ. Sometimes $f(u)$ is supposed to belong to the Lebesque class L_1 on $(-\infty, \infty)$. It is used as the weight function in the smoothing in the continuous parameter case.

$$F(\lambda) = \begin{cases} P(e^{-i\lambda}) = \Sigma a_r e^{-ir\lambda} \text{ in the discrete case} \\ \text{and} \\ \int_{-\infty}^{\infty} e^{-iu\lambda} f(u)\, du \text{ in the continuous case.} \end{cases}$$

Q: set of real zeros of $F(\lambda)$.

Though $\Phi(\lambda)$ stands generally for $\int_{-\infty}^{\infty} e^{-iv\lambda}\,\varphi(v)\,dv$, it is not uniformly so. Hence the meaning must be sought in relation to the context. $\varphi(v)$ is supposed to be bounded, and $\Phi(\lambda)$ is supposed to exist and be bounded.

A bar above an expression stands for the complex conjugate.

$X_1(t)$ and $X_2(t)$ are two stationary and mutually orthogonal processes into which $X(t)$ can be decomposed. $X_1(t)$ is uniquely determined, while $X_2(t)$ is more or less arbitrary.

$m_X(Q) = \int_Q d\,\sigma_X(\lambda)$, and $m_\xi(Q)$ has a similar meaning.

Re stands for the "real part of".

1.3. Stochastic processes

Any process that is analyzable in terms of probability distributions in a functional space is generally referred to as a "stochastic process". When we treat of random variables in a finite number of dimensions, we consider a variable point in a Euclidean space R_n with a probability measure defined on it. When we consider an infinite dimensional space, the variable will in the two most important cases have either a denumerable number of coordinates or a continuous infinity of them. In the former case, a point of the space under consideration will be an infinite sequence, whereas in the latter, the points of the space will be functions of a continuous variable. In either case t will stand for the real variable (which will be referred to as the time variable) and its range of variation corresponds to the number of coordinates which a variable point of our space is to have. Thus we regard t as taking an infinity of integral values when the space has a denumerable number of dimensions, and as taking the values of a finite or infinite interval when the dimensionality is the continuous infinity.

From a fundamental theorem of A. Kolmogoroff [1] it follows that a probability measure is uniquely defined on all the Borel sets in the functional space, when the probability of all finite combinations of arbitrarily chosen interval sets is known in a consistent manner. Such a P-measure is entirely adequate in the denumerable case. In the case of the continuous infinity the P-measure defined on the Borel sets of the function space leaves out many interesting probabilities undetermined. In this connection it is proved by J. L. Doob [3] that a P_0-measure can still be defined on a smaller sample space restricted to contain the appropriate functions, provided that the outer measure of the chosen subset of functions is unity (see also H. Cramér [2]).

A stochastic process may then be written as a function of a real variable t and a random variable ω connected with the fundamental probability field with which we are concerned. Usually the variable ω is dropped in writing, and the process is spoken of as a random function $X(t)$. An element of the sample space is a sequence or a function as the case may be, and is called a "realization". Due to the dependence of the process on t and ω, it can be regarded as an ensemble of functions or as a one-parameter family of chance variables. (For an exposition see H. Cramér [2], J. E. Moyal [1], U. Grenander [1]).

1.4 Hilbert space of a process

If $X(t)$ is a stochastic process, we shall consider the linear manifold containing all elements of the form

$$g = \sum_{r=1}^{n} c_r X(t_r),$$

where the c's are complex numbers and n is any arbitrary positive integer. Following CRAMÉR and KARHUNEN, we define the inner product of two elements g_1 and g_2 of this manifold by

$$(g_1, g_2) = \int g_1 \bar{g}_2 \, dP$$

which being the expectation of $g_1 \bar{g}_2$ is denoted by $E(g_1 \bar{g}_2)$. We call $+\sqrt{E(g\bar{g})}$ the norm of the element g and denote it by $\|g\|$. If

$$\lim_{n\to\infty} \|g_n - g\| = 0,$$

we say that the sequence $\{g_n\}$ "converges in the mean" to g as n tends to infinity, and write

$$\underset{n\to\infty}{\text{l.i.m.}}\; g_n = g.$$

We shall sometimes speak of convergence in the mean also as "convergence in norm". The above linear manifold closed with respect to convergence in the mean is termed the "closed Hilbert space" of a process. If only the values of $X(t)$ occurring up to the time instant t are considered, the corresponding closed Hilbert space is denoted by

$$L_2(X; -\infty, t),$$

while $L_2(X)$ will be used to denote the entire closed Hilbert space, the parameter t ranging over all the possible values.

1.5. Random spectral function

Let W be a set of elements (λ), and let $\sigma(s)$ be a measure defined on the subsets (s) of W. Let $Z(s)$ be a random set function defined on the elements of W such that if s_1 and s_2 are two disjoint sets

$$Z(s_1) + Z(s_2) = Z(s_1 + s_2).$$

Without loss of generality we may assume

$$E[Z(s)] = 0.$$

If for any measurable sets s_1 and s_2

$$E[Z(s_1)\overline{Z(s_2)}] = \sigma(s_1 \cdot s_2),$$

such a set function $Z(s)$ is called a "random spectral function". The process $Z(s)$ is also referred to as an orthogonal process. In our later considerations W consists of the interval $(-\pi, \pi)$ or $(-\infty, \infty)$ on the real axis. In such a case we shall denote the random set function corresponding to $(-\pi, \lambda)$ or $(-\infty, \lambda)$ (as the case may be) by $Z(\lambda)$ (see **1.9**).
If for all s in W

$$E\,|Z(s)|^2$$

is bounded, then the process $Z(s)$ is of bounded norm, and $\sigma(W)$ is totally finite. For a process of this type one can define following KARHUNEN [1] the integral

$$\int_W f(\lambda)\, dZ(\lambda)$$

as the limit in the mean of the corresponding Riemann-Stieltjes sums, if $f(\lambda)$ is any complex valued function of the variable λ such that

$$\int_W |f(\lambda)|^2\, d\sigma(\lambda)$$

is bounded. (This definition can be extended to the case where W is the sum of a denumerable number of sets each of finite σ-measure.) The following theorem of KARHUNEN [1] deals with the representation of a process in the form of an integral of the type just mentioned.

Suppose that $X(t)$ is a process with mean value zero and that its covariance function has the representation

$$R(t, u) = E\,(X(t)\,\overline{X(u)}) = \int_W f(t, \lambda)\,\overline{f(u, \lambda)}\, d\sigma(\lambda),$$

then there exists a random spectral function (or an orthogonal process) $Z(s)$ such that the process $X(t)$ has the representation

$$X(t) = \int_W f(t, \lambda)\, dZ(\lambda).$$

1.6. Stationary processes

Suppose that the stochastic process $X(t)$ is such that

$$E\,(X(t)) = M, \text{ a constant,}$$

and

$$E\,(X(t) - M)\,\overline{(X(u) - M)} = R(t, u)$$

is a function of $t - u$ only, say $R(t-u)$. Then $X(t)$ is said to be stationary in the wide sense. As against this we have the strict stationarity of a process, when all the finite dimensional probability distributions are invariant with respect to each translation on the time axis. We shall generally concern ourselves with processes which are stationary in the wide sense. Let $R(t)$ be the covariance function of a (wide sense) stationary process of continuous parameter, and let $R(t)$ be continuous at the origin. Then it is continuous for all t, and $X(t)$ is continuous in the mean. From a theorem of A. KHINTCHINE [1] we have

$$R(t) = \int_{-\infty}^{\infty} e^{it\lambda}\, d\sigma(\lambda),$$

where $\sigma(\lambda)$ is real, bounded, and never decreasing function of λ which is called the "spectrum" of the stationary process $X(t)$. We have

$$\sigma(+\infty) - \sigma(-\infty) = R(0) = \varrho^2.$$

Under these conditions

$$X(t) = \int_{-\infty}^{\infty} e^{it\lambda}\, dZ(\lambda)$$

in accordance with Karhunen's theorem mentioned in **1.5**. This is Cramér's spectral representation of a stationary process (see H. CRAMÉR [1]).

For a stationary process of discrete parameter

$$R(n) = \int_{-\pi}^{\pi} e^{in\lambda}\, d\sigma(\lambda)$$

where the spectrum $\sigma(\lambda)$ is real, bounded, and non-decreasing. Then

$$X(n) = \int_{-\pi}^{\pi} e^{in\lambda}\, dZ(\lambda).$$

1.7. Integration of a process

Let a process $X(t)$ be such that the inner product of $X(t)$ with any given element of its Hilbert space is a Lebesque measurable function of t. Following U. GRENANDER [1], we then call the process as K-measurable. If S is a measurable subset on the real axis, and

$$E(Z\overline{X(t)})$$

for $Z \in L_2(X)$ is Lebesque integrable over S, and if

$$\sup_{Z \in L_2(X)} \frac{1}{\|Z\|} \left| \int_S E(Z\overline{X(t)})\, dt \right| < \infty,$$

it has been shown by KARHUNEN [1] that there exists a unique element I in $L_2(X)$ such that

$$E(Z\bar{I}) = \int_S E(Z\overline{X(t)})\, dt.$$

Then the integral of $X(t)$ over the set S with respect to t is defined by the relation

$$I = \int_S X(t)\,dt.$$

If $g(t)$ is an ordinary function of the real variable t, one can in a similar manner define

$$\int_S g(t)\,X(t)\,dt,$$

provided $g(t)$ satisfies suitable conditions. If further $g(t)$ is continuous, and $X(t)$ is continuous in the mean, this integral will be the same as the integral in the sense of Cramér (see U. GRENANDER [1]) which is defined as the limit in the mean of the corresponding Riemann sums.

We shall be interested in Chapter III and subsequently in integrals of the form

$$\int_{-\infty}^{\infty} X(t-u)\,f(u)\,du = \xi(t),$$

where $X(t)$ is a continuous parameter stationary process which is continuous in the mean, and $f(u)$ is a bounded function such that

$$F(\lambda) = \int_{-\infty}^{\infty} e^{-iu\lambda} f(u)\,du$$

exists and is bounded. (We shall sometimes consider the case where $f(u)$ belongs to Lebesque class L_1 on $(-\infty, \infty)$). For each fixed t for which the above integral $\xi(t)$ exists, we get a chance variable. Thus when $f(u)$ is such that $\xi(t)$ exists for all t, we have the process $\xi(t)$. In this connection, we shall consider the following theorem of KARHUNEN [1].

If the function $\alpha(t, \lambda)$ is measurable on $T \times W$, $t \in T$, $\lambda \in W$, then the random function

$$X(t) = \int_W \alpha(t, \lambda)\,dZ(\lambda)$$

is measurable. T being a measurable subset on the real axis of u, $X(u)$ is integrable with respect to u on T if and only if

$$\int_W \left| \int_T \alpha(u, \lambda)\,du \right|^2 d\sigma(\lambda) \text{ is bounded} \qquad (A).$$

Then

$$\int_T X(u)\,du = \int_W \left(\int_T \alpha(u, \lambda)\,du \right) dZ(\lambda).$$

Reverting to our integral $\xi(t)$, we have

$$f(u)\,X(t-u) = f(u)\int_{-\infty}^{\infty} e^{i(t-u)\lambda}\,dZ(\lambda)$$

$$= \int_{-\infty}^{\infty} f(u)\,e^{i(t-u)\lambda}\,dZ(\lambda),$$

$$= \int_{-\infty}^{\infty} \varphi(u,\lambda)\,dZ(\lambda) \text{ when } t \text{ is fixed.}$$

Also

$$\int_{-\infty}^{\infty}\Big|\int_{-\infty}^{\infty}\varphi(u,\lambda)\,du\Big|^2 d\sigma(\lambda) = \int_{-\infty}^{\infty}\Big|\int_{-\infty}^{\infty} e^{it\lambda}e^{-iu\lambda}f(u)\,du\Big|^2 d\sigma(\lambda) = \int_{-\infty}^{\infty}|F(\lambda)|^2\,d\sigma(\lambda).$$

Since $F(\lambda)$ is bounded and $\sigma(W)$ is totally finite, being the spectrum of a stationary process, we have for the process

$$f(u)\,X(t-u) = \int_{-\infty}^{\infty}\varphi(u,\lambda)\,dZ(\lambda)$$

that the function $\varphi(u,\lambda)$ satisfies the condition (A). Hence the process $f(u)\,X(t-u)$ is integrable with respect to u over $(-\infty,\infty)$. Then

$$\xi(t) = \int_{-\infty}^{\infty} f(u)\,X(t-u)\,du = \int_{-\infty}^{\infty}\Big[\int_{-\infty}^{\infty} e^{it\lambda}e^{-iu\lambda}f(u)\,du\Big]\,dZ(\lambda) = \int_{-\infty}^{\infty} e^{it\lambda}F(\lambda)\,dZ(\lambda).$$

From this relation it follows that the ξ-process is stationary.

1.8. A lemma

If $Z(s)$ is an orthogonal process with the associated measure $\sigma(s)$ on the subsets s of the elements (λ) of W, and if $g_1(\lambda)$ and $g_2(\lambda)$ are complex valued functions of the variable λ such that each of them is quadratically integrable on W with respect to the σ-measure, then we have from KARHUNEN [1] (see his formula (5.13) on page 39) that

$$E\Big[\int_W g_1(\lambda)\,dZ(\lambda)\,\overline{\int_W g_2(\lambda)\,dZ(\lambda)}\Big] = \int_W g_1(\lambda)\,\overline{g_2(\lambda)}\,d\sigma(\lambda).$$

This lemma is still true if the set W is replaced by any subset s of W, and it is of frequent application in our subsequent work.

1.9. An integral transformation

This section is devoted to showing that if $X(t)$ and $\xi(t)$ be two stationary processes continuous in the mean which are related by

$$\int_{-\infty}^{\infty} X(t-u)\, f(u)\, du = \xi(t), \tag{A}$$

and if $f(u)$ is bounded, and

$$\int_{-\infty}^{\infty} e^{-iu\lambda} f(u)\, du = F(\lambda) \text{ exists and is bounded,} \tag{B}$$

then

$$\int_S dZ_\xi(\lambda) = \int_S F(\lambda)\, dZ_X(\lambda)$$

for every Borel set S.

It may be remarked that an exactly analogous relation holds in the case of the discrete parameter processes.

In what follows we shall denote the expressions corresponding to the two processes $X(t)$ and $\xi(t)$ by the suffixes X and ξ respectively.

From the spectral representation of the X-process we get by inversion

$$Z_X(\mu_2) - Z_X(\mu_1) = \underset{T\to\infty}{\text{l.i.m.}} \int_{-T}^{T} \frac{e^{-i\mu_2 t} - e^{-i\mu_1 t}}{-2\pi i t} X(t)\, dt.$$

The right side exists as the limit in the mean (see J. L. DOOB [1]), and the chance variable $Z_X(\mu)$ has zero mean. If (μ_1, μ_2) is a continuity interval of $\sigma_X(\lambda)$,

$$E\,|Z_X(\mu_2) - Z_X(\mu_1)|^2 = \sigma_X(\mu_2) - \sigma_X(\mu_1). \tag{C}$$

Following DOOB we may define

$$Z(\mu) = Z(\mu - 0),$$

and

$$\sigma(\mu) = \sigma(\mu - 0)$$

at a point of discontinuity. Then the relation (C) holds good for all μ_1 and μ_2. From the inversion it follows that for a stationary process $X(t)$

$$L_2(X) = L_2(Z). \tag{D}$$

The results (C) and (D) are true for discrete stationary processes also, and the former can be obtained in a similar manner by using Fourier series instead of Fourier integrals.

Let $y(\mu_1, \mu_2)$ stand for $Z_X(\mu_2) - Z_X(\mu_1)$.
Then

$$E[y(\mu_1, \mu_2) \cdot \overline{Z_\xi(\mu_4) - Z_\xi(\mu_3)}]$$

$$= E\left(\int_{-\infty}^{\infty} \frac{e^{-i\mu_2 t} - e^{-i\mu_1 t}}{-2\pi i t} X(t)\, dt\; \overline{\int_{-\infty}^{\infty} \frac{e^{-i\mu_4 u} - e^{-i\mu_3 u}}{-2\pi i u} \xi(u)\, du}\right)$$

$$= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-i\mu_2 t} - e^{-i\mu_1 t}}{t} \cdot \frac{e^{i\mu_4 u} - e^{i\mu_3 u}}{u} \cdot E\left(X(t)\,\overline{\xi(u)}\right) dt\, du.$$

Now

$$E(X(t)\overline{\xi(u)}) = E\left(X(t)\overline{\int_{-\infty}^{\infty} X(u-v) f(v)\,dv}\right) = E\left(\int_{-\infty}^{\infty} e^{it\lambda}\,dZ_X(\lambda)\,\overline{\int_{-\infty}^{\infty} e^{iu\lambda} F(\lambda)\,dZ_X(\lambda)}\right)$$

using the result at the end of section **1.7.**

By lemma **1.8** this becomes

$$\int_{-\infty}^{\infty} e^{i(t-u)\lambda}\,\overline{F(\lambda)}\,d\sigma_X(\lambda).$$

Thus

$$E\left(y(\mu_1, \mu_2)\cdot \overline{Z_\xi(\mu_4) - Z_\xi(\mu_3)}\right)$$

$$= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-i\mu_2 t} - e^{-i\mu_1 t}}{t}\,\frac{e^{i\mu_4 u} - e^{i\mu_3 u}}{u}\left\{\int_{-\infty}^{\infty} e^{i(t-u)\lambda}\,\overline{F(\lambda)}\,d\sigma_X(\lambda)\right\} dt\,du$$

$$= \frac{1}{4\pi^2}\int_{-\infty}^{\infty} \overline{F(\lambda)}\,d\sigma_X(\lambda)\int_{-\infty}^{\infty} \frac{e^{it(\lambda-\mu_2)} - e^{it(\lambda-\mu_1)}}{t}\cdot dt\cdot \int_{-\infty}^{\infty} \frac{e^{-iu(\lambda-\mu_4)} - e^{-iu(\lambda-\mu_3)}}{u}\cdot du$$

$$= \int \overline{F(\lambda)}\,d\sigma_X(\lambda),$$

this last integral being taken over the common part of the two intervals (μ_1, μ_2) and (μ_3, μ_4). If μ_r is any discontinuity point, it will be replaced by $\mu_r - 0$ in our work, and the result is still true. From this relation for any interval, we get that it is true for all Borel sets.

