Read 6 June, 1951

Behavior of solutions of linear second order differential equations

By Einar Hille

1. Introduction. The present note is concerned with the differential equation

$$w'' = \lambda F(x) w, \tag{1.1}$$

where $F(x)$ is defined, positive and continuous for $0 \leqq x < \infty$, while λ is a complex parameter which, except in section 2, is not allowed to take on real values $\leqq 0$. We are mainly interested in qualitative properties of the solutions for large positive values of x including integrability properties on the interval $(0, \infty)$. In section 6 we shall discuss certain extremal problems for this class of differentiel equations.

The results are of some importance for the theory of the partial differential equations of the Fokker-Planck-Kolmogoroff type corresponding to temporally homogeneous stochastic processes. These applications will be published elsewhere. The results also admit of a dynamical formulation and interpretation. This will be used frequently in the following for purposes of exposition. With $x = t$, the equation

$$w'' = \lambda F(t) w \tag{1.2}$$

is the equation of motion in complex vector form of a particle

$$w = u + i v = r e^{i\theta} \tag{1.3}$$

under the influence of a force of magnitude

$$|P| = \varrho F(t) r, \quad \lambda = \varrho e^{i\varphi} = \mu + i\nu, \tag{1.4}$$

making the constant angle φ with the radius vector. We can also write the equations of motion in the form

$$r'' - r(\theta')^2 = \mu F(t) r, \tag{1.5}$$

$$\frac{d}{dt}[r^2 \theta'] = \nu F(t) r^2, \tag{1.6}$$

where the left sides are the radial and the transverse accelerations respectively.

2. Almost uniform motion. As a preliminary step in the discussion we eliminate the fairly trivial case in which $x F(x) \in L(0, \infty)$. This case is basic for the applications referred to above, however.

Theorem 1. *A necessary and sufficient condition that* (1.1) *have a fundamental system of the form*

$$w_1(x) = x[1 + o(1)], \; w_2(x) = 1 + o(1), \; x \to \infty, \tag{2.1}$$

for some fixed $\lambda \neq 0$ *is that* $x F(x) \in L(0, \infty)$. *If this condition is satisfied, then* (2.1) *holds for all* λ *and we have also*

$$w_1'(x) = 1 + o(1), \; w_2'(x) = o(1), \; x \to \infty. \tag{2.2}$$

In the dynamical interpretation we could refer to this case as *almost uniform motion* (uniform motion corresponds to $F(t) \equiv 0$).

The sufficiency of the condition has been traced back to M. BÔCHER [1], p. 47, the necessity for $\lambda > 0$ to H. WEYL [5], p. 42, but it does not occur explicity in either place. A direct proof for $\lambda = 1$, $F(x)$ real was given by the author [3], pp. 237—238, in 1948; the sufficiency argument is valid also for complex-valued $F(x)$ and the necessity was proved when $F(x)$ is real and keeps a constant sign for large x. A considerably less restrictive sufficient condition for complex-valued $F(x)$ was given by A. WINTNER [7] in 1949 who also gave an alternate proof for the necessity of the condition $x F(x) \in L(0, \infty)$ when $F(x)$ is real and ultimately of constant sign. Wintner referred to the case he studied as *almost free linear motion.* Theorem 1 is a special case of the following more general result:

Theorem 2. *If* $G(x)$ *is continuous for* $0 \leqq x < \infty$ *and there exist a real* β *and a positive* δ *such that* $|\arg [e^{-i\beta} G(x)]| \leqq \frac{1}{2}\pi - \delta$ *for all large* x, *then the differential equation*

$$w'' = G(x) w \tag{2.3}$$

has a fundamental system of the form (2.1) *if and only if* $x G(x) \in L(0, \infty)$. *This system also satisfies* (2.2).

Proof. It is only the necessity that calls for a proof. Set

$$e^{-i\beta} G(x) = G_1(x) + i G_2(x), \; w_2(x) = x[1 + \eta_1(x) + i \eta_2(x)]$$

and suppose that a is so large that for $x \geqq a$ we have $G_1(x) \geqq 0$,

$$|G_2(x)/G_1(x)| \leqq \cot \delta, \; |\eta_\nu(x)| \leqq \tfrac{1}{3} \min(1, \tan \delta), \; \nu = 1, 2.$$

From

$$\Re \{e^{-i\beta} [w_2'(x) - w_2'(a)]\} = \int_a^x s G_1(s) \left\{1 + \eta_1(s) - \frac{G_2(s)}{G_1(s)} \eta_2(s)\right\} ds,$$

it follows that

$$\tfrac{5}{3} \int_a^x s G_1(s)\, ds \geqq \Re \{e^{-i\beta} [w_2'(x) - w_2'(a)]\} \geqq \tfrac{1}{3} \int_a^x s G_1(s)\, ds.$$

Integration of this inequality shows that $w_2(x)$ cannot be $O(x)$ unless $x G_1(x) \in L(0, \infty)$ and this implies and is implied by $x G(x) \in L(0, \infty)$. This completes the proof of Theorems 1 and 2.

In Theorem 1 we may take as our hypothesis the existence of a solution satisfying any one of the three conditions

$$\text{(i)}\ \ w_1(x) = x[1 + o(1)], \quad \text{(ii)}\ \ w_2(x) = 1 + o(1), \quad \text{(iii)}\ \ w_1'(x) = 1 + o(1).$$

They are equivalent and imply

$$\text{(iv)}\ \ w_2'(x) = o(1)$$

as well as $x F(x) \in L(0, \infty)$. On the other hand, (iv) does not imply (i)—(iii) or the integrability condition.

3. Direct motion. In the remainder of this paper it will be assumed that $x F(x)$ is not in $L(0, \infty)$. Further λ will be restricted to the domain Λ obtained by deleting the origin and the negative real axis from the λ-plane. If λ is real and negative, the solutions of (1.1) are normally oscillatory and their behavior is entirely different from that holding for $\lambda \in \Lambda$. This fairly well known case will not be considered in the following.

