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Quasi-analytic vectors

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1. Introduction

If S is a symmetric operator in a Hilbert space $\mathbf{H}$ and x is an element in $\mathbf{H}$ which belongs to $\bigcap_{n\geqslant 1} D(S^n)\,(D(A))$ denotes the domain of an operator A acting in $\mathbf{H}$, then the sequence of real numbers $\mu_n(x)=(S^n x/x)$, $n=0, 1, 2, \ldots$, is of positive type in the following sense: Given any finite sequence of complex numbers $(\alpha_0, \alpha_1, \ldots, \alpha_n)$, then

$$\sum_{i=0}^{n}\sum_{j=0}^{n}\alpha_i\bar{\alpha}_j\mu_{i+j}(x)=\Big\|\sum_{i=0}^{n}\alpha_i S^i x\Big\|^2\geqslant 0.$$

Hence the sequence $(\mu_n(x))$ is a Hamburger moment sequence (cf. [9]). That is, there exists a bounded positive Radon measure ν on the real line such that

$$\mu_n(x)=\int_{-\infty}^{\infty} t^n d\nu(t) \quad \text{for} \quad n=0, 1, 2, \ldots.$$

The moment sequence is said to be *determined* if the measure ν is uniquely determined. Accordingly we shall call the vector x a *vector of uniqueness for* S or a *determining vector for* S in case the moment sequence $((S^n x|x))$, $n=0, 1, 2, \ldots$ is determined. Now, T. Carleman has shown that a Hamburger moment sequence (μ_n) is determined if $\sum_{n=1}^{\infty}\mu_{2n}^{-1/2n}$ diverges (cf. [2]). If $\mu_n=(S^n x|x)$, this means that

$$\sum_{n=1}^{\infty}\frac{1}{\|S^n x\|^{1/n}}=\infty.$$

A vector $x\in\bigcap_{n\geqslant 1} D(S^n)$ such that

$$\sum_{n=1}^{\infty}\frac{1}{\|S^n x\|^{1/n}}=\infty$$

will be called a *quasi-analytic vector for* S. Thus a quasi-analytic vector for S is a vector of uniqueness for S. In [7] E. Nelson has introduced the notion of an *analytic vector for* S. A vector x in $\bigcap_{n\geqslant 1} D(S^n)$ is called an analytic vector for S if

$$\sum_{n=0}^{\infty}\frac{\|S^n x\|}{n!}t^n<\infty \quad \text{for some} \quad t>0;$$

that is, in case there exists a constant $p>0$ such that $\|S^n x\| \leq p^n n!$ for $n=1, 2, \dots$. Since $n! \leq n^n$, it follows that

$$\sum_{n=1}^{\infty} \frac{1}{\|S^n x\|^{1/n}} \geq \sum_{n=1}^{\infty} \frac{1}{pn} = \infty.$$

Thus every analytic vector for S is a fortiori a quasi-analytic vector for S and hence a vector of uniqueness for S.

E. Nelson has shown [7], using Stone's theorem, that a closed symmetric operator S is self-adjoint if and only if it has a dense set of analytic vectors. In § 2 we shall prove by completely different methods the following theorem: Let S be a closed symmetric operator in a Hilbert space $\mathbf{H}$ and D_0 the set of all vectors of uniqueness for S. Let $\tilde{D}_0$ be the vector space spanned by the vectors $\{S^k x\}$, $k=0, 1, 2, \dots$; $x \in D_0$. Then S is self-adjoint if and only if $\tilde{D}_0$ is dense in $\mathbf{H}$. As a corollary we obtain Nelson's theorem and the theorem that a closed symmetric operator is self-adjoint if and only if it has a total set of quasi-analytic vectors.

In § 3 we derive various permutability theorems for symmetric operators and in § 4 we apply the results of § 2–3 to obtain various theorems of the two parameter moment problem. Further applications will be considered in another publication.

2. The main theorem

Theorem 1. *Let S be a closed symmetric operator in a Hilbert space* $\mathbf{H}$. *Let D_0 be the set of all vectors of uniqueness for S and $\tilde{D}_0$ the vector space spanned by the vectors $\{S^k x\}$, $k=0, 1, 2, \dots$; $x \in D_0$. Then S is self-adjoint if and only if $\tilde{D}_0$ is dense in* $\mathbf{H}$.(1)

Proof. If S is self-adjoint, then S has a dense set of analytic vectors and hence a dense set of vectors of uniqueness (and hence also a dense set of quasi-analytic vectors).

By a theorem of M. Naimark (cf. [13] p. 4) S has a self-adjoint extension in the extended sense. That is, there exists a Hilbert space $\mathbf{H}_1$, which contains $\mathbf{H}$ as a Hilbert subspace and a self-adjoint operator T in $\mathbf{H}_1$ which extends S (i.e. $Sx = Tx$ for all $x \in D(S)$) and which is minimal in the following sense: If $E(\sigma)$ is the canonical spectral measure of T, then the set $\{E(\sigma)x\}$ where x ranges over $\mathbf{H}$ and σ over all the Borel sets of the real line R, is total in $\mathbf{H}_1$ (i.e. the vector space spanned by $\{E(\sigma)x\}$ is dense in $\mathbf{H}_1$).