The conditions corresponding to (A) and (B) in the discrete parameter case are respectively

$$\Sigma a_r X(t-r) = \xi(t) \qquad \text{(A')}$$

and

$$\Sigma a_r e^{-i\lambda r} = F(\lambda), \qquad \text{(B')},$$

where the series Σa_r is convergent.

Let now s_1 and s_2 be any two Borel sets, in the interval W which is $(-\pi, \pi)$ in the discrete parameter case and $(-\infty, \infty)$ in the continuous parameter case, and let

$$y = \int_{s_1} dZ_X(\lambda).$$

Then

$$E\left[y\overline{\int_{s_2} dZ_\xi(\lambda)}\right] = \int_{s_1\cdot s_2} \overline{F(\lambda)}\,d\sigma_X(\lambda)$$

$$= E\left[\int_{s_1} dZ_X(\lambda)\,\overline{\int_{s_2} F(\lambda)\,dZ_X(\lambda)}\right]$$

$$= E\left[y\,\overline{\int F(\lambda)\,dZ_X(\lambda)}\right].$$

Also the set of elements (y) from a basis in $L_2(Z_X) = L_2(X)$ to which $L_2(\xi)$ belongs by virtue of the relation (A).

Hence

$$\int_{s_2} dZ_\xi(\lambda) = \int_{s_2} F(\lambda)\, dZ_X(\lambda).$$

This we write as

$$dZ_\xi(\lambda) = F(\lambda)\, dZ_X(\lambda), \tag{E}$$

from which we obtain

$$d\sigma_\xi(\lambda) = |F(\lambda)|^2\, d\sigma_X(\lambda). \tag{F}$$

These last two results are of fundamental importance in the work that follows in subsequent chapters.

1.10. Derivative of a process

We define the strong derivative of a process at the value t as

$$\underset{h\to 0}{\text{l.i.m.}} \frac{X(t+h) - X(t)}{h},$$

when it exists. It is alternatively known as the mean square derivative also (cf. J. E. MOYAL [1]). As we shall exclusively deal with convergence in the mean, it is not proposed to discuss the idea of weak convergence and weak derivative. When the derivative exists for all t, we can construct the derived process of $X(t)$. This belongs to $L_2(X)$.

1.11. Non-autocorrelated process

In the discrete parameter case a special type of stationary process is the non-autocorrelated process, characterized by its covariance sequence $\{R(n)\}$ which is such that only $R(0)$ is non-vanishing. For such a process the spectrum is given by $d\sigma(\lambda) = C\, d\lambda$, where C is a constant.

If

$$\sum_{r=0}^{n} a_r X(t-r) = \xi(t),$$

and $\xi(t)$ is a stationary non-autocorrelated process, and all the roots of the characteristic equation $\Sigma a_r Z^{n-r} = 0$ lie inside the unit circle, then the relation is known as an "autoregression". If however $X(t)$ is a stationary and non-autocorrelated process, then $\xi(t)$ given by the above equation is known as a "moving average process" generated by $X(t)$.

1.12. Deterministic nature

If $X(t)$ is a stationary process, we may classify it into two main types according to the manner in which $L_2(X;\, -\infty, t)$ unfolds with t. If two different values t_1 and t_2 exist such that

$$L_2(X; -\infty, t_1) = L_2(X; -\infty, t_2),$$

then as a result of stationarity, the Hilbert space at any instant is identical with that at any other instant, so that it is non-developing with t (see KARHUNEN [2] and O. HANNER [1]). The values of $X(t)$ occurring up to any instant form a basis for any element belonging to the entire Hilbert space $L_2(X)$. No new elements of positive norm get added as time flows. Such a process is termed a "deterministic process".

The analytical conditions for a stationary process to be deterministic in this sense is that

$$\int\limits_W \frac{|\log \sigma'(\lambda)|}{1+\lambda^2}\, d\lambda = \infty$$

where W is $(-\pi, \pi)$ for the discrete parameter case and $(-\infty, \infty)$ for the continuous parameter process, and $\sigma'(\lambda)$ is the absolutely continuous part of the spectrum.

If for a stationary process $\sigma'(\lambda)$ vanishes in an interval, the above integral is obviously divergent, and the process is deterministic. We shall have occasion to consider such processes later on.

When this is violated the process is termed "non-deterministic". The condition for non-determinism is then obviously

$$\int\limits_W \frac{|\log \sigma'(\lambda)|}{1+\lambda^2}\, d\lambda < \infty.$$

In this case in each interval of time new elements of positive norm are added to the Hilbert space which thus goes on developing with time.

CHAPTER II

Inversion: Discrete parameter

2.1. An outline of the chapter

In this chapter we shall study the inversion of a smoothing relation between two wide sense stationary discrete parameter processes. For the most part we deal with the case of a finite smoothing.

A set of necessary and sufficient conditions is obtained for a stationary solution for $X(t)$ to exist when the weights and the spectrum of the resulting process are given. When these conditions hold, we say that the smoothing is "consistent". In our study it is assumed that these conditions of consistency are satisfied.

Given the weights, a certain set Q is defined on the real axis. If this set is empty, or the spectral mass of the primary process in it vanishes, we have that $L_2(X) = L_2(\xi)$. If not, $L_2(\xi)$ is a proper subspace of $L_2(X)$.

When Q is empty, the solution for the primary process is unique. When Q is non-empty, there exists an infinity of solutions for $X(t)$, and of these there is but one which belongs to $L_2(\xi)$, and the spectral mass of this process in the set Q is zero.

The set Q corresponds to the roots of modulus unity of the characteristic equation (cf. H. WOLD [1]). When Q is empty, the position of the roots in relation to the unit circle determines the nature of the linear representation of $X(t)$ in terms of the values (past and future) of $\xi(t)$.

The difference between any two solutions of the primary process (in the case of finite smoothing) consists of stochastic periodic terms of the form $w_K e^{it\lambda_K}$.

If Q is non-empty and the spectral mass of the ξ-process vanishes in an interval or a sum of intervals to whose interior Q belongs, a linear representation in a summability sense can be given to $X(t)$, provided the roots on the unit circle are not repeated.

If the spectral mass of $\xi(t)$ vanishes only in the set Q and not in intervals covering it, we can construct a sequence of processes $\{X_r(t)\}$, each of which has a linear summability representation in terms of $\xi(t)$, and such that as r tends to infinity $X_r(t)$ converges in the mean to the primary process belonging to $L_2(\xi)$.

Again, when Q is non-empty, the resulting process is necessarily autocorrelated. The order of smallness of the spectrum of the ξ-process in the neighbourhood of Q is determined by the third condition of consistency.

The result of applying the same smoothing to the covariance sequence of the primary process is also obtained at the end of the chapter.

2.2. General

Let $X(t)$ be an unknown wide sense stationary process of discrete parameter, and let a sequence of numbers $a_0, a_1, \ldots, a_n$, and a wide sense stationary process $\xi(t)$ be given. Let L stand for the linear operator

$$L\,X(t) = \sum_{r=0}^{n} a_r X(t-r).$$

We describe this linear operation as smoothing in this thesis, and the sequence $\{a_r\}$ as the "weights" of smoothing. Sometimes for the sake of explicitness we write the operator also as $L : \{a_r\}$. Writing

$$L\,X(t) = \xi(t),$$

we call $X(t)$ the primary process, and $\xi(t)$ the resulting process. Our object is to study the unknown primary process in terms of $\xi(t)$ and $\{a_r\}$. If the number of weights in the smoothing is infinite, it is clear (by taking the expectations of both sides) that Σa_r is to be convergent. We shall presently see that besides this, the spectrum $\sigma_\xi(\lambda)$ of the given ξ-process and the weights $\{a_r\}$ must be inter-related in two other ways to validate our assumptions.

The case where the number of weights is infinite is treated towards the end of the chapter. Until then the smoothing to be considered will be by means of a finite number of weights. In what follows we generally make the inconsequential assumption that $a_0 = 1$. When the smoothing relation is an autoregression, the ξ-process is non-autocorrelated, and

$$Z^n + \sum_{r=1}^{n} a_r Z^{n-r} = 0$$

is the well-known characteristic equation connected with the stochastic difference equation (cf. H. WOLD [1]). It will be seen that in the general case as well one encounters the same equation; and as such, we shall in all cases construct the same equation which may still be called the "characteristic equation of the smoothing relation".

As already mentioned in chapter I, no assumption of the autocorrelatedness will be made with reference to either process when we discuss the problem in general. Let the operator inverse to L be L^{-1}, so that

$$L\,X(t) = \xi(t), \quad \text{and} \quad L^{-1}\,\xi(t) = X(t).$$

Then the following questions naturally arise:

i) Can L^{-1} always be found?
ii) When is L^{-1} in the form $L^{-1} : \{b_s\}$?
iii) What can be said about the uniqueness of L^{-1}?
iv) When does the suffix s in the sequence of weights in the inverse operator range over only positive integers, thus involving only the past values of the process?

Before addressing ourselves to a consideration of these and other related questions, we shall briefly summarize the nature of the earlier work up till now bearing upon this problem of inversion.

2.3. Earlier work on inversion

The following problem, formulated by H. CRAMÉR in 1933, has been discussed by H. WOLD [1] in the discrete parameter case:

> given the covariance sequence of a stationary process of moving average, to find the primary process as a linear combination of the values of the other process.

In the study we are now going to make we take the weights of smoothing as known. Then obviously we shall not be concerned with the questions relating to the various alternative ways in which the weights may be chosen for a given covariance sequence. However, in view of what has been remarked earlier regarding the characteristic polynomial equation in the general case as well, despite the fact that we do not restrict ourselves to examining only the moving average process, the discussion on the position of roots of the characteristic equation with reference to the unit circle has naturally points of contact with the treatment of the moving average case by H. WOLD [1] and [2].

When there is a root on $|Z| = 1$, exact linear inversion fails, a fact which has led H. WOLD [2] to consider the summability aspect and R. FRISCH [1] to consider the question of estimation of $X(t)$ linearly in terms of a specified number of values of the resulting process.

Lastly we refer to some remarks of K. KARHUNEN [3]. He considers a continuous parameter stationary process smoothed over a finite number of its values at equi-distant time instants to yield the resulting process, and makes the observation that the primary process can be uniquely recovered if the characteristic polynomial equation has no root of modulus unity.

2.4. Two lemmas relating to spectral measures

Given a finite smoothing as

$$L : \{ a_0 = 1, a_1, a_2, \ldots, a_n \},$$

and also the wide sense stationary process $\xi(t)$ arising out of the smoothing of the unknown wide sense stationary process $X(t)$, we proceed to find L^{-1} so that the primary process is obtained as $X(t) = L^{-1} \xi(t)$. To achieve this end, we apply CRAMÉR's spectral representation of a wide sense stationary process to both the sides of the smoothing relation. If $Z_X(\lambda)$ and $Z_\xi(\lambda)$ denote the orthogonal processes in the spectral representation associated respectively with the X- and the ξ-processes, we have

$$\begin{aligned}\int_{-\pi}^{\pi} e^{it\lambda}\, d Z_\xi(\lambda) &= \xi(t) \\ &= \sum_{r=0}^{n} a_r X(t-r) \\ &= \sum_{r=0}^{n} \int_{-\pi}^{\pi} a_r e^{i(t-r)\lambda}\, d Z_X(\lambda) \\ &= \int_{-\pi}^{\pi} e^{it\lambda} P(e^{-i\lambda})\, d Z_X(\lambda)\end{aligned}$$

where

$$P(Z) \equiv \sum_{r=0}^{n} a_r Z^r.$$

We have seen in chapter I that from such an integral relationship it follows that

$$P(e^{-i\lambda})\, d Z_X(\lambda) = d Z_\xi(\lambda)$$

in the sense that the integral of both sides over any Borel set in $(-\pi, \pi)$ is the same element in $L_2(X)$. If we suppose that there exists another function $\psi(\lambda)$ such that

$$\psi(\lambda)\, d Z_X(\lambda) = d Z_\xi(\lambda),$$

then

$$\int_S \{ \psi(\lambda) - P(e^{-i\lambda}) \}\, d Z_X(\lambda) = 0.$$

The necessary and sufficient condition for this is that

$$\int_S |\psi(\lambda) - P(e^{-i\lambda})|^2\, d\sigma_X(\lambda) = 0.$$

Hence $\psi(\lambda)$ differs from $P(e^{-i\lambda})$ possibly only over set of σ_X-measure zero.

Let the interval $(-\pi, \pi)$ be denoted by W and divided into two sets as follows: the set Q of the real zeros of

$$F(\lambda) = P(e^{-i\lambda}),$$

and let the complementary set be $W - Q$. Then we write

$$X(t) = \int_{W-Q} e^{it\lambda}\, d Z_X(\lambda) + \int_Q e^{it\lambda}\, d Z_X(\lambda).$$

As a result of the orthogonality of the two parts it follows that

$$\| X(t) \|^2 = \Big\| \int_{W-Q} e^{it\lambda}\, d Z_X(\lambda) \Big\|^2 + \Big\| \int_Q e^{it\lambda}\, d Z_X(\lambda) \Big\|^2.$$

Writing $m_X(Q)$ for the square of the norm of the second term on the right side, we get

$$m_X(Q) = \int_Q d\sigma_X(\lambda) \geq 0.$$

Lemma 2.4.1. *It is necessary for the consistency of the smoothing relation that the spectral mass of the ξ-process in the set Q vanishes.*

This follows directly from the fact that as a consequence of the smoothing relation and the nature of the set Q,

$$\begin{aligned} m_\xi(Q) &= \Big\| \int_Q e^{it\lambda}\, d Z_\xi(\lambda) \Big\|^2 \\ &= \Big\| \int_Q e^{it\lambda} P(e^{-i\lambda})\, d Z_X(\lambda) \Big\|^2 \\ &= \int_Q |P(e^{-i\lambda})|^2\, d\sigma_X(\lambda) \\ &= 0. \end{aligned}$$

Lemma 2.4.2. *Every set of λ in $(-\pi, \pi)$ of σ_X-measure zero is of σ_ξ-measure zero, and every set disjoint from Q and of σ_ξ-measure zero is also of σ_X-measure zero.*

Proof:

We have

$$|F(\lambda)|^2 \, d\sigma_X(\lambda) = d\sigma_\xi(\lambda)$$

where

$$F(\lambda) = P(e^{-i\lambda}) = \sum_{r=0}^{n} a_r e^{-i\lambda r},$$

and it is obvious that if

$$m_X(s) = \int_s d\sigma_X(\lambda) = 0,$$

$m_\xi(s)$ is likewise zero. To prove the latter part of the lemma, we proceed thus.

Let P be a non-empty set disjoint from Q and of σ_ξ-measure zero. Consider the sequence of sets $\{P_r\}$ such that P_r consists of all points $\lambda \in P$ satisfying

$$|F(\lambda)| \geq \frac{1}{r}.$$

For every $\lambda \in P$ we can find an r_0 such that λ is in sets $\{P_r\}$ for all $r > r_0$. In fact, if this were not possible for a certain $\lambda \in P$, we should have $|F(\lambda)| < \frac{1}{r}$ for all r, and hence $F(\lambda) = 0$, contrary to our hypothesis that P is disjoint from Q. Thus $\{P_r\}$ is a non-decreasing sequence of sets tending to the limiting set P and accordingly

$$m_X(P_r) \to m_X(P)$$

as $r \to \infty$. But

$$m_X(P_r) = \int_{P_r} d\sigma_X(\lambda) \leq r^2 \int_{P_r} d\sigma_\xi(\lambda) \leq r^2 m_\xi(P) = 0.$$

Hence

$$m_X(P) = 0.$$

2.5. On the existence of a stationary solution for $X(t)$.

We shall consider under what conditions there exists a stationary process $X(t)$ satisfying the smoothing relation.

Theorem 2.5. *For a stationary process $X(t)$ to exist as a solution of the smoothing relation, it is both necessary and sufficient that the following "conditions of consistency" be satisfied by the spectrum of the given ξ-process and the weights of smoothing:*

i) $\Sigma a_r \neq 0$, *whenever* $M_\xi \neq 0$,

ii) $m_\xi(Q) = 0$, *and*

iii) $\displaystyle\int_{W-Q} \frac{1}{|F(\lambda)|^2} d\sigma_\xi(\lambda) < \infty,$

where W is the interval $(-\pi, \pi)$.

