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Semi-groups of isometries and the representation and multiplicity of weakly stationary stochastic processes

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1. Introduction and Summary

It is the purpose of this paper to examine the work of J. L. B. Cooper on the representation of a continuous semigroup $\{S_t, t \geq 0\}$ of isometries on a separable Hilbert space $\mathcal{Y}$ and to show how it can be adapted to give a complete discussion of the representation theory of a very general class of continuous parameter, weakly stationary stochastic processes which include finite as well as infinite dimensional processes [12]. The possibility of such a connection between Cooper's results and representations of stationary processes has been noted by P. Masani and J. Robertson [15] (also [14]). However, the approach of these authors has been to reduce the study of continuous parameter processes to certain discrete parameter processes associated with them ([15], § 4).

The point of view adopted in this paper enables us to dispense with the associated discrete parameter process and to give a time domain analysis based directly on the stochastic process itself. A significant tool in our analysis is the fundamental notion of multiplicity of a stochastic process introduced recently by H. Cramér [2] and T. Hida [10], and studied extensively by the former author in subsequent papers ([3], [4]). Before we can bring out the relevance of Cooper's ideas to our present aims, it is necessary to complete his basic result in two essential respects: firstly, to introduce the definition of multiplicity of Cooper's representation and secondly to show that it is equal to the dimension of the deficiency subspace $R^{\perp}$, R being the range of the Cayley transform V of the maximal symmetric operator H, where iH is the infinitesimal generator of $\{S_t\}$. Cooper's result thus completed and amplified is presented as Theorem 2.1 in section 2.

In sections 3 and 4 we obtain some interesting points of contact with more recent work on isometric operators in Hilbert space. We show in section 3 that Theorem 2.1 immediately yields in a simple and natural way a direct integral representation in terms of "differential innovation" subspaces obtained earlier by Masani [14]. Indeed, the vector valued integral of [14] turns out to be nothing other than the orthogonal sum of N "stochastic integrals", N being the multiplicity of the representation. Section 4 carries the study of the differential innovation subspaces further. Each such subspace is shown to be a "weighted" orthogonal sum of $V^n(R^{\perp})$ $(n=0,1,\ldots)$ which are the innovation subspaces of the associated discrete representation (1.1) of [14]. We believe that this theorem (Theorem 4.1) puts in better perspective, the intrinsic relationship between the given continuous parameter process and its associated discrete parameter process.

In sections 5 and 6 we apply Theorem 2.1 to the semigroup of isometries induced by the unitary group of a stationary stochastic process $\mathrm{x}_t(-\infty<t<\infty)$ defined on a separable Hausdorff space Φ and satisfying certain continuity requirements (Condition (5.1)). We obtain the Wold decomposition of such a process, together with the desired representation for the purely non-deterministic component (Theorem 6.3). A consequence of our derivation is the very natural and significant role played by the Cramér–Hida multiplicity of the process. This multiplicity is, in fact, shown to be equal to the dimension of the deficiency subspace of the induced semigroup, which in this case turns out to be $L_2(\mathrm{x};0)\ominus V^{-1}L_2(\mathrm{x};0)$, V being the Cayley transform of the unitary group and $L_2(\mathrm{x};0)$, the past and present up to time 0 of the process. For finite dimensional stationary processes, this result yields the corollary (proved in [12] by a different method) that the multiplicity of the process equals its rank.

The first time domain analysis of a continuous in quadratic mean, univariate stationary process $\{x_t, -\infty<t<\infty\}$ was given by O. Hanner in a remarkably original paper [9]. Recently, with the help of the ideas of multiplicity theory we have extended his approach to obtain representations of multivariate (including infinite dimensional) stationary processes [12]. These representations are seen to be essentially the same as the ones derived in this paper, thus effecting a synthesis between the ideas of Hanner and the ideas presented in this paper.

2. Continuous semi-groups of isometries on a Hilbert space

Let $\{S_t, t\geqslant 0\}$ be a strongly continuous semi-group of isometries on a separable Hilbert space $\mathcal{Y}$ with iH as its infinitesimal generator. J. L. B. Cooper has shown that H is a maximal symmetric operator with negative deficiency index, say, $\alpha(\alpha\neq 0)$ [1]. He further proved that every such semi-group yields the following decomposition of

$$\mathcal{Y}=\sum_{i=1}^{N}\oplus\mathfrak{M}_f(i)\oplus\mathcal{Y}_\infty, \tag{2.1}$$

where (i) $\mathcal{Y}_\infty=\bigcap_{t\geqslant 0}S_t(\mathcal{Y})$ and the restriction of S_t to $\mathcal{Y}_\infty$ is unitary, (ii) each $f^{(i)}$ is chosen from $\mathcal{D}_{H^*}$, the domain of the adjoint H^* of H in such a way that $2\mathfrak{J}m(H^*f^{(i)}, f^{(i)})=-1$ and (iii) $\mathfrak{M}_f(i)$ is the closed linear subspace consisting of all elements of the form $\int_0^\infty p(u)d(S;u;f^{(i)})$, $p\in L_2(\mu,[0,\infty))$, the Hilbert space of complex-valued functions on $[0,\infty)$, square integrable with respect to the Lebesque measure μ. In the notation of [1] (pp. 837–839) the integral introduced above is defined as the limit in norm of Riemann type sums. It will be seen below that such an integral is, in fact, nothing but a "stochastic integral" with respect to an orthogonal homogeneous set function (J. Doob [5], Ch. IX, § 2, O. Hanner [9]).