If x is any element in $\bigcap_{n\geq 1} D(S^n)$, then

$$(S^n x \mid x) = (T^n x \mid x) = \int_{-\infty}^{\infty} t^n d\|E(t)x\|^2.$$

If $x \in D_0$, then the polynomials are dense in $L_2(\nu_x)$, where ν_x is the measure $\nu_x(\sigma) = \|E(\sigma)x\|^2$ on the real line (cf. [8], [9] and [11] Theorem 10.40). Let now

(1) If $x \in D_0$, $S^k x$ does in general not belong to D_0. (Cf. discussion following Theorem 3.)

x be a fixed element in D_0 and σ a Borel set on the real line. Let χ_σ be the characteristic function of σ with respect to the real line and choose a sequence of polynomials (p_n) such that $p_n(t) \to \chi_\sigma(t)t^k$ in the L_2-norm of $L_2(\nu_x)$. Now,

$$\int_{-\infty}^{\infty} |p_n(t) - \chi_\sigma(t)t^k|^2 d\|E(t)x\|^2 = \|p_n(T)x - E(\sigma)T^k x\|^2 = \|p_n(S)x - E(\sigma)S^k x\|^2.$$

Hence $p_n(S)x \to E(\sigma)S^k x$ strongly in $\mathbf{H}_1$. Since $p_n(S)x \in \mathbf{H}$ for all n, it follows that $E(\sigma)S^k x \in \mathbf{H}$. We have proved that $E(\sigma)\tilde{D}_0 \subset \mathbf{H}$ for all Borel sets σ on the real line. Suppose now that $\tilde{D}_0$ is dense in $\mathbf{H}$. Since $E(\sigma)$ is bounded it follows that $E(\sigma)\mathbf{H} \subset \mathbf{H}$ for all Borel sets σ. Hence $\mathbf{H}_1 = \mathbf{H}$ and therefore T is a self-adjoint extension of S in $\mathbf{H}$.

Suppose that $S \neq T$, then there exists another self-adjoint extension T_1 of S in $\mathbf{H}$ which is different from T. Let $E_1(\sigma)$ be the canonical spectral measure of T_1. Let x be a fixed element in D_0, Then the measures $\|E_1(\sigma)x\|^2$ and $\|E(\sigma)x\|^2 = \nu_x(\sigma)$ are identical. Let σ be a fixed Borel set on the real line and choose a sequence of polynomials (p_n) such that $p_n(t) \to \chi_\sigma(t)t^k$ in the L_2-norm of $L_2(\nu_x)$. Then

$$\begin{aligned}\int_{-\infty}^{\infty} |p_n(t) - \chi_\sigma(t)t^k|^2 d\|E_1(t)x\|^2 &= \int_{-\infty}^{\infty} |p_n(t) - \chi_\sigma(t)t^k|^2 d\|E(t)x\|^2 \\ &= \|p_n(T_1)x - E_1(\sigma)T_1^k x\|^2 = \|p_n(T)x - E(\sigma)T^k x\|^2 \\ &= \|p_n(S)x - E_1(\sigma)S^k x\|^2 = \|p_n(S)x - E(\sigma)S^k x\|^2 \to 0.\end{aligned}$$

Hence
$$E_1(\sigma)S^k x = E(\sigma)S^k x$$
and therefore
$$E_1(\sigma)u = E(\sigma)u \quad \text{for all} \quad u \in \tilde{D}_0.$$

From this follows, since $\tilde{D}_0$ is dense in $\mathbf{H}$ by hypothesis, that $E_1(\sigma)u = E(\sigma)u$ for all $u \in \mathbf{H}$. Hence $E_1(\sigma) = E(\sigma)$ and therefore $T_1 = T$. This contradiction shows that $S = T$.

Corollary 1. *A closed symmetric operator S in a Hilbert space $\mathbf{H}$ is self-adjoint if and only if it has a total set of vectors of uniqueness.*

Since every quasi-analytic vector for S is a vector of uniqueness for S we have as an immediate corollary.

Theorem 2. *Let S be a closed symmetric operator in a Hilbert space $\mathbf{H}$. Then S is self-adjoint if and only if S has a total set of quasi-analytic vectors. (Cf. Corollary 2.)*

Remark. In the proof of Theorem 1 the property of a vector x to be a vector of uniqueness for the operator S was used only to deduce that the polynomials are dense in $L_2(\nu_x)$, where ν_x is the measure $\nu_x(\sigma) = \|E(\sigma)x\|^2$ and $E(\sigma)$ is the canonical spectral measure of a self-adjoint extension T in the extended sense as described in the proof of Theorem 1. Now, it is not difficult to show that *a vector $x \in \bigcap_{n\geq 1} D(S^n)$ has the property that the polynomials are dense in $L_2(\nu_x)$ if* *and only if the closed subspace* $\mathbf{M}_0(x)$ *of* $\mathbf{H}$ *spanned by the vectors* $\{S^k x\}$, $k=0, 1, \ldots,$ *reduces* S *to a self-adjoint operator.* A vector x with this property will be called an *extremal vector for* S. *Thus, a closed symmetric operator* S *in a Hilbert space* $\mathbf{H}$ *is self-adjoint if and only if it has a total set of extremal vectors.* Furthermore we state here without proof that an extremal vector x for a closed symmetric operator S in $\mathbf{H}$ is a vector of uniqueness for S if and only if the self-adjoint operator $S_{\mathbf{M}_0(x)}$ to which $\mathbf{M}_0(x)$ reduces S is the closure of its restriction to the linear manifold spanned by the vectors $\{S^k x\}$, $k=0, 1, 2, \ldots$.

Whether or not a vector $x \in \bigcap_{n\geq 1} D(S^n)$ is a vector of uniquenss for S depends solely upon the moment sequence $\mu_n(x) = (S^n x|x)$, $n=0, 1, 2, \ldots$ (cf. proof of Theorem 3). In contrast, whether or not a vector $x \in \bigcap_{n\geq 1} D(S^n)$ is an extremal vector for S does not only depend upon the moments $\mu_n(x) = (S^n x|x)$. In fact, if $x \in \bigcap_{n\geq 1} D(S^n)$ is not a vector of uniqueness for S, there always exists a self-adjoint operator T in $\mathbf{M}_0(x)$ (the closed subspace of $\mathbf{H}$ spanned by the vectors $\{S^k x\}$, $k=0, 1, 2, \ldots$) such that $(S^n x|x) = (T^n x|x)$ for all n. For this reason we do not consider extremal vectors in this paper.

