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Existence and approximation theorems for solutions of complex analogues of boundary problems

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1. Introduction

The purpose of this note is to prove existence and approximation theorems for analytic solutions of a differential problem of the form

$$P(D)u=f \text{ in } \Omega, \tag{1.1}$$

$$Q_j(D)u=g_j \text{ in } H\cap\Omega,\quad 1\leqslant j\leqslant r, \tag{1.2}$$

where Ω is a convex open set in $\mathbf{C}^n$, H a complex hyperplane in $\mathbf{C}^n$, and $P(D)$, $Q_j(D)$ are constant coefficient differential operators in $D=(D_1, \ldots, D_n)$ where

$$D_k=\frac{\partial}{\partial z_k}=\frac{1}{2}\left(\frac{\partial}{\partial x_k}-i\,\frac{\partial}{\partial x_{n+k}}\right), \tag{1.3}$$

(the coordinates of $\mathbf{C}^n=\mathbf{R}^{2n}$ are written $z_k=x_k+ix_{n+k}$, where $k=1, \ldots, n$ and x_k, x_{n+k} are real). The hyperplane H is supposed to be non-characteristic with respect to $P(D)$, but no restriction is imposed on the operators $Q_1(D), \ldots, Q_r(D)$. We call (1.1), (1.2) (by abuse of language) a 'boundary problem in the complex region Ω'. Our main result is a geometric characterization (Theorems 3.2 and 3.3) of those convex open sets $\Omega\subset\mathbf{C}^n$ in which the problem (1.1), (1.2) can be solved for all choices of operators $Q_j(D)$ and all analytic functions f, g_j satisfying a natural condition. The proof depends on a division algorithm for analytic functionals (Theorem 2.3) as well as on the theory of general (overdetermined) systems of differential equations with constant coefficients.

Consider a system of differential equations in an open set $\omega\subset\mathbf{R}^s$,

$$\sum_1^m P_{jk}\left(\frac{\partial}{\partial x}\right)u_k=f_j,\quad 1\leqslant j\leqslant r, \tag{1.4}$$

where $\partial/\partial x=(\partial/\partial x_1, \ldots, \partial/\partial x_s)$ and $f_j\in\mathcal{E}(\omega)$, the space of infinitely differentiable complex-valued functions in ω, and $P_{jk}(\partial/\partial x)$ are differential operators with constant coefficients. A necessary condition for (1.4) to have a solution $u_k\in\mathcal{E}(\omega)$ is that

$$\sum_1^r G_j(\xi) P_{jk}(\xi) = 0, 1 \leqslant k \leqslant m \Rightarrow \sum_1^r G_j\left(\frac{\partial}{\partial x}\right) f_j = 0 \tag{1.5}$$

for all choices of polynomials $G_1(\xi), \ldots, G_r(\xi)$. Conversely, an existence theorem for systems states that when ω is convex and (1.5) is fulfilled, (1.4) can be solved with $u_k \in \mathcal{E}(\omega)$. This has been proved by Malgrange [5] and Hörmander (according to personal communication, the proof will be included in a forthcoming monograph based on [4]), and is also a consequence of the fundamental principle of Ehrenpreis [1], [2]. The same authors have proved that when ω is convex, any solution $(u_1, \ldots, u_m) \in \mathcal{E}(\omega)^m$ of

$$\sum_1^m P_{jk}\left(\frac{\partial}{\partial x}\right) u_k = 0, \quad 1 \leqslant j \leqslant r, \tag{1.6}$$

can be approximated in $\mathcal{E}(\omega)^m$ by linear combinations of solutions of the form

$$g_k(x) \exp(x_1 \zeta_1 + \ldots + x_s \zeta_s), \quad 1 \leqslant k \leqslant m, \tag{1.7}$$

where g_k are polynomials and ζ_j complex numbers.

Now a general system in $\Omega \subset \mathbf{C}^n$,

$$\sum_1^m Q_{jk}(D) u_k = f_j, \quad 1 \leqslant j \leqslant r, \tag{1.8}$$

where $D = (D_1, \ldots, D_n)$ is defined by (1.3) and f_j and u_k are required to be analytic in Ω, is equivalent to a system of the type (1.4) with $s = 2n$ real independent variables and $r + mn$ equations, the first r being those of (1.8), and the remaining the Cauchy–Riemann equations

$$\frac{\partial u_k}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_j} + i \frac{\partial u_k}{\partial x_{n+j}}\right) = 0, \quad 1 \leqslant k \leqslant m, \quad 1 \leqslant j \leqslant n. \tag{1.9}$$

The compatibility condition (1.5) for the system (1.8), (1.9) involving $r + mn$ polynomials in $2n$ real variables can now be replaced, as an easy calculation shows, by the equivalent conditions

$$\sum_1^r S_j(\zeta) Q_{jk}(\zeta) = 0, 1 \leqslant k \leqslant m \Rightarrow \sum_1^r S_j(D) f_j = 0 \tag{1.10}$$

for all choices of (analytic) polynomials $S_1, \ldots, S_r$ in n complex variables $\zeta = (\zeta_1, \ldots, \zeta_n)$, and

$$\frac{\partial f_j}{\partial \bar{z}_k} = 0, \quad 1 \leqslant j \leqslant r, \quad 1 \leqslant k \leqslant n. \tag{1.11}$$

Thus, if Ω is convex, (1.8) has an analytic solution in Ω if and only if (1.10) and (1.11) are fulfilled.

When $f_j = 0$ in (1.8), an analytic solution $(u_1, \dots, u_m)$ of (1.8) can be approximated (in $\mathcal{E}(\Omega)^m$) by linear combinations of solutions of (1.8), (1.9) of the form (1.7) where $s = 2n$. This means that $(u_1, \dots, u_m)$ can be approximated uniformly on compact parts of Ω by linear combinations of solutions of (1.8) of the form

$$g_k(z) \exp (z_1 \zeta_1 + \dots + z_n \zeta_n), \quad 1 \leq k \leq m, \tag{1.12}$$

where g_k are analytic polynomials in z.