Since $\mathcal{Y}$ is separable it is clear from (2.1) that N can at most be equal to $\aleph_0$. Beyond this, Cooper's method of proof does not give any information about N. We shall show that there is an intrinsic connection between N and the semigroup $\{S_t\}$. Let $V=(H-iI)(H+iI)^{-1}$ be the Cayley transform of H. (It will be often convenient to write $c(H)$ for the Cayley transform of H.) Let $R=V(\mathcal{Y})$ and $R^\perp$, the orthogonal complement of R. We first prove that $N=\dim(R^\perp)=\alpha$. The essential point involved in showing this is to recognize that $\mathcal{D}_{H^*}$ is generated by the subspaces $\mathcal{D}_H$ and $R^\perp$ (Sz. Nagy B. [20], p. 38) and to see that the elements $f^{(i)}$ in (2.1) (ii) can actually be chosen from $R^\perp$. In order to bring out the significance of this result for weakly stationary stochastic processes we introduce a spectral resolution of the identity associated with $\{S_t\}$. Let $\mathcal{X} = \mathcal{Y} \ominus \mathcal{Y}_\infty$ and let $\hat{E}(t)$ be the projection onto $S_t(\mathcal{Y})\,(t \geqslant 0)$. If we define

$$\bar{E}(t) = I - \hat{E}(t) \quad \text{if} \quad t \geqslant 0, \quad \text{and} \quad = 0 \text{ for } t < 0,$$

then $\{\bar{E}(t),\ -\infty < t < \infty\}$ is the desired resolution of the identity in $\mathcal{X}$. Our next step is to show that N is equal to the multiplicity of the maximal spectral type $\bar{\varrho}$ with respect to $\{\bar{E}(t)\}$, $\bar{\varrho}$ being of positive Lebesgue type, i.e., $\bar{\varrho}$ is equivalent to the restriction μ^+ of μ on $[0, \infty)$. (In fact, $\bar{\varrho}$ is a uniform spectral type, although this fact is not used here.) These facts, presented in Theorem 2.1, enable us to give a complete discussion of the representation and multiplicity theory of stationary processes of the most general kind and to put it in the perspective of the multiplicity theory of purely non-deterministic processes developed by H. Cramér ([2], [3], [4]) and T. Hida ([10]). It also enables us essentially to identify the multidimensional extension of the time domain analysis of Hanner [9] worked out by us [12], with the theory developed here.

We begin by recasting the elements of $\mathfrak{M}_f(i)$ as stochastic integrals. In doing so we shall freely use the properties of the integral $\int_0^\infty p(u)\,d(S;u;f^{(i)})$ obtained by Cooper ([1] p. 831 and p. 840). For each finite interval $[a,b)\,(0 \leqslant a < b < \infty)$, let $\xi^{(i)}[a,b) = \int_0^\infty I_{[a,b)}(u)\,d(S;u;f^{(i)})$, $(i = 1, 2, \ldots, N)$ where $I_{[a,b)}(u) = 1$ if $u \in [a,b)$ and $= 0$, otherwise. It can be seen that $\xi^{(i)}[a,b)$ is a homogeneous orthogonal interval function; i.e., for each i and $0 \leqslant a < b < c$,

$$\left.\begin{array}{ll} (\alpha) & \xi^{(i)}[a,b) + \xi^{(i)}[b,c) = \xi^{(i)}[a,c), \\ (\beta) & \xi^{(i)}[a,b) \text{ is orthogonal to } \xi^{(i)}[b,c), \quad \text{and} \\ (\gamma) & S_t\xi^{(i)}[a,b) = \xi^{(i)}[a+t, b+t) \quad \text{for all} \quad t \geqslant 0. \end{array}\right\} \tag{2.3}$$

Since $L_2(\mu, [0,\infty))$ is generated by the family $\{I_{[a,b)}(u)\, 0 \leqslant a < b < \infty\}$ we have $\mathfrak{M}_f(i) =$ the subspace of $\mathcal{Y}$ generated by $\{\xi^{(i)}[a,b),\ 0 \leqslant a < b < \infty\}$. By the definition of stochatic integrals ([5]) it then follows that

$$\mathfrak{M}_f(i) = \left\{\int_0^\infty p(u)\,d\xi^{(i)}(u),\ p \in L_2(\mu, [0,\infty))\right\} = \mathfrak{M}_\xi(i). \tag{2.4}$$

It is convenient to recall at this point some of the terminology of multiplicity theory in a separable Hilbert space. Let A be any self-adjoint operator with the resolution of the identity $\{E(t)\}$. For any element f in $\mathcal{Y}$ let ϱ_f be the finite measure on Borel sets of the real line (sometimes called the spectral function of f) given by $\varrho_f(\Delta) = \|E(\Delta)f\|^2$, where if $\Delta = [a,b)$, $E(\Delta) = E(b) - E(a)$. The family of all finite measures on the line is divided into equivalence classes by the relation of equivalence between measures (equivalence here means mutual absolute continuity). If ϱ is used to denote the equivalence class to which ϱ_f belongs, ϱ will be called the spectral type of f with respect to A (or $\{E(t)\}$). ϱ is also referred to as the spectral type belonging to A. If elements f and g are such that $\varrho_f \equiv \varrho_g$ they obviously have the same spectral type ϱ. We say that the spectral type ϱ dominates the spectral type $\sigma(\varrho > \sigma$ or $\sigma < \varrho)$ if any (and thus every) measure belonging to σ is absolutely continuous with respect to any measure belonging to ϱ. ϱ is called a Lebesgue type if every measure belonging to ϱ is equivalent to the Lebesgue measure. ϱ and σ are said to be independent spectral types if for any spectral type ν such that $\nu < \varrho$ and $\nu < \sigma$ we have $\nu = 0$. An element f is said to be of maximal spectral type ϱ (with respect to A or $\{E(t)\}$) if for every g in $\mathcal{Y}$ $\varrho_g \ll \varrho_f$. The closed linear subspace $\mathfrak{S}\{E(\Delta)f, \Delta$ ranging over all finite intervals$\}$ is called the cyclic subspace (with respect to A) generated by f. If this subspace coincides with $\mathcal{Y}$, f is called a cyclic or generating element of A and A is cyclic. Also if f is a generating element of A, f is of maximal spectral type and the latter is referred to as the spectral type of the cyclic operator A. It is to be noted that if A is any self-adjoint operator (since $\mathcal{Y}$ is separable) there always exists a maximal spectral type belonging to A. Any system of mutually orthogonal cyclic parts of A of type ϱ is called an orthogonal system of type ϱ relative to A. An orthogonal system of type ϱ which cannot be enlarged by adding to it more cyclic parts of A is called maximal. It is a known result of this theory that all maximal systems of type ϱ have the same cardinal number. This uniquely determined cardinal number is defined to be the multiplicity of the spectral type ϱ with respect to A.