Proof:

a) *The conditions are necessary.*

We note that condition ii) has already been shown to be necessary (lemma **2.4.1.**). Further, condition i) is necessary because

$$M_\xi = E[\xi(t)] = E[\Sigma a_r X(t-r)] = M_X(\Sigma a_r),$$

where M_X is constant. When the first condition holds good, we can write the smoothing relation as

$$\Sigma a_r \{ X(t-r) - M_X \} = \xi(t) - M.$$

Then

$$F(\lambda)\, d Z_X(\lambda) = d Z_\xi(\lambda)$$

from which we get that $d Z_X(\lambda)$ is given by

$$d Z_X(\lambda) = \frac{1}{F(\lambda)} d Z_\xi(\lambda)$$

for $\lambda \in W - Q$, and for $\lambda \in Q$, $d Z_X(\lambda)$ is subject to the conditions that $Z(s)$ is to be an orthogonal process of bounded norm in the set Q, and

$$E[d Z_X(\lambda) \overline{d Z_X(\mu)}] = 0$$

for $\lambda \in Q$, $\mu \in W - Q$. Then

$$\begin{aligned} \varrho_X^2 = \| X(t) \|^2 &= \int_{W-Q} d\sigma_X(\lambda) + \int_Q d\sigma_X(\lambda) \\ &\geq \int_{W-Q} \frac{1}{|F(\lambda)|^2} d\sigma_\xi(\lambda) \end{aligned}$$

which is to be bounded if a stationary solution $X(t)$ is to exist.

b) *The conditions are also sufficient.*

In view of the first condition we can assume without loss of generality that the processes have mean value zero. Then we define two processes $X_1(t)$ and $X_2(t)$ as follows:

$$X_1(t) = \int_{W-Q} e^{it\lambda} \frac{1}{F(\lambda)} d Z_\xi(\lambda),$$

and

$$X_2(t) = \int_Q e^{it\lambda} d Z_X(\lambda)$$

the norm of which is finite but arbitrary. By the third condition it follows that

$$\| X_1(t) \|^2 < \infty,$$

so that $X_1(t)$ exists as an element of $L_2(\xi)$. Subjecting the process $X_1(t) + X_2(t)$ to the smoothing operation, we have

$$\begin{aligned} L[X_1(t) + X_2(t)] &= \int_{W-Q} e^{it\lambda} F(\lambda) \frac{1}{F(\lambda)} d Z_\xi(\lambda) + \int_Q e^{it\lambda} F(\lambda) \, d Z_X(\lambda) \\ &= \int_{W-Q} e^{it\lambda} \, d Z_\xi(\lambda) \\ &= \int_W e^{it\lambda} \, d Z_\xi(\lambda) \text{ by the second condition} \\ &= \xi(t). \end{aligned}$$

Thus $X(t) = X_1(t) + X_2(t)$ is a solution of the primary process. Since $X(t)$ is clearly stationary, the conditions are also sufficient.

We assume in our work that the three conditions of consistency given in theorem **2.5.** for a stationary solution to exist are satisfied.

We may note at this stage that

i) $L X_1(t) = \xi(t)$, and
ii) $L X_2(t) = 0$,

so that when Q is non-empty, there exists an arbitrary part $X_2(t)$ (which may be called the "complementary part") in the solution of $X(t)$.

As we are considering only the case in which the smoothing contains a finite number of weights, the set Q consists of discrete points. Hence it follows that, if $\lambda_1, \lambda_2, \ldots, \lambda_p$ are the points constituting Q, we can put

$$X_2(t) = \sum_{k=1}^{p} w_k e^{i \lambda_k t}.$$

(See also chapter IV). The process $X_2(t)$ is deterministic.

In view of lemma **2.4.2**, we see that if $\sigma_X(\lambda)$ has any saltuses in Q, they are no longer saltus positions of $\sigma_\xi(\lambda)$.

2.6. Relation between the Hilbert spaces of the processes

We shall next prove

Theorem 2.6.

$$[m_X(Q) = 0] \rightleftarrows [L_2(X) = L_2(\xi)],$$

and

$$[m_X(Q) > 0] \rightleftarrows [L_2(\xi) \text{ is a proper subspace of } L_2(X)].$$

Proof:

Case (i): $m_X(Q) = 0$.

From the relation (D) of chapter I, we have

$$L_2(X) = L_2(Z_X).$$

Since $m_X(Q) = 0$, and for $\lambda \in W - Q$, $d Z_X(\lambda) = \frac{1}{F(\lambda)} d Z_\xi(\lambda)$, it follows that

$$L_2(Z_X) \subset L_2(Z_\xi).$$

But

$$L_2(Z_\xi) = L_2(\xi),$$

so that

$$L_2(X) \subset L_2(\xi).$$

Also $L_2(X) \supset L_2(\xi)$ by the smoothing relation. Hence

$$L_2(X) = L_2(\xi).$$

Case (ii): $m_X(Q) > 0$.

In this case

$$X(t) = X_1(t) + X_2(t),$$

where

$$\| X_2(t) \|^2 = m_X(Q) > 0.$$

Thus $X_2(t)$ is a non-null process. Further by Lemma **2.4.1**, the X_2-process is orthogonal to $L_2(Z_\xi) = L_2(\xi)$. Hence

$$L_2(X) \neq L_2(\xi).$$

Therefore $L_2(\xi)$ is a proper subspace of $L_2(X)$.

The converse inferences contained in the theorem are clearly true in view of the exclusive nature of the two alternative possibilities.

2.7. Uniqueness of inversion

The question of uniqueness of inversion is closely related to the content of the spectral mass of the X-process in the set Q.

The operation of smoothing may be regarded as a mapping of the elements of $L_2(X)$ on a subspace of itself, viz., the elements of $L_2(\xi)$ which belongs to $L_2(X)$ by virtue of the smoothing relation. Then it is but natural to conjecture that for uniqueness of inversion the transformation must carry the whole of $L_2(X)$ into its entirety, and not into a part of itself. We have just seen in theorem **2.6** that this geometrical property is in turn related to the spectral content of the X-process in the set Q. Bearing out the conjecture, we have

Theorem 2.7. *In a consistent smoothing the inversion is unique when the set Q is empty. If Q is non-empty, the inversion problem has an infinity of solutions. Out of these solutions, there exists one and only one (in the sense of equivalence in norm) which belongs to $L_2(\xi)$, and only for this process is $m_X(Q)$ zero.*

Proof: When Q is empty,

$$X(t) = \int_{-\pi}^{\pi} e^{it\lambda} \frac{1}{F(\lambda)} d Z_\xi(\lambda).$$

In this case $m_X(Q) = 0$, and inversion is uniquely given the above formula.
If Q is non-empty,

$$X(t) = X_1(t) + X_2(t),$$

where $\| X_2(t) \|$ is arbitrary and $X_1(t)$ is given by

$$X_1(t) = \int_{W-Q} e^{it\lambda} \frac{1}{F(\lambda)} d Z_\xi(\lambda).$$

Since $X_2(t)$ is of arbitrary norm, we have an infinity of choices. When we choose $X_2(t)$ as the null process, $X(t)$ is uniquely determined as $X_1(t)$ which belongs to $L_2(\xi)$. Only in this case are the two Hilbert spaces identical and $m_X(Q) = 0$.

2.8. Linear inversion

We shall have to consider the real zeros of

$$F(\lambda) = P(e^{-i\lambda})$$

in connection with linear inversion of the smoothing relation. A zero $\alpha + i\beta$ can be classified into one of the three types according as $\beta = 0$, or $\beta > 0$, or $\beta < 0$, i.e., according as the zero is real, or lies in the upper half plane, or lies in the lower half plane. The set of real zeros is denoted by Q. If we consider the Z-plane, where $Z = e^{-i\lambda}$, the real axis in the λ-plane is mapped on the circle $|Z| = 1$, and the zeros in the lower and upper half planes become respectively the roots of $P(Z) = 0$ inside and outside the unit circle, $|Z| = 1$. The set Q in the λ-plane corresponds to the roots on $|Z| = 1$. We shall frequently express our statements in terms of the roots of the characteristic equation which are the reciprocals of the roots of $P(Z) = 0$. The set Q continues to correspond to the roots on the unit circle of the characteristic equation, while the roots of $P(Z) = 0$ inside (and outside) the unit circle become the roots of the characteristic equation outside (and inside) it.

In any attempt to obtain the primary process as a linear representation in terms of the values of $\xi(t)$ the position of the zeros of $F(\lambda)$ with respect to the real axis in the λ-plane (and hence the position of the roots of the characteristic equation in relation to $|Z| = 1$) plays an important rôle.

We have seen that $X_1(t)$ is almost certainly $X(t)$ if $m_X(Q) = 0$. Otherwise it forms the best estimate of $X(t)$, viz., as much of $X(t)$ as belongs to $L_2(\xi)$. The problem of inversion is then naturally to obtain $X_1(t)$, since we are given only the ξ-process. A special case of inversion is linear inversion in which $X_1(t)$ is to be obtained as $\Sigma\, b_r\, \xi(t-r)$.

We start by examining the case when the equation $P(Z) = 0$ has no roots inside or on the unit circle. Then there exists a positive constant $R > 1$ such that for values of the complex variable Z lying inside $|Z| = R$ the function $\frac{1}{P(Z)}$ can be expanded as a Taylor series. Thus

$$X(t) = \int_{-\pi}^{\pi} e^{it\lambda} \frac{1}{P(e^{-i\lambda})}\, d Z_\xi(\lambda)$$

$$= \sum_{j=0}^{K} \int_{-\pi}^{\pi} e^{it\lambda} b_j e^{-ij\lambda}\, d Z_\xi(\lambda) + \int_{-\pi}^{\pi} e^{it\lambda} R_K\, d Z_\xi(\lambda),$$

where

$$|R_K| \to 0 \quad \text{as} \quad K \to \infty.$$

The first term becomes

$$\sum_{j=0}^{K} b_j\, \xi(t-j),$$

while

$$\left\| \int_{-\pi}^{\pi} e^{it\lambda} R_K\, d Z_\xi(\lambda) \right\|^2 = \int_{-\pi}^{\pi} |R_K|^2\, d\sigma_\xi(\lambda) \le \text{constant} \cdot |R_K|^2.$$

The constant being independent of K,

$$\left\| X(t) - \sum_{j=0}^{K} b_j\, \xi(t-j) \right\|^2 = \int_{-\pi}^{\pi} |R_K|^2\, d\sigma_\xi(\lambda)$$

tends to zero as $K \to \infty$. Hence

$$X(t) = \underset{K\to\infty}{\text{l.i.m.}} \sum_{j=0}^{K} b_j\, \xi(t-j)$$

which we denote by writing

$$X(t) = \sum_{j=0}^{\infty} b_j\, \xi(t-j).$$

The b's are the coefficients appearing in the convergent expansion of $\frac{1}{P(Z)}$ in powers of Z. It is then evident that if $\frac{1}{P(Z)}$ can be expanded on the unit circle in a Laurent's series, $X(t)$ will have the corresponding linear representation in which j will range over integers positive as well as negative. The roots of the characteristic equation being the reciprocals of the roots of the equation $P(Z) = 0$, we have the following

Theorem 2.8.1. *If $X(t)$ and $\xi(t)$ are any stationary processes connected by a consistent smoothing relation, and if the characteristic equation has all its roots inside the unit circle, the primary process $X(t)$ has a linear representation entirely in terms of the past values of $\xi(t)$.*

This result is known in the cases treated earlier by WOLD [1]. If, however, all the roots of $P(Z) = 0$ lie inside $|Z| = 1$, then it is seen that

$$X(t) = \sum_{j=-\infty}^{-n} b_j \xi(t-j),$$

and when the roots lie both inside and outside but not on $|Z| = 1$,

$$X(t) = \sum_{j=-\infty}^{\infty} b_j \xi(t-j).$$

The fact that on the unit circle $\frac{1}{P(Z)}$ has no convergent expansion only in positive powers of Z when $P(Z) = 0$ has some roots inside $|Z| = 1$ is equivalent to the following

Theorem 2.8.2. *When the characteristic equation has no root on $|Z| = 1$, but has one or more of its roots outside the unit circle, the primary process has a linear representation in $\xi(t)$, but this representation will not be in terms of only the past values of $\xi(t)$, but will involve its future values as well.*

The following examples show how L^{-1} goes to infinity in either direction.

Example 1.

If $L = L : \{1, \alpha\}$, $|\alpha| < 1$, then

$$L^{-1} = L^{-1} : \{1, -\alpha, \alpha^2, -\alpha^3, \ldots\}.$$

Example 2.

If $L = L : \{1, -(\alpha+\beta), \alpha\beta\}$, $|\alpha| < 1$, $|\beta| < 1$, then

$$L^{-1} = L^{-1} : \{1, (\alpha+\beta), (\alpha^2 + \alpha\beta + \beta^2), \ldots, (\alpha^n + \alpha^{n-1}\beta + \ldots + \beta^n), \ldots\}.$$

Example 3.

If $L = L : \{1, \alpha\}$, $|\alpha| > 1$, then

$$X(t) = \frac{1}{\alpha}\xi(t+1) - \frac{1}{\alpha^2}\xi(t+2) + \frac{1}{\alpha^3}\xi(t+3) - \ldots$$

and hence

$$L^{-1} = L^{-1} : \left\{\ldots -\frac{1}{\alpha^4}, \frac{1}{\alpha^3}, -\frac{1}{\alpha^2}, \frac{1}{\alpha}, 0\right\}.$$

Example 4.

If $L = L : \{1, -(\alpha+\beta), \alpha\beta\}, |\alpha| < 1, |\beta| > 1,$ then

$$X(t) = \frac{1}{\alpha-\beta}\left(\sum_{r=0}^{\infty} \alpha^r \xi(t-r) + \sum_{r=-1}^{-\infty} \beta^r \xi(t-r)\right).$$

Hence

$$L^{-1} = L^{-1} : \left\{\ldots, \frac{1}{\beta^2(\alpha-\beta)}, \frac{1}{\beta(\alpha-\beta)}, \frac{1}{\alpha-\beta}, \frac{\alpha}{\alpha-\beta}, \frac{\alpha^2}{\alpha-\beta}, \ldots\right\}.$$

2.9. Roots of modulus unity

In this section we shall prove

Theorem 2.9.1. *When the characteristic equation has a root on* $|Z| = 1$, *the resulting process is autocorrelated,* and

Theorem 2.9.2. *Let all the roots of the characteristic equation be of modulus less than or equal to unity, and let none of the roots on* $|Z| = 1$ *be repeated. If the roots on the unit circle be* $e^{i\lambda_1}, e^{i\lambda_2}, \ldots, e^{i\lambda_p}$, *let the points* $\lambda_1, \lambda_2, \ldots, \lambda_p$ *on the real axis be interior points of one or more intervals in which the spectral mass of the* ξ*-process vanishes. Then using* (E, q) *summability interpretation, the part* $X_1(t)$ *of the primary process belonging to* $L_2(\xi)$ *has a linear representation entirely in terms of the past values of* $\xi(t)$.

As for Theorem **2.9.1**, it follows as a consequence of condition iii) of Theorem **2.5**. To see this, let the root on the unit circle of the characteristic equation be $e^{i\lambda_1}$, where λ_1 is real, and let it be a root of multiplicity h. Then

$$F(\lambda) = P(e^{-i\lambda}) = P_1(e^{-i\lambda})\{1 - e^{i(\lambda_1-\lambda)}\}^h$$

where $P_1(e^{-i\lambda})$ is a polynomial in $e^{-i\lambda}$ which does not vanish for $\lambda = \lambda_1$. The condition iii) of the theorem then requires that the limit of

$$\left(\int_{-\pi}^{\lambda_1-\epsilon_1} + \int_{\lambda_1+\epsilon_2}^{\pi}\right) \frac{1}{|P_1(e^{-i\lambda})|^2\,|1-e^{i(\lambda_1-\lambda)}|^{2h}}\, d\sigma_\xi(\lambda)$$

is finite when ϵ_1 and ϵ_2 tend to zero. Since $P_1(e^{-i\lambda})$ is continuous at $\lambda = \lambda_1$ and does not vanish for that value, this in turn is equivalent to the condition that

$$\lim_{\epsilon_1\to 0,\ \epsilon_2\to 0} \left(\int_{-\pi}^{\lambda_1-\epsilon_1} + \int_{\lambda_1+\epsilon_2}^{\pi}\right) \frac{1}{(\lambda-\lambda_1)^{2h}}\, d\sigma_\xi(\lambda) < \infty \cdots \tag{A}$$

If the ξ-process were non-autocorrelated,

$$d\sigma_\xi(\lambda) = C \cdot d\lambda, \ (C, \text{ a constant}),$$

and the above integral diverges. Hence the $\xi(t)$-process is autocorrelated which proves the theorem.

In fact, the condition (A) on the nature of smallness of $d\sigma_\xi(\lambda)$ in the neighbourhood of λ_1 goes beyond the autocorrelatedness of $\xi(t)$, and describes the actual situation.