These existence and approximation theorems for the general system (1.8) combined with an existence theorem for the Cauchy problem (Theorem 3.1) yield similar theorems for the problem (1.1) (1.2) (see Theorems 3.3 and 4.2), provided Ω satisfies a geometric condition determined by the principal part of $P(D)$.

The subject of the present paper was suggested to the author by Professor Lars Hörmander, the questions being originally formulated for boundary problems in a real linear space. Its scope, however, is determined by the methods of proof employed. Thus Theorem 3.1 is to be regarded rather as an application of the division algorithm for analytic functionals of Section 2. To treat the real case, an analogous algorithm for distributions with compact support would be useful.

I wish to express here my gratitude to Professor Lars Hörmander for his stimulating instruction and incessant interest in my work.

2. Preliminaries concerning analytic functionals

Let $\mathcal{A}(\Omega)$ denote the Fréchet space of all analytic functions in an open set $\Omega \subset \mathbf{C}^n$ under the topology of uniform convergence on all compact parts of Ω. An element μ of the dual space $\mathcal{A}'(\Omega)$ is called an analytic functional, and a compact part K of Ω is called a carrier of μ if for all neighborhoods ω of K there is a constant C such that

$$|\mu(f)| \leq C \sup_{\omega} |f|$$

for all $f \in \mathcal{A}(\Omega)$. By definition any functional $\in \mathcal{A}'(\Omega)$ is carried by some compact set in Ω.

The Laplace transform $\hat{\mu}$ of $\mu \in \mathcal{A}'(\Omega)$ is defined by

$$\hat{\mu}(\zeta) = \mu(\exp \langle z, \zeta \rangle), \quad \zeta \in \mathbf{C}^n,$$

where $\langle z, \zeta \rangle = \sum_1^n z_j \zeta_j$. It follows that $\hat{\mu}$ is an entire function of exponential type, and that $\hat{\mu}$ determines μ uniquely if Ω is a Runge domain.

We choose $\operatorname{Re} \langle z, \zeta \rangle = \log |\exp \langle z, \zeta \rangle|$ as the bilinear form taking into duality two copies of $\mathbf{C}^n$ regarded as real linear spaces. Consequently, the supporting function φ of a non-empty part M of $\mathbf{C}^n$ is defined by

$$\varphi(\zeta) = \sup_{z \in M} \operatorname{Re} \langle z, \zeta \rangle, \quad \zeta \in \mathbf{C}^n.$$

It is clear that φ is convex and positively homogeneous:

$$\varphi(0)=0, \quad \varphi(\zeta+t\theta)\leqslant\varphi(\zeta)+t\varphi(\theta), \quad \zeta, \theta\in\mathbf{C}^n, t\geqslant 0,$$

and has values in $(-\infty, +\infty]$. As is well known,

$$\{z\in\mathbf{C}^n;\ \forall\zeta\in\mathbf{C}^n\colon \operatorname{Re}\langle z,\zeta\rangle\leqslant\varphi(\zeta)\}$$

is the closed convex hull of M. We note also that M is contained in a real linear subspace V of $\mathbf{C}^n$ if and only if $\varphi(\zeta+\theta)=\varphi(\zeta)$ for all $\zeta\in\mathbf{C}^n$ and all θ in

$$V^{\perp}=\{\theta\in\mathbf{C}^n;\ \forall z\in V\colon \operatorname{Re}\langle z,\theta\rangle=0\}.$$

In particular M lies in the complex hyperplane $\langle z, N\rangle=0$ if and only if $\varphi(\zeta+\tau N)=\varphi(\zeta)$ for all $\zeta\in\mathbf{C}^n$ and all $\tau\in\mathbf{C}^1$.

It is clear that if $\mu\in\mathcal{A}'(\Omega)$ is carried by a compact set K whose supporting function is φ, we have with $F=\hat{\mu}$

$$\forall\varepsilon>0\ \exists C\ \forall\zeta\in\mathbf{C}^n\colon\ |F(\zeta)|\leqslant C\exp(\varphi(\zeta)+\varepsilon|\zeta|), \tag{2.1}$$

for $\varphi(\zeta)+\varepsilon|\zeta|$ is the supporting function of the neighborhood $\{z\in\mathbf{C}^n;\ \exists w\in K\colon |w-z|\leqslant\varepsilon\}$ of K. (Here $|\zeta|^2=\sum|\zeta_j|^2$.) The converse of this statement (when K is convex) is an important theorem.

Theorem 2.1. (*Martineau–Ehrenpreis–Hörmander*) *Suppose that (2.1) is valid, where $F\in\mathcal{A}(\mathbf{C}^n)$ and φ is the supporting function of a convex compact set K. Then for any Runge domain $\Omega\supset K$, there is a unique analytic functional $\mu\in\mathcal{A}'(\Omega)$ such that $\hat{\mu}=F$, and this functional is carried by K.*

Proofs have been given by three different methods, see Martineau [6], Ehrenpreis [2], and Hörmander [4]. When K is a polycylinder, the theorem follows from the Pólya representation of analytic functionals in one variable, but this case is insufficient for our purposes, cf. Remark 2.4.

In the sequel we shall differentiate analytic functionals according to the formula

$$(P(D)\mu)(f)=\mu(P(-D)f), \quad f\in\mathcal{A}(\Omega), \quad \mu\in\mathcal{A}'(\Omega),$$

where $P(D)=P(D_1,\dots,D_n)$ is an arbitrary differential operator with constant coefficients, and D_k is defined by (1.3). By the Cauchy integral formula, $P(D)\mu\in\mathcal{A}'(\Omega)$ if $\mu\in\mathcal{A}'(\Omega)$. We also note that the Laplace transform of $P(D)\mu$ is $P(-\zeta)\hat{\mu}(\zeta)$, and that $\mu(Q(z)\exp\langle z,\zeta\rangle)=Q(D_\zeta)\hat{\mu}(\zeta)$ if $Q(z)$ is a polynomial.