Let S be a symmetric operator in a Hilbert space $\mathbf{H}$. Let D_0 be the set of all determining vectors for S, D_1 be the set of all quasi-analytic vectors for S and D_2 the set of all analytic vectors for S. Then $D_2 \subset D_1 \subset D_0$. D_2 is a linear set but D_0 is not. However, D_1 and D_0 are clearly both closed under the operation $x \to cx$, where c is a scalar. It is possible on the other hand to construct linear sets of quasti-analytic vectors which will in general contain properly the analytic vectors D_2. For example, the set E of all vectors $x \in \bigcap_{n\geq 1} D(S^n)$ such that $\overline{\lim}_{n\to\infty} (\|S^n x\|^{1/n}/n) < \infty$ is linear and $D_2 \subset E \subset D_1$.

Theorem 3. *If* x *is a vector of uniqueness for the symmetric operator* S *in the Hilbert space* $\mathbf{H}$ *and if* B *is a bounded operator in* $\mathbf{H}$ *such that* $Bx \in \bigcap_{n\geq 1} D(S^n)$ *and* $S^n Bx = BS^n x$ *for* $n=1, 2, 3, \ldots,$ *then* Bx *is a vector of uniqueness for* S. *(This condition is satisfied in particulav if* $BS \subset SB$*; i.e. permutes with* S.*)*

Proof. According to H. Hamburger ([6 (a)] and [6 (b)]) a necessary and sufficient condition that a moment sequence (μ_n) be determined is that at least one of the two equalities

$$\lim_{n\to\infty} \left[\min_{\alpha_0=1} \sum_{i=0}^{n} \sum_{j=0}^{n} \alpha_i \alpha_j \mu_{i+j}\right] = 0$$

$$\lim_{n\to\infty} \left[\min_{\alpha_0=1} \sum_{i=0}^{n} \sum_{j=0}^{n} \alpha_i \alpha_j \mu_{i+j+2}\right] = 0$$

is valid where the α_i are real numbers. From this follows that a vector $x \in \bigcap_{n\geq 1} D(S^n)$ Is a vector of uniqueness for S if and only if at least one of the two equalities

$$\lim_{n\to\infty} [\min_{\alpha_i} \| (I + \alpha_1 S + \alpha_2 S^2 + \ldots + \alpha_n S^n) x \|^2] = 0$$

$$\lim_{n\to\infty} [\min_{\alpha_i} \| (I + \alpha_1 S + \alpha_2 S^2 + \ldots + \alpha_n S^n) Sx \|^2] = 0$$

is valid where the α_i are real numbers. If one of the above equalities holds for a given vector $x \in \bigcap_{n\geq 1} D(S^n)$ and B is a bounded operator in $\mathbf{H}$ and that $Bx \in \bigcap_{n\geq 1} D(S^n)$ and $S^n Bx = BS^n x$ for $n = 1, 2, \dots$, then it clearly also holds for Bx instead of x, because

$$\|(I+\alpha_1 S+\dots+\alpha_n S^n)Bx\|^2 = \|B(I+\alpha_1 S+\dots+\alpha_n S^n)x\|^2$$
$$\leq \|B\|^2\|(I+\alpha_1 S+\dots+\alpha_n S^n)x\|^2$$

and similarly

$$\|(I+\alpha_1 S+\dots+\alpha_n S^n)SBx\|^2 \leq \|B\|^2\|(I+\alpha_1 S+\dots+\alpha_n S^n)Sx\|^2.$$

Theorem 3 is not valid anymore if the hypothesis that B is a bounded operator is dropped. In fact, it is in general false if we take for B the operator S, because if (μ_n), $n = 0, 1, \dots$ is a determined moment sequence, then in general the moment sequence $\nu_n = \mu_{n+2}$, $n = 0, 1, 2, \dots$ is not determined.

The theorem remains true, however, if we drop the requirement that B is bounded but assume that x is a quasi-analytic vector for S. More precisely we have the following.

Theorem 4. *If x is a quasi-analytic vector for the symmetric operator S in the Hilbert space $\mathbf{H}$ and if A and A^+ are two operators in $\mathbf{H}$ which are adjoint to each other (i.e. they satisfy the relation $(Ay|z) = (y|A^+z)$ for every $y \in D(A)$ and every $z \in D(A^+)$) and if $x \in D(A^+A)$, $Ax \in \bigcap_{n\geq 1} D(S^n)$ and $S^nAx = AS^nx$ for $n = 1, 2, 3, \dots$, then Ax is a quasi-analytic vector for S.*

Proof.

$$\|S^nAx\|^2 = (S^nAx|S^nAx) = (S^{2n}x|A^+Ax) \leq \|S^{2n}x\|\,\|A^+Ax\|.$$

If $x = 0$ there is nothing to prove. We may therefore assume that $x \neq 0$. Since a vector x is quasi-analytic for S if and only if cx, $c \neq 0$, is quasti-analytic for S, the vector $y = (1/\|x\|)x$ is quasi-analytic for S and

$$\sum_{n=1}^{\infty} \frac{1}{\|S^nAx\|^{1/n}} \geq \sum_{n=1}^{\infty} \frac{1}{\|S^{2n}y\|^{1/2n}} \cdot \frac{1}{\|\,\|x\|A^+Ax\|^{1/2n}}.$$

To show that Ax is a quasi-analytic vector for S it is therefore sufficent to show that

$$\sum_{n=1}^{\infty} \frac{1}{\|S^{2n}y\|^{1/2n}} = \infty.$$

Now $\|S^ny\|^{1/n}$ is monotonically increasing with n. This can be verified directly, but it also follows from the well-known fact that if ν is a bounded positive measure on a space X such that $\nu(X) = 1$, then $\|f\|_p = (\int_X |f(x)|^p d\nu(x))^{1/p}$ is a monotonically increasing function of p, $p \geq 1$, for any ν-measurable function f.