Finally, we need the notion of a uniform spectral type. The spectral type ϱ is said to be uniform if every non-zero type σ dominated by ϱ has the same multiplicity as ϱ itself. Most of the above definitions have been taken from the article of A. I. Plessner and V. A. Rohlin [16] to which the reader is referred for further details.

Let us denote by A, the self-adjoint operator on $\mathfrak{X}(=\mathcal{Y}_\infty^\perp)$ given by the resolution of the identity $\{\bar{E}(t)\}$. Our aim is to show that each $\mathfrak{M}_{f(i)}$ reduces A and that the restriction $A^{(i)}$ of A to $\mathfrak{M}_{f(i)}$ is cyclic with a generating element $g^{(i)} \in R^{\top}$. We need the following characterization of $R^\perp$ ([20], p. 38),

$$R^\perp = \{\varphi \mid \varphi \in \mathcal{D}_{H^*},\ H^*\varphi = -i\varphi\}, \tag{2.5}$$

and the relations due to Cooper ([1], p. 840 (5.12) (5.13));

(C 1) $\quad \hat{E}(t) = S_t S_t^*$ for each $t \geq 0$,

(C 2) $\quad S_t \int_0^\infty p(u)\, d(S; u; f^{(i)}) = \int_t^\infty p(u-t)\, d(S; u; f^{(i)})$

for $p \in L_2(\mu, [0, \infty))$,

(C 3) $\quad S_t^* \int_0^\infty p(u)\, d(S; u; f^{(i)}) = \int_0^\infty p(u+t)\, d(S; u; f^{(i)})$.

Since $\bar{E}[a,b) \int_0^\infty p(u)\, d(S; u; f^{(i)}) = \{\hat{E}(a) - \hat{E}(b)\} \int_0^\infty p(u)\, d(S; u; f^{(i)})$ we have from (C 1), $\bar{E}[a,b) \int_0^\infty p(u)\, d(S; u; f^{(i)}) = \{S_a S_a^* - S_b S_b^*\} \int_0^\infty p(u)\, d(S; u; f^{(i)})$. However, by (C 3) and (C 2)

$$S_b S_b^* \int_0^\infty p(u)\, d(S; u; f^{(i)}) = S_b \int_0^\infty p(u+b)\, d(S; u; f^{(i)}) = \int_b^\infty p(u)\, d(S: u; f^{(i)}).$$

Hence it follows that

$$\bar{E}[a,b) \int_0^\infty p(u)\, d(S; u; f^{(i)}) = \int_a^b p(u)\, d(S; u; f^{(i)}) = \int_0^\infty p(u) I_{[a,b)}(u)\, d(S; u; f^{(i)}). \tag{2.6}$$

Lemma 2.1. *For each i, there exists an element $g^{(i)} \in R^{\perp}$, such that $\mathfrak{M}_{f^{(i)}} = \mathfrak{S}\{\bar{E}[a,b)g^{(i)}$, where $[a,b)$ is any finite subinterval of the line$\}$.*

Proof. Let us define $g^{(i)} = \sqrt{2}\int_0^\infty \bar{e}^u d\xi^{(i)}$. Clearly from (2.6),

$$\bar{E}[a,b)g^{(i)} = \sqrt{2}\int_0^\infty \bar{e}^u I_{[a,b)}(u)\, d(S;u;f^{(i)})$$

for $-\infty < a < b < +\infty$. Also, $\bar{e}^u > 0$ for $u \geq 0$ and $L_2(\mu,[0,\infty))$ is generated by $\{I_{[a,b)}(\cdot), 0 \leq a < b < \infty\}$. Hence $\mathfrak{M}_{f^{(i)}} = \mathfrak{S}\{\bar{E}[a,b)g^{(i)}, -\infty < a < b < +\infty\}$. To complete the proof of the lemma, we now show that $H^* g^{(i)} = -ig^{(i)}$ so that by (2.5) $g^{(i)} \in R^{\perp}$. Let $f \in \mathcal{D}_H$, the domain of H. Then $\lim_{t\to 0}(t^{-1}[S_t^* - I]g^{(i)}, f)$ exists, since

$$\lim_{t\to 0}(t^{-1}[S_t^* - I]g^{(i)}, f) = \lim_{t\to 0}(g^{(i)}, t^{-1}[S_t - I]f) = (g^{(i)}, iHf). \tag{2.7}$$

Also, from (C 3),

$$S_t^* g^{(i)} = 2\int_0^\infty \bar{e}^{(u+t)} d(S;u;f^{(i)}) = \bar{e}^t g^{(i)}.$$

Hence

$$(g^{(i)}, iHf) = \lim_{t\to 0} t^{-1}(\bar{e}^t - 1)(g^{(i)}, f) = -(g^{(i)}, f) \quad \text{for} \quad f \in \mathcal{D}_H; \text{ i.e.,}$$

$$g^{(i)} \in \mathcal{D}_{H^*} \quad \text{and} \quad -i(H^* g^{(i)}, f) = -(g^{(i)}, f) \quad \text{for all} \quad f \in \mathcal{D}_H.$$

Thus $H^* g^{(i)} = -ig^{(i)}$ and the lemma is proved.