We can now formulate a division algorithm for analytic functionals. Given $\mu\in\mathcal{A}'(\Omega)$ and a differential operator $P(D)$ with constant coefficients, we try to find ν, $\varrho\in\mathcal{A}'(\Omega)$ such that $\mu=P(-D)\nu+\varrho$ and ϱ is orthogonal to all functions in $\mathcal{A}(\Omega)$ having vanishing Cauchy data of order $<m$ in the complex hyperplane $\langle z,N\rangle=0$, where m is the order of $P(D)$. When Ω is a Runge domain, the Laplace transformation reduces this to a division algorithm for entire functions, $\hat{\mu}(\zeta)=P(\zeta)\hat{\nu}(\zeta)+\hat{\varrho}(\zeta)$. The condition on ϱ is reformulated in terms of $\hat{\varrho}$ by

the following lemma. We say that an entire function H is of degree $<m$ in the direction N if, for any $\zeta \in \mathbf{C}^n$, the function $\mathbf{C}^1 \ni \tau \to H(\zeta+\tau N) \in \mathbf{C}^1$ is a polynomial of degree $<m$ in one variable.

Lemma 2.2. *Suppose Ω is a Runge domain and $\varrho \in \mathcal{A}'(\Omega)$. Then $\hat{\varrho}$ is of degree $<m$ in the direction $N \neq 0$ if and only if ϱ is orthogonal to all functions $u \in \mathcal{A}(\Omega)$ having zero Cauchy data of all orders $<m$ in the hyperplane $\langle z, N\rangle = 0$.*

When $N=(0, \ldots, 0, 1)$, the latter condition means that $\varrho(u)=0$ if

$$D_n^k u(z_1, \ldots, z_{n-1}, 0) = 0 \text{ when } (z_1, \ldots, z_{n-1}, 0) \in \Omega,\ 0 \leqslant k < m. \tag{2.2}$$

Proof. We may suppose that $N=(0, \ldots, 0, 1)$. If ϱ is orthogonal to the solutions of (2.2) we can in particular choose $u(z)=z_n^m \exp\langle z, \zeta\rangle$ and get

$$0=\varrho(z_n^m \exp\langle z, \zeta\rangle)=\frac{\partial^m}{\partial \zeta_n^m}\hat{\varrho}(\zeta) \tag{2.3}$$

which proves that $\hat{\varrho}$ is a polynomial in ζ_n of degree less than m.

Conversely, suppose $\hat{\varrho}$ has degree less than m in the direction $(0, \ldots, 0, 1)$. Then (2.3) follows, and also $\varrho(z_n^m v)=0$ for any $v \in \mathcal{A}(\Omega)$ since linear combinations of the exponential functions are dense in $\mathcal{A}(\Omega)$. But any solution $u \in \mathcal{A}(\Omega)$ of (2.2) is of the form $u(z)=z_n^m v(z)$ for some $v \in \mathcal{A}(\Omega)$ (also if (2.2) is empty) so this completes the proof of the lemma.

We are thus led to study representations

$$F(\zeta)=P(\zeta)\,G(\zeta)+H(\zeta) \tag{2.4}$$

of an entire function F of exponential type, where P is a given polynomial of degree m, and G, H are required to be entire, H being of degree less than m in a direction $N \neq 0$. We also want G and H to be Laplace transforms of analytic functionals in $\mathcal{A}'(\mathbf{C}^n)$, i.e. G, H shall be of exponential type, and this is possible for arbitrary F only when

$$p(N) \neq 0, \tag{2.5}$$

where p is the homogeneous part of P of degree m. We assume that (2.5) is satisfied in all what follows.

The representation (2.4) is always unique. In fact, (2.4) is equivalent to

$$\frac{G(\zeta+\tau N)}{\tau}=\frac{F(\zeta+\tau N)}{\tau P(\zeta+\tau N)}-\frac{H(\zeta+\tau N)}{\tau P(\zeta+\tau N)}, \tag{2.6}$$

where $\tau \in \mathbf{C}^1$, $\zeta \in \mathbf{C}^n$. For an arbitrary fixed $\zeta \in \mathbf{C}^n$, the zeros of $\tau \to \tau P(\zeta+\tau N)$ lie inside some circle Γ, and if H is of degree $<m$ in the direction N we get from (2.6)

$$G(\zeta)=\frac{1}{2\pi i}\int_\Gamma \frac{F(\zeta+\tau N)\,d\tau}{\tau P(\zeta+\tau N)} \tag{2.7}$$

provided only (2.5) is fulfilled. This determines G and hence H uniquely.

On the other hand, (2.7) defines indeed an entire function, and if we put $H=F-PG$ we have

$$H(\zeta)=\frac{1}{2\pi i}\int_{\Gamma}\frac{F(\zeta+\tau N)\,(P(\zeta+\tau N)-P(\zeta))\,d\tau}{\tau P(\zeta+\tau N)}. \tag{2.8}$$

We now use the identity

$$\frac{P(\zeta+\tau N)-P(\zeta+\sigma N)}{(\tau-\sigma)\,p(N)}=\sum_1^m(\sigma-\tau_1)\dots(\sigma-\tau_{k-1})\,(\tau-\tau_{k+1})\dots(\tau-\tau_m),$$

where $\tau_1,\dots,\tau_m$ are the zeros of $\tau\to P(\zeta+\tau N)$ for a fixed ζ, and obtain

$$H(\zeta+\sigma N)=\sum_1^m(\sigma-\tau_1)\dots(\sigma-\tau_{k-1})\,\frac{1}{2\pi i}\int_{\Gamma}\frac{F(\zeta+\tau N)\,d\tau}{(\tau-\tau_1)\dots(\tau-\tau_k)}$$

after a change of variable in the line integral (2.8). This proves that H is of degree $<m$ in the direction N, and so the existence of a representation (2.4) is established.