The behavior for real positive values of λ is also well known (see, for instance H. Weyl [5], § 1), but it sets the pattern for the rest of Λ so we shall summarize the results. Let $w_0(x) = w_0(x, \lambda)$, $w_1(x) = w_1(x, \lambda)$ be the fundamental system determined by the initial conditions

$$w_0(0) = 0, \ w_0'(0) = 1; \quad w_1(0) = 1, \ w_1'(0) = 0. \tag{3.1}$$

For $x > 0$, $\lambda > 0$, these solutions are positive, monotone increasing and convex downwards so that $w_k(x, \lambda)/x \to \infty$ with x by Theorem 1. Simple counterexamples show that no stronger assertion can be made concerning the rate of growth (cf. section 6 below). In passing we note the Liapounoff-Birkhoff inequalities which in the present case may be given the form

$$\begin{aligned} C_k \exp\left\{-\sqrt{\varrho}\int_0^x [1 + F(s)]\,ds\right\} &\leq [w_k(x, \varrho)]^2 + \frac{1}{\varrho}[w_k'(x, \varrho)]^2 \\ &\leq C_k \exp\left\{\sqrt{\varrho}\int_0^x [1 + F(s)]\,ds\right\}, \end{aligned} \tag{3.2}$$

where C_k is the initial value of the second member for $x = 0$.

For fixed x the solutions $w_k(x, \lambda)$ are entire functions of λ given by the power series

$$w_k(x, \lambda) = \sum_{n=0}^{\infty} u_{k,n}(x)\,\lambda^n, \quad u_{k,n}(x) = \int_0^x (x - s)\,F(s)\,u_{k,n-1}(s)\,ds \tag{3.3}$$

with $u_{0,0}(x)=x$, $u_{1,0}(x)=1$. The coefficients being never negative, one sees that

$$(3.4) \qquad |w_k(x,\lambda)| \leqq w_k(x,\varrho), \quad |w_k'(x,\lambda)| \leqq w_k'(x,\varrho).$$

This in conjunction with (3.2) shows that $w_k(x,\lambda)$ is an entire function of λ of order $\leqq \frac{1}{2}$. This could also be inferred from (3.3).

For $\lambda > 0$ the formula

$$(3.5) \qquad w_+(x,\lambda) = w_1(x,\lambda) \int_x^\infty \frac{ds}{[w_1(s,\lambda)]^2}$$

is obviously meaningful and defines a *subdominant solution* of (1.1). It is positive, monotone decreasing, convex downwards and tends to zero when $x \to \infty$. The first property is obvious; the second follows from

$$w_+'(x,\lambda) = w_1'(x,\lambda) \int_x^\infty \frac{ds}{[w_1(s,\lambda)]^2} - \frac{1}{w_1(x,\lambda)}$$
$$< \int_x^\infty \frac{w_1'(s,\lambda)\,ds}{[w_1(s,\lambda)]^2} - \frac{1}{w_1(x,\lambda)} = 0,$$

and the convexity is implied by (1.1). Finally

$$w_+(x,\lambda) < \frac{w_1(x,\lambda)}{w_1'(x,\lambda)} \int_x^\infty \frac{w_1'(s,\lambda)\,ds}{[w_1(s,\lambda)]^2} = \frac{1}{w_1'(x,\lambda)} \to 0.$$

as $x \to \infty$.

The descriptive properties of $w_k(x,\lambda)$ for complex λ are more complicated than for $\lambda > 0$, but the following results hold.

Theorem 3. *If $xF(x) \notin L(0,\infty)$, if $\lambda = \mu + i\nu \in \Lambda$ and $\nu \neq 0$, then $w_k(x,\lambda)$ describes a spiral $S_k(\lambda)$ from k to ∞ in the complex w-plane as x goes from 0 to $+\infty$, $k=0, 1$, and $\frac{1}{\nu}\arg w_k(x,\lambda)$ increases steadily from 0 to $+\infty$ with x. $S_k(\lambda)$ has a positive radius of curvature everywhere and is concave towards the origin. If $\mu \geqq 0$, $|w_k(x,\lambda)|$ is monotone increasing and $|w_k(x,\lambda)|/x \to \infty$ with x when $\mu > 0$. For all $\lambda \in \Lambda$, $|w_k(x,\lambda)|^{-1} \in L_2(1,\infty)$.*

Proof. We set $|w_k(x,\lambda)| = r_k(x,\lambda) = r_k(x) = r_k$ and $\arg w_k(x,\lambda) = \theta_k(x,\lambda) = \theta_k(x) = \theta_k$ with $\theta_k(0,\lambda) \equiv 0$. The functions $r_k(t)$ and $\theta_k(t)$ satisfy equations (1.5) and (1.6). From the former we see that for $\mu \geqq 0$ we have $r_k'' > 0$, r_k' increasing, from 1 if $k=0$, from 0 if $k=1$. Thus $r_k' > 0$ and r_k is increasing. Further r_k' tends to a limit, γ_k say, as $t \to \infty$. If γ_k is finite, $r_k(t) \sim \gamma_k t$. But $tF(t) \notin L(1,\infty)$ and we have

$$r_k'(t) - r_k'(0) > \mu \int_0^t F(s)\, r_k(s)\, ds,$$

which tends to infinity with t if $\mu > 0$. Hence $r_k'(t) \to \infty$ if $\mu > 0$ and $r_k(t)/t \to \infty$ with t. If $k=0$ the ratio $r_k(t)/t$ is increasing for $t > 0$ and if $k=1$ it is increasing for all large t for

$$t r_k'(t) - r_k(t) > \mu \int_0^t s\, F(s)\, r_k(s)\, ds - r_k(0).$$

If $\mu = 0$, we still have $r_k'' > 0$, $r_k' > 0$ and increasing so that $r_k(t) > Ct$. An example in section 6 below shows that $r_k(t)$ may actually be $O(t)$ when $\mu = 0$.

A Sturmian comparison argument applied to the equations

$$R'' = \mu F(x) R, \quad r'' = \{\mu F(x) + [\theta'(x)]^2\} r$$

gives

$$w_k(x, |\mu + i\nu|) > |w_k(x, \mu + i\nu)| > w_k(x, \mu), \; x > 0, \tag{3.6}$$

provided $\mu > 0$, $\nu \neq 0$.

The expression for the transverse acceleration in (1.6) gives

$$[r_k(t)]^2 \theta_k'(t) = \nu \int_0 F(s) [r_k(s)]^2 ds \tag{3.7}$$

so that $(1/\nu)\, \theta_k'(t)$ is positive. This equation gives the angular momentum of a particle of unit mass at the time t; if $\mu > 0$ it clearly becomes infinite with t and, as we shall see, the same is true everywhere in Λ for $\nu \neq 0$. We note in passing that the left side of (3.7) is never zero for $t > 0$ and this implies that neither $w_k(t, \lambda)$ nor $w_k'(t, \lambda)$ can be zero for $t > 0$, $\lambda \in \Lambda$.

