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Fourier transforms and probability theory on a non-commutative locally compact topological group

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1. Introduction

The usefulness of the characteristic function in probability investigations on the real line, and even in an Abelian locally compact topological group, has led Grenander [1] to define an analogous object as the characteristic function or Fourier transform in the non-commutative case.

Although it is then an operator-valued function he has shown (for regular measures on a group with a countable base of neighbourhoods) that a probability measure and its characteristic function uniquely determine each other, and that convolution of measures corresponds to multiplication of characteristic functions, and has given a version of the continuity theorem. The latter states that if a sequence of probability measures converges to a probability measure, the sequence of characteristic functions also converges, and conversely that if a sequence of characteristic functions converges to a characteristic function, the sequence of probability measures also converges, provided the group satisfies a certain further condition. The significance of this condition is not known, but it is certainly a restriction.

In section 3 of this paper we shall show that this restriction is unnecessary, and incidentally that the group need only be supposed to have a countable base of neighbourhoods at each point. Even with this improvement, however, the theorem is not as strong as in the classical case on the real line, in which it is sufficient to suppose that a sequence of characteristic functions converges to a continuous function, to conclude the convergence of the measures. The problem of obtaining a similar strengthening in the present case will therefore also be considered briefly.

The properties of Fourier transforms will be used in section 4 to show that the only idempotent probability measures on the group are Haar measures on compact subgroups, thus extending a result which is well-known in the commutative case (see, for example, Rudin [2]).

2. Definitions

Throughout the paper the group G, of elements g, that we deal with will be a locally compact (Hausdorff) topological group.

The measurable sets will be supposed to include the smallest Borel field containing the open and closed sets, and all measures considered will be finite, positive and regular, so that for all measurable sets A

$$\mu\{A\} = \sup \mu\{C\},$$

where the supremum is taken over the compact sets C contained in A. Regularity is not, of course, a restriction if G has a countable base of neighbourhoods.

Let L be the set of continuous functions on G which vanish outside a compact set (depending on the particular function), and let $C(G)$ be the set of all bounded continuous functions on G; then a sequence of measures μ_n tends to a measure μ vaguely, if

$$\int f(g)\,d\mu_n(g) \to \int f(g)\,d\mu(g) \quad (f \in L),$$

and weakly if this is true for $f \in C(G)$. Weak convergence implies vague convergence, and if a sequence of probability measures converges weakly to a measure μ, μ is also a probability measure. Furthermore if G is compact, weak and vague convergence coincide.

To define the Fourier transform we take any complete set B of continuous irreducible unitary representations of G (which is then, however, held fixed throughout the entire discussion). Let

$$g \to U_g^\beta \quad (\beta \in B)$$

be a typical representation belonging to B, U_g^β being unitary operators in the Hilbert space H_β. Then the Fourier transform of the finite positive measure μ is $\hat{\mu}(\beta)$, where $\hat{\mu}(\beta)$ is the bounded-operator-valued function uniquely defined by

$$(\hat{\mu}(\beta)\,y, z) = \int (U_g^\beta y, z)\,d\mu(g) \quad (y, z \in H_\beta;\ \beta \in B).$$

3. The continuity theorem

We first prove the following lemma, where P is the set of normalised elementary continuous positive definite functions $\phi(g)$. (The terminology here, and indeed throughout the paper, is that of Naimark [3].)

Lemma 1. *Suppose that G has a countable base of neighbourhoods at each point. Then if P is given the structure of a measurable space in such a way that $\phi(g)$ is measurable for fixed g, $\phi(g)\,\chi_C(g)$ is product-measurable on $P \times G$, χ_C being the characteristic function of any compact set $C \subset G$.*

Corollary. *If in addition μ and ϱ are (finite) measures on G and P respectively*

$$\int \left\{ \int \phi(g)\,d\mu(g) \right\} d\varrho(\phi) = \int \left\{ \int \phi(g)\,d\varrho(\phi) \right\} d\mu(g).$$

To prove the lemma we construct a sequence of product-measurable functions $f_n(\phi, g)$ which converges at every point (ϕ, g) of $P \times C$ to $\phi(g)$, and the result then follows at once.

Let N_n be a countable base of neighbourhoods at the identity of G. Then for each n there exists a finite set of points $g_j^{(n)}$ such that $g_j^{(n)} N_n$ cover C. Still keeping n fixed, sets $N_{n,j} \subset N_n$ can be found so that $g_j^{(n)} N_{n,j}$ are mutually disjoint and cover C, and we then write

$$f_n(\phi, g) = \phi(g_j^{(n)}) \quad (g \in g_j^{(n)} N_{n,j}).$$

It is easy to see that these functions have the properties described above.