Lemma 2.1 immediately implies that A is reduced by $\mathfrak{M}_{f^{(i)}}$, and that

$$A = A^{(1)} + A^{(2)} + \dots + A^{(N)}, \tag{2.8}$$

where $A^{(i)}$, the restriction of A to $\mathfrak{M}_{f^{(i)}}$, is cyclic with the generating element $g^{(i)} = \sqrt{2}\int_0^\infty \bar{e}^u d(S;u;f^{(i)})$. If, further, $\mu^+(\Delta) = \mu(\Delta \cap [0,\infty))$ for each Δ (Lebesgue) measurable on the real line, we get

$$\varrho_{g^{(1)}} \equiv \varrho_{g^{(2)}} \equiv \dots \varrho_{g^{(N)}} \equiv \mu^+. \tag{2.9}$$

Since $\mathfrak{X} = \sum_{i=1}^N \oplus \mathfrak{M}_{f^{(i)}}$, it follows from Lemma 2.1 that

$$\mathfrak{X} = \sum_{i=1}^N \oplus\, \mathfrak{S}\{\bar{E}[a,b)g^{(i)}, -\infty < a < b < +\infty\} = \sum_{i=1}^N \oplus\, \mathfrak{S}\{\bar{E}[a,b)g^{(i)}, 0 \leq a < b < \infty\}. \tag{2.10}$$

The last equality in (2.10) is a consequence of the fact that $\bar{E}[a,b)g^{(i)} = 0$ for $-\infty < a < b < 0$. We now state the main theorem of this section.

Theorem 2.1. *Let $\{S_t, t \geq 0\}$ be a strongly continuous semigroup of isometries on a separable Hilbert space $\mathcal{Y}$. Then*

(1) $\mathcal{Y} = \sum_1^N \oplus \mathfrak{M}_{\xi^{(i)}} \oplus \mathcal{Y}_\infty$, *where (i)* $\mathcal{Y}_\infty = \bigcap_{t\geq 0} S_t(\mathcal{Y})$ *and the restriction of* S_t *to* $\mathcal{Y}_\infty$ *is unitary;*

(2) $\mathfrak{M}_{\xi^{(i)}} = \{\int_0^\infty p(u)\,d\xi^{(i)}(u),\ p \in L_2(\mu,[0,\infty))\}$ *where* $\xi^{(i)}$ *is a homogeneous, orthogonal interval function and the integral* $\int_0^\infty p(u)\,d\xi^{(i)}(u)$ *is a "stochastic integral";*

(3) *For each* i, $(i=1,2,\dots,N)$, $\mathfrak{M}_{\xi^{(i)}}$ *is a cyclic subspace of* A *with generating element* $g^{(i)} \in R^\perp$ *such that* $\varrho_{g^{(i)}} \equiv \mu^+$, *where* μ^+ *is the restriction of the Lebesgue measure* μ *to* $[0,\infty)$*;*

(4) N *is equal to the multiplicity of the common spectral type* $\bar{\varrho}$ *of* $\varrho_{g^{(i)}}$ *with respect to* A*;*

(5) *Finally,* $N =$ *the dimension of the deficiency subspace* $R^\perp$.

Proof. Conclusions 1 and 2 follow from (2.1) and (2.4) respectively. Conclusion 3 is precisely Lemma 2.1. To prove concl. 4 observe that from (2.8) and (2.9) $\{A^{(i)}\}$ is an orthogonal system of type $\bar{\varrho}$. To show that this is a maximal system of type $\bar{\varrho}$, we have recourse to an argument based on the ideas of A. I. Plessner and V. A. Rohlin [16] and used by us in [12] (Theorem 5.2).

Let $\{A'_\beta\}$ be an orthogonal system of type $\bar{\varrho}$ and cardinality M'; i.e., a system of orthogonal cyclic restrictions A'_β of the operator A, the spectral type of each A'_β being $\bar{\varrho}$. According to our definition N is the multiplicity of $\bar{\varrho}$ if we prove $M' \leq N$. By the separability of $\mathcal{Y}$ neither N nor M' can exceed $\aleph_0$. There is obviously nothing to prove if $N = \aleph_0$. Thus the only case to be considered is where N is a finite cardinal. If possible let $M' > N$. Let g'_β be a generating element of A'_β. Clearly there is no loss of generality in assuming that all these elements have the same spectral function ϱ' (i.e., $\varrho' = \varrho_{g'_\beta} = \varrho_{g^{(i)}}$). From (2.10) it follows that

$$g'_\beta = \sum_{i=1}^N \int_0^\infty F_{i\beta}(u)\,d\bar{E}(u)\,g^{(i)}, \quad \text{where} \quad \sum_{i=1}^N \int_0^\infty |F_{i\beta}(u)|^2\,d\varrho'(u) \quad \text{is finite.}$$

For every finite interval Δ we obtain

$$(\bar{E}(\Delta)\,g'_\beta, g'_\gamma) = \int_\Delta \sum_1^N F_{i\beta}(u)\,F_{i\gamma}(u)\,d\varrho'(u).$$

The left hand side of the above relation is zero if $\beta \neq \gamma$ and equals $\varrho'(\Delta)$ if $\beta = \gamma$. Hence for u not in a set $\mathcal{N}_{\beta\gamma}$ of zero ϱ'-measure we have

$$\sum_{i=1}^N F_{i\beta}(u)\,F_{i\gamma}(u) = \delta_{\beta\gamma} \text{ for all } \beta,\ \gamma.$$

Since M' is at most $\aleph_0$ the set $\mathcal{N} = \bigcup_{\beta,\gamma} \mathcal{N}_{\beta\gamma}$ is measurable and $\varrho'(\mathcal{N}) = 0$. Choosing a fixed point u_0 in the complement of $\mathcal{N}$ we see that

$$\sum_1^N F_{i\beta}(u_0)\,F_{i\gamma}(u_0) = \delta_{\beta\gamma} \quad \text{for all } \beta, \gamma. \tag{2.11}$$

If $\mathbf{a}_\beta = (F_{1\beta}(u_0), \dots, F_{N\beta}(u_0))$ the relations (2.11) imply that the a_β's are M' orthonormal vectors in N dimensional space. Hence $M' \leq N$. In other words $\bar{\varrho}$ has multiplicity N.