We shall now prove that $(1/\nu) \arg w_k(x, \lambda)$ tends to infinity with x. Suppose, contrariwise, that it tends to a finite limit ω_k instead and set

$$W_k(x) = e^{-i\nu\omega_k} w_k(x) = U_k(x) + i\, V_k(x).$$

For a given $\varepsilon > 0$, we can find an $a = a_\varepsilon$ such that $U_k(x) > 0$ and

$$0 \leqq |V_k(x)| \leqq \varepsilon\, U_k(x)$$

for $x \geqq a$. We have

$$W_k'(x) - W_k'(a) = \lambda \int_a^x F(s)\, W_k(s)\, ds. \tag{3.8}$$

Here there are two possibilities. First, the integral may tend to a finite limit as $x \to \infty$ so that $W_k'(x)$ tends to a finite limit. This limit cannot be different from zero since otherwise the case of almost uniform motion would be present. If $W_k'(\infty) = 0$, a second integration gives

$$W_k(x) = W_k(a) - \lambda \int_a^x ds \int_s^\infty F(t)\, W_k(t)\, dt.$$

Here the repeated integral must become infinite with x since otherwise the almost uniform case would turn up again. This implies that $W_k(x)$ becomes infinite in such a manner that its argument tends to zero while in the right member of the equation the term that becomes infinite has an argument differing from that of $-\lambda$ by at most ε. Thus the first possibility leads to a contradiction. Secondly, the integral in (3.8) may become infinite with x. Integration of (3.8) then shows that $W_k(x)$ differs from

$$\lambda \int_a^x (x-t) F(t) W_k(t)\, dt$$

by (linear) terms of lower order. This alternative gives rise to the same type of contradiction: $W_k(x)$ has an argument close to zero, while that of the dominating term in the second member is close to arg λ. Thus the assumption that arg $w_k(x)$ stays bounded must be rejected.

Let us now introduce some notation. We set

$$M_k(x, \lambda) = \int_0^x |w_k'(s, \lambda)|^2 \, ds, \tag{3.9}$$

$$N_k(x, \lambda) = \int_0^x F(s) |w_k(s, \lambda)|^2 \, ds, \tag{3.10}$$

$$L_k(x, \lambda) = M_k(x, \lambda) + \varrho N_k(x, \lambda), \quad \varrho = |\lambda|. \tag{3.11}$$

These are obviously positive increasing functions of x and it will be shown later that they all become infinite with x. We also have

$$\overline{w_k(x, \lambda)}\, w_k'(x, \lambda) = M_k(x, \lambda) + \lambda N_k(x, \lambda). \tag{3.12}$$

This is the Green's transform of the equation (1.1) corresponding to the interval $(0, x)$. See E. HILLE [3], p. 3, and E. L. INCE [4], p. 508. This formula also shows that the product occurring on the left cannot vanish for $x \neq 0$ and $\lambda \in \Lambda$.

As a first application of (3.12) let us verify the assertion concerning the radius of curvature For a complex curve $w = w(x)$ the radius of curvature is given by

$$R = \frac{|w'|^3}{\Im[\bar{w}' w'']}.$$

Using (1.1) and (3.12) this reduces to

$$R_k = \frac{|w_k'(x, \lambda)|^3}{\nu F(x) M_k(x, \lambda)} \tag{3.13}$$

if $\nu > 0$. For $\nu < 0$, the sign should be reversed since the spirals $S_k(\lambda)$ are then described in the negative sense. Note that there are no points of inflection.

Another application of (3.12) is the observation that if

$$\arg \lambda = \varphi = 2\gamma, \quad |\gamma| < \tfrac{1}{2}\pi,$$

then

$$\cos \gamma\, L_k(x, \lambda) = \Re\,[e^{-i\gamma}\, \overline{w_k(x, \lambda)}\, w_k'(x, \lambda)]. \tag{3.14}$$

Recombined with (3.12) this leads to the basic double inequality

$$\cos \gamma\, L_k(x, \lambda) \leqq |w_k(x, \lambda)\, w_k'(x, \lambda)| \leqq L_k(x, \lambda). \tag{3.15}$$

As a first consequence of (3.15) we note that

$$\frac{\cos^2 \gamma}{|w_k(x)|^2} \leqq \frac{|w_k'(x)|^2}{[L_k(x)]^2} \leqq \frac{L_k'(x)}{[L_k(x)]^2}.$$

Hence

$$\cos^2 \gamma \int_x^\infty \frac{ds}{|w_k(s, \lambda)|^2} < \frac{1}{L_k(x, \lambda)}, \tag{3.16}$$

so that $|w_k(x, \lambda)|^{-1} \in L_2(1, \infty)$ as asserted. From this fact, together with the observation that $\arg w_k(x, \lambda)$ becomes infinite with x, one concludes from (3.7) that $N_k(x, \lambda)$ and hence also $L_k(x, \lambda)$ become infinite with x for every $\lambda \in \Lambda$. The same is true for $M_k(x, \lambda)$ by virtue of (3.16) and the estimate

$$|w_k(x, \lambda) - k|^2 \leqq x\, M_k(x, \lambda) \tag{3.17}$$

which follows from Schwarz's inequality applied to

$$w_k(x, \lambda) - k = \int_0^x w_k'(s, \lambda)\, ds.$$

We also note the inequality

$$|w_k(x, \lambda)|^2 \leqq k + 2 \int_0^x L_k(s, \lambda)\, ds \tag{3.18}$$

which may be read off from (3.15). This completes the proof of Theorem 3.

Some further consequences of (3.15) are listed in

Theorem 4. *Let $\Phi(u)$ be any non-negative function integrable over every finite interval $(0, \omega)$. Let $0 < a < b < \infty$, $A = L_k(a, \lambda)$, $B = L_k(b, \lambda)$. Let ξ, η, ζ be arbitrary non-negative constants. Then*

$$\begin{aligned} \int_a^b \left\{ \xi \frac{\cos^2 \gamma}{|w_k(x, \lambda)|^2} + \eta\, \varrho \frac{\cos^2 \gamma\, F(x)}{|w_k'(x, \lambda)|^2} + \zeta \frac{2\sqrt{\varrho}\, \cos \gamma \sqrt{F(x)}}{L_k(x, \lambda)} \right\} \Phi\,[L_k(x, \lambda)]\, dx \\ \leqq (\xi + \eta + \zeta) \int_A^B \Phi(u) \frac{du}{u^2}. \end{aligned} \tag{3.19}$$