If
$$\sum_{n=1}^{\infty} \frac{1}{\|S^{2n}y\|^{1/2n}}$$

were convergent, then

$$\sum_{n=1}^{\infty} \frac{1}{\| S^{2n+1} y \|^{1/2n+1}}$$

would be convergent by the comparison test and it would follow that

$$\sum_{n=1}^{\infty} \frac{1}{\| S^{n} y \|^{1/n}}$$

is convergent. Hence

$$\sum_{n=1}^{\infty} \frac{1}{\| S^{2n} y \|^{1/2n}} = \infty .$$

Corollary 2. *If x is a quasi-analytic vector for S and $p(t)$ a polynomial, then $p(S)x$ is a quasi-analytic vector for S.*

Corollary 3. *A vector $x \in \bigcap_{n\geq 1} D(S^n)$ is quasi-analytic for S if and only if*

$$\sum_{n=1}^{\infty} \frac{1}{\| S^{2n} x \|^{1/2n}} = \infty \quad \left(\textit{or equivalently} \quad \sum_{n=1}^{\infty} \frac{1}{\| S^{2n+1} x \|^{1/2n+1}} = \infty \right).$$

3. Permutability theorems for symmetric operators

Theorem 5. *Let S and T be symmetric operators in a Hilbert space $\mathbf{H}$ and let D_0 be the set of all vectors x in $\mathbf{H}$ which are vectors of uniqueness for both S and T and which are in the domain of the operators T^nS^m, S^mT^n for $n = 1, 2, \ldots, m = 1, 2, \ldots$ and such that $T^nS^mx = S^mT^nx$ for all n and m. If D_0 is dense in $\mathbf{H}$, then S and T are essentially self-adjoint and $\bar{S}$ and $\bar{T}$ permute. ($\bar{S}$ denotes the closure of S. $\bar{S}$ and $\bar{T}$ permute means that their spectral resolutions permute.)*

Proof. If D_0 is dense in $\mathbf{H}$, $\bar{S}$ and $\bar{T}$ are self-adjoint by Theorem 1. Let $E(\sigma)$ and $F(\sigma)$ be the spectral resolutions of $\bar{T}$ and $\bar{S}$, respectively. Let x be a fixed element in D_0 and σ and τ Borel sets on the real line. Then there exist two sequences of polynomials (p_n) and (q_n) with *real* coefficients such that

$$p_n(T)x \to E(\sigma)x \quad \text{and} \quad q_n(S)x \to F(\tau)x$$

(cf. proof of Theorem 1). Hence

$$(E(\sigma)x \,|\, F(\tau)x) = \lim_{n\to\infty} (p_n(T)x \,|\, q_n(S)x) = \lim_{n\to\infty} (q_n(S)x \,|\, p_n(T)x = (F(\tau)x \,|\, E(\sigma)x).$$

Therefore $((E(\sigma)F(\tau) - F(\tau)E(\sigma))x \,|\, x) = 0$ for all $x \in D_0$. Since D_0 is dense in $\mathbf{H}$, it follows that

$$((E(\sigma)F(\tau) - F(\tau)E(\sigma))x \,|\, x) = 0 \quad \text{for all} \quad x \in \mathbf{H}.$$

But this implies by the polarization identity that $E(\sigma)F(\tau) - F(\tau)E(\sigma) = 0$; i.e. $E(\sigma)F(\tau) = F(\tau)E(\sigma)$ for all Borel sets σ and τ.

Theorem 6. *Let T and S be symmetric operators in a Hilbert space* $\mathbf{H}$ *and* D *a set of vectors in* $\mathbf{H}$ *which are quasi-analytic for both* T *and* S *and which are in the domain of the operators* T^nS^m, S^mT^n *for* $n=1, 2, \ldots$, $m=1, 2, \ldots$, *and such that* $T^nS^mx=S^mT^nx$ *for all* n *and* m. *If the set* $\{T^nS^mx\}$, $n=0, 1, \ldots$, $m=0, 1, \ldots$, $x \in D$ *is total in* $\mathbf{H}$, *then* $\bar{T}$ *and* $\bar{S}$ *are self-adjoint and they permute.*

Proof. Let $\tilde{D}$ be the vector space spanned by the set of vectors $\{T^nS^mx\}$, $n=0, 1, \ldots$, $m=0, 1, \ldots$, $x \in D$. If $y \in \tilde{D}$, then $y=Ax$, where A is an operator of the form $A=\sum_i a_i T^{n_i}S^{m_i}x$, $x \in D$. Let $A^+=\sum_j \bar{a}_i T^{n_i}S^{m_i}$. Then Theorem 4 applies and shows that y is a quasi-analytic vector for T and S. Thus every vector y in $\tilde{D}$ is quasi-analytic for T and S and $T^nS^my=S^mT^ny$ for $n=1, 2, \ldots$, $m=1, 2, \ldots$. Hence, if $\tilde{D}$ is dense in $\mathbf{H}$ it follows from Theorem 5 that $\bar{T}$ and $\bar{S}$ are self-adjoint and that $\bar{T}$ and $\bar{S}$ permute.

