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A bilateral Tauberian theorem

By Åke Pleijel

This note contains the proof of a bilateral Tauberian theorem which in incomplete form was used in my paper [2].

By the elementary proof the Tauberian theorem is reduced to two well-known theorems by Hardy and Littlewood, see [1].

1. The Hardy and Littlewood theorems

Let σ be a real function of bounded variation on every finite subinterval of $0 \leqslant \lambda < +\infty$.

The following Abelian theorems are easily proved.

Theorem $A\{q\}$. *If* $0<q<1$ *and* $\sigma(\lambda)=o(\lambda^q)$ *when* $\lambda \to +\infty$, *then*

$$\int_0^\infty (\lambda+t)^{-1}\,d\sigma(\lambda)=o(t^{q-1}),\ t\to+\infty.$$

Theorem $A\{0\}$. *If* $\sigma(\lambda)=\sigma(+\infty)+o(1)$ *when* $\lambda\to+\infty$, *then*

$$\int_0^\infty (\lambda+t)^{-1}\,d\sigma(\lambda)=(\sigma(+\infty)-\sigma(0))\,t^{-1}+o(t^{-1}),\ t\to+\infty.$$

The conclusions are still valid if t tends to infinity along half rays from the origin in the complex t-plane which are different from the real negative axis. To see this one has only to use the inequality $|\lambda+t| \geqslant (\lambda+|t|)\sin\frac{1}{2}\delta$, $|\arg t| \leqslant \pi-\delta$, when the necessary estimations are performed.

To formulate the corresponding Tauberian theorems we need

Definition 1. *The function* σ *belongs to* T^s *if there is a real constant* C *such that* $d\sigma(\lambda)+C\lambda^{s-1}d\lambda$ *is definite,* $\geqslant 0$ *or* $\leqslant 0$, *for sufficiently large values of* λ.

The theorems are

Theorem $T\{q\}$. *If* $0<q<1$, $\sigma \in T^q$ *and*

$$\int_0^\infty (\lambda+t)^{-1}\,d\sigma(\lambda)=o\,(t^{q-1}),\ t\to+\infty,$$

then $\sigma(\lambda)=o(\lambda^q)$ *when* $\lambda\to+\infty$.

Theorem $T\{0\}$. *If* $\sigma\in T^0$ *and*

$$\int_0^\infty (\lambda+t)^{-1}\,d\sigma(\lambda)= Ht^{-1}+o(t^{-1}),\ t\to+\infty,$$

then $\sigma(\lambda)=\sigma(+\infty)+o(1)$ *when* $\lambda\to+\infty$. *The constant* H *is related to* σ *by* $H=\sigma(+\infty)-\sigma(0)$.

2. Certain lemmas

An addition of a constant to the function σ of the previous section is evidently irrelevant. The same holds for the functions of this section since we are essentially only interested in their differentials. Thus the definition

$$I^k\varphi(\lambda)=\int^\lambda \mu^k d\varphi(\mu)$$

determines $I^k\varphi$ up to an additive constant. It is always possible to avoid divergence difficulties by taking the lower limit of integration positive. Obviously

$$I^j(I^k\varphi)=I^{j+k}\varphi.$$

Definition 2. *We write* $\varphi(\lambda)\in\omega^s$ *if* $\varphi(\lambda)=o(\lambda^s)$ *when* $\lambda\to+\infty$ *and* $s>0$ *or if* $\varphi(\lambda)=\varphi(+\infty)+o\,(\lambda^s)$ *when* $\lambda\to+\infty$ *and* $s\leqslant 0$.