Summability in norm

When a root of modulus unity exists, it is sometimes still possible to represent $X(t)$ in the form

$$X(t) = \sum_{r=0}^{\infty} b_r \xi(t-r),$$

if this interpreted in the sense of summability in norm with respect to Hilbert space built by the process involved. The question of summability has been considered by WOLD [2]. As our present purpose is only to make the idea of summability in norm precise, we shall confine our attention to the simple case of (E, q)-summability.

If the real part of Z is less than unity, we can find a $q > 0$ such that

$$|q+Z| < 1+q.$$

For such Z

$$\frac{1}{1-Z} = \frac{1}{1+q}\left(1-\frac{q+Z}{1+q}\right)^{-1} = \sum_{n=0}^{\infty} \frac{(q+Z)^n}{(1+q)^{n+1}}.$$

Whenever $\sum_{n=0}^{\infty} Z^n$ is convergent,

$$\sum_{n=0}^{\infty} Z^n = \sum_{n=0}^{\infty} \frac{(q+Z)^n}{(1+q)^{n+1}}.$$

The latter series converges for some Z for which the former does not. We say that, whenever the latter series is convergent, ΣZ^n is summable (E, q) to $\frac{1}{1-Z}$. The meaning is as follows: consider the transformation T such that

$$T Z^n = \frac{(q+Z)^n}{(1+q)^{n+1}}.$$

When $\Sigma(T Z^n)$ is convergent, we say that ΣZ^n is summable to the sum of $\Sigma(T Z^n)$.

The same notion can be transferred to a series whose terms are elements of the Hilbert space of a process. Suppose that ξ's and η's are elements belonging to the space built by the same process, and T is a transformation giving

$$T \xi_r = \eta_r \text{ for all } r.$$

If $\|\Sigma \xi_r - \Sigma \eta_r\| = 0$, whenever $\|\Sigma \xi_r\| < \infty$, then we shall say that by method (T), $\Sigma \xi_r$ is summable in norm to the sum of $\Sigma \eta_r$, this last mentioned sum existing.

Let

$$X(t) - e^{i\lambda_1} \cdot X(t-1) = \xi(t),$$

λ_1 being real, and let the spectral mass of the ξ-process vanish in an interval Q_1 of which λ_1 is an interior point. Then

$$\begin{aligned} X_1(t) &= \int_{W-Q_1} e^{it\lambda} \frac{1}{1 - e^{i\lambda_1} e^{-i\lambda}} \, dZ_\xi(\lambda) \\ &= \int_{W-Q_1} \frac{e^{it\lambda}}{1+q} \left(1 - \frac{q + e^{i\lambda} e^{-i\lambda_1}}{1+q}\right)^{-1} dZ_\xi(\lambda). \end{aligned}$$

If we put

$$\eta_r = \int_{W-Q_1} e^{it\lambda} \frac{(q + e^{i\lambda_1} e^{-i\lambda})^n}{(1+q)^{n+1}} \, dZ_\xi(\lambda),$$

and write as before

$$\left\| X_1(t) - \sum_{r=0}^{K} \eta_r \right\|^2 = \left\| \int_{W-Q_1} e^{it\lambda} R_K \, dZ_\xi(\lambda) \right\|^2,$$

it follows from the uniform convergence of the series $\sum_{n=0}^{\infty} \frac{(q + e^{i\lambda_1} e^{-i\lambda})^n}{(1+q)^{n+1}}$ for $\lambda \in (W - Q_1)$ that $|R_K| \to 0$ as $K \to \infty$, the same K holding good for all λ. Hence it follows that

$$\underset{K\to\infty}{\text{l.i.m.}} \sum_{r=0}^{K} \eta_r = X_1(t).$$

The transformation T of the elements of $L_2(\xi)$ corresponding to above summability is given by $T(e^{i\lambda_1 r} \xi(t-r)) = \eta_r$. We have that

$$\sum_{r=0}^{\infty} e^{i\lambda_1 r} \xi(t-r) = \sum_{r=0}^{\infty} \eta_r$$

whenever the former series converges in norm, and we say as before that

$$\sum_{r=0}^{\infty} e^{i\lambda_1 r} \xi(t-r)$$

is summable (E, q) to

$$\begin{aligned} \sum_{r=0}^{\infty} T(e^{i\lambda_1 r} \xi(t-r)) &= \sum_{r=0}^{\infty} \eta_r \\ &= X_1(t). \end{aligned}$$

When none of the roots of the characteristic equation lying on $|Z| = 1$ are repeated, resolution by partial fractions will help us to carry the proof of theorem **2.9.2** to completion.

More comprehensive forms of summability can be considered in the same way.

From lemma **2.4.2** it follows that the spectrum of the X_1-process has also to vanish in Q_1 when $m_\xi(Q_1) = 0$. Thus the processes become restricted if the ξ-process is to fulfill such a condition.

Let none of the roots on $|Z| = 1$ be repeated, and let them be denoted by $\{e^{i\lambda_K}\}$ as before. With each λ_K we can associate the closed interval

$$\left(\lambda_K - \frac{1}{r},\ \lambda_K + \frac{1}{r}\right),$$

thus getting p intervals corresponding to the p roots on the unit circle. Let the sum set of these intervals be Q_r. As r tends to infinity, Q_r tends to the set of p points $\lambda_1, \lambda_2, \ldots \lambda_p$. We may consider the process $X_r(t)$ defined by

$$X_r(t) = \int\limits_{W-Q_r} e^{it\lambda} \frac{1}{P(e^{-i\lambda})} d\,Z_\xi(\lambda).$$

Resolving $\dfrac{1}{P(e^{-i\lambda})}$ by partial fractions, and using (E, q) summability we get that

$$X_r(t) = \sum_{j=-\infty}^{\infty} b_{rj}\,\xi(t-j).$$

If there are no roots of the characteristic equation outside $|Z| = 1$, the above representation will involve only the past values, i.e., $b_{rj} = 0$, for $j < 0$. Further

$$\underset{r\to\infty}{\text{l.i.m.}}\ X_r(t) = X(t).$$

Hence we can approach the primary process through a sequence of processes as nearly as desired (in the sense that the norm of the difference can be made as small as wished for). Each of these approximating processes has a linear summability representation in $\xi(t)$.

2.10. Elementary Gaussian processes

In connection with inversion where the primary process is given linearly in terms of the past values of the resulting process, we may consider the processes studied by J. L. Doob [2] under the name of elementary Gaussian process. Let $X(t)$ be an "elementary t. h. G. $\mathrm{M_N}$" one-dimensional process of the non-deterministic type in the discrete parameter case. Such a process satisfies a difference equation

$$X(t) + \sum_{r=1}^{N} a_r X(t-r) = \xi(t),$$

where $(\xi(t))$ are independent chance variables. It has been shown by Doob how the non-deterministic nature of $X(t)$ requires that all the roots of the characteristic equation of the above smoothing relation lie inside the unit circle, if the difference equation is expressed with the least possible number of terms.

If we take the smoothing relation with the weights (a_r) occurring in it where the number of terms is least, and invert, we get

$$X(t) = \sum_{r=0}^{\infty} b_r \xi(t-r),$$

where $\xi(t)$ is a stationary process of independent chance variables.

2.11. Infinite smoothing

Let us now consider smoothing by means of an infinite sequence of weights, going to infinity in one or both the directions. In order to ensure the existence of the first and second moments of the process, we impose the conditions

i) $\Sigma a_r < \infty$ and
ii) $\Sigma |a_r|^2 < \infty$.

The process $X_1(t)$ is obtained as

$$X_1(t) = \int\limits_{W-Q} e^{it\lambda} \frac{1}{P(e^{-i\lambda})} dZ_\xi(\lambda),$$

where $P(e^{-i\lambda}) = \Sigma a_r e^{-ir\lambda}$ is now an infinite series, and Q is the set of real λ for which $P(e^{-i\lambda})$ vanishes. The problem of linear inversion is then equivalent to that of expanding $\dfrac{1}{P(e^{-i\lambda})}$ as a convergent series on the unit circle in powers of $e^{i\lambda}$. If we now impose the restriction that

$$L_2(X) = L_2(\xi),$$

then $X(t)$ is given almost certainly by $X_1(t)$.

2.12. Covariance sequence of the primary process

It is well known that the covariance sequence of a stationary autoregressive process satisfies the homogeneous difference equation

$$\sum_{j=0}^{n} a_j R_X(K-j) = 0, \quad K > 0,$$

where $\{R_X(p)\}$ denotes the covariance sequence of the X-process which is autoregressive.

It will be of interest to examine the effect of subjecting $R_X(t)$ to the same smoothing operation L in the general case.

Theorem 2.12. *The covariance sequence of the primary process satisfies the following relation:*

$$\sum_{j=0}^{n} a_j R_X(t-j-s) = \int\limits_{W-Q} \frac{e^{i(t-s)\lambda}}{P(e^{-i\lambda})} d\sigma_\xi(\lambda),$$

provided $m_X(Q) = 0$.

Proof: Consider the smoothing relation

$$\sum_{j=0}^{n} a_j X(t-j) = \xi(t)$$

and the result of inversion, viz.,

$$\overline{X(s)} = \overline{\int_{W-Q} \frac{e^{is\lambda}}{P(e^{-i\lambda})} d Z_\xi(\lambda)}.$$

Then

$$\begin{aligned}\sum_{j=0}^{n} a_j R_X(t-j-s) &= E[\{\sum_{j=0}^{n} a_j X(t-j)\}\overline{X(s)}] \\ &= E[\xi(t)\overline{X(s)}] \\ &= E\left[\int_{W-Q} e^{it\lambda} d Z_\xi(\lambda) \int_{W-Q} \frac{e^{-is\lambda}}{\overline{P(e^{-i\lambda})}} \overline{d Z_\xi(\lambda)}\right] \\ &= \int_{W-Q} e^{i(t-s)\lambda} \frac{1}{\overline{P(e^{-i\lambda})}} d\sigma_\xi(\lambda).\end{aligned}$$

CHAPTER III

Inversion. Continuous parameter

3.1. An outline of the chapter

The following is a brief summary of the results of this chapter.

The conditions of consistency are mainly the same as in the case of the discrete parameter processes. Theorems **2.5**, **2.6**, and **2.7** are still true.

Sufficient conditions for inversion to be given as again a smoothing relationship are obtained. When the weight functions in such dual smoothing relationships are restricted to belong to the Lebesque class L_1, as in a finite smoothing, the processes have to belong to a subclass of the deterministic type.

It is seen that sometimes it is possible to approximate to a non-deterministic process by a sequence of deterministic processes.

In theorem **3.5** the following question is considered. If the primary process is obtainable as a smoothing of $\xi(t)$, are the two parts of the smoothing over its past values and future values orthogonal? Sufficient conditions are given for their orthogonality.

We obtain the result of effecting the same smoothing on the covariance function of the primary process as the process itself is subjected to. Lastly the nature of the singular process $X_2(t)$ is also discussed.

3.2. Conditions for obtaining inversion as a smoothing

In chapter I the meaning of integration of a process has been explained. Passing on to the continuous parameter case we naturally consider smoothing by means of a weight function which is a function of a real variable. The study of inversion will now have to deal with some new questions not met with in the earlier case of the discrete parameter. Having restricted the weight function $f(u)$ suitably, we form the resulting process by the integral smoothing relation

$$\int_{-\infty}^{\infty} X(t-u) f(u)\, du = \xi(t).$$

The range of integration is taken to comprise the entire real axis even when the interval over which the smoothing is effected is finite by regarding the weight function as vanishing outside this range, the main reason for this being that smoothing over a finite range is effectively covered by supposing the weight function to belong to the Lebesque class L_1 on $(-\infty, \infty)$.

The following are some aspects of interest in the study of the problem of inversion in the continuous parameter case:

i) Can $X(t)$ have the representation

$$X(t) = \int_{-\infty}^{\infty} \xi(t-v)\, \varphi(v)\, dv,$$

and if so, what is $\varphi(v)$?

ii) When is inversion unique?

iii) When can the primary process be obtained as a smoothing of the resulting process over its past values only?

As before $X(t)$ is the unknown wide sense stationary process and $\xi(t)$ is the completely known also wide sense stationary process. Taking the expectation of both sides of the smoothing relation, we get (see: Additional Note at the end)

$$M_X \cdot \int_{-\infty}^{\infty} f(u)\, du = M_\xi.$$

Hence if $M_\xi \neq 0$, then

$$\int_{-\infty}^{\infty} f(u)\, du \neq 0. \tag{i}$$

Condition (i) has to be satisfied as the first condition of consistency. The other two conditions necessary and sufficient for a stationary solution $X(t)$ of the smoothing relation to exist are in the same form as in the discrete parameter case.

In chapter I we have seen that the resulting process $\xi(t)$ obtained by the smoothing is given by

$$\xi(t) = \int_{-\infty}^{\infty} e^{it\lambda} F(\lambda)\, d Z_X(\lambda);$$

when $f(u)$ is bounded, and

$$F(\lambda) = \int_{-\infty}^{\infty} e^{-iu\lambda} f(u)\, du,$$

exists and is bounded. Also

$$d Z_\xi(\lambda) = F(\lambda)\, d Z_X(\lambda). \tag{A}$$

It is then clear that if $F_1(\lambda)$ is any bounded function of λ, we can replace $F(\lambda)$ in (A) by F_1 over a set of σ_X-measure zero. As before

$$\begin{aligned} X(t) &= \int_{W-Q} e^{it\lambda} d Z_X(\lambda) + \int_{Q} e^{it\lambda} d Z_X(\lambda) \\ &= X_1(t) + X_2(t). \end{aligned}$$

Then $X_1(t)$ and $X_2(t)$ are stationary processes with associated random spectral functions

$$d Z_{X_1}(\lambda) = \begin{cases} d Z_X(\lambda) \text{ for } \lambda \in W - Q \\ \text{and} \\ 0 \text{ otherwise} \end{cases}, \quad d Z_{X_2}(\lambda) = \begin{cases} d Z_X(\lambda) \text{ for } \lambda \in Q \\ \text{and} \\ 0 \text{ otherwise.} \end{cases}$$

By the orthogonality of the two parts it follows that

$$\| X(t) \|^2 = \| X_1(t) \|^2 + \| X_2(t) \|^2.$$

When the third condition of consistency holds good, $X_1(t)$ defined above is completely specified by the relation

$$X_1(t) = \int_{W-Q} e^{it\lambda} \frac{1}{F(\lambda)}\, d Z_\xi(\lambda).$$

Exactly as in chapter II, we can prove for the continuous parameter case also theorems **2.5**, **2.6**, and **2.7**.

In what follows it is assumed that the smoothing is consistent with the given ξ-process and the weight function $f(u)$. We seek in our problem of inversion the primary process belonging to $L_2(\xi)$, so that $L_2(X) = L_2(\xi)$ and $m_X(Q) = 0$. Under these conditions the primary process is almost certainly the process $X_1(t)$, and we shall write it without the suffix 1. Since $m_X(Q) = 0$, every set of σ_X-measure zero is also of σ_ξ-measure zero and vice versa.

Suppose that we now want to obtain the primary process again as an integral smoothing of the ξ-process in the dual form as

$$X(t) = \int_{-\infty}^{\infty} \xi(t - v)\, \varphi(v)\, dv,$$

where $\varphi(v)$ is bounded and

$$\Phi(\lambda) = \int_{-\infty}^{\infty} e^{-iv\lambda} \varphi(v)\, dv$$

exists and is bounded for all λ. It is evident at the very outset that, when inversion is obtainable as a smoothing of $\xi(t)$, the entire closed Hilbert spaces spanned by the two processes are identical, as each belongs to the other. Regarding sufficient conditions for the existence of such an inverse smoothing relationship we state

Theorem 3. 2. *Let $\xi(t)$ be a given stationary process with zero mean and with spectrum $\sigma_\xi(\lambda)$, and let $f(u)$ be a known real or complex valued bounded function of the real variable u such that*

$$F(\lambda) = \int_{-\infty}^{\infty} e^{-iu\lambda} f(u)\, du$$

exists and is bounded. Let Q be the set of real zeros of $F(\lambda)$, and let

$$\int_{W-Q} \frac{1}{|F(\lambda)|^2}\, d\sigma_\xi(\lambda) \text{ be finite} \tag{A}$$

and

$$m_\xi(Q) = 0. \tag{B}$$

Let the processes $X(t)$ and $\xi(t)$ be related by the smoothing relation

$$\int_{-\infty}^{\infty} X(t-u) f(u)\, du = \xi(t).$$

Let

i) $L_2(X) = L_2(\xi)$.