The corollary now follows since, because of the regularity of μ, each side may be approximated arbitrarily closely by integrals of $\phi(g)\chi_C(g)$ for suitable compact C, and to these Fubini's theorem may be applied

It is now possible to prove the following version of the continuity theorem.

Theorem 1. *If the measures μ_n converge weakly to the measure μ, then the Fourier transforms $\hat{\mu}_n(\beta)$ converge weakly to $\hat{\mu}(\beta)$ for each β. If G has a countable base of neighbourhoods at each point and a sequence of Fourier transforms $\hat{\mu}_n(\beta)$ converges weakly for each β to an operator $\psi(\beta)$, the measures μ_n converge vaguely to some measure ν. If in addition $\psi(\beta) = \hat{\mu}(\beta)$, the Fourier transform of a measure μ, then μ_n converges weakly to μ.*

The first part of the theorem follows directly from the fact that $(U_g^\beta y, z)$ is a continuous function of g for fixed y and z in H_β.

Let $p(g)$ be a continuous positive definite function, which may be normalised by dividing by a suitable constant. According to [3], Theorem 1, p. 393, there exists a cyclic unitary representation U_g and a unit vector x in some Hilbert space H such that

$$p(g) = (U_g x, x).$$

By Theorem 8, p. 519 of [3], H may be expressed as a topological direct integral,

$$H = \int_D H_f \sqrt{d\varrho_1}$$

over a domain D, such that

$$U_g = \int \oplus U_g^f$$

where, for almost all f of D, U_g^f is a continuous irreducible unitary representation of G. It follows that

$$p(g) = \int \phi^f(g)\, d\varrho_1(f)$$

where, for almost all f, $\phi^f(g)$ is an elementary normalised continuous positive definite function, and $\phi^f(g)$ is continuous in f for fixed g. The correspondence from f to ϕ^f allows the set P to be given the structure of a measurable space in which $\phi(g)$ is measurable for fixed g, and ϱ_1 induces a measure ϱ in P, for which

$$p(g) = \int \phi(g)\, d\varrho(\phi).$$

Now suppose that G has a countable base of neighbourhoods at each point and that the sequence of Fourier transforms $\hat{\mu}_n(\beta)$ converges weakly to an operator $\psi(\beta)$ for each β. Then in particular $\int \phi(g)\, d\mu_n(g)$ converges for each $\phi \in P$.

Since among the set of representations B must be a one-dimensional β for which $U_g^\beta = I (g \in G)$, to which corresponds the positive definite function $\phi(g) = 1$, the assumed convergence of $\hat{\mu}_n$ implies that the sequence of measures μ_n is bounded, and it follows that for some constant K

$$\int \phi(g)\, d\mu_n(g) < K \quad (\phi \in P, n \geq 1).$$

For any continuous positive definite function $p(g)$ we have just seen that there is a measure ϱ such that

$$p(g) = \int \phi(g)\, d\varrho(\phi).$$

Thus
$$\int p(g)\, d\mu_n(g) = \int \left\{ \int \phi(g)\, d\varrho(\phi) \right\} d\mu_n(g),$$

which by the corollary to Lemma 1 is equal to

$$\int \left\{ \int \phi(g)\, d\mu_n(g) \right\} d\varrho(\phi).$$

It has just been shown that from the hypothesis of the convergence of $\hat{\mu}_n$ follows the dominated convergence of the integrand, and thus $\int p(g)\, d\mu_n(g)$ also converges.

In the proof of Theorem 5, p. 403 of [3] it is shown (essentially) that a function belonging to L can be approximated uniformly on G by continuous positive definite functions, and hence $\int f(g)\, d\mu_n(g)$ converges for all $f \in L$. Since such a limit is clearly a positive linear functional on L, the sequence μ_n does converge vaguely.

If the limit $\psi\ (\beta)$ of the sequence $\hat{\mu}_n(\beta)$ is in fact a Fourier transform $\hat{\mu}\ (\beta)$, then the previous argument may be extended to show that the vague limit of μ_n is μ. Since for $f \in C(G)$ we can write

$$\int f(g)\, d\mu_n(g) = \int f(g)\, \theta(g)\, d\mu_n(g) + \int f(g)\, [1 - \theta(g)]\, d\mu_n(g),$$

where $\theta(g)$ is a member of L bounded by 0 and 1, equal to unity on a compact set which contains all but an arbitrarily small amount of the mass of μ, the weak convergence of μ_n to μ follows from the vague convergence and the fact that $\mu_n(G)$ converges to $\mu(g)$. The proof of the theorem is then complete.

As we have already observed, the theorem in the classical case on the real line (and even on an Abelian locally compact group) is appreciably stronger, and thereby much more useful in that continuity of the limit of a sequence of characteristic functions is enough for the weak convergence of the measures. The question therefore arises, is it possible to find a simple sufficient (and preferably also necessary) condition for the limit of a sequence of Fourier transforms to be a Fourier transform? (Note that on a compact group, since vague and weak convergence coincide, the theorem already shows that no extra condition is necessary.)