Consider $I^k\varphi$ when $\varphi\in\omega^s$. It is no restriction to assume $\varphi=o(\lambda^s)$ also when $s\leqslant 0$. From the relation

$$I^k\varphi(\lambda)-I^k\varphi(A)=\frac{\varphi(\lambda)}{\lambda^s}\,\lambda^{k+s}-\frac{\varphi(A)}{A^s}\,A^{\,k+s}-k\int_A^\lambda \frac{\varphi(\mu)}{\mu^s}\,\mu^{k+s-1}d\mu \tag{1}$$

it follows when $k+s>0$ that

$$I^k\varphi(\lambda)=o(\lambda^{k+s}),\ \lambda\to+\infty.$$

If $k+s<0$ relation (1) shows the convergence of $I^k\varphi(\lambda)$ when $\lambda\to+\infty$. The transition to the limit $A\to+\infty$ then leads to a formula from which one concludes that

$$I^k\varphi(\lambda)-I^k\varphi(+\infty)=o(\lambda^{k+s}),\ \lambda\to+\infty.$$

Thus we have proved

Lemma 1. *If* $\varphi \in \omega^s$ *then* $I^k\varphi \in \omega^{k+s}$ *provided* $k \neq -s$.

That the lemma is not valid without the condition $k \neq -s$ is seen by the example $\varphi(\lambda) = I^s \log \log \lambda$. This function belongs to ω^s when $s \neq 0$ but $I^{-s}\varphi(\lambda) = \log \log \lambda$ is not in ω^0.

Definition 3. *We write* $\varphi \in I^s$ *if* $I^k\varphi \in \omega^{k+s}$ *for all* $k \neq -s$.

Observe that it follows from lemma 1 that if $I^k\varphi \in \omega^{k+s}$ holds for an arbitrary value of k it holds for all values $k \neq -s$. This leads to

Lemma 2. $\omega^s = I^s$ *if* $s \neq 0$ *and* $\omega^0 \subset I^0$.

The function $\log \log \lambda$ belongs to I^0 but is not in ω^0.

Lemma 3. $\varphi \in I^s$ *implies* $I^k\varphi \in I^{k+s}$.

Lemma 3 is an immediate consequence of the definition of I^s. The lemma is closely related to lemma 1 but is free from supplementary condition.

Lemma 4. *The class* I^s *is linear.*

Lemma 5. *If* $\varphi(\lambda) \in I^s$ *then* $\varphi(\lambda^k) \in I^{ks}$.

Lemma 4 and 5 follow from the definition of ω^s and I^s.

Lemma 6. *If* $\varphi \in T^s$ *then* $I^k\varphi \in T^{k+s}$.

For if $d\varphi + C\lambda^{s-1}d\lambda$ is definite, the same is true for the differential obtained when $d\varphi + C\lambda^{s-1}d\lambda$ is multiplied by λ^k.

Lemma 7. *If* $\varphi_1, \varphi_2 \in T^s$ *then either* $\varphi_1 + \varphi_2 \in T^s$ *or* $\varphi_1 - \varphi_2 \in T^s$.

If $d\varphi_1 + C_1\lambda^{s-1}d\lambda \geqslant 0$, $d\varphi_2 + C_2\lambda^{s-1}d\lambda \geqslant 0$ it follows by addition that $\varphi_1 + \varphi_2 \in T^s$. If $d\varphi_1 + C_1\lambda^{s-1}d\lambda \geqslant 0$, $d\varphi_2 + C_2\lambda^{s-1}d\lambda \leqslant 0$ subtraction yields $\varphi_1 - \varphi_2 \in T^s$.

Lemma 8. *If* $\varphi(\lambda) \in T^s$ *then* $\varphi(\lambda^k) \in T^{ks}$.

Lemma 9. *If* $s < r$ *then* $T^s \subset T^r$, $T^s \neq T^r$. *Also* $\omega^s \subset \omega^r, \omega^s \neq \omega^r$ *and* $I^s \subset I^r$, $I^s \neq I^r$.

Lemma 8 is obtained by a change of the independent variable and Lemma 9 is easily deduced from the definitions of T^s, ω^s and I^s.