Further, let $\varphi(v)$ be a bounded function (real or complex) of the real variable v such that

$$\Phi(\lambda) = \int_{-\infty}^{\infty} e^{-iv\lambda} \varphi(v)\, dv$$

exists and is bounded, and let

ii) $\int_{-\infty}^{\infty} |1 - F(\lambda)\Phi(\lambda)|^2\, d\sigma_\xi(\lambda) = 0.$

The conditions i) *and* ii) *are sufficient for the primary process to be given by the inverse smoothing relationship*

$$\int_{-\infty}^{\infty} \xi(t-v)\varphi(v)\, dv = X(t).$$

Before taking up the proof we may note the following facts:

a) In view of the conditions (A) and (B), a stationary solution for $X(t)$ exists.

b) From condition i) we have that

$$m_X(Q) = m_\xi(Q) = 0$$

so that the failure of the condition

$$F(\lambda)\,\Phi(\lambda) = 1$$

in the set Q is not any inconsistency for a $\Phi(\lambda)$ to exist.

c) The condition ii) can be written with the σ_ξ-measure being replaced by σ_X-measure because of condition i).

d) There is a considerable amount of symmetry in the conditions of this theorem except that the conditions corresponding to (A) and (B) involving the set Q' of the real zeros of $\Phi(\lambda)$ do not occur explicitly. Once the theorem is proved, it follows from the stationarity of the known ξ-process that

$$\int\limits_{W-Q'} \frac{1}{|\Phi(\lambda)|^2}\, d\sigma_X(\lambda) \text{ is finite} \tag{A'}$$

and that

$$m_X(Q') = 0\,. \tag{B'}$$

As a consequence of the condition i), it follows that $m_\xi(Q') = 0$.

e) When we take the X-process, and the function $\varphi(v)$ as known, we shall impose the conditions (A′) and (B′) instead of (A) and (B) and then seek the ξ-process as a smoothing of the X-process.

f) It is obvious that condition i) is necessary.

Proof: We have from the given smoothing relationship that almost certainly with respect to σ_X- and σ_ξ-measure that

$$d Z_\xi(\lambda) = F(\lambda)\, d Z_X(\lambda)$$

so that

$$\Phi(\lambda)\, d Z_\xi(\lambda) = \Phi(\lambda)\, F(\lambda)\, d Z_X(\lambda)\,.$$

The right side reduces to $d Z_X(\lambda)$ almost certainly, as a result of condition ii). Hence

$$d Z_X(\lambda) = \Phi(\lambda)\, d Z_\xi(\lambda)$$

and

$$X(t) = \int_{-\infty}^{\infty} e^{it\lambda}\, \Phi(\lambda)\, d Z_\xi(\lambda)\,.$$

The conditions imposed on $\varphi(v)$ and $\Phi(\lambda)$ are sufficient for the relation

$$\int_{-\infty}^{\infty} \xi(t-v)\, \varphi(v)\, d v = \int_{-\infty}^{\infty} e^{it\lambda}\, \Phi(\lambda)\, d z_\xi(\lambda)$$

to hold. Thus

$$X(t) = \int_{-\infty}^{\infty} \xi(t-v)\,\varphi(v)\,dv,$$

which proves our theorem.

The following example is illustrative.

Let u and v be real variables, and let

$$f(u) = \frac{\sin a u}{\pi u}, \quad a > 0,$$

and let the spectral mass of the ξ-process be vanishing outside the interval (α, β), where a exceeds the greater of the two numbers $|\alpha|$ and $|\beta|$. Let us seek the primary process whose Hilbert space is the same as $L_2(\xi)$. Now

$$F(\lambda) = \begin{cases} 1 \text{ for } |\lambda| < a \\ \text{and} \\ 0 \text{ otherwise} \end{cases}$$

and $W - Q$ is $(-a, a)$. $F(\lambda)$ is bounded, and

$$\lim_{\epsilon \to 0} \int_{-(a-\epsilon)}^{a-\epsilon} \frac{1}{|F(\lambda)|^2}\,d\sigma_\xi(\lambda) = \varrho_\xi^2 < \infty.$$

If $\varphi(v)$ is chosen as

$$\varphi(v) = \frac{\sin b v}{\pi v}, \quad b > 0,\ b > |\alpha|,\ b > |\beta|,$$

then

$$\Phi(\lambda) = \begin{cases} 1 \text{ for } |\lambda| < b \\ \text{and} \\ 0 \text{ otherwise} \end{cases}$$

and the conditions of theorem 3. 2 are satisfied, for in (α, β) we have $F(\lambda)\,\Phi(\lambda) = 1$. Hence in this case we obtain from the relation

$$\int_{-\infty}^{\infty} X(t-u)\,\frac{\sin a u}{\pi u}\,du = \xi(t)$$

the inverse smoothing relation

$$\int_{-\infty}^{\infty} \xi(t-v)\,\frac{\sin b v}{\pi v}\,dv = X(t).$$

Let us next examine the nature of the processes for which such an inverse smoothing relationship can exist, when the weight functions $f(u)$ and $\varphi(v)$ are both restricted to belong to the Lebesque class L_1 on $(-\infty, \infty)$. By the Riemann-Lebesque lemma on the Fourier transforms of functions belonging to the class L_1 we have

$$\lim_{|\lambda| \to \infty} F(\lambda) = 0,$$

and a similar result for $\Phi(\lambda)$. Therefore, when the numerical value of λ is sufficiently large, the product $F(\lambda)\,\Phi(\lambda)$ becomes small and cannot become unity. Then condition ii) of the theorem will be satisfied if and only if

$$\int_{|\lambda| > \Lambda_0} d\sigma_X(\lambda) = 0.$$

Having regard to condition i), we can write a similar result involving the spectral measure of the ξ-process. *Hence the only processes that can have a dual smoothing relationship with such weight functions are those whose spectral energy is entirely confined to a finite region of the real axis.* As such processes then have intervals of vanishing spectral mass the integral

$$\int_{-\infty}^{\infty} \frac{|\log \sigma'(\lambda)|}{1 + \lambda^2}\, d\lambda$$

diverges, $\sigma'(\lambda)$ standing for the absolutely continuous part of the spectrum. Hence these processes are deterministic in the sense explained in Chapter I. As every deterministic process need not have its spectral mass confined to a finite part of the real axis, the processes having a dual smoothing relationship with weight functions belonging to the L_1 class form but a subclass of the deterministic ones. Also, these processes are completely characterized from the point of view of spectral properties by some discrete parameter stationary processes, since the spectral mass can by a change of origin and units be repacked in the interval $(-\pi, \pi)$.

In view of the fact that the primary process is derivable as a smoothing of the ξ-process only in special instances, the integral equation derived on the assumption of the existence of an inverse smoothing relation is of limited validity. However, notwithstanding this apparent drawback, one can sometimes construct a sequence of stationary processes $\{X_r(t)\}$ by a repeated application of the integral equation so as to converge to the primary process. The following section deals with this topic.

3.3. Sequence of processes converging to the primary process

In any general case the spectral mass of the processes will spread over the entire real axis. Hence we shall here consider a procedure of sucessively enlarging the region on the real axis so as to bring into the picture by going to the limit the entire spectrum of the processes. In this connection we state

Theorem 3.3. *Let $\{S_r\}$ be a convergent sequence of sets on the real axis to which the set Q and successively receding tail ends of the real axis belong. Let us suppose that we are able to effect a solution of $\varphi_r(v)$ for every r from the integral equation*

$$\int_{W-S_r} |1 - F(\lambda)\,\Phi_r(\lambda)|^2\, d\,\sigma_\xi(\lambda) = 0,$$

where

$$\Phi_r(\lambda) = \int_{-\infty}^{\infty} e^{-iv\lambda}\,\varphi_r(v)\,d\,v,$$

and $\varphi_r(v)$ and $\Phi_r(\lambda)$ are bounded.

Further let

$$\int_{S_r} d\,\sigma_X(\lambda) \to 0 \quad as \quad r \to \infty, \tag{i}$$

and

$$\int_{S_r} |\Phi_r(\lambda)|^2\, d\,\sigma_\xi(\lambda) \to 0 \quad as \quad r \to \infty. \tag{ii}$$

Let the given smoothing relation be consistent. Then the sequence of processes $\{X_r(t)\}$ given by

$$X_r(t) = \int_{-\infty}^{\infty} \xi(t-v)\,\varphi_r(v)\,d\,v$$

converges in the mean to the primary process belonging to $L_2(\xi)$.

Proof: The process $X_r(t)$ constructed is given by

$$X_r(t) = \int_{W-S_r} e^{it\lambda}\,\Phi_r(\lambda)\,d\,Z_\xi(\lambda) + \int_{S_r} e^{it\lambda}\,\Phi_r(\lambda)\,d\,Z_\xi(\lambda).$$

Since by hypothesis the integral equation is satisfied for every r, we can write for $\lambda \in W - S_r$

$$\Phi_r(\lambda) = \frac{1}{F(\lambda)}$$

in respect of each of the spectral measures under consideration. For the same λ it follows that

$$\Phi_r(\lambda)\,d\,Z_\xi(\lambda) = d\,Z_X(\lambda).$$

Hence

$$X(t) - X_r(t) = \int_{S_r} e^{it\lambda}\,d\,Z_X(\lambda)$$

$$- \int_{S_r} e^{it\lambda}\,\Phi_r(\lambda)\,d\,Z_\xi(\lambda).$$

As r tends to infinity, the norm of each term on the right side of the above equation tends to zero, and hence the theorem is proved.

An important case is when the set Q is empty. We can then take $W - S_r$ as the interval $(-r, r)$ and condition (i) of the theorem holds automatically. Further let us assume that $\frac{1}{F(\lambda)}$ is differentiable in every interval $(-r, r)$. Then we define $\psi_r(\lambda)$ and $\varphi_r(v)$ as follows:

$$\psi_r(\lambda) = \begin{cases} \frac{1}{F(\lambda)} \text{ for } \lambda \in (-(r+1), (r+1)) \\ \text{and} \\ 0 \text{ otherwise} \end{cases}$$

and

$$\varphi_r(v) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iv\lambda} \psi_r(\lambda)\, d\lambda.$$

Then

$$\begin{aligned} \Phi_r(\lambda) &= \int_{-\infty}^{\infty} e^{-iv\lambda} \varphi_r(v)\, dv \\ &= \psi_r(\lambda) \text{ for } \lambda \text{ belonging to } (-r, r) \text{ because of the differentiability} \\ &\qquad \text{of } \frac{1}{F(\lambda)} \text{ in } (-(r+1), (r+1)). \end{aligned}$$

Hence for λ in $(-r, r)$ including the end points,

$$\Phi_r(\lambda) F(\lambda) = 1,$$

so that $\Phi_r(\lambda)$ is a solution of the integral equation with the range $W - S_r$. Therefore the sequence of processes are given by

$$\{X_r(t)\} = \left\{ \int_{-\infty}^{\infty} \xi(t-v)\, \varphi_r(v)\, dv \right\},$$

and

$$\underset{r\to\infty}{\text{l.i.m.}}\; X_r(t) = X(t).$$

This method of truncation can be evidently employed when the set Q can be covered by a convergent sequence of sets on the real axis, each set being comprised of a finite number of intervals such that conditions analogous to (i) and (ii) hold.

Let us now turn our attention to the case where the set Q is empty and the given ξ-process has a continuous spectrum. Then the primary process has also a continuous spectrum for otherwise the spectrum of the ξ-process will have to have a saltus part (cf. chapter IV, section **4. 1**). Constructing $\psi_r(\lambda)$ as before as the truncated form of the reciprocal of $F(\lambda)$, we note that $\psi_r(\lambda)$ belongs to the L_1 class. Thus it can be recovered as the Fourier inversion of its Fourier transform except possibly for a set of Lebesque measure zero. As a consequence of the continuity of the spectra, every set of Lebesque measure zero is also of σ_X- and σ_ξ-measure zero. Smoothing $\xi(t)$ with the weight function $\varphi_r(v)$ of the previous paragraph, we get $X_r(t)$. The difference process of $X(t)$ and $X_r(t)$ tends to zero in norm unconditionally. The difference process has been seen earlier to be made up of two terms each of which depends on S_r. The first term tends to zero in norm as $r \to \infty$ because of the convergence of the spectral mass, and the second term

$$\int_{S_r} e^{it\lambda}\, \Phi_r(\lambda)\, d Z_\xi(\lambda)$$

is of vanishing norm, for, when $\lambda \in S_r$

$$\Phi_r(\lambda) = \psi_r(\lambda) = 0, \text{ almost everywhere.}$$

In these cases, each process of the sequence $\{X_r(t)\}$ belongs to that subclass of deterministic processes whose spectral range is but finite and is obtained as a smoothing of the ξ-process. When we focus our attention on processes which have their spectrum spread out over the entire real axis, it is clear that the limit function of weighting does not exist. As the primary process need not be necessarily deterministic, we have incidentally that *it is sometimes possible to approximate to a non-deterministic process by a sequence of deterministic processes with their spectral mass vanishing outside a finite range though continually expanding.*

The following example is illustrative:

Let

$$f(u) = e^{-|u|}.$$

Then

$$\frac{1}{F(\lambda)} = \frac{1+\lambda^2}{2}$$

is differentiable for all λ.

We now have

$$\psi_r(\lambda) = \begin{cases} \dfrac{1+\lambda^2}{2} \text{ for } |\lambda| \le r+1 \\ \text{and} \\ 0 \text{ otherwise} \end{cases}$$

and

$$\varphi_{r-1}(v) = \frac{1}{4\pi}\left\{\frac{2}{v}\left(1+r^2-\frac{2}{v^2}\right)\sin r v + \frac{4r}{v^2}\cos r v\right\}.$$

The two conditions that the set Q is empty and $\frac{1}{F(\lambda)}$ is differentiable are sufficient for the sequence of processes

$$\{ X_r(t) \} = \left\{ \int_{-\infty}^{\infty} \xi(t-v)\, \varphi_r(v)\, d\, v \right\}$$

to converge to $X(t)$ in norm as r tends to infinity.

It may be noted that in this case the sequence of functions $\{ \varphi_r(v) \}$ does not converge to a limit function as r tends to infinity. Yet, the sequence of processes obtained by smoothing the resulting process with these functions as the weight functions converges in norm to the primary process.

3. 4. Uniqueness of inversion

Following an earlier notation, we write

$$X(t) = X_1(t) + X_2(t)$$

where the second process on the right side is arbitrary but for constancy of norm, while the first process is completely specified in the sense of norm. Therefore the primary process will be unique if and only if the arbitrary part is of vanishing norm, which condition is in turn equivalent to the identity of the entire closed Hilbert spaces of the primary and the resulting processes.

3. 5. An orthogonal decomposition

In this section we shall consider a decomposition into two parts of the primary process of a consistent smoothing for which there exists a duality of the smoothing relationship. Then we can form the primary process as the sum of two smoothings of the resulting process, viz., smoothing over the past values of the resulting process and smoothing over the future values of the same. Then it is natural to inquire if the two parts are orthogonal. Relating to this we state

Theorem 3. 5. *Let the smoothing be consistent and let the given ξ-process have absolutely continuous spectrum. If the weight function appearing in the inversion is real and belongs to the Lebesque class L_2 on $(-\infty, \infty)$, and the real and imaginary parts of $\Phi_1 \bar{\Phi}_2$ (occurring below) do not change their sign, then the two parts in question are orthogonal.*

Proof: Putting

$$\varphi(v) = \varphi_1(v) + \varphi_2(v),$$

where

$$\varphi_1(v) = \begin{cases} \varphi(v) \text{ for } v \geq 0 \\ \text{and} \\ 0 \text{ otherwise} \end{cases}$$

$$\varphi_2(v) = \begin{cases} \varphi(v) \text{ for } v < 0 \\ \text{and} \\ 0 \text{ otherwise,} \end{cases}$$

the two parts

$$\int_{v \geq 0} \xi(t-v)\, \varphi_1(v)\, d\, v, \quad \text{and} \quad \int_{v<0} \xi(t-v)\, \varphi_2(v)\, d\, v$$

are orthogonal if

$$E\left[\int \xi(t-v)\, \varphi_1(v)\, d\, v \overline{\int \xi(t-v)\, \varphi_2(v)\, d\, v}\right] = 0$$

which is equivalent to the condition (to be proved)

$$\int_{-\infty}^{\infty} \Phi_1(\lambda)\, \overline{\Phi_2(\lambda)}\, d\, \sigma_\xi(\lambda) = 0,$$

where

$$\Phi_k(\lambda) = \int_{-\infty}^{\infty} e^{-iv\lambda}\, \varphi_k(v)\, d\, v, . \qquad k = 1, 2.$$

Since $\varphi_2(v)$ is real,

$$\overline{\Phi_2(\lambda)} = \Phi_2(-\lambda),$$

and by the assumption that $\varphi(v)$ belongs to the Lebesque class L_2 we know that $\varphi_1(v)$ and $\varphi_2(v)$ also belong to that class. Hence by Parseval's relation we have

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_1(\lambda)\, \overline{\Phi_2(\lambda)}\, d\, \lambda &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_1(\lambda)\, \Phi_2(-\lambda)\, d\, \lambda \\ &= \int_{-\infty}^{\infty} \varphi_1(v)\, \varphi_2(v)\, d\, v \\ &= 0 \text{ by definition of } \varphi_1(v) \text{ and of } \varphi_2(v).\end{aligned}$$

Now we use the hypothesis that the spectrum of the ξ-process is absolutely continuous which implies that all sets of Lebesque measure zero are also of σ_ξ-measure zero. Then from the conditions imposed we have

$$\int_{-\infty}^{\infty} \Phi_1(\lambda)\, \overline{\Phi_2(\lambda)}\, d\, \sigma_\xi(\lambda) = 0$$

which proves the required orthogonality.