Proof. The inequality is obviously the sum of three inequalities obtained by setting two of the parameters equal to zero. Here the inequality for $\eta=\zeta=0$ is proved exactly as formula (3.16) which is the special case $\Phi(u)\equiv 1$ and the case $\xi=\zeta=0$ is handled in a similar manner using the inequality

$$\varrho F(x)\,|w_k(x,\lambda)|^2 < L_k'(x,\lambda).$$

The case $\xi=\eta=0$ follows from the inequality between the geometric and the arithmetic means which gives

$$2\sqrt{\varrho F(x)}\,|w_k(x,\lambda)\,w_k'(x,\lambda)| \leqq L_k'(x,\lambda)$$

whence

$$2\sqrt{\varrho}\cos\gamma\sqrt{F(x)} < \frac{L_k'(x,\lambda)}{L_k(x.\lambda)}. \tag{3.20}$$

Incidentally, the last inequality also shows that

$$L_k(x,\lambda) > L_k(1,\lambda)\exp\left\{2\sqrt{\varrho}\cos\gamma\int_1^x\sqrt{F(s)}\,ds\right\}. \tag{3.21}$$

Since the three fractions within the braces in (3.19) appear to have similar integrability properties, one might expect them to have similar behavior for large values of x. By analogy with the case of $\lambda>0$ (cf. A. WIMAN [6] p. 17) one might expect that

$$\lim_{x\to\infty}\frac{\sqrt{F(x)}\,w_k(x,\lambda)}{w_k'(x,\lambda)}=\frac{1}{\sqrt{\lambda}}=\varrho^{-\frac{1}{2}}e^{-i\gamma}, \tag{3.22}$$

at least if $F(x)$ is suitably limited. We are not able to prove any such relation, but we can show that

$$\limsup_{x\to\infty}\frac{1}{x}\int_1^x\sqrt{F(s)}\left|\frac{w_k(s,\lambda)}{w_k'(s,\lambda)}\right|ds \leqq \varrho^{-\frac{1}{2}}\sec^2\gamma. \tag{3.23}$$

Indeed, using (3.18) and (3.20) one sees that $\sqrt{\varrho}\cos^2\gamma$ times the integrand does not exceed

$$\frac{1}{2}\frac{L_k'(s,\lambda)}{[L_k(s,\lambda)]^2}\left[k+2\int_0^s L_k(t,\lambda)\,dt\right]$$

and an integration by parts gives the desired inequality.

4. Retrograde motion. The results of the preceding section hold, at least for sufficiently large values of x, for all solutions with one striking exception. Formula (3.5) makes sense for all $\lambda\in\Lambda$ and defines a solution of (1.1). This will be referred to as the *exceptional* or the *subdominant solution.*

Theorem 5. *If $x F(x) \notin L(0, \infty)$ and if $\lambda \in \Lambda$, $\nu \neq 0$, then the subdominant solution $w_+(x, \lambda)$ describes a spiral $S_+(\lambda)$ in the complex w-plane as x goes from 0 to infinity. The sense of rotation is opposite to that of $S_1(\lambda)$ and $-(1/\nu)$ $\arg w_+(x, \lambda)$ increases to $+\infty$ with x. $S_+(\lambda)$ has a positive radius of curvature and is concave towards the origin. $|w_+(x, \lambda)|$ is monotone decreasing if $\mu \geqq 0$ and tends to zero if $\mu > 0$. For each $\lambda \in \Lambda$ we have that $w_+(x, \lambda) w_+'(x, \lambda) \to 0$ as $x \to \infty$ and $F(x) |w_+(x, \lambda)|^2$ and $|w_+'(x, \lambda)|^2$ belong to $L(0, \infty)$.*

Proof. That $F(x) |w_+(x, \lambda)|^2 \in L(0, \infty)$ follows from (3.5) and (3.16) together with $\varrho F(x) |w_1(x, \lambda)|^2 < L_1'(x, \lambda)$. Further

$$(4.1) \qquad \begin{aligned} |w_+'(x, \lambda)| &= \left| w_1'(x, \lambda) \int_x^\infty \frac{ds}{[w_1(s, \lambda)]^2} - \frac{1}{w_1(x, \lambda)} \right| \\ &< \sec^2 \gamma \frac{|w_1'(x, \lambda)|}{L_1(x, \lambda)} + \frac{1}{|w_1(x, \lambda)|} \end{aligned}$$

and both terms in the third member are in $L_2(1, \infty)$ so the same is true for the first member. We have also

$$(4.2) \qquad w_+(x, \lambda) w_+'(x, \lambda) = w_1(x, \lambda) w_1'(x, \lambda) \left\{ \int_x^\infty \frac{ds}{[w_1(s, \lambda)]^2} \right\}^2 - \int_x^\infty \frac{ds}{[w_1(s, \lambda)]^2}$$

and by (3.15) and (3.16) both terms on the right are $O\{[L_1(x, \lambda)]^{-1}\}$ and thus tend to zero when $x \to \infty$.

The Green's transform for the interval (x, ∞) gives

$$(4.3) \qquad \begin{aligned} \overline{w_+(x, \lambda)}\, w_+'(x, \lambda) &= -\int_x^\infty |w_+'(s, \lambda)|^2 ds - \lambda \int_x^\infty F(s) |w_+(s, \lambda)|^2 ds \\ &\equiv - M_+(x, \lambda) - \lambda N_+(x, \lambda), \end{aligned}$$

so the first member is different from zero for $0 \leqq x < \infty$, $\lambda \in \Lambda$. In particular, the integral occurring in formula (3.5) is never zero. Further

$$(4.4) \qquad \frac{d}{dx} |w_+(x, \lambda)|^2 = -2 [M_+(x, \lambda) + \mu N_+(x, \lambda)]$$

so that $|w_+(x, \lambda)|$ is monotone decreasing for $\mu \geqq 0$ and tends to a positive limit or zero. If $F(x) \notin L(0, \infty)$ and $\mu > 0$, the first possibility is obviously excluded. It is easily verified, however, that

$$(4.5) \qquad w_0(x, \lambda) = w_+(0, \lambda) w_+(x, \lambda) \int_0^x \frac{ds}{[w_+(s, \lambda)]^2}$$

so that if $|w_+(x, \lambda)| \to C > 0$ it would follow that $w_0(x, \lambda) = O(x)$ and this is false for $\mu > 0$, but it may be true for $\mu = 0$ as shown by an example in section 6. Next we observe that

$$\arg w_+(x, \lambda) = \arg w_+(0, \lambda) - \nu \int_0^x \frac{N_+(s, \lambda)}{|w_+(s, \lambda)|^2} ds,$$

whence the monotony properties of the argument follow. If now the left member should tend to a finite limit as $x \to \infty$, formula (4.5) may be used to show that $\arg w_0(x, \lambda)$ must also tend to a finite limit and this we know is not true. Finally the radius of curvature R_+ of $S_+(\lambda)$ is obtained from formula (3.13), replacing the subscript k by $+$. This completes the proof.