3. Unilateral theorems

Let σ be a function of the type in section 1 and assume that the integral of $\lambda^{-h}d\sigma(\lambda)$ converges absolutely when taken over a right side neighbourhood of the origin. Under this condition consider

$$\int_0^\infty \lambda^{-h}(\lambda+t)^{-1}\,d\sigma(\lambda). \tag{2}$$

If this integral converges at infinity for one value $t=t_0$, for instance for $t=0$, it is seen by introducing in (2)

$$\beta(\lambda)=\int_a^\lambda \mu^{-h}(\mu+t_0)^{-1}\,d\sigma(\mu), \quad a>0,$$

and by partial integration that (2) converges at infinity for every t. If

$$\int_0^\infty \lambda(\lambda+t)^{-1}\,dI^{-h-1}\sigma(\lambda)$$

is integrated by parts it follows that (2) is $o(1)$ when t tends to infinity along half rays different from the negative real axis. Thus we have

Theorem 1. *Necessary and sufficient for the convergence of* (2) *is* $I^{-h-1}\sigma\in\omega^0$. *The integral is then* $o(1)$ *when* $t\to\infty$ *along half rays from the origin which are different from the negative real axis.*

In the Abelian theorem for (2) which will be considered presently, it is assumed that $\sigma\in I^s$ when $s-h\neq$ integer or $I^{-s}\sigma\in\omega^0$ when $s-h=$ integer. These relations appear as conclusions in the corresponding Tauberian theorem. It follows that it is natural to suppose $s<h+1$. If $s=h+1$ the condition $I^{-s}\sigma\in\omega^0$ is equivalent to the existence of the integral (2) and also implies that this integral is $o(1)$ when $t\to\infty$. The Abelian and Tauberian theorems given below can therefore be extended in a trivial way to include also the case $s=h+1$ in which the Tauberian theorem holds without the Tauberian condition $\sigma\in T^s$. We restrict ourselves to the non-trivial case $s<h+1$.

The following Abelian and Tauberian theorems are proved in section 4 and 5.

Theorem A $(0,\infty)$. *If* $s<h+1$ *and* $\sigma\in I^s$ *or in case* $s-h$ *is an integer, if* $I^{-s}\sigma\in\omega^0$, *then*

$$\int_0^\infty \lambda^{-h}(\lambda+t)^{-1}d\sigma(\lambda)=t^{-1}p(t^{-1})+o(t^{s-h-1}), \quad t\to+\infty, \tag{3}$$

where p *is a polynomial.*

Recall that if $s\neq 0$ the condition $\sigma\in I^s$ coincides with $\sigma\in\omega^s$ i.e. $\sigma(\lambda)=o(\lambda^s)$ when $s>0$ and $\sigma(\lambda)=\sigma(+\infty)+o(\lambda^s)$ when $s<0$. The condition $I^{-s}\sigma\in\omega^0$ is equivalent to the convergence at infinity of

$$\int^\infty \lambda^{-s}d\sigma(\lambda).$$

According to lemma 1 and definition 3 the relation $I^{-s}\sigma\in\omega^0$ implies that $\sigma\in I^s$ but the converse is not true.

The conclusion of $A(0, \infty)$ also holds when t tends to infinity along arbitrary half rays from the origin different from the negative real axis.

Theorem $T(0, \infty)$. *If $s<h+1$, if $\sigma \in T^s$ and if, with a polynomial p,*

$$\int_0^\infty \lambda^{-h}(\lambda+t)^{-1}d\sigma(\lambda) = t^{-1}p(t^{-1}) + o(t^{s-h-1}), \quad t \to +\infty, \tag{4}$$

then $\sigma \in I^s$. If $s-h$ is an integer the result can be sharpened to $I^{-s}\sigma \in \omega^0$.

Remark. The assumption $\sigma \in T^s$ can be replaced by the slightly more general condition that $\sigma + \varphi \in T^s$ for a φ satisfying $\varphi \in I^s$ or, in case $s-h=$ integer, $I^{-s}\varphi \in \omega^0$.