Proof of 5. Let us consider for $\alpha(=1,2,\dots,N)$ $h\in\mathfrak{D}_{H(\alpha)}=\mathfrak{D}_H\cap\mathfrak{M}_{f^{(\alpha)}}$. Then $\mathfrak{D}_{H(\alpha)}$ is dense in $\mathfrak{M}_{f^{(\alpha)}}$ and $h=\int_0^\infty q_{(\alpha,h)}(u)\,d(S;u;f^{(\alpha)})$. It follows that the set $\{q_{(\alpha,h)},h\in\mathfrak{D}_{H(\alpha)}\}$ is dense in $L_2(\mu,[0,\infty))$ for every α. It is known (see N. Dunford and J. Schwarz [6], p. 1258) that $(Hh,\varphi)=\langle iDq_{\alpha,h},p_\alpha\rangle$, where $\varphi=\sum_{\alpha=1}^N\int_0^\infty p_\alpha(u)\,d(S;u;f^{(\alpha)})$, $\langle\,,\rangle$ denotes the inner product in $L_2(\mu,[0,\infty))$ and iD is the differential operator $i(d/du)$. If, further, $\varphi\in R^\perp$, from (2.5) we have,

$$(Hh,\varphi)=(h,H^*\varphi)=(h,-i\varphi)=\langle q_{\alpha,h},-ip_\alpha\rangle. \tag{2.12}$$

But since the operator iD is formally self-adjoint ([6], p. 1287),

$$\langle iDq_{\alpha,h},p_\alpha\rangle=\langle q_{\alpha,h},iDp_\alpha\rangle=\langle q_{\alpha,h},-ip_\alpha\rangle \tag{2.13}$$

from (2.12). The set $\{q_{\alpha,h},h\in\mathfrak{D}_{(\alpha)}\}$ being dense in $L_2(\mu,[0,\infty))$, from (2.13) we get $(d/du)\,p_\alpha(u)=-p_\alpha(u)$ for every α. The above differential equation has the solution in $L_2(\mu,[0,\infty))$ given by $p_\alpha(u)=a_\alpha\bar{e}^u$. Hence $\varphi\in R^\perp$ implies that

$$\varphi=\sum_{\alpha=1}^N a_\alpha\int_0^\infty \bar{e}^u d(S;u;f^{(\alpha)})=\sum_{\alpha=1}^N a_\alpha/\sqrt{2}\,g^{(\alpha)};\ \text{i.e.,}$$

the orthonormal system $\{g^{(\alpha)},\alpha=1,2,\dots,N\}$ in $R^\perp$ is complete, giving $N=\dim(R^\perp)$. The proof of Theorem 2.1 is complete.

3. An alternative derivation of a direct integral representation of P. Masani

It is well known (P. R. Halmos [8]) that if V is an isometry on a Hilbert space $\mathcal{Y}$ onto a subspace R of $\mathcal{Y}$, then

$$\mathcal{Y}=\sum_{k=0}^\infty\oplus V^k(R^\perp)\oplus\bigcap_{k\geq 0}V^k(\mathcal{Y}), \tag{3.1}$$

where for $j\neq k$ $V^k(R^\perp)\perp V^j(R^\perp)$ and restriction of V to $\bigcap_{k\geq0}V^k(\mathcal{Y})$ is a unitary operator. Recently P. Masani [14] has obtained a continuous parameter generalization of the decomposition (3.1) as follows.

Theorem M ([14], Theorem 6.5). *Let $\{S_t\}$ $(t\geq 0)$ be a strongly continuous semi-group of isometries on $\mathcal{Y}$ into $\mathcal{Y}$, iH its infinitesimal generator and V the Cayley transform of H. Then for every a, non-negative,*

$$S_a(\mathcal{Y})=\int_a^\infty T_{dt}(R^\perp)\oplus\mathcal{Y}_\infty,\quad \int_0^\infty T_{dt}(R^\perp)\perp\mathcal{Y}_\infty; \tag{3.2}$$

where $R=V(\mathcal{Y})$, $\mathcal{Y}_\infty=\bigcap_{t\geq0}S_t(\mathcal{Y})$ and for each a,b $(0\leq a<b)$

$$T_{ab}=1/\sqrt{2}\left\{S_b-S_a-\int_a^b S_h\,dh\right\}$$

is an operator-valued measure and $\int_a^b T_{dt}(R^\perp)$ *is defined as a direct integral of differential innovation subspaces.*

In this section we deduce Theorem M as a consequence of Theorem 2.1. In fact, Theorem 2.1 even enables us to give an explicit representation for each "differential innovation subspace" in terms of the subspaces $V^n(R^\perp)$. The latter result furnishes another generalization of (3.1) and as such we treat it in the next section.

Noting that the subspace $\mathfrak{M}_{\xi^{(i)}}$ of the preceding section reduces S_a for each $a \geq 0$, we can write the representation obtained in Theorem 2.1 in the following somewhat more general form:

$$S_a(\mathcal{Y}) = \sum_1^N \oplus\, S_a \mathfrak{M}_{\xi^{(i)}} \oplus \mathcal{Y}_\infty. \tag{3.3}$$

The elements of $S_a\mathfrak{M}_{\xi^{(i)}}$ are "stochastic integrals". Hence, we can rewrite (3.3) as,

$$S_a(\mathcal{Y}) = \left\{ \nu \,|\, \nu = \sum_{k=1}^N \int_a^\infty c_k(u)\, d\xi^{(k)}(u),\ c_k \in L_2(\mu, [0,\infty)) \right\} \oplus \mathcal{Y}_\infty; \tag{3.4}$$

where, if $N(=\dim R^\perp)$ is infinite then the sum of stochastic integrals representing ν converges in norm.