It remains to estimate the growth of G and H. Suppose F satisfies

$$\forall\varepsilon>0\,\exists C\,\forall\zeta\in\mathbf{C}^n:\ |F(\zeta)|\leqslant C\exp\,(\varphi(\zeta)+\varepsilon|\zeta|) \tag{2.9}$$

for some continuous and positively homogeneous function φ. If H happens to be zero we can apply the inequality

$$|p(N)\,G(\zeta)|\leqslant\sup_{|\tau|\leqslant 1}|P(\zeta+\tau N)\,G(\zeta+\tau N)|=\sup_{|\tau|\leqslant 1}|F(\zeta+\tau N)| \tag{2.10}$$

(see e.g. [3, Lemma 3.1.2]) to prove that $G=F/P$ satisfies (2.9) with the same function φ. In the general case we choose coordinates so that $N=(0,\dots,0,1)$ and write $\zeta=(\zeta',\zeta_n)$, $\zeta'=(\zeta_1,\dots,\zeta_{n-1})$. Now (2.8) takes the form

$$H(\zeta)=\frac{1}{2\pi i}\int_{\Gamma}\frac{F(\zeta',\tau)\,S(\zeta,\tau)\,d\tau}{P(\zeta',\tau)}, \tag{2.11}$$

where $S(\zeta,\tau)=(P(\zeta',\tau)-P(\zeta))/(\tau-\zeta_n)$ is a polynomial in $n+1$ variables of degree $m-1$ and Γ is any sufficiently large circle. Let $\sigma_1,\dots,\sigma_m$ be the zeros of $\sigma\to p(\zeta',\sigma)$ for some arbitrary $\zeta'\in\mathbf{C}^{n-1}$, and let γ be the boundary of the union of the disks with centers at $\sigma_1,\dots,\sigma_m$ and radii equal to $\delta|\zeta'|$, where $\delta\in(0,1]$ is specified later. In view of (2.5) there is a constant A such that $p(\zeta',\sigma)=0$ implies $|\sigma|\leqslant A|\zeta'|$, and hence $\tau\in\gamma$ implies $|\tau|\leqslant(A+1)|\zeta'|$. We conclude that

$$\tau\in\gamma\Rightarrow|P(\zeta',\tau)-p(\zeta',\tau)|\leqslant B(1+|\zeta'|^{m-1})$$

for some constant B independent of ζ' and δ. Hence we obtain if $\tau\in\gamma$

$$|P(\zeta',\tau)|\geqslant|p(\zeta',\tau)|-B(1+|\zeta'|^{m-1})\geqslant|p(N)|\,(\delta|\zeta'|)^m-B(1+|\zeta'|^{m-1}),$$

for $p(\zeta',\tau)=p(N)\prod_1^m(\tau-\sigma_k)$ where each factor in the product has absolute value not less than $\delta|\zeta'|$. Hence, for some constant a_δ depending only on P and δ, $|\zeta'|>a_\delta$ implies that the zeros of $\sigma\to P(\zeta',\sigma)$ lie inside γ, and that $|P(\zeta',\tau)|\geqslant 1$ when $\tau\in\gamma$. Using (2.9) and (2.11) with Γ replaced by the cycle γ we now obtain for $\varepsilon>0$

$$|H(\zeta)|\leqslant C_\varepsilon \sup_{\tau\in\gamma}|\zeta|^m \exp(\varphi(\zeta',\tau)+\varepsilon|(\zeta',\tau)|)$$

if $|\zeta'|>a_\delta$. Since φ is positively homogeneous and uniformly continuous in $\{\theta;|\theta|\leqslant 2\}$, we can to each $\varepsilon>0$ choose $\delta\in(0,1]$ such that $|\eta-\theta|\leqslant\delta|\theta|$ implies $|\varphi(\eta)-\varphi(\theta)|\leqslant\varepsilon|\theta|$, in particular $|\varphi(\zeta',\tau)-\varphi(\zeta',\sigma_k)|\leqslant\varepsilon(A+1)|\zeta'|$ for some k when $\tau\in\gamma$, and hence

$$|H(\zeta)|\leqslant C_\varepsilon \sup_k|\zeta|^m \exp(\varphi(\zeta',\sigma_k)+\varepsilon(2A+3)|\zeta'|)$$

when $|\zeta'|>a_\delta$. This, together with the fact that H is a polynomial in ζ_n of degree less than m, gives

$$\forall\varepsilon>0\,\exists C\,\forall\zeta\in\mathbf{C}^n:\ |H(\zeta)|\leqslant C\exp(\psi'(\zeta)+\varepsilon|\zeta|), \tag{2.12}$$

where ψ' is positively homogeneous and continuous,

$$\psi'(\zeta)=\sup_\sigma(\varphi(\zeta',\sigma);\ \ p(\zeta',\sigma)=0). \tag{2.13}$$

Putting

$$\psi(\zeta)=\sup(\varphi(\zeta),\psi'(\zeta)) \tag{2.14}$$

we get the same estimate for $F-H$ with ψ' replaced by ψ and hence from (2.10) also for $(F-H)/P=G$,

$$\forall\varepsilon>0\,\exists C\,\forall\zeta\in\mathbf{C}^n:\ |G(\zeta)|\leqslant C\exp(\psi(\zeta)+\varepsilon|\zeta|). \tag{2.15}$$

The estimates (2.9), (2.12), and (2.15) can now be interpreted in terms of analytic functionals by means of Theorem 2.1.