4. Proof of the unilateral Abelian theorem

Theorem $A(0, \infty)$ is easily reduced to $A\{q\}$ and $A\{0\}$. If $0<s-h<1$ the polynomial $t^{-1}p(t^{-1})$ is irrelevant in (3) and can be included in the remainder. When $s-h=0$ only the first term $a_1 t^{-1}$ of $t^{-1}p(t^{-1})$ is relevant. The left hand side of (3) can be written

$$\int_0^\infty (\lambda+t)^{-1} dI^{-h}\sigma(\lambda).$$

According to lemma 3 it follows from the condition $\sigma \in I^s$ of $A(0, \infty)$ that $I^{-h}\sigma \in I^{s-h}$ and if $s-h \neq 0$ we know from lemma 2 that $I^{s-h} = \omega^{s-h}$. Thus $I^{-h}\sigma \in \omega^{s-h}$ when $s-h \neq 0$ which shows that if $0<s-h<1$ theorem $A(0, \infty)$ reduces to $A\{q\}$ with $q=s-h$ and with σ replaced by $I^{-h}\sigma$. When $s-h=0$ it is required in $A(0, \infty)$ that $I^{-s}\sigma \in \omega^0$ which is the condition of theorem $A\{0\}$ if σ in this theorem is replaced by $I^{-h}\sigma = I^{-s}\sigma$. Thus $A(0, \infty)$ is proved when $0 \leqslant s-h<1$.

To prove the theorem when $s-h<0$ we use the identity

$$(\lambda+t)^{-1} = \sum_{j=0}^{k-1} (-\lambda)^j t^{-j-1} + (-t)^{-k}\lambda^k(\lambda+t)^{-1}$$

with $h-s \leqslant k < h-s+1$. Thus the left hand side of (3) equals a polynomial in t^{-1} without constant term plus

$$(-t)^{-k}\int_0^\infty \lambda^{k-h}(\lambda+t)^{-1}d\sigma(\lambda). \tag{5}$$

Since $\sigma \in I^s$ it follows that $I^{-h+j}\sigma \in I^{s-h+j} \subset I^{s-h+k-1} = \omega^{s-h+k-1} \subset \omega^0$ which shows the convergence of the occurring integrals. In (5) $0 \leqslant s-(h-k)<1$ and the integral can be treated in the same way as the left hand side of (3) when $0 \leqslant s-h<1$. This completes the proof.

5. Proof of the unilateral Tauberian theorem

When $0 \leqslant s-h<1$ the Tauberian theorem $T(0, \infty)$ is reduced to $T\{q\}$ and $T\{0\}$. The assumption $\sigma \in T^s$ gives $I^{-h}\sigma \in T^{s-h}$ according to lemma 6. Hence if $0<s-h<1$ it follows from $T\{q\}$, $q=s-h$, that $I^{-h}\sigma \in \omega^{s-h}=I^{s-h}$. Lemma 3 shows that $\sigma \in I^s$. When $s-h=0$ the theorem $T\{0\}$ gives $I^{-h}\sigma \in \omega^0$ or $I^{-s}\sigma \in \omega^0$.

If $s-h<0$, the basic relation (4) implies that

$$\int_0^\infty (\lambda+t)^{-1} dI^{-h}\sigma = a_1 t^{-1} + o(t^{-1}), \quad t \to +\infty.$$

Since $I^{-h}\sigma \in T^{s-h}$ and $s-h<0$ it follows from lemma 9 that $I^{-h}\sigma \in T^0$. Thus $T\{0\}$ shows that $I^{-h}\sigma \in \omega^0$. Because of this and theorem 1 formula (4) can be written

$$\int_0^\infty \lambda^{-h} d\sigma - \int_0^\infty \lambda^{1-h}(\lambda+t)^{-1} d\sigma = a_1 + a_2 t^{-1} + \dots + o(t^{s-h}).$$

