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Addendum to "The Wiener-Hopf equation in an algebra of Beurling"

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In a paper [1] by the author the main statement (Theorem 3, page 107) states, for a kernel f in the Beurling algebra A^2 (defined on page 90) a complete set of solutions φ in the dual space B^2 of A^2 to the Wiener-Hopf equation

$$\int_{-\infty}^{\infty} f(x-y)\varphi(y)\,dy = \varphi(x)+\psi(x), \quad \varphi(x)=0 \text{ for } x<0,\ \psi(x)=0 \text{ for } x\geqslant 0,$$

under certain conditions.

1. The function f is hermitean symmetric $(f(-x)=\overline{f(x)})$ so that the fourier transform $\hat{f}(\xi)=\int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx$ is real valued.

2. The function $\log|\hat{f}-1|$ is locally integrable, so that the Szegö factorization $\hat{f}(\xi)-1=e^{h_1(\xi)}e^{-h_2(\xi)}$ can be defined (page 99); one can continue h_1 and h_2 analytically to the upper respectively lower half-plane; also, on every interval on the real axis contained in the complement of $\hat{f}^{-1}(1)$, h_1 and h_2 are in a sense defined by Lemma 3, page 100, "locally" in A^2.

3. The set $\hat{f}^{-1}(1)$ is countable, and if $\{b_\nu\}$ is the set of points for which $(\xi-b_\nu)\hat{f}(\xi)\leqslant 0$ in some neighborhood of b_ν then $\sum|b_\nu|<\infty$.

Under such conditions the "formal solution" $E(\zeta)=c\zeta^{-n}\prod(1-\beta_\nu/\zeta)(1-b_\nu/\zeta)^{-1}$, c real, n pos. integer, $\sum|\beta_\nu|<\infty$, gives the complete set of solutions, back-transforming the formulae

$$\hat{\varphi}(\xi-i\eta)=e^{h_2(\xi-i\eta)}E(\xi-i\eta), \quad \hat{\psi}(\xi+i\eta)=e^{h_1(\xi+i\eta)}E(\xi+i\eta)$$

provided one can show by means of the criteria of Theorem 2, page 95, and Proposition 6, page 96, that $\hat{\varphi}$ is the analytic continuation into the negative half-plane of a transformation of a function in B^2 vanishing on the negative half-axis and that $\hat{\psi}$ is the analytic continuation into the positive half-plane of a function in B_0^2 (the "sub-dual" of A^2) vanishing on the positive half-axis. The purpose of this note is to show that the condition on $\hat{\psi}$ is superfluous. The application of the above cited criteria to $\hat{\varphi}$ and $\hat{\psi}$ yield the explicit conditions written out the first two rows of page 107, so we will demonstrate here that the second of these conditions is dispensable.

The proof of this fact follows to a great extent the original proof of Theorem 3 in [1], and we here skip some of the details that were carried out there. We suppose then that E is given subject to the same conditions as in [1] except for the one insuring that $\hat{\psi}$ is the analytic continuation of a B_0^2 function (line 2, page 107).

Then $\varphi \in B^2$ can be found satisfying $\varphi(x)=0$ for $x<0$ and $\hat{\varphi}(\xi - i\eta) = \int_0^\infty \varphi(x)\, e^{-\eta x} e^{-i\xi x} dx = e^{h_2(\xi - i\eta)} E(\xi - i\eta)$, and the goal is to show that $\delta(x) = \int_0^\infty f(x-y)\varphi(y)\,dy - \varphi(x)$ vanishes for $x>0$.

We define the function V in the complement of the real line by

$$V(\zeta) = \begin{cases} \displaystyle\int_{-\infty}^{0} \left\{ \int_0^\infty f(x-y)\,\varphi(y)\,dy \right\} e^{-i\zeta x} dx - e^{h_1(\zeta)} E(\zeta), \ \mathrm{Im}\ \zeta > 0 \\ \displaystyle - \int_0^\infty \delta(x)\, e^{-i\zeta x} dx, \ \mathrm{Im}\ \zeta < 0. \end{cases}$$