Theorem 2.3. *Suppose $P(D)$ is a differential operator of order m with constant coefficients satisfying*

$$p(0,\ldots,0,1)\neq 0, \tag{2.16}$$

where p is the homogeneous part of P of order m. Let Ω be a convex open set in $\mathbf{C}^n$ (hence a Runge domain) such that $\Omega'=\{z\in\Omega;\ z_n=0\}$ is non-empty, and suppose that the supporting function Φ of Ω satisfies

$$\Phi(\zeta)=\Phi'(\zeta)\ \text{ if }\ p(\zeta)=0, \tag{2.17}$$

where Φ' is the supporting function of Ω'. Then for any $\mu\in\mathcal{A}'(\Omega)$ there exist uniquely determined analytic functionals $\nu,\varrho\in\mathcal{A}'(\Omega)$ such that

$$\mu=P(-D)\nu+\varrho \tag{2.18}$$

and ϱ is orthogonal to all functions $u \in \mathcal{A}(\Omega)$ satisfying

$$D_n^k u(z) = 0 \ \text{when} \ z \in \Omega', \ 0 \leqslant k < m.$$

Proof. We write $F = \hat{\mu}$ and define $\nu, \varrho \in \mathcal{A}'(\mathbf{C}^n)$ by $\hat{\nu} = G, \hat{\varrho} = H$ where G, H are found by (2.7) and (2.8) respectively. According to (2.12) and Theorem 2.1, ϱ is carried by any convex compact set L' in the plane $z_n = 0$ whose supporting function ψ' satisfies

$$\psi'(\zeta) \geqslant \varphi(\zeta) \ \text{if} \ p(\zeta) = 0, \tag{2.19}$$

where φ is the supporting function of some carrier K of μ contained in Ω (note that ψ' is independent of ζ_n). Let ψ_j' be the supporting functions of convex compact sets L_j' such that $L_j' \subset L_{j+1}'$ and $\bigcup L_j' = \Omega'$. We claim that for some j, (2.19) is valid with $\psi' = \psi_j'$. Otherwise all the compact sets

$$\{\zeta \in \mathbf{C}^n;\ \psi_j'(\zeta) \leqslant \varphi(\zeta),\ p(\zeta) = 0, |\zeta| = 1\}$$

are non-empty, hence their intersection contains some point $\zeta \in \mathbf{C}^n$, i.e. $|\zeta| = 1$, $p(\zeta) = 0$, and

$$\Phi'(\zeta) = \Phi(\zeta) = \lim \psi_j'(\zeta) \leqslant \varphi(\zeta).$$

If $\Phi(\zeta)$ happens to be infinite, this inequality is impossible, and if $\Phi(\zeta) < +\infty$ we can take $z \in K$ such that

$$\operatorname{Re} \langle z, \zeta \rangle = \varphi(\zeta) \geqslant \Phi(\zeta).$$

This means in particular that z lies on the boundary of Ω, also a contradiction. Thus ϱ is carried by some compact part L' of Ω', and hence ν is carried by the convex hull of $K \cup L'$ which is a compact part of Ω. In particular, ν and ϱ can be extended to $\mathcal{A}(\Omega)$, and this proves the theorem since Lemma 2.2 shows that ϱ has the desired properties.

Remark 2.4. For any convex set Ω_1 relatively open in the plane $z_n = 0$ there exist open sets $\Omega \subset \mathbf{C}^n$ satisfying (2.17) such that $\Omega' = \Omega_1$. In fact, we may without restriction suppose that $0 \in \Omega_1$ and then define Ω as the convex hull of $\Omega_1 \cup \{(0, z_n) \in \mathbf{C}^n;\ |z_n| < a\}$, where a is specified below. Then $\Omega' = \Omega_1$ and if we denote the supporting function of Ω_1 by Φ_1, the supporting function of Ω becomes

$$\Phi(\zeta) = \sup\,(\Phi_1(\zeta),\ a|\zeta_n|),$$

and (2.17) is satisfied as soon as

$$a|\zeta_n| \leqslant \Phi_1(\zeta) \ \text{when} \ p(\zeta) = 0. \tag{2.20}$$

Since $0 \in \Omega_1$ we have $\varepsilon|\zeta'| \leqslant \Phi_1(\zeta)$ for some $\varepsilon > 0$, and since $p(0, \ldots, 0, 1) \neq 0$, we have $a|\zeta_n| \leqslant \varepsilon|\zeta'|$ when $p(\zeta) = 0$ for some $a > 0$. Hence (2.20) is valid when a is small enough. On the other hand, no bounded polycylinder Ω satisfies (2.17) (unless $n = 1$ or p is constant), and this shows also that one is not allowed to restrict attention to polycylinder carriers of the functionals occurring in (2.18). Indeed, let $\Omega=\Omega'\times\Omega_n$, where $\Omega'\subset\mathbf{C}^{n-1}$ is bounded and $\Omega_n\subset\mathbf{C}^1$. Then $\Phi(\zeta)=\Phi'(\zeta')+\Phi_n(\zeta_n)$ if Φ,Φ', and Φ_n are the supporting functions in $\mathbf{C}^n$, $\mathbf{C}^{n-1}$, and $\mathbf{C}^1$ of Ω, Ω', and Ω_n respectively. (2.17) now implies that $\Phi_n(\zeta_n)=0$ when $p(\zeta)=0$, and hence (if $m>0$) that $\Phi_n=0$, $\Omega_n=\emptyset$.

3. Existence of solutions of boundary problems in convex complex regions

We first investigate in which convex open sets in $\mathbf{C}^n$ a non-characteristic Cauchy problem (3.3), (3.4) has a solution for arbitrary analytic right-hand side and data.

Theorem 3.1. *Let Ω be a convex open set in $\mathbf{C}^n$ such that $\Omega'=\{z\in\Omega;\ z_n=0\}$ is non-empty and suppose that the supporting functions Φ and Φ' of Ω and Ω' satisfy*

$$\Phi(\zeta)=\Phi'(\zeta')\ \textit{when}\ p(\zeta)=0 \tag{3.1}$$

for some homogeneous polynomial p of degree m such that

$$p(0,\dots,0,1)\neq 0. \tag{3.2}$$

Further, let $P(D)$ be a differential operator with constant coefficients whose principal part is $p(D)$ (where $D=(D_1,\dots,D_n)$ is defined by (1.3)). Then the Cauchy problem

$$P(D)u=f\ \textit{in}\ \Omega, \tag{3.3}$$

$$D_n^k u(z',0)=g_k(z')\ \textit{in}\ \Omega',\quad 0\leqslant k<m, \tag{3.4}$$

has a unique solution $u\in\mathcal{A}(\Omega)$ for every $f\in\mathcal{A}(\Omega)$ and every $g=(g_0,\dots,g_{m-1})\in\mathcal{A}(\Omega')^m$. Here $z=(z',z_n)$, $z'=(z_1,\dots,z_{n-1})$. The solution depends continuously on f and g, i.e. for every compact part K of Ω there exist compact sets $L\subset\Omega$ and $L'\subset\Omega'$ and a constant C such that

$$\sup_K|u|\leqslant C\Big(\sup_L|P(D)u|+\sup_{L'}\sum_0^{m-1}|D_n^k u|\Big) \tag{3.5}$$

for all $u\in\mathcal{A}(\Omega)$.