Letting t tend to infinity one finds because of the second part of theorem 1 that

$$\int_0^\infty \lambda^{-h} d\sigma = a_1$$

and the formula reduces to

$$\int_0^\infty \lambda^{1-h}(\lambda+t)^{-1} d\sigma(\lambda) = -a_2 t^{-1} - \dots + o(t^{s-h}).$$

Thus, provided $s-h<0$, formula (4) can be replaced by a similar one with $h-1$ instead of h. If $s-(h-1)<0$ the procedure is repeated. Finally a formula is obtained in which h is replaced by $h-r$ with $0 \leqslant s-(h-r)<1$. We are then in the case already considered and the proof is accomplished.

6. Inhomogeneous theorems

Let $\chi = K\lambda^s$ when $s \neq 0$ and $\lambda \geqslant a > 0$ and let $\chi = K \log \lambda$ when $s=0$ and $\lambda \geqslant a>0$. Here K and a are constants. For $0 \leqslant \lambda < a$ the function χ is arbitrarily defined so as to cause no trouble about the convergence at the origin of the integral

$$\int_0^\infty \lambda^{-h}(\lambda+t)^{-1} d\chi(\lambda). \tag{6}$$

It is easy to see that when $s<h+1$ the condition $I^{-h-1}\chi \in \omega^0$ is fulfilled which guarantees the convergence at infinity of the integral. Also $\chi \in T^s$.

If σ is replaced by $\varphi-\chi$ in $A(0, \infty)$, $T(0, \infty)$ these theorems are transferred into "inhomogeneous" theorems connecting the conditions $\varphi-\chi \in I^s$ or $I^{-s}(\varphi-\chi) \in \omega^0$ (when $s-h=$ integer) on one side and

$$\int_0^\infty \lambda^{-h}(\lambda+t)^{-1}d\varphi=\int_0^\infty \lambda^{-h}(\lambda+t)^{-1}d\chi+t^{-1}p(t^{-1})+o\left(|t|^{s-h-1}\right)$$

on the other. In the inhomogeneous Tauberian theorem the condition $\varphi \in T^s$ takes the place of $\sigma \in T^s$.

Simple calculations show that in these inhomogeneous theorems the integral (6) can be replaced by

$$K\frac{\pi s}{\sin \pi(s-h)}t^{s-h-1} \text{ when } s\neq 0,\ s-h\neq \text{integer},$$

$$K s(-1)^{s-h}t^{s-h-1}\log t \text{ when } s\neq 0,\ s-h=\text{integer},$$

$$K\frac{\pi}{\sin(-\pi h)} \text{ when } s=0,\ s-h\neq\text{integer},$$

$$K(-1)^{-h}t^{-h-1}\log t \text{ when } s=0,\ s-h=\text{integer}.$$

7. A bilateral Abelian theorem

In the bilateral case functions $\sigma(\lambda)$ are considered which are defined on $-\infty<\lambda<+\infty$ and of bounded variation over every finite subinterval. It is assumed that

$$\int |\lambda|^{-h}|d\sigma(\lambda)|$$

converges over a two-sided neighbourhood of the origin.

We are concerned with the study of the integral

$$\int_{-\infty}^{+\infty}|\lambda|^{-h}(\lambda+t)^{-1}d\sigma(\lambda) \tag{7}$$

which is supposed to converge. This means that $I^{-h-1}\sigma(\lambda)$ and $I^{-h-1}\sigma(-\lambda)$ are in ω^0 when considered for positive values of λ. As in the unilateral case it is natural to assume $s\leqslant h+1$ and the case when $s=h+1$ is trivial. We therefore suppose $s<h+1$.