3.6. The covariance function of the primary process

In analogy with the discrete parameter case we shall here show that if $m_X(Q) = 0$, then the covariance function $R_X(t)$ of the primary process satisfies the relation

$$\int_{-\infty}^{\infty} R_X(t-u)\, f(u)\, d\,u = \int_{W-Q} e^{it\lambda} \frac{1}{\overline{F(\lambda)}}\, d\,\sigma_\xi(\lambda).$$

Since $m_X(Q) = 0$,

$$X(t) = \int_{W-Q} e^{it\lambda} \frac{1}{F(\lambda)}\, d\,Z_\xi(\lambda)$$

almost certainly. Then

$$\int_{-\infty}^{\infty} R_X(t-u)\, f(u)\, d\,u = \int_{-\infty}^{\infty} E\,\{\, X(t)\, \overline{X(u)}\,\}\, f(u)\, d\,u$$

$$= \int_{-\infty}^{\infty} \left\{ \int_{W-Q} e^{it\lambda}\, e^{-iu\lambda} \frac{1}{|\,F(\lambda)\,|^2}\, d\,\sigma_\xi(\lambda) \right\} f(u)\, d\,u$$

$$= \int_{W-Q} \frac{e^{it\lambda}\, F(\lambda)}{|\,F(\lambda)\,|^2}\, d\,\sigma_\xi(\lambda)$$

on interchanging the order of integrations,

$$= \int_{W-Q} \frac{e^{it\lambda}}{\overline{F(\lambda)}}\, d\,\sigma_\xi(\lambda).$$

3.7. On obtaining $X(t)$ by smoothing $\xi(t)$ over its past values

We next make a brief reference as to when inversion yields the primary process as a smoothing of the resulting process over only the past values. If the function $\Phi(\lambda)$ in the integral equation

$$\int_{-\infty}^{\infty} |\,1 - F(\lambda)\, \Phi(\lambda)\,|^2\, d\,\sigma_\xi(\lambda) = 0$$

can be solved for and has the representation

$$\Phi(\lambda) = \int_0^{\infty} e^{-iv\lambda}\, \varphi(v)\, d\,v,$$

where $\varphi(v)$ and $\Phi(\lambda)$ are bounded, then it is clear that

$$X(t) = \int_0^{\infty} \xi(t-v)\, \varphi(v)\, d\,v,$$

it being however assumed that $m_X(Q) = 0$.

3.8. On the nature of the arbitrary part in inversion

We shall conclude this chapter with some remarks on the nature of the arbitrary stationary process $X_1(t)$ appearing in the solution of $X(t)$ by inversion. We shall treat both cases of the discrete as well as the continuous parameters together by using symbols in a form suitable to either. The smoothing relation is written in a brief way as

$$L X(t) = \xi(t) \tag{A}$$

irrespective of the nature of the parameter. The range W and the function $F(\lambda)$ have their significance appropriate to the instance under consideration.

Associated with the smoothing is the equation

$$L X(t) = 0 \tag{B}$$

and the stationary process which is a solution of this relation is what we call the singular process or the complementary part. This latter equation can be written as

$$\int_Q F(\lambda) e^{it\lambda} d Z_X(\lambda) + \int_{W-Q} F(\lambda) e^{it\lambda} d Z_X(\lambda) = 0$$

from which we get that

$$\int_{W-Q} F(\lambda) e^{it\lambda} d Z_X(\lambda) = 0.$$

Denoting the stationary solution of (B) by $X^1(t)$, we have

$$X^1(t) = \int_W e^{it\lambda} d Z_{X^1}(\lambda)$$

where

$$d Z_{X^1}(\lambda) = \begin{cases} d Z_X(\lambda) \text{ for } \lambda \in Q \\ \text{and} \\ 0 \text{ otherwise.} \end{cases}$$

Thus the part $X_2(t)$ of $X(t)$ previously considered is the same as the singular process $X^1(t)$ associated with the smoothing relation.

We shall next show that the part $X_2(t)$ is deterministic, when the weight function $f(u)$ belongs to the L_1-class. $F(\lambda)$ is now continuous, and does not vanish identically. Hence the set Q of the real zeros of $F(\lambda)$ is not everywhere dense in W, and $W - Q$ contains one or more non-degenerate intervals.

Since $d Z_{X_2}(\lambda) = 0$ in $W - Q$, it follows that

$$\int_W \frac{|\log \sigma'_{X_2}(\lambda)|}{1+\lambda^2} d\lambda = \infty.$$

Therefore $X_2(t)$ is a deterministic process. In particular if the set Q consists only of discrete points, and $m_X(Q) > 0$, the singular process now consists of terms of the type

$$w_k e^{i \lambda_k t},$$

(see **4. 3**), and is thus deterministic.

Chapter IV

Primary and resulting processes compared

4. 1. Mean values, variances, spectral saltuses, and Gaussian nature

In this chapter we shall occupy ourselves mainly with a consideration of the extent to which smoothing transmits to the resulting process some important characteristics which the primary process may have.

We start by noting that the strict or wide sense stationarity of the X-process implies that the ξ-process has the same property. The mean values, spectra and variances of the processes are related by

i) $M_X \cdot (\Sigma a_r) = M_\xi$ in the discrete parameter case and

$$M_X \cdot \int_{-\infty}^{\infty} f(u)\, d u = M_\xi$$

in the continuous parameter case.

ii) $| F(\lambda) |^2 d \sigma_X(\lambda) = d \sigma_\xi(\lambda)$, and

iii) $E | \xi(t) - M_\xi |^2 = \int_W | F(\lambda) |^2 d \sigma_X(\lambda)$, and

$$E | X(t) - M_X |^2 = \int_{W-Q} \frac{1}{| F(\lambda) |^2} d \sigma_\xi(\lambda),$$

provided $m_X(Q) = 0$.

We have also noted earlier that every set of σ_X-measure zero is also of σ_ξ-measure zero, while for the converse to be true it is both necessary and sufficient that $m_X(Q) = 0$.

Let λ_1 be a saltus position of $\sigma_X(\lambda)$ and this point be not in the set Q. Using Lebesque-decomposition of an additive set function in respect of both $\sigma_X(\lambda)$ and $\sigma_\xi(\lambda)$, we find that λ_1 is also a saltus point of $\sigma_\xi(\lambda)$. If on the contrary any λ_1 should bolong to Q, then $m_X(Q) > 0$, and this saltus does not reappear as a saltus of $\sigma_\xi(\lambda)$ having been now obliterated by smoothing. Hence the condition that $m_X(Q) = 0$ ensures that none of the saltuses of $\sigma_X(\lambda)$ belong to Q, and then the saltus positions are common to both the spectra. If the spectrum of the primary process is continuous, so is that of the resulting process, while $m_X(Q) = 0$ is a sufficient condition for the validity of the converse.

If $X(t)$ is a Gaussian process, the same is true of the resulting process. When the smoothing is consistent, and $m_X(Q) = 0$, and the $\xi(t)$-process is Gaussian, it follows from

$$X(t) = \int_{W-Q} e^{it\lambda} \frac{1}{F(\lambda)} d Z_\xi(\lambda)$$

that the $X(t)$-process is also Gaussian.

4.2. A case of metric transitivity

Let the primary process be a Gaussian stationary process with a continuous spectrum, and let the weight function $f(u)$ be suitably chosen so that the smoothing relation is consistent. Then the ξ-process has the same features as $X(t)$. In this case we infer the sameness of the nature of the two processes in respect of metric transitivity from the following theorem of U. GRENANDER [1]: In order that a stationary normal process with a continuous covariance function shall be metrically transitive, it is both necessary and sufficient that the spectrum be continuous.

4.3. Periodicities

From a result of KHINTCHINE [1] which is sometimes called "the statistical ergodic theorem" (see HOPF [1]) we have that for real λ and a stationary process $X(t)$,

$$z_\lambda = \underset{|v-u|\to\infty}{\text{l.i.m.}} \frac{1}{v-u} \int_u^v e^{-it\lambda} X(t)\, d t$$

exists. When z_λ is different from zero, λ is called an "eigen frequency". The eigen frequencies constitute the saltus points of the spectrum of the process and form at most a denumerable set. Further

$$X(t) = \sum_k z_k e^{i\lambda_k t} + \zeta(t),$$

where $\zeta(t)$ and the terms in the summation on the right side are all mutually orthogonal and where $\zeta(t)$ has no eigen frequencies. (See K. KARHUNEN [1]). Hence every periodic term of non-zero norm comes from a saltus in the spectrum.

With this background let us examine the effect of smoothing on periodic terms. It is evident that if $m_X(Q) > 0$, $X(t)$ may have more periodic terms than $\xi(t)$, while if $m_X(Q) = 0$, the periodic terms present in both the processes are the same in respect of frequencies though with differing associated energies (or the squares of norms). Thus *a periodic term may sometimes be obliterated by smoothing, while inversion sometimes leads to the introduction of a periodic term into the primary process with a frequency not present in the resulting process.*

In the case when Q consists of a discrete set of points, and $m_X(Q) > 0$, it must happen that some or all of the points of Q are saltus positions of the spectrum of the X-process. Writing $X(t)$ as in the previous chapters as the sum of two stationary processes $X_1(t)$ and $X_2(t)$, we now see that $X_2(t)$, the singular process forming the complementary part, is almost periodic, being composed entirely of harmonic terms.

The search for periodicities has figured prominently in a number of studies on time series. The rest of this section touches upon some aspects of this in relation to our smoothing problem.

Firstly the relation between the spectra shows how an unstressed frequency in the spectrum of the primary process near the maximum of the function $|F(\lambda)|$ appears as a stressed frequency in the spectrum of the resulting process as a consequence of smoothing. (See also J. L. DOOB [1]). When talking about a single smoothing in contradistinction to continued iteration of it, it is not true to say that smoothing may introduce a periodicity into the resulting process not originally to be found in the primary process, except as being intended to convey in a loosely worded way the idea that there is a shift in the emphasis laid on the spectral frequencies (not necessarily the eigen frequencies), caused by operation of smoothing.

Coming to the topic of continued iteration of the same smoothing operation, it has been recently shown by P. A. P. MORAN [1] that a discrete parameter stationary process when put through the same smoothing again and again will for some types of smoothing yield a process whose spectrum tends to a pure step function as the number of repetitions of the smoothing tends to infinity. To the same set of ideas also belongs the sinusoidal limit theorem of E. SLUTSKY [1], differing however from the result of MORAN in that an iteration of two types of smoothing are under consideration in the treatment of SLUTSKY.

Notwithstanding the foregoing theoretical possibility of the existence of stochastic periodic terms in a process, in any practical instance relating to observed data one generally fails to find exact periodic components, and hence, as has been pointed out by N. WIENER [1], the only spectra that become relevant in applications are almost always those belonging to the continuous type.

4.4. Gaussian Markoff nature

Suppose we consider real Gaussian stationary processes with the additional condition that the primary process is of the Markoff type. We desire to know if this Markoff nature is transmitted to the resulting process in spite of the operation of smoothing.

a) *Discrete parameter case:*

With the hypothesis made, the covariance sequence of the primary process is given by

$$R_X(p) = h^2 c^{|p|},$$

where h^2 is the variance of the X-process and c is a real constant numerically less than or equal to unity. Without any loss of generality we may suppose that the X-process has been standardized to have its variance unity so that $h^2 = 1$. In the case of the discrete parameter processes we have three extreme types of Gaussian Markoff processes, viz., those corresponding to

$$c = -1,\ 1, \text{ and } 0.$$

We shall exclude them from our consideration here.

Let us take the simple example where

$$L = L : \{1, a\}.$$

Then

$$R_\xi(0) = (1 + a\bar{a}) + (a + \bar{a}) \cdot c,$$

$$R_\xi(1) = (1 + a\bar{a})\, c + a + \bar{a}\, c^2,$$

and

$$R_\xi(2) = (1 + a\bar{a})\, c^2 + a\, c + \bar{a}\, c^3.$$

If possible let $\xi(t)$ be also a Markoff process. As a necessary condition we must have

$$[R_\xi(1)]^2 = R_\xi(0) \cdot R_\xi(2),$$

which gives

$$a = 0, \text{ or } c = \pm 1, \text{ or } a + c = 0, \text{ or } \bar{a}\, c = -1.$$

Leaving out the first case where the smoothing is the identity operation, and the second one referring to excluded types, the other alternatives are seen by some calculations to lead to

$$R_\xi(p) = 0$$

which is again an excluded case. Hence *the resulting process cannot belong to the proper Markoff class.* If the weights in the general case are unrelated to the number c, the same procedure shows that smoothing destroys the Markoff property. Nevertheless it is a moot question whether there may not be some types of smoothing with more than two terms in the sequence of weights standing in a special relation to the number c such as lead to resulting processes in which the Markoff property is preserved. It appears likely that the Markoff property is not transmitted to $\xi(t)$.

b) *Continuous parameter case:*

Turning to the continuous parameter case, we answer the question in the negative whenever the weight function $f(u)$ used in the smoothing relation belongs to the L_1 or the L_2 class. The covariance function of the standardized X-process is $e^{-c|t|}$, so that the spectral density is given by

$$\frac{c}{\pi(c^2 + \lambda^2)}.$$

If possible let the resulting process possess the Markoff property. Then its spectral density must be of the form

$$K^2 \cdot \frac{\alpha}{\pi(\alpha^2 + \lambda^2)},$$

and by virtue of the smoothing relation it follows that

$$|F(\lambda)|^2 \cdot \frac{c}{\pi(c^2 + \lambda^2)} = K^2 \cdot \frac{\alpha}{\pi(\alpha^2 + \lambda^2)}.$$

Taking note of the fact that $F(\lambda)$ is a Fourier transform and is hence small at infinity, we find that the two sides of the above equation cannot be equal for numerically large values of λ. Hence it is not possible that the resulting process is also of the Markoff type.

4.5. Deterministic and non-deterministic nature

When the resulting process is formed by smoothing the primary process only over its past values, and when inversion yields the primary process as again a smoothing of the resulting process over only its past values or even as the limit of a sequence of such smoothings, it is clear that

$$L_2\{X; -\infty, t\} = L_2\{\xi; -\infty, t\}.$$

Hence in such cases either both the processes are deterministic or both are non-deterministic. As an example we may consider the instance of a smoothing relation in the discrete parameter case, the operator L being of the form

$$L = L : \{a_0, a_1, \ldots, a_n\},$$

and the roots of the characteristic equation of the smoothing all lying within the unit circle.

Next let us consider the case in which the resulting process is formed by smoothing the primary process over its values in the range $(-\infty, t+h)$. Further, let the resulting process be deterministic, and let us consider the primary process which belongs to $L_2(\xi)$. Then

$$\begin{aligned} L_2(X, -\infty, \infty) &= L_2(\xi, -\infty, \infty) \\ &= L_2(\xi, -\infty, t) && \text{by the assumption of determinism,} \\ &= L_2(X, -\infty, t+h) && \text{due to the nature of the smoothing} \end{aligned}$$

from which we conclude that $X(t)$ is also deterministic.

Lastly, let us consider the weights or the weight function to be such that

$$\int_W \frac{|\log |F(\lambda)||}{1+\lambda^2}\, d\lambda < \infty.$$

(In the continuous parameter case we note that this condition holds good by theorem XII of PALEY and WIENER [1] whenever the weight function $f(u)$ vanishes over a half axis, and $F(\lambda)$ belongs to the Lebesque class L_2 on $(-\infty, \infty)$). If, in such a smoothing, we have the primary process as non-deterministic, the following shows that the resulting process must also be non-deterministic:

$$\begin{aligned}\int_W \frac{|\log \sigma'_\xi(\lambda)|}{1+\lambda^2}\, d\lambda &= \int_W \frac{|\log |F(\lambda)|^2 \sigma'_X(\lambda)|}{1+\lambda^2}\, d\lambda \\ &\leq \int_W 2\frac{|\log |F(\lambda)||}{1+\lambda^2}\, d\lambda + \int_W \frac{|\log \sigma'_X(\lambda)|}{1+\lambda^2}\, d\lambda \\ &< \infty\end{aligned}$$

since each term on the right side is finite as a result of our hypothesis.

CHAPTER V

Estimation and prediction

5.1. An outline of the chapter

In this chapter the following aspects of estimation and prediction are studied:

a) Relationship between the topics of inversion, filtering and estimation is explained.

b) Observing the resulting process over a stretch of time (usually some or all its past values), we desire to construct a linear, unbiassed, and minimum variance estimate of the mean value of the primary process. The case of discrete parameter and finite smoothing has been treated. The relation of this problem to the construction of similar estimates of the mean value of the resulting process by observations on itself considered by U. GRENANDER is explained.

c) The result of inversion may lead to $X(t)$ with a linear representation requiring more than the given set of values of $\xi(t)$. Yet we can sometimes form a linear combination of the given values which may be termed the best estimate of the primary process. This difference between inversion and estimation is illustrated by considering two examples of moving averages, viz., the problem of FRISCH and the case when the characteristic equation has a root equal to 2. In the continuous parameter case we have to construct a function of a real variable and smooth by it the given values of $\xi(t)$ to obtain an estimate of $X(t)$. The function to be constructed for obtaining the best estimate is to be a solution of an integral equation, which has been derived, assuming that the past values of $\xi(t)$ are all known.

d) The relation of our present considerations to WIENER's filtering is discussed. Also the integral equation for the function to be used in prediction of the primary process is obtained, and it is shown that under suitable restrictions this equation yields LEVINSON's form of the corresponding one in WIENER's theory.