In passing we note that

$$w_0(x, \lambda) + w_+(x, \lambda) = w_1(x, \lambda) \int_0^\infty \frac{ds}{[w_1(s, \lambda)]^2}, \tag{4.6}$$

where the integral is finite and different from zero for $\lambda \in \Lambda$.

The solutions $w_1(x, \lambda)$ and $w_+(x, \lambda)$ are evidently linearly independent when $\lambda \in \Lambda$. From this we conclude that the subdominant is characterized uniquely up to a multiplicative constant by anyone of the properties listed in Theorem 5. In particular, $C\, w_+(x, \lambda)$ *is the only solution describing a retrograde spiral if* $S_1(\lambda)$ *is considered as defining the direct motion.* Any other solution will describe a spiral which is ultimately direct.

We note that formula (3.6) has an analogue for $w_+(x, \lambda)$. This is proved in essentially the same manner, but requires the use of part one of Theorem 8 below. The resulting inequality is

$$\left| \frac{w_+(x, \mu + i\nu)}{w_+(0, \mu + i\nu)} \right| < \frac{w_+(x, \mu)}{w_+(0, \mu)}, \quad \mu > 0, \tag{4.7}$$

The other formulas of section 3 may also be extended. Introducing

$$L_+(x, \lambda) = M_+(x, \lambda) + \varrho N_+(x, \lambda), \tag{4.8}$$

we obtain inequalities like

$$\cos\gamma\, L_+(x, \lambda) \leqq |w_+(x, \lambda) w'_+(x, \lambda)| \leqq L_+(x, \lambda), \tag{4.9}$$

$$L_+(x, \lambda) \leqq L_+(0, \lambda) \exp\left\{-2\sqrt{\varrho}\cos\gamma \int_0^x \sqrt{F(s)}\, ds\right\}, \tag{4.10}$$

as well as analogues of formulas (3.19) and (3.20).

The three functions L_0, L_1, and L_+ are not unrelated. We conclude from (4.6) that $w_0(x, \lambda)/w_1(x, \lambda)$ and $w'_0(x, \lambda)/w'_1(x, \lambda)$ tend to the same limit as $x \to \infty$, namely the integral in the right member. It follows that $L_0(x, \lambda)/L_1(x, \lambda)$ is bounded away from zero and infinity with bounds depending upon γ. Similarly, (4.2) shows that $L_1(x, \lambda)\, L_+(x, \lambda)$ is bounded above, the bound depending upon γ. Thus there is essentially only one L-function governing the rate of growth of the solutions of (1.1) or, more precisely, of the products

$$w(x, \lambda)\, w'(x, \lambda) = \tfrac{1}{2} \frac{d}{dx} [w(x, \lambda)]^2$$

which apparently behave in a more regular manner than the solutions themselves when $\mu < 0$.

5. Further study of the subdominant. In the study of Cauchy's problem for the generalized heat equation

$$\frac{\partial^2 U}{d x^2} = F(x) \frac{\partial U}{\partial t}$$

and the adjoint (Fokker-Planck) equation one needs solutions of (1.1) having special properties. These properties are satisfied by the subdominant solution for $\mu > 0$ and cannot possibly be satisfied by any other solution. For $\mu \leqq 0$ we need more information than what is given by Theorem 5. In particular, we want to know if

$$\lim_{x \to \infty} w_+(x, \lambda) = 0, \tag{5.1}$$

$$F(x)\, w_+(x, \lambda) \in L(0, \infty) \tag{5.2}$$

for values of λ in the left half-plane at some distance from the negative real axis. The present section is concerned with these and related questions. We start by introducing some notation and definitions.

We set

$$F_1(x) = \int_0^x F(s)\, ds, \quad F_{\frac{1}{2}}(x) = \int_0^x \sqrt{F(s)}\, ds. \tag{5.3}$$

A positive function $G(x)$ will be said to be of *upper order* ω_2 and *lower order* ω_1 (at infinity) if

$$\lim_{x \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log G(x)}{\log x} = \begin{cases} \omega_2 \\ \omega_1 \end{cases}. \tag{5.4}$$

$F(x)$ satisfies the condition $M(a)$ if

$$\limsup_{x \to \infty} \frac{\log F_1(x)}{F_{\frac{1}{2}}(x)} = a < \infty. \tag{5.5}$$

Finally $P(\eta)$ shall denote the part of the complex λ-plane to the right of the parabola

$$4 \varrho \cos^2 \gamma = \eta^2, \quad \lambda = \varrho e^{2 i \gamma}, \tag{5.6}$$

having its focus at the origin and its axis directed along the negatixe real axis.

Theorem 6. *If* $L_+(x, \lambda) \in L(0, \infty)$, *then the area swept by the radius vector of* $w_+(x, \lambda)$ *is finite and* (5.1) *holds.*

Proof. From the expression for the transverse acceleration in (1.6) we get

$$|w_+(x, \lambda)|^2 \theta'_+(x, \lambda) = -\nu N_+(x, \lambda)$$

so that a necessary and sufficient condition that the radius vector sweep a finite area is that $N_+(x, \lambda) \in L(0, \infty)$ and this is certainly satisfied if $L_+(x, \lambda) \in L(0, \infty)$ or, what is equivalent, that $[L_1(x, \lambda)]^{-1} \in L(1, \infty)$. Formula (4.9) shows then that $w_+(x, \lambda)$ tends to a limit which must be zero since $x F(x) \notin L(0, \infty)$. Formula (4.10) gives a

Corollary. *The area swept by the radius vector of* $w_+(x, \lambda)$ *is finite and* (5.1) *holds, both for* λ *in* $P(a)$, *if* $\exp[F_{\frac{1}{2}}(x)]$ *is of lower order* $1/a$.

Theorem 7. *A sufficient condition that* $S_+(\lambda)$ *be of finite length is that* $|w_1(x, \lambda)|^{-1} \in L(1, \infty)$. *In this case* (5.1) *also holds.*

This follows from (4.1) combined with (3.15). The condition is not necessary even for real positive λ.