The part of (7) which comes from $-\infty<\lambda<0$ is transformed into an integral from 0 to $+\infty$ by the substitution $\lambda=-\mu$. Afterwards μ is replaced by λ. With

$$S(\lambda)=\sigma(\lambda)+\sigma(-\lambda),$$

$$A(\lambda)=\sigma(\lambda)-\sigma(-\lambda),$$

the result reads

$$\int_0^\infty \lambda^{1-h}(\lambda^2-t^2)^{-1}dS(\lambda)-t\int_0^\infty \lambda^{-h}(\lambda^2-t^2)^{-1}dA(\lambda).$$

From this it follows that a relation

$$\int_{-\infty}^{+\infty} |\lambda|^{-h}(\lambda+t)^{-1}d\sigma(\lambda)=t^{-1}p(t^{-1})+o(|t|^{s-h-1}),$$

holding when $t\to\infty$ along two opposite non real half rays, can be split into formulas

$$\int_0^\infty \Lambda^{-h/2}(\Lambda+T)^{-1}\,dA(\sqrt{\Lambda})=T^{-1}P_1(T^{-1})+o(|T|^{s/2-h/2-1}) \tag{8}$$

$$\int_0^\infty \Lambda^{-\frac{h-1}{2}}(\Lambda+T)^{-1}dS(\sqrt{\Lambda})=T^{-1}P_2(T^{-1})+o\left(|T|^{\frac{S}{2}-\frac{h-1}{2}-1}\right), \tag{9}$$

valid when $T\to\infty$ along a half ray different from the negative real axis. Here $T=-t^2, \Lambda=\lambda^2$ and the polynomials P_1, P_2 are determined by $p(x)+p(-x)=2P_1(-x^2)$, $p(x)-p(-x)=-2xP_2(-x^2)$ when the polynomial p is known. Evidently $\frac{1}{2}s<\frac{1}{2}h+1$, $\frac{1}{2}s<\frac{1}{2}(h-1)+1$ since $s<h+1$.

The bilateral Abelian theorem is now an immediate consequence of $A(0,\infty)$ applied to (8), (9).

Theorem $A(-\infty,+\infty)$. *If $s<h+1$ and $s-h$ is not an integer it follows from $\sigma(\lambda)\in I^s$, $\sigma(-\lambda)\in I^s$ that*

$$\int_{-\infty}^{+\infty} |\lambda|^{-h}(\lambda+t)^{-1}d\sigma(\lambda)=t^{-1}p(t^{-1})+o(|t|^{s-h-1})$$

when $t\to\infty$ along non real half rays from the origin. Here p is a polynomial. If $s-h$ is an integer the same result is valid under the supplementary conditions $I^{-s}(\sigma(\lambda)+\sigma(-\lambda))\in\omega^0$ when $s-h$ is odd, $I^{-s}(\sigma(\lambda)-\sigma(-\lambda))\in\omega^0$ when $s-h$ is even.

8. The bilateral Tauberian theorem

Theorem $T(-\infty,+\infty)$. *Let $s<h+1$ and assume that $\sigma(\lambda)\in T^s$, $\sigma(-\lambda)\in T^s$ for positive values of λ. If with a polynomial p*

$$\int_{-\infty}^{+\infty} |\lambda|^{-h}(\lambda+t)^{-1}d\sigma(\lambda)=t^{-1}p(t^{-1})+o(|t|^{s-h-1}) \tag{10}$$

as $t\to\infty$ along non real half rays from the origin, then $\sigma(\lambda)\in I^s$ and $\sigma(-\lambda)\in I^s$ for $\lambda\to+\infty$.

Observe that the couple $A(-\infty,+\infty)$, $T(-\infty,+\infty)$ does not show the same symmetric reciprocity as $A(0,\infty)$, $T(0,\infty)$.

If we wanted to include also the case when $s=h+1$ we would conclude from the mere existence of the integral in (10) that $I^{-s}\sigma(\lambda)\in\omega^0$, $I^{-s}\sigma(-\lambda)\in\omega^0$ which implies $\sigma(\lambda)\in I^s$, $\sigma(-\lambda)\in I^s$.