Theorem 7. *Let T and S be symmetric operators in a Hilbert space* $\mathbf{H}$ *and* D *a set of vectors* x *which are in the domain of the operators* T^nS, ST^n *for* $n=0, 1, 2, \ldots$ *such that* $T^nSx=ST^nx$ *for* $n=1, 2, \ldots$, *and such that* $(S+iI)x$ *is a vector of uniqueness for* T. *Let* $\tilde{D}$ *be the vector space spanned by the vectors* $\{T^nx, T^mSx\}$, $n=0, 1, \ldots$, $m=0, 1, 2, \ldots$, $x \in D$. *Suppose that* $\tilde{D}$ *is dense in* $\mathbf{H}$, *then* $\bar{T}$ *is self-adjoint and* $\bar{T}$ *permutes with* $\bar{S}_1$, *where* S_1 *is the restriction of* S *to the vector space* $\tilde{D}_1$ *generated by the vectors* $\{T^nx\}$, $n=0, 1, 2, \ldots$, $x \in D$. ($\bar{T}$ *permutes with* $\bar{S}_1$ *means that* $E(\sigma)\bar{S}_1 \subset \bar{S}_1 E(\sigma)$ *for all Borel sets* σ *on the real line, where* $E(\sigma)$ *is the canonical spectral measure of* $\bar{T}$.)

Proof. If $x \in D$, then

$$(T^n(S+iI)x \,|\, (S+iI)x) = (T^nSx \,|\, Sx) + (T^nx \,|\, x), \quad n=0, 1, 2, \ldots,$$

is a determined moment sequence. Hence clearly $(T^nSx\,|\,Sx)$ and $(T^nx\,|\,x)$, $n=0, 1, \ldots$, are determined moment sequences. Hence, since $\tilde{D}$ is dense in $\mathbf{H}$, $\bar{T}$ is self-adjoint by Theorem 1. Let $E(\sigma)$ be the spectral resolution of $\bar{T}$, x be an element in D, σ a fixed Borel set on the real line and k a non-negative integer. Since

$$(T^nSx\,|\,Sx) + (T^nx\,|\,x) = \int_{-\infty}^{\infty} t^n d\,\|E(t)Sx\|^2 + \int_{-\infty}^{\infty} t^n d\,\|E(t)x\|^2, \quad n=0, 1, \ldots,$$

is a determined moment sequence there exists a sequence of polynomials (p_n) such that if χ_σ is the characteristic function of σ, then

$$\int_{-\infty}^{\infty} |p_n(t)-\chi_\sigma(t)t^k|^2 d\,\|E(t)Sx\|^2 + \int_{-\infty}^{\infty} |p_n(t)-\chi_\sigma(t)t^k|^2 d\,\|E(t)x\|^2$$
$$= \|p_n(T)Sx - E(\sigma)T^kSx\|^2 + \|p_n(T)x - E(\sigma)T^kx\|^2 \to 0$$

(cf. proof of Theorem 1). That is,

$$p^n(T)x \to E(\sigma)T^kx$$

and $$Sp_n(T)x = p_n(T)Sx \to E(\sigma)T^kSx = E(\sigma)ST^kx.$$

That is $$p_n(T)x \to E(\sigma)T^kx \quad \text{and} \quad \bar{S}_1 p_n(T) \to E(\sigma)\bar{S}_1 T^kx.$$

Hence $E(\sigma)T^kx \in D(\bar{S}_1)$ and $\bar{S}_1(E(\sigma)T^kx = E(\sigma)\bar{S}_1 T^kx$ since $\bar{S}_1$ is closed. From this follows that $E(\sigma)y \in D(\bar{S}_1)$ and $E(\sigma)\bar{S}_1 y = \bar{S}_1 E(\sigma)y$ for all $y \in D(\bar{S}_1)$; i.e. $E(\sigma)\bar{S}_1 \subset \bar{S}_1 E(\sigma)$.

Corollary 4. *If every vector in $\tilde{D}_1$ is also a vector of uniqueness for S, then $\bar{S}$ is self-adjoint and $\bar{T}$ and $\bar{S}$ permute.*

Proof. If every vector in $\tilde{D}_1$ is a vector of uniqueness for S, then $\bar{S}$ is self-adjoint by Theorem 1. $\tilde{D}_1$ is then also a dense set of determining vectors for $\bar{S}_1$. Hence $\bar{S}_1$ is self-adjoint. But $\bar{S}_1 \subset \bar{S}$ and hence $\bar{S}_1 = \bar{S}$.

Corollary 5. *Let T and S be symmetric operators in a Hilbert space* **H** *and D a set of vectors x which are in the domain of the operators T^nS, ST^n for $n = 0, 1, 2, \dots$ such that $T^nSx = ST^nx$ for $n = 1, 2, \dots$, and such that x is quasi-analytic for T. Let $\tilde{D}$ be the vector space spanned by the vectors $\{T^nx, T^mSx\}$, $n = 0, 1, \dots$, $m = 0, 1, \dots$, $x \in D$. Suppose that $\tilde{D}$ is dense in* **H**, *then $\bar{T}$ is self-adjoint and $\bar{T}$ permutes with $\bar{S}_1$, where S_1 is the restriction of S to the vector space $\tilde{D}_1$ generated by the vectors $\{T^nx\}$, $n = 0, 1, 2, \dots$, $x \in D$.*

Proof. The Corollary is an immediate consequence of Theorem 4 and Theorem 7.