On the real line there are several ways of obtaining the result, but there is basically only one which appears likely to generalise. This is to take a set of necessary and sufficient conditions for a function to be a characteristic function, and show that most of them are necessarily satisfied by a limit of characteristic functions. For instance, a function is a characteristic function if and only if it is continuous and positive definite, and the limit of a sequence of characteristic functions is necessarily positive definite, so that the usual conclusion follows at once.

From this point of view, then, the problem is to characterise those (operator-valued) functions which are Fourier transforms. Now among the characterisations known on the real line, there appears to be only one which does not involve sums or differences of two values of the independent variable: that due to Bochner [4]. For instance, positive definiteness involves $f(s-t)$, or in the general Abelian locally compact case $f(st^{-1})$, where s and t are elements of the dual group. As this corresponds in the general case to "multiplication" of two different representations, an operation which would be very difficult to handle no matter how it was defined, it is probably more profitable to consider Bochner's conditions. On an Abelian group these take the form that a function $\psi(\gamma)$ defined on the dual group is the Fourier transform of a (not necessarily positive) measure if and only if it is continuous and if there is a constant C such that

$$|\sum a_n \psi(\gamma_n)| \leqslant C \sup_g |\sum a_n \gamma_n(g)|$$

for all finite sets of real numbers a_n and characters γ_n. The prospects of generalising even this do not seem very hopeful, however.

4. Idempotent measures

By an idempotent measure, we mean a measure μ with the property $\mu \ast \mu = \mu$, where $\ast$ is the convolution operation. Then we have the following result.

Theorem 2. *A positive (finite, regular) idempotent measure on a locally compact group is the normed Haar measure on a compact sub-group.*

The fact that convolution of measures is equivalent to multiplication of Fourier transforms, shown by Grenander with a countability condition on G, is clearly true in general. We have therefore to find those positive measures μ for which

$$\hat{\mu}(\beta)\,\hat{\mu}(\beta) = \hat{\mu}(\beta) \quad (\beta \in B).$$

During the proof we shall need to use the concept of the support S of the measure μ, defined as the smallest closed set which has the same measure as the whole group G. The definition is meaningful because μ is regular.

It is clear that $\mu(G)=1$. Let us for the moment fix β, and take x as an arbitrary fixed element of H_β. If $z=\hat{\mu}(\beta)\,x$, then

$$\hat{\mu}(\beta)\,z = z.$$

Thus

$$0 = (z,z) - (\hat{\mu}(\beta)\,z, z) = \int [(z,z) - (U_g^\beta z, z)]\, d\mu(g),$$

from which it follows that with probability one

$$U_g^\beta z = z;$$

as the set of points g for which this is true is obviously closed, it follows that if g belongs to the support S

$$U_g^\beta \hat{\mu}(\beta)\, x = \hat{\mu}(\beta)\, x.$$

Recalling that x is an arbitrary element of H_β, and that β itself is arbitrary, we have

$$S \subset \bigcap_{\beta \epsilon \beta} \{g : U_g^\beta \hat{\mu}(\beta) = \hat{\mu}(\beta)\},$$

where the set on the right-hand side is a closed sub-group G_1. If $h \in G_1$, let us define a new measure μ_h by the relationship

$$\mu_h(A) = \mu(h^{-1}A)$$

for all measurable sets A. Clearly μ_h is also finite, positive, and regular. Then evaluating the Fourier transform we find

$$\begin{aligned}(\hat{\mu}_h(\beta)\, y, z) &= \int (U_g^\beta\, y, z)\, d\mu_h(g) \\ &= \int (U_{hg}^\beta\, y, z)\, d\mu(g) \\ &= \int (U_h^\beta\, U_g^\beta\, y, z)\, d\mu(g) \\ &= (U_h^\beta\, \hat{\mu}(\beta)\, y, z) \\ &= (\hat{\mu}(\beta)\, y, z)\end{aligned}$$

for all $y, z \in H_\beta$, and all β. Hence by the uniqueness theorem for Fourier transforms $\mu_h = \mu$.

Now G_1 with its relative topology is also a locally compact group, whose open subsets are measurable subsets of G. Thus μ restricted to G_1 is a left invariant measure, and hence Haar measure on G_1, which is therefore compact, since μ is finite.

Returning now to the original group G, let A be a measurable set. Then

$$\mu(A) = \mu(A \cap G_1) + \mu(A \cap G_1^c).$$

The second term on the right-hand side is not greater then $\mu(S^c)$ and therefore vanishes, and the resulting equality completes the proof of the theorem.

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