Let us now turn to the validity of (5.2) the dynamical interpretation of which is that *the length of the hodograph of* $S_+(\lambda)$ *is finite.*

Theorem 8. (5.3) *is valid if anyone of the following conditions is satisfied:*

(i) $\mu > 0$,
(ii) $F(x) \in L(0, \infty)$, $\lambda \in \Lambda$,
(iii) $F(x)$ *satisfies condition* $M(a)$ *and* $\lambda \in P(a)$.

Proof. Case (i) follows from (1.5) with $r = r_+ = |w_+(t, \lambda)|$. If $\mu > 0$, we know that $r''_+ > 0$, $r'_+ < 0$, so that r'_+ tends to a finite limit when $t \to \infty$. Hence $r''_+ \in L(0, \infty)$ and the conclusion is immediate. The fact that also $(\theta'_+)^2 r_+ \in L(0, \infty)$ is used in the proof of (4.7). Case (ii) follows from Theorem 5 and Schwarz's inequality.

Case (iii) is based upon the implications of formula (3.15). We have

$$\int_a^\beta F(x)\,|w_+(x, \lambda)|\,dx < \sec^2\gamma \int_a^\beta \frac{F(x)\,|w_1(x, \lambda)|\,dx}{L_1(x, \lambda)} < \frac{1}{\varrho}\sec^2\gamma \int_a^\beta \frac{L'_1(x, \lambda)\,dx}{|w_1(x, \lambda)|\,L_1(x, \lambda)}.$$

This means that if $|w_1(x, \lambda)| \geqq [L_1(x, \lambda)]^\varepsilon$ with a fixed $\varepsilon > 0$ in the interval (a, β), then the first member is dominated by $(\varrho\varepsilon)^{-1} \sec^2\gamma\,[L_1(a, \lambda)]^{-\varepsilon}$. For a given $\varepsilon > 0$, let

$$S_\varepsilon = [x \,\|\, w_1(x, \lambda)| \geqq [L_1(x, \lambda)]^\varepsilon,\ x \geqq 0]$$

and let E_ε be the complementary set in $(0, \infty)$. Then the integral of $F(x)\,|w_+(x, \lambda)|$ over the set S_ε is finite. The assertion is consequently proved if we can show that condition $M(a)$ implies the existence for every $\lambda \in P(a)$ of an $\varepsilon_0 = \varepsilon_0(\lambda)$ such that the exceptional set E_ε is bounded for $\varepsilon < \varepsilon_0$. This is proved by showing that the assumptions give two estimates of $|w'_1(x, \lambda)|$ in E_ε and these become inconsistent if ε is small and E_ε is unbounded.

For x in E_ε we have by (3.15)

$$|w_1'(x,\lambda)| > \cos\gamma\,[L_1(x,\lambda)]^{1-\varepsilon}.$$

But

$$w_1'(x,\lambda) = \lambda\int_0^x F(s)\,w_1(s,\lambda)\,ds,$$

so

$$|w_1'(x,\lambda)|^2 \leqq \varrho^2 F_1(x)\,N_1(x,\lambda) < \varrho\,F_1(x)\,L_1(x,\lambda).$$

On the other hand, if $\delta > 0$ is given, condition $M(a)$ combined with (3.2) show that for large x, $x \geqq x_\delta$,

$$F_1(x) \leqq e^{(a+\delta)F_{\frac{1}{2}}(x)} < C_0(\lambda)\,[L_1(x,\lambda)]^{\frac{1}{2}(a+\delta)\varrho^{-\frac{1}{2}}\sec\gamma}.$$

The resulting double inequality for $|w_1'(x,\lambda)|$ implies that a certain power of $L_1(x,\lambda)$ is bounded away from zero on the set E_ε. If E_ε is unbounded, this requires that the exponent of $L_1(x,\lambda)$ be non-negative. Since δ is arbitrary, this gives

$$4\,\varrho\,\cos^2\gamma \leqq \left(\frac{a}{1-2\,\varepsilon}\right)^2,$$

provided $\varepsilon < \frac{1}{2}$, as we may assume. But if $\lambda = \varrho\,e^{2i\gamma}$ is a point in $P(a)$, this inequality cannot hold for arbitrarily small values of ε. This shows the existence of an $\varepsilon_0(\lambda)$ such that E_ε is actually bounded for $\varepsilon < \varepsilon_0$ and that

(5.7) $$|w_1(x,\lambda)| \geqq [L_1(x,\lambda)]^{\varepsilon}$$

holds for all large x when $\varepsilon < \varepsilon_0$. This completes the proof.

It shold be observed that (5.2) also implies

(5.8) $$w_+(x,\lambda)\left[\frac{d}{dx}\arg w_+(x,\lambda)\right]^2 \in L(0,\infty),$$

(5.9) $$\lim w_+'(x,\lambda) = 0.$$

The first relation follows readily from (1.5), the second is implied by

$$w_+'(x,\lambda) - w_+'(0,\lambda) = \lambda\int_0^x F(s)\,w_+(s,\lambda)\,ds$$

and $x\,F(x) \notin L(0,\infty)$.

The peculiar condition $M(a)$ serves to exclude functions $F(x)$ of highly irregular behavior. It is not necessarily satisfied by increasing functions $F(x)$; for such a function the inferior limit of the quotient in (5.5) is always zero, but the superior limit may very well be infinite. This happens, for instance, if

$$\sqrt{F(x)} = \begin{cases} G_{n-1}, & n-1 \leqq x \leqq n - g_n^{-1}, \\ G_{n-1} + g_n^2\,(x - n + g_n^{-1}), & n - g_n^{-1} < x < n, \end{cases} \qquad n = 1, 2, 3, \ldots$$

where

$$G_n = \sum_{k=0}^{n} g_k, \quad g_k = e^{k\, g_{k-1}}, \quad g_0 = 1.$$

While the sudden spurts of $F(x)$ have very little local effect on $F_{\frac{1}{2}}(x)$, they do affect $F_1(x)$ so strongly that $\log F_1(n) > C\, n\, F_{\frac{1}{2}}(n)$.