But for certain exceptional cases occurring only when $s-h$ is an integer the proof of $T(-\infty, +\infty)$ is obtained by replacing t in (10) by it, the new t being real. If the resulting relation is split as in section 7 we obtain (8), (9) with T real positive.

According to lemma 7 and 8 either $S(\sqrt{\Lambda})$ or $A(\sqrt{\Lambda})$ belongs to $T^{s/2}$. Let us first assume $S(\sqrt{\Lambda}) \in T^{s/2}$. Then $T(0, \infty)$ can be applied to (9) with the result $S(\sqrt{\Lambda}) \in I^{s/2}$. If $s-h$ is an odd integer $T(0, \infty)$ gives in addition the convergence of

$$\int_0^\infty \lambda^{-s} d(\sigma(\lambda)+\sigma(-\lambda))$$

which, however, is of no use in the rest of the proof. Provided $s-h$ is not an even integer theorem $A(0, \infty)$ yields

$$\int_0^\infty \Lambda^{-\frac{h}{2}} (\Lambda+T)^{-1} d\, S(\sqrt{\Lambda}) = T^{-1} P_2'(T^{-1}) + o(|T|^{\frac{s}{2}-\frac{h}{2}-1}), \tag{11}$$

where P_2' is a new polynomial. Addition and subtraction of (11), (8) lead to separate formulas for $\sigma(\sqrt{\Lambda})$ and $\sigma(-\sqrt{\Lambda})$. Application of $T(0, \infty)$ to these formulas finally shows that $\sigma(\sqrt{\Lambda})$, $\sigma(-\sqrt{\Lambda}) \in I^{s/2}$ and the conclusion of $T(-\infty, +\infty)$ follows from lemma 5.

If instead $A(\sqrt{\Lambda}) \in T^{s/2}$ the same method is applied but with starting point in (8). The same result is obtained provided $s-h$ is not an odd integer. If $s-h$ is an even integer we find the additional result that

$$\int_0^\infty \lambda^{-s} d(\sigma(\lambda)-\sigma(-\lambda))$$

exists.

The exceptional cases $s-h=$ even integer, $S(\lambda) \in T^s$ and $s-h=$ odd integer, $A(\lambda) \in T^s$ remain to be considered.

9. Proof of the bilateral Tauberian theorem in the exceptional cases

To take care also of these cases we must use (10) not only along one couple of opposite half rays but along two. Since it is not more complicated to consider n couples we do this.

The point of departure is the partition in partial fractions

$$2n\, t^{2n-\alpha-1} \lambda^\alpha (\lambda^{2n}+t^{2n})^{-1} = -\sum_{k=0}^{2n-1} \varepsilon_k^{\alpha+1} (\lambda-\varepsilon_k t)^{-1}, \tag{12}$$

where $\alpha = 0, 1, 2, \ldots (2n-1)$ and

$$\varepsilon_k = \exp\left(\frac{\pi i}{2n} + k\frac{\pi i}{n}\right).$$

In (10) t is now replaced by $-\varepsilon_k t$, $k=0,1,2,\dots(2n-1)$, after which the resulting formulas are multiplied by $\varepsilon_k^{\alpha+1}$ and added. The result is condensed by help of (12) and occurring integrals from $-\infty$ to 0 are transformed into integrals from 0 to $+\infty$. At last the new variables $\Lambda=\lambda^{2n}$, $T=t^{2n}$ are introduced.

The formulas deduced in this way are

$$\int_0^\infty \Lambda^{-\frac{h-\alpha}{2n}}(\Lambda+T)^{-1}d\left(\sigma(\Lambda^{\frac{1}{2n}})-(-1)^\alpha\sigma(-\Lambda^{\frac{1}{2n}})\right)=T^{-1}P_\alpha(T^{-1})+o\left(|T|^{\frac{s}{2n}-\frac{h-\alpha}{2n}-1}\right), \tag{13}$$

$\alpha=0,1,2,\dots(2n-1)$. They are certainly valid when $T\to\infty$ along the positive real axis.