Proof. We define an operator T from the Fréchet space $\mathfrak{F}=\mathcal{A}(\Omega)\times\mathcal{A}(\Omega')^m$ to $\mathcal{A}(\Omega)$ by $TF=T(f,g_0,\dots,g_{m-1})=u$ if u is a solution of (3.3), (3.4). According to the Cauchy–Kovalevsky theorem (see the proof of Theorem 5.1.1 in Hörmander [3]), the Cauchy problem has a solution $u\in\mathcal{A}(\mathbf{C}^n)$ when f and g_k are entire, i.e. when $F\in\mathfrak{F}_1=\mathcal{A}(\mathbf{C}^n)\times\mathcal{A}(\mathbf{C}^{n-1})^m\subset\mathfrak{F}$. Thus T is densely defined in $\mathfrak{F}$, and T is obviously closed. We are going to prove that T is continuous in $\mathfrak{F}_1$ (with respect to the topology induced by $\mathfrak{F}$). This is only possible when T is everywhere defined in $\mathfrak{F}$, and hence T is continuous, i.e. (3.5) is valid.

Now for an arbitrary compact set $K\subset\Omega$, let $\mathcal{B}_K$ be the Banach space

$$\mathcal{B}_K=\{\mu\in\mathcal{A}'(\Omega);\ \|\mu\|=\sup_{u\in\mathcal{A}(\Omega)}(|\mu(u)|/\sup_{z\in K}|u(z)|)<+\infty\},$$

and let $U_j, j=1,2,\ldots,$ be a fundamental system of neighborhoods of 0 in $\mathfrak{F}_1$. We set if $F\in\mathfrak{F}_1$,

$$M_F=\{\mu\in\mathcal{B}_K;\ |\mu(TF)|\leqslant 1\}.$$

This is a closed convex symmetric set in $\mathcal{B}_K$. Thus the same is true of the intersection

$$M_j=\bigcap(M_F;\ F\in U_j).$$

If $\mu\in\mathcal{B}_K$ we have by Theorem 2.3

$$\mu(u)=\nu(P(D)u)+\varrho(u),$$

where $\nu,\varrho\in\mathcal{A}'(\Omega)$ and ϱ is zero on all entire functions u such that $D_n^k u(z',0)=0$, $0\leqslant k<m$. Defining $\varrho_k\in\mathcal{A}'(\Omega')$ by $\varrho_k(v)=\varrho(v(z')z_n^k/k!)$ we can write

$$\varrho(u)=\sum_0^{m-1}\varrho_k(D_n^k u(z',0)),\quad u\in\mathcal{A}(\mathbf{C}^n),$$

which follows from the Taylor expansion of u. Hence

$$\mu(TF)=\nu(f)+\sum_0^{m-1}\varrho_k(g_k)$$

which proves that for some sufficiently large j, $|\mu(TF)|\leqslant 1$ as soon as $F\in U_j$. We have proved that $\bigcup_1^\infty M_j=\mathcal{B}_K$.

Since $\mathcal{B}_K$ is a complete metric space, the Baire cathegory theorem shows that 0 is in the interior of some M_j, i.e. for some constant C, $\|\mu\|<1/C$ implies that $|\mu(TF)|\leqslant 1$ for all $F\in U_j$. This means that

$$|\mu(TF)|=C\|\mu\|\left|\frac{\mu}{C\|\mu\|}(TF)\right|\leqslant C\|\mu\|$$

for all $\mu\in\mathcal{B}_K$ and all $F\in U_j$. For some compact sets $L\subset\Omega$ and $L'\subset\Omega'$ and some constant B we now have

$$\sup_L|f|+\sup_{L'}\sum_0^{m-1}|g_k|<1/B\Rightarrow F\in U_j,$$

and this implies that for all $\mu\in\mathcal{B}_K$, $F\in\mathfrak{F}_1$

$$|\mu(TF)|\leqslant BC\|\mu\|\ (\sup_L|f|+\sup_{L'}\sum_0^{m-1}|g_k|).$$

Applying this estimate to the Dirac measures $\mu(u)=u(z)$ where $z\in K$, we have $\|\mu\|\leqslant 1$, and hence

$$\sup_K |TF| \leqslant BC(\sup_L |f| + \sup_{L'} \sum_0^{m-1} |g_k|)$$

for all $F \in \mathfrak{F}_1$ which proves that T is continuous in $\mathfrak{F}_1$. The proof is complete.

The following theorem is a converse of Theorem 3.1.

Theorem 3.2. *Let Ω be a convex open set in $\mathbf{C}^n$ such that $\Omega' = \{z \in \Omega;\, z_n = 0\}$ is non-empty. Suppose that a non-characteristic Cauchy problem*

$$P(D)\,u = 0 \ \text{ in } \ \Omega, \tag{3.6}$$

$$D_n^k\, u(z', 0) = g_k(z') \ \text{ in } \ \Omega', \quad 0 \leqslant k < m, \tag{3.7}$$

where $P(D)$ is a constant coefficient differential operator of order m, always has a solution $u \in \mathcal{A}(\Omega)$ when $g_k \in \mathcal{A}(\Omega')$. Then (3.1) is valid for the supporting functions Φ and Φ' of Ω and Ω' if $p(D)$ denotes the principal part of $P(D)$.