Condition $M(a)$ is satisfied if there exists a monotone increasing function $Q(x)$ such that

$$(5.10) \qquad [Q'(x)]^2 \leqq F(x) \leqq Q'(x)\, e^{a\, Q(x)}$$

as is easily seen. Another sufficient condition is the existence of a positive integrable function $G(u)$ such that

$$(5.11) \qquad F(x) \leqq G\,[F_{\frac{1}{2}}(x)], \quad \int_0^u G(s)\, d\,s \leqq e^{a\,u}.$$

In particular, taking $G(u)$ as a constant, we see that *if $F(x)$ is bounded then* (5.2) *holds in some parabolic domain $P(a)$ if $F_{\frac{1}{2}}(\infty) < \infty$ and everywhere in Λ if $F_{\frac{1}{2}}(\infty) = \infty$.*

Theorem 9. *There exists a parabolic domain $P(c)$ in which* (5.1) *and* (5.2) *are both satisfied if either*

(1) *$F(x)$ satisfies condition $M(a)$, $F_1(x)$ is of lower order $1/\sigma$, and $c \geqq \max(a, \sigma\, a)$, or*

(2) *$\exp[F_{\frac{1}{2}}(x)]$ is of lower order $1/\tau$, $F(x)$ is of upper order ω, and $c \geqq \tau(\omega + 1)$.*

Proof. In the first case $[F_1(x)]^{-\sigma-\varepsilon} \in L(1, \infty)$ for every $\varepsilon > 0$ and condition $M(a)$ gives

$$[F_1(x)]^{-\sigma-\varepsilon} \geqq \exp\,[-(\sigma + \varepsilon)(a + \delta)\, F_{\frac{1}{2}}(x)]$$

for $x \geqq x_\delta$. But if $\lambda \in P(\sigma a)$, then $2\,\varrho^{\frac{1}{2}} \cos\gamma \geqq (\sigma + \varepsilon)(a + \delta)$, provided δ and ε are sufficiently small. It then follows from (4.10) that $L_+(x, \lambda) \in L(0, \infty)$ so the conclusion of Theorem 6 applies in $P(\sigma\, a)$ while that of Theorem 8 holds in $P(a)$.

In the second case the Corollary of Theorem 6 shows that (5.1) holds for $\lambda \in P(\tau)$. For (5.2) we have to go back to the proof of Theorem 8. If E_ε is the exceptional set in which (5.7) fails to hold for a particular fixed ε, then

$$\int_{E_\varepsilon} F(x)\,|\,w_+(x, \lambda)\,|\,d\,x < \sec^2\gamma \int_{E_\varepsilon} F(x)\,[L_1(x, \lambda)]^{\varepsilon-1}\, d\,x.$$

For large values of x we have $F(x) < x^{\omega+\delta}$, $\delta > 0$, while

$$\log L_1(x, \lambda) > \log L_1(1, \lambda) + \left(\frac{1}{\tau} - \eta\right) 2\,\varrho^{\frac{1}{2}} \cos\gamma \log x$$

by (3.21) and the hypothesis. Since $\varepsilon > 0$ is at our disposal, a simple calculation shows that the integral over the exceptional set is finite as soon as $\lambda \in P\,[\tau(\omega + 1)]$. This completes the proof.

We shall see in the next section that $w_+(x, \lambda)$ need not tend to zero when $x \to \infty$ and λ is purely imaginary, but we have no example of an equation for which $F(x)\, w_+(x, \lambda)$ fails to be in $L(0, \infty)$ for any $\lambda \in \Lambda$. We have spoken above of the exceptional set E_ε in which (5.7) fails to hold for a particular ε, $0 < \varepsilon < \frac{1}{2}$. We do not know if E_ε can be unbounded no matter how small ε is; at any rate the exceptional intervals must be quite short and far apart because for every $\delta > 0$ the integral of $[L_1(x, \lambda)]^{1-2\varepsilon-\delta}$ over E_ε exists as may be shown with the aid of Theorem 4 taking $\Phi(u) = u^{1-\delta}$.

6. Extremal problems. We shall discuss briefly the question of the extremals for the rate of growth of the solutions. To measure the rate of growth of a solution $w(x, \lambda)$ we use its *upper order*

$$\sigma(\lambda) = \sigma(\lambda, w) = \limsup_{x\to\infty} \frac{\log |w(x, \lambda)|}{\log x}, \tag{6.1}$$

which may be finite or infinite. For a given $F(x)$, the upper order of a solution has only two possible values, $\sigma_n(\lambda)$ and $\sigma_s(\lambda)$, the former corresponding to $w_1(x, \lambda)$, the latter to $w_+(x, \lambda)$. These orders satisfy the following inequalities:

$$\sigma_s(\lambda) \leqq \tfrac{1}{2} \leqq \sigma_n(\lambda) \leqq \sigma_n(\varrho), \qquad \lambda \in \Lambda. \tag{6.2}$$

$$\sigma_s(\lambda) \leqq 0, \;\; 1 \leqq \sigma_n(\lambda), \qquad \mu \geqq 0, \tag{6.3}$$

$$\sigma_s(\lambda) \leqq \sigma_s(\mu), \;\; \sigma_n(\mu) \leqq \sigma_n(\lambda), \qquad \mu \geqq 0, \tag{6.4}$$

where as usual $\lambda = \mu + i\nu$, $\varrho = |\lambda|$.

The first part of (6.2) follows from

$$|w_+(x, \lambda) - w_+(0, \lambda)|^2 \leqq x \int_0^x |w_+'(s, \lambda)|^2\, ds \leqq C x,$$

the second from (3.16), and the third from (3.4); (6.3) follows from Theorems 3 and 5, while (6.4) is derived from (3.6) and (4.7).

Here (6.2) is a best possible inequality, for if μ is given, $\mu < 0$, we can choose $a > 0$ and ν such that when $F(x) = (1 + a x)^{-2}$, the orders $\sigma_s(\mu + i\nu)$ and $\sigma_n(\mu + i\nu)$ both differ from $\frac{1}{2}$ by less than any preassigned number (see also below). This of course also implies that $|w_+(x, \lambda)|$ may be monotone increasing to infinity when $\mu < 0$, a behavior totally different from what takes place for $\mu \geqq 0$. Formula (6.3) is also a best possible estimate and here we can show that the bounds are reached everywhere in the right half-plane for a suitable choice of $F(x)$. Thus if

$$F(x) = (x + 2)^{-2} [\log(x + 2)]^{-1} \{1 + (\alpha - 1)[\log(x + 2)]^{-1}\}$$

with $\alpha \geqq 1$ and $|\lambda| \leqq \alpha$, we have

$$|w_1(x, \lambda)| \leqq w_1(x, |\lambda|) \leqq w_1(x, \alpha).$$

But for $\lambda = \alpha$ the equation has the solution $(x+2)[\log(x+2)]^{\alpha}$ and this is manifestly not the subdominant solution. Its order being unity, it follows that $\sigma_n(\lambda) = 1$ for $\mu \geqq 0$. Further

$$w_+(x, \alpha) = [C_\alpha + o(1)][\log(x+2)]^{-\alpha},$$

so that $\sigma_s(\alpha) = 0$ and a simple argument shows that $\sigma_s(\lambda) = 0$ holds for $\mu \geqq 0$.