5.2. Inversion, estimation and filtering

To begin with we may see how the problem of estimation is connected with the earlier one of inversion. Given the resulting process over a stretch of time, the problem of estimation will consist of constructing a sequence of numbers or a function, as the case may be, using which as the weights or the weight function of smoothing the resulting process, we can recover the primary process to the best possible extent according to some desirable criterion of "the best". The condition of minimum variance is here employed to yield the best estimate. We have seen that the problem of inversion is to obtain $X_1(t)$, and that $X_2(t) \perp L_2(\xi)$. Hence in any attempt at estimation of $X(t)$ with a knowledge of the $\xi(t)$-process, we shall be concerned only with estimating $X_1(t)$; and when estimation gives $X_1(t)$, the results of estimation and inversion coincide. Thus whenever the process formed by the difference of the primary process and its estimate has the square of its norm equal to $m_X(Q)$, estimation solves the problem of inversion. There are however aspects which distinguish them. Linear inversion and linear estimation have not much common ground when the norm of the difference process mentioned above exceeds $+\sqrt{m_X(Q)}$. Consider a finite moving average relation in the discrete parameter case with each root of the characteristic equation being of modulus greater than unity. We have seen earlier that in such a case the part of linear inversion expressed in terms of the past values of the resulting process is zero. As we shall see in **5.4**, it is still possible to construct a linear estimate of non-zero norm in terms of a specified set of the past values of the resulting process.

The process $\xi(t)$ being more or less known, we have at our disposal a certain mix-up of the primary process with itself, and from such a mixture we try to disentangle $X(t)$ in the best possible manner. Hence the construction of an estimate $X^*(t)$ of $X(t)$ is here also an instance of "filtering". When the entire closed Hilbert spaces of the processes are identical, inversion filters out the primary process completely. Again, when we estimate the value of the primary process at the time instant $t+h$, $h>0$, in terms of the values of $\xi(t)$ up to t we deal with "prediction" or prognosis. Putting $h=0$ in this, we can also get an estimate of the primary process in terms of the past values of the resulting process, whenever the integral equation for prediction can be solved.

5. 3. Mean value of the primary process

The object of this section is to form an estimate of the mean value of the primary process using observations made on the resulting process.

Consider the case of finite smoothing in the discrete parameter case, the conditions of consistency of smoothing being satisfied. Let the covariance sequence of the ξ-process be known, *but not its mean value*, except that we have the knowledge that it is non-zero. Let the observations consist of k consecutive values of $\xi(t)$ which we shall denote by $\xi(t), \xi(t-1), \ldots, \xi(t-k+1)$. Under these conditions we wish to form an estimate M_X^* of the mean value M_X of the primary process such that

i) the estimate is linear in the observations,

ii) unbiassed, and

iii) has minimum variance.

If the required estimate is

$$M_X^* = \sum_{j=0}^{k-1} c_j \xi(t-j),$$

the condition of unbiassedness gives

$$\begin{aligned} M_X = E[M_X^*] &= E\left[\sum_{j=0}^{k-1} c_j \xi(t-j)\right] \\ &= E\left[\sum_{j=0}^{k-1} c_j \left\{\sum_{r=0}^{n} a_r X(t-j-r)\right\}\right] \\ &= M_X \cdot \left(\sum_{j=0}^{k-1} c_j\right) \cdot \left(\sum_{r=0}^{n} a_r\right). \end{aligned}$$

Also $\Sigma a_r \neq 0$, since $M_\xi \neq 0$ and the smoothing is consistent. Hence Σc_j is known being the reciprocal of Σa_r. The condition of minimum variance gives that

$$\begin{aligned} E[M_X^* - M_X]^2 &= E[\Sigma c_j \xi(t-j) - M_X]^2 \\ &= H - 2 M_X M_\xi \cdot \Sigma c_j + M_X^2. \end{aligned}$$

is to be a minimum, where H is a quadratic form in the c's with known coefficients as a consequence of our knowledge of the covariance sequence of ξ-process. Further the quadratic form is positive definite. The minimization in question is secured by minimizing H, subject to the linear constraint of unbiassedness. As is known, this problem has a unique solution. Thus the c's and hence M_X^* can be found.

The quadratic form takes the simple diagonal form when the ξ-process is non-autocorrelated, i.e., when the smoothing is an autoregression. Also the coefficient of each c in the linear constraint is the same. Thus in this case the solution is

$$c_1 = c_2 = \cdots = c_k = \frac{1}{k \cdot (\Sigma\, a_r)},$$

and the estimate of the mean value of the primary process is now *equi-distributed* in the observed values of the other process.

In a recent study U. GRENANDER [1] has considered the problem of constructing such estimates of M_ξ in terms of observations on the values of the ξ-process in the continuous parameter case. Let us now examine the connection between the two problems, namely of constructing estimates of the mean values of the two processes, the observations being made on the ξ-process in either case. As the smoothing is a linear relation, the properties of linearity, unbiassedness, and minimum variance are possessed in common by the estimates M_X^* and M_ξ^* whenever they are related by

$$(\Sigma\, a_r) \cdot M_X^* = M_\xi^*.$$

Hence the problem of constructing M_X^* can be solved by constructing first M_ξ^* and then multiplying it by $\frac{1}{\Sigma\, a_r}\cdot$ The same is true in the continuous parameter case as well, $\Sigma\, a_r$ being now replaced by $\int\limits_{-\infty}^{\infty} f(u)\, d\, u$.

5.4. A problem of Frisch

Let the smoothing be a finite moving average in the discrete parameter case with a root of the characteristic equation on $|z| = 1$. Though we could not in this case obtain linear inversion in the strict sense, we can still construct

$$X^*(t) = \xi(t) + b_1\, \xi(t-1) + \cdots + b_N\, \xi(t-N),$$

such that for a given N the norm of $X^*(t) - X(t)$ is a minimum. This determination of the b's is what is here called the problem of R. FRISCH, having been treated by him earlier [1].

The restriction of the case to a moving average specifies the spectral density of the primary process. As such we shall here treat this problem by the spectral method to show that it is at once simple and effective. Writing the ξ's in terms of the primary process, and using the spectral representation of it, we have

$$X(t) - X^*(t) = \int\limits_{-\pi}^{\pi} e^{it\lambda}(1 - P \cdot B)\, d\, Z_X(\lambda),$$

where

$$P = \sum_{r=0}^{n} a_r e^{-ir\lambda}, \text{ with } a_0 = 1,$$

and

$$B = \sum_{j=0}^{N} b_j e^{-ij\lambda}, \text{ with } b_0 = 1.$$

Also

$$d\sigma_X(\lambda) = \frac{1}{2\pi} d\lambda,$$

so that the square of the norm of the error is

$$\| X(t) - X^*(t) \|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |(1 - P \cdot B)|^2 \, d\lambda$$

which is a quadratic polynomial expression in the b's. To select the b's so as to minimize the integral on the right side, we differentiate the integral under the sign of integration with respect to the real and imaginary parts of each of the b's and equate to zero each of the integrals so obtained. They are seen to give

$$\int_{-\pi}^{\pi} e^{ir\lambda} \overline{P} (1 - P \cdot B) \, d\lambda = 0, \qquad r = 1, 2, \ldots, N$$

which are N linear equations to determine the N unknown parameters.

For illustration and comparison we shall consider the case where $L = L$: $\{1, -1\}$, discussed by FRISCH.

The N equations now become

$$2b_1 = 1 + b_2,$$

$$2b_r = b_{r-1} + b_{r+1}, \qquad r = 2, 3, \ldots, N-1,$$

and

$$2b_N = b_{N-1}.$$

These give the unique solution

$$b_k = 1 - \frac{k}{N+1}$$

derived by FRISCH by another method. Using these values of the b's we find that

$$\| X(t) - X^*(t) \|^2 = \frac{1}{2(N+1)} \varrho_\xi^2.$$

Whenever the solution is unique, it constitutes the minimum solution, as there exists just one element to within equivalence in norm which is nearest to $X(t)$ out of all those possible for various values of b's.

The same method can be pressed into service even when the roots of the characteristic equation of a moving average relation all lie outside the unit circle, in spite of the inversive linear representation of $X(t)$ being entirely in terms of the future values of $\xi(t)$. If we consider the equation

$$X(t) - 2X(t-1) = \xi(t),$$

the best two term linear estimate of the primary process in the past values of $\xi(t)$ is obtained as

$$X^*(t) = \xi(t) + \frac{2}{5}\xi(t-1).$$

In this case it will be seen that

$$\|X(t) - X^*(t)\|^2 = \frac{16}{25}\varrho_\xi^2.$$

5.5. The continuous parameter case

Even when estimation of $X(t)$ is made in terms of all the values of $\xi(t)$, i.e., over the entire past and future, the problem is not often equivalent to inversion. When inversion is solvable as a dual smoothing, the inverse weight function to be constructed in the estimation problem has already been seen to satisfy the integral equation arising in that connection.

Suppose that the values of $\xi(t)$ up to the instant t are known, and we desire to form estimates of $X(t)$ in the form

$$X^*(t) = \int_0^\infty \xi(t-v)\, d\varphi(v),$$

the function $\varphi(v)$ being of bounded variation. Then this inverse weight function $\varphi(v)$ has to be the solution of an integral equation, if it is to yield the best estimate of the primary process. The integral equation in question is obtainable by putting $h = 0$ in the integral equation of prediction which we are going to derive in **5.7**. This gives the equation for $\varphi(v)$ as

$$Re\left\{\int_{-\infty}^{\infty}\left(\psi(\lambda) - \int_0^\infty e^{-iu\lambda} d\varphi(u)\right) e^{iv\lambda}\, d\sigma_\xi(\lambda)\right\} = 0, \qquad v > 0,$$

where Re stands for "the real part of", and

$$\psi(\lambda) = \begin{cases} \dfrac{1}{F(\lambda)} & \text{for } \lambda \in W - Q \\ \text{and} & \\ 0 & \text{otherwise.} \end{cases}$$

5.6. Relation to Wiener's filtering

N. WIENER [1] has considered the following problem. Two stationary time series $f(t)$ and $g(t)$ are taken to stand for a message and a superposed disturbance respectively, and the message has to be recovered in the best possible way. This can be achieved if one can find an integral operator which when operating on the combined series gives the best approximation to the message at a required time instant $t + h$. For solving this problem it is supposed that the necessary information regarding the auto-correlation and cross-correlation of the series concerned is available. The point of view adopted in the present study has been somewhat different in respect of the starting assumptions in that we regard that $F(\lambda)$ and $\xi(t)$ as known. However, in view of our earlier study of the subject of inversion, the gap is seen to disappear. It has been observed previously that inversion itself is a special case of filtering. To make this more specific, let us write the smoothing relation in all cases symbolically as

$$L X(t) = \xi(t).$$

This can be written alternatively as

$$(L-1) X(t) + X(t) = \xi(t),$$

and we may designate $X(t)$ and $(L-1)\ X(t)$ respectively as the message and the disturbance. Then our known ξ-process constitutes their sum function. The disentanglement of $X(t)$ (which is inversion) is now described in the terminology of filtering. In view of this, the question of examining inversion in relation to WIENER's problem of filtering has been suggested to me by Professor M. S. BARTLETT in a discussion of my notes on the topic of inversion of a smoothed process. As mentioned in **5. 2**, filtering coincides with inversion if and only if

$$\| X^*(t) - X(t) \|^2 = m_X(Q).$$

The next section is directed towards a clarification of the relationship that exists between the prediction problem for the primary process in our present study and the filtering considered by WIENER. Naturally the two corresponding integral equations are under suitable conditions variant forms of one another, each adapted to the hypothesis made in that treatment. In the present study the resulting process and the weight function $f(u)$ are supposed to be known, and our integral equation for $\varphi(v)$ is expressed in terms of the $F(\lambda)$ and $\sigma_\xi(\lambda)$ which are known. Under suitable restrictions our equation can be made to yield N. LEVINSON's form of the corresponding equation in WIENER's theory. The restrictions needed are evidently such as will recast the integrals with respect to the spectral measure of the ξ-process into the time average functions in the other approach.

5. 7. Integral equation for prediction

We shall here concern ourselves with the continuous parameter case, the modifications required for the discrete case being mentioned at the end. Suppose we take a function $\varphi(v)$ of bounded variation and smooth the resulting process over its past values with this function to obtain an estimate $X^*(t+h)$ of the primary process at the instant $t+h$, i.e.,

$$X^*(t+h) = \int_0^\infty \xi(t-v)\, d\varphi(v).$$

As before we shall assume that the entire closed Hilbert spaces of the two processes are identical, and derive the condition which the function $\varphi(v)$ has to satisfy in order that the norm of the error process may be the least out of all the estimates formed by such smoothings of the ξ-process over its past values. In this connection we shall write

$$\Phi(\lambda) = \int_0^\infty e^{-iv\lambda} d\varphi(v),$$

while $F(\lambda)$ has the same meaning as hitherto. The minimization of the norm of the error process is given by the usual variational considerations leading to the equation

$$\lim_{\epsilon \to 0} \frac{d}{d\epsilon} \| X(t+h) - \int_0^\infty \xi(t-v)\, d\{\varphi(v) + \epsilon\,\delta\varphi(v)\} \|^2 = 0$$

i.e.,

$$E[I_1 - I_2 + \bar{I}_1 - \bar{I}_2] = 0,$$

where

$$I_1 = X(t+h) \overline{\int_0^\infty \xi(t-v)\, d\,\delta\varphi(v)}$$

and

$$I_2 = \xi(t-v)\, d\varphi(v) \overline{\int_0^\infty \xi(t-v)\, d\,\delta\,\varphi(v)},$$

the bars as usual standing for complex conjugates. Using the spectral representation of the processes and the condition of the identity of their Hilbert spaces the expectation of the first two terms is

$$\int_{-\infty}^{\infty} \left[e^{ih\lambda} \psi(\lambda) \int_0^\infty e^{iv\lambda} d\,\delta\overline{\varphi(v)} - \Phi(\lambda) \int_0^\infty e^{iv\lambda} d\,\delta\overline{\varphi(v)} \right] d\sigma_\xi(\lambda)$$

while that of the remaining two terms is its complex conjugate. Hence we have

$$Re\left[\int_{-\infty}^{\infty} \left\{ e^{ih\lambda} \psi(\lambda) \int_0^\infty e^{iv\lambda} d\,\delta\,\overline{\varphi(v)} - \Phi(\lambda) \int_0^\infty e^{iv\lambda} d\,\delta\,\overline{\varphi(v)} \right\} d\sigma_\xi(\lambda) \right] = 0, \quad v > 0,$$

where the prefix symbol *Re* stands for denoting the real part, the function $\psi(\lambda)$ occurring in the above being the same as in **5.5**. As $\varphi(v)$ is a function of bounded variation, and the function $\psi(\lambda)$ is integrable with respect to σ_ξ-measure, the order of integrations can be interchanged by FUBINI's theorem. Then the fact that $\delta\varphi(v)$ is arbitrary gives the condition which $\varphi(v)$ has to satisfy in the form of the following integral equation

$$Re\left[\int_{-\infty}^{\infty} \left\{ e^{ih\lambda} \psi(\lambda) - \int_0^\infty e^{-iu\lambda} d\varphi(u) \right\} e^{iv\lambda} d\sigma_\xi(\lambda) \right] = 0, \qquad v > 0,$$

which we shall refer to as the "integral equation for prediction", if h is a positive number. When $\varphi(v)$ is a solution of the integral equation, we have

$$\left\| X(t+h) - \int_0^\infty \xi(t-v)\, d\left\{\varphi(v) + \epsilon\, \delta\varphi(v)\right\} \right\|^2 = \left\| X(t+h) - \int_0^\infty \xi(t-v)\, d\varphi(v) \right\|^2$$

$$+ \epsilon^2 \left\| \int_0^\infty \xi(t-v)\, d\,\delta\varphi(v) \right\|^2 \geq \left\| X(t+h) - \int_0^\infty \xi(t-v)\, d\varphi(v) \right\|^2.$$

Hence the integral equation is both a necessary and sufficient condition for the norm of the error to be a minimum. Thus we can state.

Theorem 5.7. *If $\varphi(v)$ is a solution of the integral equation of prediction, it constitutes the best weight function to be used in the inverse smoothing to obtain $X(t+h)$ from the values of $\xi(t)$ up to the instant t.*

If the processes and the function $\varphi(v)$ are taken to be real, the prefix symbol Re drops out.

Let us now specialize the processes as follows:

i) The processes be real,
ii) the resulting process $\xi(t)$ be metrically transitive, and
iii) for each fixed t the process

$$\zeta(\tau) = X(t+\tau)\,\xi(\tau)$$

be continuous and stationary in the wide sense with its spectrum continuous in the origin.