Proof. The mapping

$$\{u \in \mathcal{A}(\Omega);\, P(D)\,u = 0\} \ni u \to (u(z', 0), \ldots, D_n^{m-1}\,u(z', 0)) \in \mathcal{A}(\Omega')^m$$

is continuous, defined in a Fréchet space, and according to the Cauchy–Kovalevsky theorem it is one-to-one. The hypothesis in the theorem means that its range is all of $\mathcal{A}(\Omega')^m$, a Fréchet space. From an application of Banach's theorem it follows that the inverse mapping is also continuous, i.e. for any compact part K of Ω there exist a compact set $L' \subset \Omega'$ and a constant C such that

$$\sup_K |u| \leqslant C \sup_{0 \leqslant k < m} \sup_{L'} |D_n^k\, u| \tag{3.8}$$

for all $u \in \mathcal{A}(\Omega)$ satisfying (3.6). Now let $\zeta \in \mathbf{C}^n$, $|\zeta| = 1$, be given with $p(\zeta) = 0$. We can choose a sequence $(\theta^{(j)})$ of points in $\mathbf{C}^n$ such that $P(\theta^{(j)}) = 0$, $\theta^{(j)}/|\theta^{(j)}| \to \zeta$, and $|\theta^{(j)}| \to +\infty$ as $j \to +\infty$. Applying (3.8) to the solution $u(z) = \exp \langle z, \theta^{(j)} \rangle$ of (3.6) and $K = \{z\} \subset \Omega$, we obtain

$$\operatorname{Re} \langle z, \theta^{(j)} \rangle \leqslant \log C + \sup_{0 \leqslant k < m} k \log |\theta_n^{(j)}| + \psi'(\theta^{(j)})$$

if ψ' is the supporting function of $L' \subset \Omega'$. Dividing this inequality by $|\theta^{(j)}|$ and letting j tend to infinity gives

$$\operatorname{Re} \langle z, \zeta \rangle \leqslant \psi'(\zeta) \leqslant \Phi'(\zeta).$$

Since $z \in \Omega$ was arbitrary we have proved that

$$\Phi(\zeta) \leqslant \Phi'(\zeta) \ \text{ if } \ p(\zeta) = 0,$$

which together with the obvious inequality $\Phi' \leqslant \Phi$ gives (3.1). The proof is complete.

We now turn to the case when (3.4) are replaced by more general conditions in the hyperplane $z_n = 0$.

Theorem 3.3. *Let Ω, Ω', and $P(D)$ satisfy the hypotheses made in Theorem 3.1, and let $Q_1(D), \ldots, Q_r(D)$ be arbitrary differential operators with constant coefficients. Then the 'boundary problem'*

$$P(D)u = f \ \textit{in} \ \Omega, \tag{3.9}$$

$$Q_j(D)\, u = g_j \ \textit{in} \ \Omega', \quad 1 \leqslant j \leqslant r, \tag{3.10}$$

where $f \in \mathcal{A}(\Omega), g_1(z'), \ldots, g_r(z') \in \mathcal{A}(\Omega')$ has a solution $u \in \mathcal{A}(\Omega)$ if and only if the compatibility condition

$$S_0(\zeta)\, P(\zeta) + \sum_1^r S_j(\zeta')\, Q_j(\zeta) = 0 \Rightarrow S_0(D) f + \sum_1^r S_j(D')\, g_j = 0 \ \textit{in} \ \Omega', \tag{3.11}$$

is satisfied for all choices of polynomials $S_0(\zeta), S_1(\zeta'), \ldots, S_r(\zeta')$. Here $\zeta' = (\zeta_1, \ldots, \zeta_{n-1})$ and $D' = (D_1, \ldots, D_{n-1})$.

Proof. The condition is obviously necessary, and we are going to prove its sufficiency using the theory of general systems of differential equations with constant coefficients mentioned in the introduction. In view of (3.2) we can find polynomials R_j such that $Q_j - R_j P$ has degree $< m$ in ζ_n. We then replace Q_j by $Q_j - R_j P$ and g_j by $g_j - R_j(D) f$ to obtain a problem equivalent to (3.9), (3.10) where the new conditions in the hyperplane $z_n = 0$ involve only differentiations of order less than m in z_n. We may suppose in the sequel that this substitution has already been achieved. In this situation the condition (3.11) is equivalent to

$$\sum_1^r S_j(\zeta')\, Q_j(\zeta) = 0 \Rightarrow \sum_1^r S_j(D')\, g_j = 0 \ \text{ in } \Omega' \tag{3.12}$$

for all polynomials $S_1, \ldots, S_r$ in $n-1$ variables. The conditions (3.10) can now be rewritten as a system with $n-1$ independent variables

$$\sum_0^{m-1} Q_{jk}(D')\, u_k(z') = g_j(z') \ \text{ in } \Omega', 1 \leqslant j \leqslant r, \tag{3.13}$$

where $Q_j(D) = \sum_0^{m-1} D_n^k\, Q_{jk}(D')$ and $u_k(z') = D_n^k\, u(z', 0)$. Introducing Q_{jk} in (3.12) we obtain

$$\sum_1^r S_j\, Q_{jk} = 0, 0 \leqslant k < m \Rightarrow \sum_1^r S_j(D')\, g_j = 0 \ \text{ in } \Omega' \tag{3.14}$$

for all polynomials $S_1, \ldots, S_r$ in $n-1$ variables. This is the compatibility condition (1.10) for the system (3.13), so by the existence theorem for general systems we can find a solution $(w_0, \ldots, w_{m-1}) \in \mathcal{A}(\Omega')^m$ of (3.13). Now Theorem 3.1 shows that the Cauchy problem

$$P(D)u = f \text{ in } \Omega,$$

$$D_n^k u = w_k \text{ in } \Omega', \quad 0 \leqslant k < m,$$

has a solution $u \in \mathcal{A}(\Omega)$. This completes the proof since u obviously satisfies (3.10).