4. Two parameter moment problems

Let $(\mu(n, m))$, $n, m = 0, 1, 2, \dots$, be a two parameter sequence of real numbers. We wish to find sufficient conditions so that the sequence $(\mu(n, m))$ be a moment sequence; that is, may be represented by an integral

$$\mu(n, m) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} t^n s^m d\nu(t, s), \quad n, m = 0, 1, 2, \dots,$$

where ν is a bounded positive Radon measure on R^2. (Cf. [9] and [4].) An obvious necessary condition is that $(\mu(n, m))$ be of positive type, denoted by $\mu(m, n) \gg 0$, in the following sense: Given any finite sequence $((n_i, m_i))$, $i = 1, 2, 3, \dots, k$ of pairs of non-negative integers and a sequence (α_i), $i = 1, 2, \dots, k$, of complex numbers, then

$$\sum_{j=1}^{k} \sum_{i=1}^{k} \alpha_i \bar{\alpha}_j \mu(n_i + n_j, m_i + m_j) \geq 0.$$

R. B. Zarhina [14] has shown (using the well-known theorem of Hilbert that not every non-negative polynomial in two variables can be written as a sum of squares of polynomials) that this condition is not sufficient.

In this section we shall apply the results of § 3 to obtain various sufficient conditions for a two parameter sequence $(\mu(n,m))$ of positive type to be a moment sequence. These conditions had previously been obtained by A. Devinatz [4] and G. I. Eskin [5] by different methods. All the results of this section can be extended to n-parameter sequences for $n>2$. The proofs are identical with those for $n=2$.

Let $(\mu(n,m))$ be a two parameter sequence of positive type. We associate with $(\mu(n,m))$ a reproducing hernel space $\mathbf{H}$ in the well-known fashion (cf. [1]): Let $\mathbf{H}_0$ be the linear space which consists of all functions $f(n,m)$, $n,m=0,1,2,\dots$ of the form $f(n,m)=\sum_{i=1}^{k}\alpha_i\mu(n+n_i,\,m+m_i)$. If $g(n,m)$ is another such function, i.e. $g(n,m)=\sum_{j=1}^{l}\beta_j\mu(n+\bar{n}_j,\,m+\bar{m}_j)$, we introduce into $\mathbf{H}_0$ a bilinear form by setting $(f|g)=\sum_{i=1}^{k}\sum_{j=1}^{l}\alpha_i\bar{\beta}_j\mu(n_i+\bar{n}_j,\,m_i+\bar{m}_j)$. It is not hard to verify that $(f|g)$ is well defined; i.e. independent of the particular representations of f and g and that is actually it an inner product on $\mathbf{H}_0$. $\mathbf{H}_0$ can be completed to a Hilbert space $\mathbf{H}$ such that the elements in $\mathbf{H}$ are also double sequences $(h(n,m))$, $n,m=0,1,2,\dots,$ of complex numbers and such that $(\mu(n,m))$ is a reproducing kernel for $\mathbf{H}$. That is, if $h\in\mathbf{H}$ then

$$h(r,s)=(h\,|\,\mu_{(r,s)}),$$

where $\mu_{(r,s)}$ is the function $\mu_{(r,s)}(n,m)=\mu(n+r,\,m+s)$, $n,m=0,1,2,\dots$ (for details cf. [1] and [3]).

Let $T_1\,(T_2)$ be the linear operator in $\mathbf{H}$ whose domain $D(T_1)\,(D(T_2))$ consists of all $h\in\mathbf{H}$ such that $h_{(1,0)}\,(h_{(0,1)})$ belongs to $\mathbf{H}$. (If $h\in\mathbf{H}$, we denote by $h_{(r,s)}$ the double sequence $h_{(r,s)}(n,m)=h(n+r,\,m+s)$.) Then T_1 and T_2 are closed operators and $T=T_1^*$ and $S=T_2^*$ are symmetric operators in $\mathbf{H}$. Furthermore, $T(S)$ is the closure of its restriction to $\mathbf{H}_0$ (for details cf. [1], [3]). If T and S have self-adjoint extensions H_1 and H_2 respectively, which permute, then $(\mu(n,m))$ is a moment sequence. Indeed, let $E_1(\sigma)$ and $E_2(\sigma)$ be the spectral resolutions of H_1 and H_2 respectively and $\mu_0=\mu_{(0,0)}$, then

$$\mu(n,m)=(\mu_{(n,m)}\,|\,\mu_0)=(H_1^nH_2^m\mu_0\,|\,\mu_0)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}t^ns^m d\,\|E_1(t)E_2(s)\mu_0\|^2$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}t^ns^m d\nu(t,s).$$

If the operators T and S are both self-adjoint and permute, then the sequence $(\mu(n,m))$ is a determined moment sequence; i.e. the measure ν representing $(\mu(n,m))$ is uniquely determined (cf. [4] p. 487).

The following lemma will be needed.

Lemma 8. *Let S be a closed symmetric operator in a Hilbert space* $\mathbf{H}$ *and T a self-adjoint operator in* $\mathbf{H}$ *which permutes with S and suppose that T and S are both real with respect to a conjugation J in* $\mathbf{H}$ *(cf.* [11] p. 360)*. Then S has a self-adjoint extension $\tilde{S}$ in* $\mathbf{H}$ *which permutes with T.*