For each finite subinterval $[a,b)$ $(0 \leq a < b)$ let τ_{ab} be the bounded linear operator on $R^\perp$ which transforms the complete orthonormal system $\{g^{(k)}\}$ in $R^\perp$ as follows:

$$\tau_{ab}\, g^{(k)} = \xi^{(k)}[a,b) \quad (k = 1, 2, \ldots, M). \tag{3.5}$$

Then τ_{ab} is an operator valued measure on intervals which has the following properties ([14] p. 627 (4.2));

$$\left.\begin{aligned} &(\alpha) \quad \tau_{ab} + \tau_{bc} = \tau_{ac} \quad (0 \leq a < b \leq c), \\ &(\beta) \quad S_t \tau_{ab} = \tau_{a+t,\, b+t} \quad (t \geq 0, 0 \leq a < b), \\ &(\gamma) \quad \text{For every } r_1, r_2 \in R^\perp \text{ and finite intervals } J_1, J_2 \\ &\qquad (\tau_{J_1} r_1, \tau_{J_2} r_2) = \mu(J_1 \cap J_2)\, (r_1, r_2). \end{aligned}\right\} \tag{3.6}$$

We shall show (Theorem 3.1 a) that τ_{ab} is identical with the operator valued measure $-T_{ab}$ on $R^\perp$. Following Masani we denote by $L_2([a,b], R^\perp)$, the Hilbert space of all strongly (Lebesgue) measurable functions x on $[a,b]$ with values $x(t)$ in $R^\perp$ and such that $\int_a^b \|x(t)\|^2 dt$ is finite. Since each element x of $L_2([a,b], R^\perp)$ has the form $x(t) = \sum_{n=1}^N c_n(t)\, g^{(n)}$ where $c_n \in L_2(\mu, [a,b])$ the integral $\int_a^b \tau_{dt}(x(t))$ can be naturally defined as follows,

$$\int_a^b \tau_{dt}(x(t)) = \sum_1^N \int_a^b c_n(t)\, d\xi^{(n)}(t). \tag{3.7}$$

The above definition is unambiguous because the functions $c_n \in L_2(\mu, [a,b])$ are uniquely determined by x. From the properties of stochastic integrals, one can show that (3.7) is a "generalized vector-valued integral" of the kind introduced by Masani [14] and satisfies the following properties:

$$\left.\begin{array}{ll}\text{(i)} & \left(\int_a^b \tau_{dt}(x(t)), \int_a^b \tau_{dt}(y(t))\right) = \int_a^b (x(t), y(t))\,dt, \\ & \text{for all } x, y \in L_2([a,b), R^\perp); \\ \text{(ii)} & \int_a^b \tau_{dt}(\lambda_1 x(t) + \lambda_2 y(t)) = \lambda_1 \int_a^b \tau_{dt}(x(t)) + \lambda_2 \int_a^b \tau_{dt}(y(t)), \\ & \text{for } \lambda_1, \lambda_2 \text{ complex numbers;} \\ \text{(iii)} & \left\| \int_a^b \tau_{dt}(x^{(n)}(t)) - \int_a^b \tau_{dt}(x(t)) \right\| \text{ converges to zero as} \\ & \int_a^b \|x^{(n)}(t) - x(t)\|^2\,dt \text{ tends to zero;} \\ \text{(iv)} & S_s \int_a^b \tau_{dt}(x(t)) = \int_{a+s}^{b+s} \tau_{du}(u-s) \quad \text{for} \quad s \geqslant 0. \end{array}\right\} \tag{3.8}$$

Let us now define

$$\int_a^b \tau_{dt}(R^\perp) = \left\{ \nu \,\middle|\, \nu = \int_a^b \tau_{dt}(x(t)),\ x \in L_2([a,b], R^\perp) \right\}. \tag{3.9}$$

Then $\int_a^b \tau_{dt}(R^\perp)$ is a closed linear subspace of $\mathcal{Y}$ isomorphic to $L_2([a,b], R^\perp)$. From (3.4) and definitions (3.7), (3.9) it follows that

$$S_a(\mathcal{Y}) = \int_a^\infty \tau_{dt}(R^\perp) \oplus \mathcal{Y}_\infty, \quad \int_0^\infty \tau_{dt}(R^\perp) \perp \mathcal{Y}_\infty. \tag{3.10}$$

The direct integral representation of Masani is identical with the one obtained in (3.10). We show this by proving that $\tau_{ab}\, r = -T_{ab}\, r$ for all a, b $(0 \leqslant a < b)$ and $r \in R^\perp$ and $\int_a^b \tau_{dt}(x(t)) = -\int_a^b T_{dt}(x(t))$ for all $x \in L_2([a,b], R^\perp)$. Since $\int_a^b \tau_{dt}(R^\perp)$ is a subspace the result will then follow.

Theorem 3.1. (a) *If* $\tau_t = \tau_{0t}$ *is defined as in* (3.5) *then* $\tau_t x = -T_t x$ *for all* $x \in R^\perp$ *and* $t \geqslant 0$, *where* $T_{0t} = 1/\sqrt{2}\,(S_t - I - \int_0^t S_h\,dh)$.

(b) *For* $x \in L_2([a,b], R^\perp)$ *and all* a, b $(0 \leqslant a < b)$

$$\int_a^b \tau_{dt}(x(t) = -\int_a^b T_{dt}(x(t)), \tag{3.11}$$

where $\int_a^b T_{dt}(x(t))$ *is the generalized vector-valued integral due to Masani* ([14], (5.2) (a)).