4. Approximation of solutions of homogeneous boundary problems in convex complex regions

Using Theorem 3.1 we derive from the approximation theorem for general systems of differential equations a corresponding result for boundary problems in a convex open set in $\mathbf{C}^n$.

Lemma 4.1. *Let $P(D)$ be a differential operator with constant coefficients of order m such that the plane $z_n = 0$ is non-characteristic, and $f_0, \ldots, f_{m-1}$ polynomials in $n-1$ variables. Then the solution u of the Cauchy problem in $\mathbf{C}^n$*

$$P(D)\,u = 0, \tag{4.1}$$

$$D_n^k\, u(z', 0) = f_k(z') \exp \langle z', \theta' \rangle, \quad 0 \leqslant k < m, \tag{4.2}$$

has the form

$$u(z) = \exp \langle z', \theta' \rangle \sum_1^m g_j(z) \exp (z_n \tau_j), \tag{4.3}$$

where g_j are polynomials and τ_j complex numbers. $(z = (z', z_n),\ z' = (z_1, \ldots, z_{n-1}).)$

Proof. We define

$$E_k(z; \zeta') = \frac{1}{2\pi i} \int_\Gamma \frac{P_k(\zeta', \tau) \exp \langle z, (\zeta', \tau) \rangle \, d\tau}{P(\zeta', \tau)}, \quad 0 \leqslant k < m,$$

where Γ is a circle in $\mathbf{C}^1$ surrounding the zeros of $\tau \to P(\zeta', \tau)$ and

$$P_k(\zeta', \tau) = \sum_{k+1}^m \tau^{j-k-1} a_j(\zeta')$$

if a_j are defined by

$$P(\zeta', \tau) = \sum_0^m \tau^j a_j(\zeta').$$

Arguing as in the proof of Lemma 5.7.4 in [3], we get

$$P(D)\,E_k = 0,$$

$$D_n^j\, E_k(z', 0; \zeta') = 0 \text{ if } 0 \leqslant j < k \text{ or } k < j < m,$$

and

$$D_n^k\, E_k(z', 0; \zeta') = \exp \langle z', \zeta' \rangle.$$

It is easy to see that E_k are entire functions of $(z;\zeta')\in \mathbf{C}^{2n-1}$ and of form (4.3) for a fixed $\zeta'=\theta'$. The same is true of

$$u(z)=\sum_0^{m-1} f_k\left(\frac{\partial}{\partial\zeta_1},\dots,\frac{\partial}{\partial\zeta_{n-1}}\right)E_k(z;\zeta')\Big|_{\zeta'=\theta'}$$

which is the solution of the Cauchy problem (4.1), (4.2).

Theorem 4.2. *Let Ω, Ω', and $P(D)$ be as in Theorem 3.1, and let $Q_1(D),\dots,Q_r(D)$ be arbitrary differential operators with constant coefficients. Then the linear combinations of solutions of type (4.3) of the boundary problem*

$$P(D)\,u=0 \ \text{ in } \ \Omega, \tag{4.4}$$

$$Q_j(D)\,u=0 \ \text{ in } \ \Omega', \quad 1\leqslant j\leqslant r, \tag{4.5}$$

are dense (under the topology induced by $\mathcal{A}(\Omega)$) in the set of all solutions in $\mathcal{A}(\Omega)$ of the same problem.

Proof. As in the proof of Theorem 3.3 we may assume that $Q_1(\zeta),\dots,Q_r(\zeta)$ are all of degree less than m in ζ_n. We also write (4.5) as a homogeneous system

$$\sum_0^{m-1} Q_{jk}(D')\,u_k=0 \ \text{ in } \ \Omega', \quad 1\leqslant j\leqslant r, \tag{4.6}$$

using the notation of (3.13). Now the approximation theorem for general systems of the type (4.6) (see Section 1) shows that to any given solution

$$(u_0,\dots,u_{m-1})\in\mathcal{A}(\Omega')^m$$

of (4.6) there exist solutions of (4.6) of the form

$$(w_{0sj},\dots,w_{m-1,s,j})=\exp\langle z',\theta'_{sj}\rangle\,(f_{0sj},\dots,f_{m-1,s,j}),$$

where f_{ksj} are polynomials in $n-1$ variables (cf. (1.12)) such that

$$w_{ks}=\sum_{j=1}^{N_s} w_{ksj}, \quad 0\leqslant k<m, \quad s=1,2,\dots,$$

tend to u_k in $\mathcal{A}(\Omega')$ when $s\to+\infty$. The unique function $v_{sj}\in\mathcal{A}(\Omega)$ satisfying

$$P(D)\,v_{sj}=0,$$

$$D_n^k v_{sj}(z',0)=w_{ksj}(z'), \quad 0\leqslant k<m,$$

is according to Lemma 4.1 of type (4.3), and is also a solution of (4.4), (4.5). Now set

$$v_s=\sum_{j=1}^{N_s} v_{sj}, \quad s=1,2,\dots .$$

The Cauchy data of v_s of order $< m$ tend to those of u (in the topology of $\mathcal{A}(\Omega')^m$) when $s \to +\infty$, and by Theorem 3.1 we conclude that $v_s \to u$ in $\mathcal{A}(\Omega)$. The theorem is proved.

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REFERENCES

1. EHRENPREIS, L., A fundamental principle for systems of linear differential equations with constant coefficients, and some of its applications. Proc. Internat. Sympos. Linear Spaces, 161–174. Jerusalem 1961.
2. ——, The structure of solutions of systems of partial differential equations. Stanford University 1961.
3. HÖRMANDER, L., Linear partial differential operators. Springer 1963.
4. ——, Lectures on functions of several complex variables. Stanford University 1964.
5. MALGRANGE, B., Sur les systèmes différentiels à coefficients constants. Colloques internationaux du C.N.R.S. no. 117. Les équations aux dérivées partielles, 113–122. Paris 1963.
6. MARTINEAU, A., Sur les fonctionelles analytiques et la transformation de Fourier–Borel. J. Analyse Math. *11* (1963), 1–164.

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