Proof. If $x\in\mathbf{H}$ we denote the element Jx by $\bar{x}$. Let $E(\sigma)$ be the spectral resolution of T. If x is any element in $\mathbf{H}$ we denote by $\mathbf{M}(x)$ the closed subspace of $\mathbf{H}$ generated by the vectors $\{E(\sigma)x\}$, where σ ranges over all the Borel sets on the real line. $\mathbf{M}(x)$ is the set of all elements y of the form $y=f(T)x$, where $f\in L_2(\nu_x)$ and ν_x is the measure $\nu_x(\sigma)=\|E(\sigma)x\|^2$ (cf. [11] p. 243). Let $\mathbf{H}_i$ and $\mathbf{H}_{-i}$ be the deficiency spaces of S; i.e. $\mathbf{H}_i\,(\mathbf{H}_{-i})$ is the set of vectors x in $D(S^*)$ such that $S^*x=ix(S^*x=-ix)$. Since S is a real transformation with respect to J, the mapping $x\to\bar{x}$ is an isometric mapping of $\mathbf{H}_i$ onto $\mathbf{H}_{-i}$. Now choose a family $(\varphi_i)_{i\in I}$ of distinct vectors in $\mathbf{H}_i$ such that $\mathbf{H}_i=\sum_{i\in I}\oplus\mathbf{M}(\varphi_i)$ ($\oplus$ denotes orthogonal direct sum). Then $\mathbf{H}_{-i}=\sum_{i\in I}\oplus\mathbf{M}(\bar{\varphi}_i)$. Let U be the mapping on $\mathbf{H}_i$ which maps the element $x=\sum_{i\in I}x_i$, $x_i\in\mathbf{M}(\varphi_i)$, $x_i=f_i(T)\varphi_i$ onto the element $\sum_{i\in I}f_i(T)\bar{\varphi}_i$. Now,

$$\|f_i(T)\varphi_i\|^2=\int_{-\infty}^{\infty}|f_i(t)|\,d\|E(t)\varphi_i\|^2=\int_{-\infty}^{\infty}|f_i(t)|^2d\|E(t)\bar{\varphi}_i\|^2=\|f_i(T)\bar{\varphi}_i\|^2,$$

since $E(\sigma)\bar{\varphi}_i=E(\sigma)\varphi_i$ (this is true because T is real with respect to J cf. [11] p. 362). Hence U is an isometric mapping of $\mathbf{H}_i$ onto $\mathbf{H}_{-i}$. Since each $\mathbf{M}(\varphi_i)$ reduces $E(\sigma)$ (cf. [11] p. 243) it follows that $E(\sigma)U=UE(\sigma)$ for all Borel sets σ. Finally, let $D(\tilde{S})=D(S)\dotplus(I-U)\mathbf{H}_i$ ($\dotplus$ denotes direct sum) and define $\tilde{S}$ as the operator whose domain is $D(\tilde{S})$ and which maps the element $\tilde{x}=x+(I-U)\varphi$, $x\in D(S)$, $\varphi\in\mathbf{H}_i$ into the element $S^*x=Sx+i(I+U)\varphi$. $\tilde{S}$ is a self-adjoint extension of S which permutes with T (for details on the Cayley transform of a symmetric operator cf. [11] and [12]).

Theorem 9. (G. I. Eskin [5]). *Let $(\mu(n,m))$, $n,m=0,1,2,\dots,$ be a two parameter sequence of real numbers such that $\mu(n,m)\gg 0$. Suppose that for every fixed m_0 the one parameter moment sequence*

$$(\mu(n,2(m_0+1))+\mu(n,2m_0)),\quad n=0,1,2,\dots,$$

is determined, then $(\mu(n,m))$ is a two parameter moment sequence. If in addition the moment sequence $(\mu(2n_0,m))$, $m=0,1,2,\dots,$ is determined for each n_0, then the moment sequence $(\mu(n,m))$ is determined.

Proof.

$$\begin{aligned}\mu(n,2(m_0+1))+\mu(n,2m_0)&=(T^nS^{m_0+1}\mu_0\,|\,S^{m_0+1}\mu_0)+(T^nS^{m_0}\mu_0\,|\,S^{m_0}\mu_0)\\&=(T^n(S+iI)S^{m_0}\mu_0\,|\,(S+iI)S^{m_0}\mu_0).\end{aligned}$$

Let $D=\{S^{m_0}\mu_0\}$, $m_0=0,1,2,\dots,$ and apply Theorem 7. Since the vector space $\tilde{D}$ spanned by the vectors $\{T^nS^{m_0}\mu_0\}$, $n,m_0=0,1,2,\dots,$ is precisely $\mathbf{H}_0$, it follows that T is self-adjoint and that T permutes with S. Since T and S are real operators with respect to the conjugation J which maps an element of $\mathbf{H}$ into its complex conjugate, it follows from Lemma 8 that S has a self-adjoint extension $\tilde{S}$ which permutes with T, This proves the first part of the theorem. If $\mu(2n_0,m)=(S^mT^{n_0}\mu_0\,|\,T^{n_0}\mu_0)$, $m=0,1,2,\dots,$ is a determined moment sequence, then the set $\{T^{n_0}\mu_0\}$, $n_0=0,1,\dots,$ is a set of determining vectors for S and S is self-adjoint by Theorem 1.

J. A. Shohat and J. D. Tamarkin have proved the following theorem in their book [9] p. 21: Let $(\mu(n_1, n_2, \ldots, n_k))$ be a k-parameter sequence of real numbers of positive type and suppose that $(\mu(n_1, n_2, \ldots, n_k))$ is a k-parameter moment sequence. Let

$$\lambda(2n) = \mu(2n, 0, \ldots, 0) + \mu(0, 2n, 0, \ldots, 0) + \ldots + \mu(0, 0, 0, \ldots, 2n)$$

and suppose that
$$\sum_{n=1}^{\infty} \lambda(2n)^{-1/2n} = \infty,$$

then the moment sequence $(\mu(n_1, n_2, \ldots, n_k))$ is determined.

Using the results of § 3 (specifically Theorem 6) we can prove the following very much stronger sesult.