Proof of (a). It suffices to prove that $T_t(-g^{(k)}) = \xi^{(k)}[0,t)$ for each $t \geqslant 0$ and $k = 1, 2, \ldots, N$. Now,

$$T_t\, g^{(k)} = \frac{1}{\sqrt{2}} \left\{ S_t\, g^{(k)} - g^{(k)} - \left[\int_0^t S_h\, dh \right] g^{(k)} \right\}.$$

For each $t \geqslant 0$ (see Section 2 (C 3))

$$S_t(-g^{(k)}) = -\sqrt{2}\, e^t \int_t^\infty \bar{e}^u \, d(S; u; f^{(k)}) = -\sqrt{2}\, e^t \int_t^\infty \bar{e}^u \, d\xi^{(k)}(u). \tag{3.12}$$

But

$$\sqrt{2}\, e^t \int_t^\infty \bar{e}^u \, d\xi^{(k)}(u) = e^t g^{(k)} - \sqrt{2}\, e^t \int_0^t \bar{e}^u \, d\xi^{(k)}(u) = e^t g^{(k)} - \sqrt{2}\, e^t \zeta(t), \tag{3.13}$$

where $\zeta(t) = \int_0^t \bar{e}^u \, d\xi^{(k)}(u)$.

Since

$$\left[\int_0^t S_h \, dh\right](-g^{(k)}) = \int_0^t S_h(-g^{(k)}) \, dh = -g^{(k)} \int_0^t e^h \, dh + \sqrt{2} \int_0^t e^h \zeta(h) \, dh$$
$$= -g^{(k)}(e^t - 1) + \sqrt{2}\, e^t \zeta(t) - \sqrt{2} \int_0^t e^h \, d\zeta(h),$$

we get from (3.12), (3.13) and the definition of T_t that

$$T_t(-g^{(k)}) = -\frac{1}{\sqrt{2}} \left\{ e^t g^{(k)} - \sqrt{2}\, e^t \zeta(t) - g^{(k)} - g^{(k)}(e^t - 1) + \sqrt{2}\, e^t \zeta(t) - \sqrt{2} \int_0^t e^h \, d\zeta(h) \right\},$$

i.e., $-T_t g^{(k)} = T_t(-g^{(k)} = \int_0^t e^h \, d\zeta(h)$. But

$$\left\| \int_0^t e^h \, d\zeta(h) - \int_0^t d\xi^{(k)}(h) \right\|^2 = 0. \quad \text{Hence} \quad -T_{0t}\, g^{(k)} = \xi^{(k)}[0, t) = \tau_{0t}\, g^{(k)},$$

for $k = 1, 2, \ldots, N$.

Proof of (b). It suffices to prove (3.11) for the functions $x \in L_2([a, b], R^\perp)$ of the form $x(t) = \sum_1^N I_{J_k}(t) g^{(k)}$, where I_{J_k} is the indicator function of the subinterval J_k of $[a, b]$. By (3.7),

$$\int_a^b \tau_{dt}(x(t)) = \sum_1^N \int_a^b I_{J_k}(t) \, d\xi^{(k)}(t) = \sum_1^N \int_{J_k} d\xi^{(k)}(t) = \sum_1^N \tau_{J_k}(g^{(k)}).$$

But $\tau_{J_k}(g^{(k)}) = T_{J_k}(-g^{(k)})$ and hence $\int_a^b \tau_{dt}(x(t)) = -\sum_1^N T_{J_k}(g^{(k)}) = -\int_a^b T_{dt}(g^{(k)})$. (See [14] (5.2) (a).) The proof is complete.

4. A representation of T_{ab} in terms of $\{V^n, n \geqslant 0\}$

The representation (3.1) of Section 3 closely resembles the Wold decomposition of a weakly stationary stochastic process into a sum of innovation subspaces and its remote past. Interpreting k as the time, $V^k(R^\perp)$ can be regarded as an innovation subspace of $\mathcal{Y}$. In the continuous parameter situation we shall refer to $T_{ab}(R^\perp)$,

$(0 \leqslant a < b < \infty)$ as the differential innovation subspace of the continuous semi-group $\{S_t, t \geqslant 0\}$ ([14], p. 624). The purpose of this section is to express each $T_{ab}(R^\perp)$ in terms of the discrete subspaces $V^n(R^\perp)$, $(n=0,1,2,\ldots)$.

From Theorem 2.1 we recall that $\mathcal{Y} = \sum_1^N \oplus \mathfrak{M}_{\xi^{(i)}} \oplus \mathcal{Y}_\infty$ where $N = \dim(R^\perp)$ and the restriction of S_t to $\mathcal{Y}_\infty$ is unitary. Let us set $\mathfrak{X} = \sum_1^N \oplus \mathfrak{M}_{\xi^{(i)}}$. It is well known (see, e.g., Sz. Nagy [20], p. 40) that $\mathfrak{X}$ also has the decomposition

$$\mathfrak{X} = \sum_1^N \oplus \mathfrak{M}_i^*, \quad \text{where} \quad \mathfrak{M}_i^* = \mathfrak{S}\{g^{(i)}, Vg^{(i)}, V^2 g^{(i)}, \ldots\}, \quad \text{where} \quad \{g^{(i)}.\ i = 1, 2, \ldots, N\}$$

is the complete orthonormal system in $R^\perp$ introduced earlier. Since H is reduced by $\mathfrak{M}_{\xi^{(i)}}$, its Cayley transform V is also reduced by $\mathfrak{M}_{\xi^{(i)}}$, and thus for each i $\mathfrak{M}_{\xi^{(i)}} = \mathfrak{M}_i^*$. Hence, for each finite subinterval $[a, b)$ $(0 \leqslant a < b)$

$$\xi^{(i)}[a, b) = \sum_{n=0}^{\infty} C_n(a, b) V^n g^{(i)} \tag{4.1}$$

and
$$V^n g^{(i)} = \int_0^\infty \psi_n^{(i)}(u)\, d\xi^{(i)}(u), \quad \text{where} \quad \psi_n^{(i)} \in L_2(\mu, [0, \infty)). \tag{4.2}$$