We observe that the proof on page 108 of the analytic continuation to points outside $\hat{f}^{-1}(1)$ on the real line carries over almost verbatim for this new definition of V. It is true that ψ appears in the formulas, but that is only superficial, since it comes from $e^{h_1(\zeta)} E(\zeta)$ that is transformed back and forth, and this function could equally well have been kept till line 7 from below (there are a couple of misprints on this page 108, line 9 from below should begin $\int \{ \int f \ldots$ and on lines 7 and 8 from below one should have

$$\ldots e^{h_2(\xi - i\theta) - h_2(\xi)} E(\xi - i\theta) - e^{h_1(\xi + i\theta) - h_1(\xi)} E(\xi + i\theta) \ldots).$$

The isolated singularities off the origin are removable as before, since at these the function $e^{h_1(\zeta)} E(\zeta)$ grows at most as $(\mathrm{Im}\,\zeta)^{-\frac{1}{2}}$ (and for the other terms we have the previous estimates). Hence $V(\zeta)$ is an entire function of $1/\zeta$, and it is clear from the Riemann-Lebesgue lemma applied to the definition that V has a zero at infinity. The other zeros of V can be estimated by means of Jensen's formula and, as was the case with B on pages 104 and 105, this analysis shows that $V(\zeta) = c\zeta^{-n} \prod (1 - \gamma_\nu / \zeta)$, with c real, n natural number and $\sum |\gamma_\nu| < \infty$.

Noticing that this means that $V(1/\zeta)$ is an entire function of order at most one, and that if its order equals one then its type will be zero, we expand it in a power series, $V(\zeta) = \sum_1^\infty v_n \zeta^{-n}$ and back transform to get $\delta(x) = -\sum_0^\infty (v_{n+1}(i)^{n+1}/n!)\,x^n$. Using the growth theory for entire functions we see that δ may be continued to an entire function $\delta(z)$ of order at most $\frac{1}{2}$, and that if the order of δ is $\frac{1}{2}$ then its type will be zero (the relevant growth theorems are listed in Boas [2] as Theorem 2.2.2, page 9, and Theorem 2.2.10, page 11). Now we know that the restriction of δ to the positive real axis shall be a function in B^2. Hence the function

$$u(z) = \frac{1}{z} \int_0^z \delta(z)\, \overline{\delta(\bar{z})}\, dz$$

is, by Proposition 2 of [1] (page 93) bounded on the positive real axis, and since u is also of growth order at most $\frac{1}{2}$, minimal type it is of course standard that u is a constant (Theorems 1.4.2 and 1.4.3 of Boas [2]: Phragmén-Lindelöf type theorems) and that so is δ.

Now we have that $V(\zeta) = c/\zeta$ for some constant c and we will make use of the definition of V for small negative pure imaginary values of ζ. Fix an arbitrary $\varepsilon > 0$ and decompose the kernel function $f = f_1 + f_2$ so that f_1 has compact support and $\|f_2\|_{A^2} < \varepsilon$. Then $\hat{f}_1$ is an entire function and $|\hat{f}_1(0) - 1| < \varepsilon$. Let ζ tend to zero along the negative imaginary axis. Then

$$-V(\zeta)=\int_0^\infty\int_0^\infty f_1(x-y)\varphi(y)(e^{-|\zeta|x}-e^{-|\zeta|y})\,dx\,dy+(\hat{f}_1(0)-1)\hat{\varphi}(\zeta)$$

$$+\int_0^\infty (f_2*\varphi)(x)e^{-|\zeta|x}dx$$

and in view of the estimates

$$\left|\int_0^\infty\int_0^\infty f_1(x-y)\varphi(y)(e^{-|\zeta|x}-e^{-|\zeta|y})\,dx\,dy\right|\leqslant\|f_1\|\,\|\varphi\|,$$

$$|(\hat{f}_1(0)-1)\hat{\varphi}(\zeta)|\leqslant\frac{2\varepsilon}{|\zeta|}(1+|\zeta|)^{\frac{1}{2}}\|\varphi\|$$

(Proposition 7 and Theorem 2, pages 96 and 95 respectively of [1]), and

$$\left|\int_0^\infty (f_2*\varphi)(x)e^{-|\zeta|x}dx\right|\leqslant\|f_2\|\,\|\varphi\|\,\|e^{-|\zeta|x}\|_{A^2}\leqslant\varepsilon\|\varphi\|\left(1+\frac{2}{|\zeta|}\right)$$

it is clear that, since ε was arbitrary, the function $V(\zeta)$ does in fact vanish identically. The proof is then complete.

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2. Boas, R. P., Entire Functions. New York, 1954.

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