Let $(\mu(n_1, n_2, \ldots, n_k))$ be a k-parameter sequence of real numbers of positive type and suppose that

$$\sum_{n=1}^{\infty} \lambda_i(2n)^{-1/2n} = \infty \quad \text{for} \quad i = 1, 2, \ldots, k,$$

where $\lambda_1(n) = \mu(n, 0, 0, \ldots)$, $\lambda_2(n) = \mu(0, n, 0, \ldots, 0)$, $\ldots$, $\lambda_k(n) = \mu(0, 0, 0, \ldots, n)$ then $(\mu(n_1, n_2, \ldots, n_k))$ is a determined k-parameter moment sequence.

We shall prove the theorem for $k = 2$.

Theorem 10. *Let $(\mu(n, m))$, $n, m = 0, 1, 2, \ldots$, be a two parameter sequence of real numbers such that $\mu(n, m) \gg 0$. Suppose that*

$$\sum_{n=1}^{\infty} \frac{1}{\mu(2n, 0)^{1/2n}} = \infty, \tag{1}$$

then $(\mu(n, m))$ is a two parameter moment sequence. If in addition

$$\sum_{m=1}^{\infty} \frac{1}{\mu(0, 2m)^{1/2m}} = \infty,$$

then the moment sequence $(\mu(n, m))$ is determined.

Proof. $\mu(n, 0) = (T^n \mu_0 \,|\, \mu_0)$. (1) implies that μ_0 is a quasi-analytic vector for T. Hence $S^m \mu_0$ is a quasi-analytic vector for T for $m = 0, 1, 2, \ldots$, by Theorem 4. Let $D = \{S^m \mu_0\}$, $m = 0, 1, 2, \ldots$ and apply Corollary 5. It follows that T is self-adjoint and that T permutes with S. The remainder of the proof is identical with the proof of Theorem 9.

Theorem 11. (A. Devinatz [4]). *Let $(\mu(n, m))$, $n, m = 0, 1, 2, \ldots$, be a two parameter sequence of real numbers such that $\mu(n, m) \gg 0$ and such that $(\mu(2n_0, m))$, $m = 0, 1, 2, \ldots$, is a determined moment sequence for each n_0. Suppose furthermore that the one parameter moment sequence*

$$(\mu(n, 2m_0) + \mu(n, 0)), \quad n = 0, 1, 2, \ldots,$$

is determined. Then $(\mu(n, m))$ is a determined moment sequence.

Proof. $$\mu(n, 2m_0) + \mu(n, 0) = (T^n S^{m_0}\mu_0 \mid S^{m_0}\mu_0) + (T^n\mu_0 \mid \mu_0).$$

Since (2) is determined for each m_0, it follows that $S^{m_0}\mu_0$ is a vector of uniqueness for T for every m_0. Hence T is self-adjoint by Theorem 1. The assumption that $\mu(2n_0, m) = (S^m T^{n_0}\mu_0 \mid T^{m_0}\mu_0)$, $m = 0, 1, 2, \ldots$, is determined for each n_0 implies that $T^{m_0}\mu_0$ is a vector of uniqueness for S for each n_0. Hence S is self-adjoint by Theorem 1. Let $E(\sigma)$ be the spectral resolution of T, σ a fixed Borel set on the real line and m_0 and k two fixed non-negative integers. Since

$$(T^n S^{m_0}\mu_0 \mid S^{m_0}\mu_0) + (T^n\mu_0 \mid \mu_0) = \int_{-\infty}^{\infty} t^n d\,\|E(t)\,S^{m_0}\mu_0\|^2 + \int_{-\infty}^{\infty} t^n d\,\|E(t)\,\mu_0\|^2,$$

$n = 0, 1, 2, \ldots$, is a determined moment sequence, there exists a sequence of polynomials (p_n) such that

$$\int_{-\infty}^{\infty} |p_n(t) - \chi_\sigma(t)\,t^k|^2 d\,\|E(t)\,S^{m_0}\mu_0\|^2 + \int_{-\infty}^{\infty} |p_n(t) - \chi_\sigma(t)\,t^k|^2 d\,\|E(t)\,\mu_0\|^2$$
$$= \|p_n(T)\,S^{m_0}\mu_0 - E(\sigma)\,T^k S^{m_0}\mu_0\|^2 + \|p_n(T)\,\mu_0 - E(\sigma)\,T^k\mu_0\|^2 \to 0.$$

That is, $$p_n(T)\,\mu_0 \to E(\sigma)\,T^k\mu_0$$

and $$S^{m_0}p_n(T)\,\mu_0 = p_n(T)\,S^{m_0}\mu_0 \to E(\sigma)\,T^k S^{m_0}\mu_0.$$

S^{m_0} is self-adjoint and hence closed. Therefore

$$E(\sigma)\,T^k\mu_0 \in D(S^{m_0}) \quad \text{and} \quad S^{m_0}E(\sigma)\,T^k\mu_0 = E(\sigma)\,T^k S^{m_0}\mu_0$$

for $m_0 = 0, 1, 2, \ldots$.

From this follows that

$$E(\sigma)\,T^k S^{m_0}\mu_0 = S^{m_0}E(\sigma)\,T^k\mu_0 \in D(S)$$

and $$SE(\sigma)\,T^k S^{m_0}\mu_0 = S^{m_0+1}E(\sigma)\,T^k\mu_0 = E(\sigma)\,T^k S^{m_0+1}\mu_0$$

for $k, m_0 = 0, 1, 2, \ldots$.

Hence $E(\sigma)g \in D(S)$ and $SE(\sigma)g = E(\sigma)Sg$ for all $g \in \mathbf{H}_0$.

From this follows, since S is the closure of its restriction to $\mathbf{H}_0$, that $E(\sigma)g \in D(S)$ and $SE(\sigma)g = E(\sigma)Sg$ for all $g \in D(S)$. That is,

$$E(\sigma)\,S \subset SE(\sigma)$$

for all Borel sets σ.

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