With these restrictions we shall rewrite the integral equation for prediction in a form in which $F(\lambda)$ and $\sigma_\xi(\lambda)$ are eliminated by being expressed in terms of the time average functions which are taken as known in the other approach. As our equation will be presently seen to reduce to that of LEVINSON, we shall refer to a matter concerning the notation to prevent a possible misunderstanding. LEVINSON follows in the appendix C of WIENER [1] a different notation from that of chapter III of the same book. In this notation the message is denoted by $g(t)$ and the disturbance by $f(t) - g(t)$, so that $f(t)$ is our $\xi(t)$. Hereafter we follow the notation of the appendix. The first term of the integral equation can be written as

$$E\Big[\int_{-\infty}^{\infty} e^{i(h+v+\tau)\lambda}\,\psi(\lambda)\, dZ_\xi(\lambda) \overline{\int_{-\infty}^{\infty} e^{i\tau\lambda}\, dZ_\xi(\lambda)}\Big] = E\left[X(h+v+\tau)\,\xi(\tau)\right].$$

According to the mean ergodic theorem

$$\underset{T\to\infty}{\text{l.i.m.}} \frac{1}{2T} \int_{-T}^{T} X(t+\tau)\,\xi(\tau)\, d\tau = \chi(t)$$

is a random variable with variance equal to the discrete spectral mass of the process in the origin which is zero by our hypothesis. Hence the limit is almost certainly a constant, being the mean value of the process for each fixed t, so that

$$\chi(t+v) = E[X(t+v+\tau)\,\xi(\tau)].$$

Further on our hypothesis of metric transitivity of the ξ-process we have

$$\Psi(t) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \xi(t+s)\,\xi(s)\,ds = R_\xi(t)$$

almost certainly for all t.

The same method of recasting as has been employed in the case of the first term can be used to obtain the second term of the equation in the form

$$\int_0^\infty \Psi(v-u)\,d\varphi(u).$$

Setting

$$d\varphi(u) = K(u)\,du,$$

the integral equation becomes

$$\chi(h+v) = \int_0^\infty \Psi(v-u)\,K(u)\,du, \qquad v > 0$$

which is equation 2.2 of LEVINSON in the appendix C.

The method of solution in WIENER's theory is still capable of being used (under the condition of the existence of certain Fourier transforms, cf. appendix C) even in the general case when the processes are not specialized to make the expectations previously considered convertible to the time average forms, provided however

$$d\varphi(u) = K(u)\,du$$

where $K(u)$ is real. For, we shall show now that if we employ the same notation in the general case as well, we can arrive at the same form of the equation when $K(u)$ is real. Let

$$Re\Big\{\int_{-\infty}^{\infty} e^{ih\lambda}\psi(\lambda)\,d\sigma_\xi(\lambda)\Big\} \text{ be denoted by } \chi(h)$$

and

$$Re\{R_\xi(h)\} \text{ by } \Psi(h).$$

Then the functions $\chi(h)$ and $\Psi(h)$ are known. We have

$$\begin{aligned}\chi(h+v) &= Re\left\{\int_{-\infty}^{\infty}\psi(\lambda)\,e^{i(h+v)\lambda}\,d\sigma_\xi(\lambda)\right\}\\ &= Re\left\{\int_{-\infty}^{\infty}\Phi(\lambda)\,e^{iv\lambda}\,d\sigma_\xi(\lambda)\right\} \text{ by the integral equation,}\\ &= Re\left\{\int_{0}^{\infty}R_\xi(v-u)\,K(u)\,du\right\} \text{ as before}\\ &= \int_{0}^{\infty}Re\,\{R_\xi(v-u)\,K(u)\,du\} \text{ since } K(u) \text{ is real}\\ &= \int_{0}^{\infty}\Psi(v-u)\,K(u)\,du.\end{aligned}$$

Hence when $K(u)$ is real, we may adopt the same method of solution of the integral equation as in WIENER's theory.

Turning to the case of the discrete parameter we shall need the following changes to be made. The range for λ is to be altered to $(-\pi, \pi)$, and $F(\lambda)$ now becomes the familiar $P(e^{-i\lambda})$,

while

$$\Phi(\lambda) = \int_0^\infty e^{-iv\lambda}\,d\varphi(v)$$

stands for a function with the representation

$$\sum_{r=0}^{\infty} b_r\,e^{-ir\lambda}.$$

The number h will in this case be a positive integer. The sequence of numbers required to be constructed in our estimation problem is then given by $\{b_r\}$.

CHAPTER VI

Some generalizations

6.1. Inversion in terms of $\xi(t)$ and its derived processes

In chapter III we have considered the inversion of a smoothing relation in the case of stationary processes of continuous parameter. In theorem **3.2** we obtained the necessary and sufficient conditions for each of the processes to be a smoothing of the other. This was seen to be applicable only to instances of processes with restricted types of spectra. To enlarge the class of relevant processes, inversion was attempted not as a single smoothing of the given $\xi(t)$-process, but as the limit in the mean of a sequence of processes each of which was a smoothing of the resulting process (see theorem **3.3**). Here we shall consider yet another way of obtaining a solution of the primary process as a combined smoothing of the resulting process and its derived processes.

Let $f(u)$ be bounded and the associated $F(\lambda)$ exist and be bounded, and let the smoothing relation

$$\int_{-\infty}^{\infty} X(t-u) f(u)\, d u = \xi(t)$$

be consistent, and consequently let the given ξ-process have derived processes (as explained in chapter I) up to order p. $\left(\text{These will be denoted by } \frac{d\xi(t)}{dt}, \frac{d^2\xi(t)}{dt^2}, \ldots, \frac{d^p\xi(t)}{dt^p}\right)$. The present purpose is to seek inversion of the smoothing relation to obtain $X(t)$ in the form

$$X(t) = \int_{-\infty}^{\infty} \xi(t-v)\varphi(v)\, d v + \sum_{r=0}^{p} b_r \frac{d^r\xi(t)}{dt^r},$$

where $\varphi(v)$ is bounded and

$$\Phi(\lambda) = \int_{-\infty}^{\infty} e^{-iv\lambda}\varphi(v)\, d v$$

exists and is bounded. Each of the derived processes is given by

$$\frac{d^r\xi(t)}{dt^r} = \int_{-\infty}^{\infty} e^{it\lambda}(i\lambda)^r\, d Z_\xi(\lambda),$$

so that, if $X(t)$ has the above representation, then

$$X(t) = \int_{-\infty}^{\infty} e^{it\lambda}\{\Phi(\lambda) + \sum_{r=0}^{p} b_r (i\lambda)^r\}\, d Z_\xi(\lambda).$$

By just adopting the same line of argument as employed in proving theorem **3.2**, we can prove the following

Theorem 6.1. *If $\xi(t)$ is a given continuous parameter stationary process which is continuous in the mean and if the smoothing relation*

$$\int_{-\infty}^{\infty} X(t-u) f(u)\, d u = \xi(t)$$

is consistent and necessitates the existence of the derived processes of $\xi(t)$ up to the order p, then for obtaining the inversion as

$$X(t) = \int_{-\infty}^{\infty} \xi(t-v)\,\varphi(v)\,dv + \sum_{r=0}^{p} b_r \frac{d^r \xi(t)}{dt^r}$$

it is sufficient that

i) $L_2(X) = L_2(\xi)$ *and*

ii) $\int_{-\infty}^{\infty} \left| 1 - F(\lambda)\left\{\Phi(\lambda) + \sum_{r=0}^{p} b_r (i\lambda)^r\right\}\right|^2 d\,\sigma_\xi(\lambda) = 0,$

where $F(\lambda)$ *has the usual meaning.*

The following example is of particular interest in that it has been already considered in connection with theorem **3.3**.

Example:

Let

$$f(u) = e^{-|u|},$$

and let the smoothing be consistent. Then

$$F(\lambda) = \frac{2}{1+\lambda^2}$$

the set Q of real zeros of which is empty. Then the third condition of consistency gives that

$$\int_{-\infty}^{\infty} \frac{1}{|F(\lambda)|^2}\,d\,\sigma_\xi(\lambda) = \int_{-\infty}^{\infty} \frac{(1+\lambda^2)^2}{4}\,d\,\sigma_\xi(\lambda)$$

is finite. From this it follows that the ξ-process has derived processes up to the second order. Let us put

$$\varphi(v) = 0 \text{ so that } \Phi(\lambda) = 0,$$

and choose

$$b_1 = 0, \text{ and } b_0 = -b_2 = \tfrac{1}{2}.$$

Then

$$1 - F(\lambda)\left\{\Phi(\lambda) + \sum_{r=0}^{2} b_r (i\lambda)^r\right\} = 0.$$

Further since Q is empty,

$$m_X(Q) = 0, \text{ i.e., } L_2(X) = L_2(\xi).$$

Hence by theorem **6.1**

$$X(t) = \tfrac{1}{2}\,\xi(t) - \tfrac{1}{2}\,\frac{d^2\xi(t)}{dt^2}. \tag{B}$$

Therefore the sequence of processes

$$\{X_r(t)\} = \left\{\int_{-\infty}^{\infty} \xi(t-v)\,\varphi_r(v)\,dv\right\}$$

constructed in **3.3** converges in norm to the right side of (B).

It is clear that we can generalize the smoothing relation itself into the form

$$\int_{-\infty}^{\infty} X(t-u)\,f(u)\,du + \sum_{s=0}^{l} a_s \frac{d^s X(t)}{dt^s} = \xi(t)$$

and seek inversion in a similar form, viz.,

$$X(t) = \int_{-\infty}^{\infty} \xi(t-v)\,\varphi(v)\,dv + \sum_{r=0}^{n} b_r \frac{d^r \xi(t)}{dt^r}.$$

Then the conditions will read as

i) $L_2(X) = L_2(\xi)$ and

ii) $\displaystyle\int_{-\infty}^{\infty} \left|1 - \{F(\lambda) + \sum_{s=0}^{l} a_s (i\lambda)^s\}\, \{\Phi(\lambda) + \sum_{r=0}^{l} b_r (i\lambda)^r\}\right|^2 d\sigma_\xi(\lambda) = 0.$

6.2. Vector processes

Suppose $(X_r(t))$ to be a set of k stationary processes such that $L_2(X_r) \perp L_2(X_s)$, $r \neq s$. Let k^2 smoothings (L_{lm}) be given, and let

$$L_{lr} X_r(t) = \xi_l(t),$$

the repeated index r standing for summation from 1 to k. Such a summation convention will be used in what follows. We now suppose the vector process $(\xi_l(t))$ is made up of k stationary processes which are stationarily correlated with each other. The $(\xi_r(t))$-process and the k^2 smoothing operators (L_{rs}) are assumed to be known. Then the problem of inversion will consist in obtaining the primary vector process $(X_r(t))$. It is not proposed to go into any details of this and related problems here, except to point out that they are more or less analogous to those treated in this thesis. Corresponding to each smoothing operator L_{rs} we have as before a function $F(\lambda)$ which we shall denote by $F_{rs}(\lambda)$. The set of real zeros of $F_{rs}(\lambda)$ is denoted by Q_{rs}, while the set of real λ for which

$$\det |F_{rs}(\lambda)| = 0$$

is denoted by Q. Also with the smoothing operation L_{rs} we associate the constant

$$C_{rs} = \begin{cases} \sum_l a_{(rs)\,l}, & \text{the sum of the weights of the smoothing, in the discrete parameter case} \\ \text{and} & \\ \int_{-\infty}^{\infty} f_{(rs)}(u)\,du & \text{in the continuous parameter case, } f_{rs}(u) \text{ being the weight function used in the smoothing } L_{rs}. \end{cases}$$

The determinant $|C_{rs}|$ is denoted by Δ. We shall denote the k mean values of the given resulting vector process by (M_{ξ_r}). Then the first condition of consistency of the smoothing vector relationship is seen to be that

$$\text{if } (M_{\xi_r}) \text{ is not the null vector, } \Delta \neq 0. \tag{i}$$

The method of **1.9** combined with the assumption of the orthogonality of the Hilbert spaces of the components of the primary vector process leads us to the relation

$$d Z_{\xi_s}(\lambda) = F_{sl}(\lambda)\, d Z_{X_l}(\lambda) \tag{A}$$

from which we have as the second condition of consistency that for each fixed s

$$m_{\xi_s}(Q_{s1} \cdot Q_{s2} \cdot \cdots \cdot Q_{sk}) = 0, \qquad s = 1, 2, \ldots, k. \tag{ii}$$

(Note that s (being fixed in any equation) is not a summation symbol).

When $\lambda \in W - Q$, we can solve the linear equations (A) and obtain

$$d Z_{X_r}(\lambda) = G_{rs}(\lambda)\, d Z_{\xi_s}(\lambda).$$

The third condition of consistency will be that

$$\Big(\int\limits_{W-Q} e^{it\lambda} G_{rs}(\lambda)\, d Z_{\xi_s}(\lambda) \Big)$$

is composed of mutually orthogonal processes of finite norm.

When these conditions are satisfied, we obtain the primary process belonging to the Hilbert space of the given resulting vector process uniquely (in the sense of norm) as

$$X_r(t) = \int\limits_{W-Q} e^{it\lambda} G_{rs}(\lambda)\, d Z_{\xi_s}(\lambda).$$

6.3. Certain processes of bounded norm

Lastly let us consider the case where the resulting process $\xi(t)$ is assumed to be of bounded norm but not stationary, is of continuous parameter, and is adjusted to have its mean value function $M_\xi(t) = 0$ for all t. Further let this process have KARHUNEN's representation

$$\xi(t) = \int\limits_{-\infty}^{\infty} \beta(t, \lambda)\, d Z(\lambda).$$

We take the smoothing relation to be as before

$$\int\limits_{-\infty}^{\infty} X(t-u) f(u)\, du = \xi(t).$$

In our attempts to recover the primary process we have in our previous work obtained by use of Fourier transforms a linear relationship between the random spectral functions of the two processes from the given smoothing relationship. As this is no longer possible, we shall specify the problem of inversion as follows:

to find $X(t)$ of bounded norm and having the representation

$$X(t) = \int_{-\infty}^{\infty} \alpha(t, \lambda)\, dZ(\lambda).$$

In this specification we have $L_2(X) = L_2(\xi)$. Also the mean value of $X(t)$ may be taken to be zero for all t. We must then have

$$\int_{-\infty}^{\infty} |\alpha(t, \lambda)|^2\, d\sigma(\lambda) \text{ bounded}. \tag{i}$$

Also from KARHUNEN's theorem of **1.7**, it is necessary that

$$\int_{-\infty}^{\infty} \Big| \int_{-\infty}^{\infty} \alpha(t-u, \lambda) f(u)\, du \Big|^2 d\sigma(\lambda) \text{ is bounded}. \tag{ii}$$

Then

$$\begin{aligned} \int_{-\infty}^{\infty} \Big\{ \int_{-\infty}^{\infty} \alpha(t-u, \lambda) f(u)\, du \Big\} dZ(\lambda) &= \int_{-\infty}^{\infty} X(t-u) f(u)\, du \\ &= \xi(t) \\ &= \int_{-\infty}^{\infty} \beta(t, \lambda)\, dZ(\lambda) \end{aligned}$$

so that

$$\int_{-\infty}^{\infty} \Big| \beta(t, \lambda) - \int_{-\infty}^{\infty} \alpha(t-u, \lambda) f(u)\, du \Big|^2 d\sigma(\lambda) = 0 \tag{iii}$$

for all t.

When (iii) furnishes a solution of $\alpha(t, \lambda)$ which satisfies (i) and (ii), it follows that

$$X(t) = \int_{-\infty}^{\infty} \alpha(t, \lambda)\, dZ(\lambda)$$

exists as an element of $L_2(Z) = L_2(\xi)$, and constitutes a solution of the primary process.

Suppose further that a function $\varphi(v)$ of the real variable v can be found such that

$$\int_{-\infty}^{\infty} \Big| \alpha(t, \lambda) - \int_{-\infty}^{\infty} \beta(t-v, \lambda)\, \varphi(v)\, dv \Big|^2 d\sigma(\lambda) = 0$$

for all t, and

$$\int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \beta(t-v, \lambda)\, \varphi(v)\, dv \right|^2 d\sigma(\lambda)$$

is bounded. Then it is seen that

$$X(t) = \int_{-\infty}^{\infty} \xi(t-v)\, \varphi(v)\, dv.$$

Additional note

The object of appending this note is to make explicit what is meant by taking the expectation of the process obtained by integrating a stationary process. Let us start by considering the case where the stationary process $X(t)$ has the mean value zero, and the resulting process $\xi(t)$ is given by the integral smoothing

$$\xi(t) = \int_{-\infty}^{\infty} X(t-u)\, f(u)\, du = \underset{\substack{a \to -\infty \\ b \to \infty}}{\text{l.i.m.}} \int_a^b X(t-u)\, f(u)\, du.$$

Then

$$\xi(t) = \int_{-\infty}^{\infty} e^{it\lambda} F(\lambda)\, dZ_X(\lambda),$$

where the random spectral function $Z_X(s)$ is such that $E(Z_X(s)) = 0$. In this case $E(\xi(t)) = 0$.

We now turn our attention to the case in which the mean value of the process $X(t)$ is a constant $M_X \neq 0$. In this case we define the integral

$$\int_{-\infty}^{\infty} X(t-u)\, f(u)\, du$$

as

$$\int_{-\infty}^{\infty} (X(t-u) - M_X)\, f(u)\, du + \int_{-\infty}^{\infty} M_X \cdot f(u)\, du,$$

where the first of the integrals is the integral of a stationary process of mean value zero in the sense already explained, and the second one is an ordinary infinite integral. Thus in this case we have

$$\xi(t) = \int_{-\infty}^{\infty} e^{it\lambda} F(\lambda)\, dZ_X(\lambda) + M_X \cdot \int_{-\infty}^{\infty} f(u)\, du.$$

Taking the expectation of both sides we have

$$M_\xi = E(\xi(t)) = M_X \cdot \int_{-\infty}^{\infty} f(u)\, du.$$

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