Theorems 3 and 5 suggest the possibility of $w_1(x, \nu i) = O(x)$ and $w_+(x, \nu i) = O(1)$, not $o(1)$, for suitable choices of $F(x)$. This is indeed the case, but examples are harder to construct. We may take, however, as argument the function

$$\theta_+(x) = [\log c]^{\frac{1}{3}} - [\log(x+c)]^{\frac{1}{3}}, \quad c > 1,$$

and determine the corresponding absolute value $r_+(x) = r_+(x, 1)$ as the subdominant solution of the differential equation

$$r'' = \tfrac{1}{9}(x+c)^{-2}[\log(x+c)]^{-\frac{4}{3}}\, r.$$

By Theorem 1, $r_+(x) = 1 + o(1)$ (with a suitable normalization of the solution) and the function $r_+(x)\, e^{i\theta_+(x)}$ satisfies a differential equation of type (1.1) with $\lambda = i$ and $F(x) > 0$ for $x \geqq 0$, provided c is sufficiently large to start with. For large x we have

$$F(x) = \tfrac{1}{3}(x+c)^{-2}[\log(x+c)]^{-\frac{5}{3}}[1 + o(1)].$$

This means that for the corresponding equation (1.1) the solution $w_+(x, i)$ is a constant multiple of a function which for large x is of the form

$$[1 + o(1)]\, e^{-i[\log(x+c)]^{\frac{1}{3}}}$$

so that the spiral $S_+(i)$ has an *asymptotic circle*. Naturally such a circle can arise only when $F(x) \in L(0, \infty)$. The corresponding solution $w_1(x, i)$ may be shown to be $O(x)$, the minimal order, with the aid of (4.5) and (4.6).

Let us finally consider the class $\mathbf{L}$ of all linear differential equations of type (1.1) with $x F(x) \notin L(0, \infty)$. For each equation in $\mathbf{L}$ there are two indices $\sigma_n(\lambda; F)$ and $\sigma_s(\lambda; F)$. For a given $\lambda \in \Lambda$ we consider $\inf \sigma_n(\lambda; F)$ and $\sup \sigma_s(\lambda; F)$ where $F(x)$ ranges over all admissible functions. These two quantities depend only on $\arg \lambda = \varphi$ and not on $|\lambda| = \varrho$ since

$$\sigma(\varrho e^{i\varphi}; F) = \sigma(e^{i\varphi}; \varrho F).$$

Therefore we define

$$\omega_s(\varphi) = \sup \sigma_s(e^{i\varphi}; F), \quad \omega_n(\varphi) = \inf \sigma_n(e^{i\varphi}; F). \tag{6.5}$$

Evidently

$$\omega_s(\varphi) \equiv 0, \quad \omega_n(\varphi) \equiv 1, \qquad -\tfrac{1}{2}\pi \leqq \varphi \leqq \tfrac{1}{2}\pi. \tag{6.6}$$

In the remainder of Λ the situation is different, but we are unable to determine the exact values of the extremal functions. We shall prove merely the following inequalities:

$$\omega_s(\varphi) \geqq \tfrac{1}{2}[1 - \sin|\varphi|], \quad \omega_n(\varphi) \leqq \tfrac{1}{2}[1 + \sin|\varphi|], \quad \tfrac{1}{2}\pi < |\varphi| < \pi. \tag{6.7}$$

For $F(x) = (1 + ax)^{-2}$ the solutions are linear combinations of powers $(1 + ax)^{\alpha}$ where α satisfies the indicial equation $a\alpha(\alpha - 1) - \lambda = 0$. Here we set $\lambda = \varrho e^{i\varphi}$ and keep φ fixed, $\frac{1}{2}\pi < |\varphi| < \pi$. There is a root α_1 with $\Re(\alpha_1) > \frac{1}{2}$ and $\Re(\alpha_1)$ will be a minimum if $\varrho = -\frac{1}{2}a^2 \cos\varphi$ and simultaneously the real part of the other root will reach its maximum. Calculating the values of these extrema we obtain (6.7). We have no means of telling if these inequalities are actually equalities; at least the right hand sides have the growth and convexity properties as functions of φ as we expect the extremal functions $\omega_s(\varphi)$ and $\omega_n(\varphi)$ to have.

If it should really turn out that the functions of the form $(ax + b)^{-2}$ give the solution of the extremal problems defined by (6.5), it might be of some interest to observe that the same class of functions is connected with condition $M(a)$ of (5.5). One might ask if the related functional equation

$$\log [1 + a F_1(x)] = a F_{\frac{1}{2}}(x)$$

has a solution. Here we have replaced $F_1(x)$ by $1 + aF_1(x)$ to get the appropriate normalization at the origin. Twofold differentiation gives a first order differential equation for $F(x)$ and

$$F(x) = (1 - ax)^{-2}$$

satisfies the equation for $0 \leqq x < 1/a$. The fact that the solution has only a finite range of existence, underlies the fact observed above that the inferior limit of the quotient in (5.5) is zero for increasing functions.

REFERENCES. [1] **M. Bôcher,** On regular singular points of linear differential equations of the second order whose coefficients are not necessarily analytic, Trans. Amer. Math. Soc. 1, 39—52 (1900). — [2] **E. Hille,** On the zeros of Sturm-Liouville functions, Arkiv för Mat., Astr. o. Fysik, 16, no. 17, 20 pp. (1922). — [3] ——, Non-oscillation theorems, Trans, Amer. Math. Soc., 64, 234—252 (1948). — [4] **E. L. Ince,** Ordinary Differential Equations. Dover, New York, 1944, viii + 558 pp. — [5] **H. Weyl,** Über gewöhnliche lineare Differen, tialgleichungen mit singulären Stellen und ihre Eigenfunktionen, Göttinger Nachrichten, Math.-phys. Klasse, 1909, pp. 37—63. — [6] **A. Wiman,** Über die reellen Lösungen der linearen Differentialgleichungen zweiter Ordnung, Arkiv för Mat., Astr. o. Fysik, 12. no. 14. 22 pp. (1917). — [7] **A. Wintner,** On almost free linear motions. Amer. J. of Math., 71, 595—602 (1949).

Tryckt den 10 december 1951

Uppsala 1951. Almqvist & Wiksells Boktryckeri AB