According to lemma 8 and 7 we know that

$$\sigma(\Lambda^{\frac{1}{2n}})-(-1)^\alpha\sigma(-\Lambda^{\frac{1}{2n}})\in T^{\frac{s}{2n}} \tag{14}$$

holds either for α odd or even. Assume first that it holds for α odd. Theorem $T(0,\infty)$ can then be applied to any of the formulas (13) with α odd. The result is

$$\sigma(\Lambda^{\frac{1}{2n}})+\sigma(-\Lambda^{\frac{1}{2n}})\in I^{\frac{s}{2n}}. \tag{15}$$

Next chose α even and such that

$$\frac{s}{2n}-\frac{h-\alpha}{2n}$$

is not an integer. This is possible if only $n>1$ i.e. if there is more than one even α, $0\leqslant\alpha<2n$. With the chosen even α theorem $A(0,\infty)$ shows because of (15) that

$$\int_0^\infty \Lambda^{-\frac{h-\alpha}{2n}}(\Lambda+T)^{-1}d\left(\sigma(\Lambda^{\frac{1}{2n}})+\sigma(-\Lambda^{\frac{1}{2n}})\right)=T^{-1}P'_\alpha(T^{-1})+o\left(|T|^{\frac{s}{2n}-\frac{h-\alpha}{2n}-1}\right),$$

where P'_α is a polynomial. This relation and formula (13) with the same even α lead to relations for

$$\int_0^\infty \Lambda^{-\frac{h-\alpha}{2n}}(\Lambda+T)^{-1}d\sigma(\pm\Lambda^{\frac{1}{2n}})$$

from which it follows by help of $T(0,\infty)$ that

$$\sigma(\pm\Lambda^{\frac{1}{2n}})\in I^{\frac{s}{2n}}.$$

According to lemma 5 this is equivalent to

$$\sigma(\lambda)\in I^s,\quad \sigma(-\lambda)\in I^s.$$

The case when (14) is valid with α even is similarly treated with the same result.

10. Inhomogeneous form of the bilateral Tauberian theorem

With

$$\chi(\lambda)=\begin{cases} K_1\lambda^s & \text{for } 0<a\leqslant\lambda \\ K_2(-\lambda)^s & \text{for } \lambda\leqslant -a \end{cases}$$

when $s\neq 0$ and

$$\chi(\lambda)=\begin{cases} K_1\log\lambda & \text{for } 0<a\leqslant\lambda \\ K_2\log(-\lambda) & \text{for } \lambda\leqslant -a \end{cases}$$

when $s=0$ the theorem $T(-\infty,+\infty)$ takes the inhomogeneous form

Theorem $T(-\infty,+\infty)$. *If* $s<h+1$ *and* $\varphi(\lambda)\in T^s$, $\varphi(-\lambda)\in T^s$ *and if with a polynomial* p

$$\int_{-\infty}^{+\infty}|\lambda|^{-h}(\lambda+t)^{-1}d\varphi(\lambda)=\int_{-\infty}^{+\infty}|\lambda|^{-h}(\lambda+t)^{-1}d\chi(\lambda)+t^{-1}p(t^{-1})+o\left(|t|^{s-h-1}\right)$$

as $t\to\infty$ *along non real half rays from the origin, then* $\varphi(\lambda)-\chi(\lambda)\in I^s$, $\varphi(-\lambda)-\chi(-\lambda)\in I^s$ *for* $\lambda\to+\infty$.

In this theorem the integral on the right hand side

$$\int_0^{\infty}\lambda^{-h}(\lambda+t)^{-1}d\chi(\lambda)+\int_0^{\infty}\lambda^{-h}(\lambda-t)^{-1}d\chi(-\lambda) \tag{16}$$

can be replaced by expressions directly obtained from section 6. The expression for the first integral in (16) shall be real when t is real positive (we assume K_1 and K_2 real) and the one replacing the second integral shall be real when t is real negative.

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