From (4.2) and the fact that $\{V^n g^{(i)}\}$ $(n = 0, 1, 2, \ldots)$ is a complete orthonormal system in $\mathfrak{M}_{\xi^{(i)}}$ we obtain the following relations:

$$\left.\begin{aligned}
&\text{(a)} \quad \int_0^\infty \psi_n^{(i)}(u)\, \overline{\psi_m^{(i)}(u)}\, d\mu(u) = \quad (m \neq n),\\
&\text{(b)} \quad \int_0^\infty |\psi_n^{(i)}(u)|^2\, d\mu(u) = 1;\\
&\text{(c)} \quad \int_0^\infty e^{-u}\, \psi_n^{(i)}(u)\, d\mu(u) = 0 \quad \text{for} \quad n \geqslant 1.
\end{aligned}\right\} \tag{4.3}$$

Further the system $\{\psi_n^{(i)}(u),\ n = 0, 1, \ldots\}$ is complete in $L_2(\mu, [0, \infty))$. Let us now denote by $L_n^{(i)*}(u) = 1/\sqrt{2}\, e^u \psi_n^{(i)}(u)$. Then $L_n^{(i)*}(u)$ satisfy the equations:

$$2\int_0^\infty e^{-2u}\, L_n^{(i)*}(u)\, \overline{L_m^{(i)*}(u)}\, d\mu(u) = \delta_{nm} \quad (m, n = 0, 1, \ldots).$$

In other words, we have

$$\int_0^\infty e^{-u}\, L_n^{(i)*}(u/2)\, \overline{L_m^{(i)*}(u/2)}\, d\mu(u) = \delta_{nm} \quad \text{for all } m, n. \tag{4.4}$$

From the completeness of the system $\{\psi_n^{(i)}(u), n = 0, 1, 2, \ldots\}$ in $L_2(\mu, [0, \infty))$ and the orthogonality relations (4.4) it follows that the functions $L_n^{(i)*}(u/2)$, $n = 0, 1, 2, \ldots$ are complete in the space of functions on $[0, \infty)$, square integrable with respect to the measure $e^{-u} d\mu(u)$. Hence for each $i = 1, 2, \ldots, N$, $L_n^{(i)*}(u/2) = L_n(u)$, where $L_n(u)$

is the nth Laguerre polynomial (G. Sansone [19]). Hence for each i, $\psi_n^{(i)}(u) = \sqrt{2}\, e^{-u} L_n(2u)$ and

$$V^n g^{(i)} = \sqrt{2} \int_0^\infty e^{-u} L_n(2u)\, d\xi^{(i)}(u). \tag{4.5}$$

From (4.1) and (4.5) we have

$$C_n(a,b) = (\xi^{(i)}[a,b), V^n g^{(i)}) = \sqrt{2} \int_a^b e^{-u} L_n(2u)\, du. \tag{4.6}$$

Theorem 4.1. (a) *If* $T_{ab} = 1/\sqrt{2}\,(S_b - S_a - \int_a^b S_h\, dh)$ $(0 \leq a < b)$, *then on* $R^\perp$

$$-T_{ab} = \sum_{n=0}^\infty C_n(a,b) V^n, \tag{4.7}$$

where $C_n(a,b) = \sqrt{2} \int_a^b e^{-u} L_n(2u)\; d\mu(u)$ *and the operator-valued power series on the right-hand side converges in the strong sense on* $R^\perp$.

(b) *The differential innovation subspace* $T_{ab}(R^\perp) = \sum_{n=0}^\infty C_n(a,b) V^n (R^\perp)$.

Proof of (a). Let us observe that

$$-T_{ab} g^{(i)} = \xi^{(i)}[a,b) = \sum_{n=0}^\infty C_n(a,b) V^n g^{(i)} \quad \text{for every } i = 1, 2, \ldots, N. \tag{4.8}$$

For any $f \in R^\perp$,

$$\|\sum_{n=0}^\infty C_n(a,b) V^n f\|^2 = \sum_{n=0}^\infty |C_n(a,b)|^2 \, \|V^n f\|^2 = \|f\|^2 \sum_{n=0}^\infty |C_n(a,b)|^2. \tag{4.9}$$

But $\sum_{n=0}^\infty |C_n(a,b)|^2 = 2\int_0^\infty e^{-2u} I_{[a,b)}(u)\, d\mu(u)$, by Parseval's identity. Hence from (4.9) it follows that the operator-valued series $\sum_{n=0}^\infty C_n(a,b) V^n$ converges strongly on $R^\perp$. Clearly the operator $\sum_{n=0}^\infty C_n(a,b) V^n$ is linear and bounded (see (4.9)). This along with (4.8) implies that $-T_{ab} = \sum_{n=0}^\infty C_n(a,b) V^n$.

Proof of (b). From (a) we have $-T_{ab}(R^\perp) = \sum_{n=0}^\infty C_n(a,b) V^n (R^\perp)$. Since $T_{ab}(R^\perp)$ is a subspace, $T_{ab}(R^\perp) = -T_{ab}(R^\perp)$. This completes the proof.

In the next two sections we shall apply the results hitherto developed to the representation and multiplicity theory of weakly stationary stochastic processes.

5. Stationary stochastic processes and the associated semigroup of isometries

We consider the stationary stochastic process (henceforth, S.P.) of the following kind. Let Φ be a Hausdorff space satisfying the second countability axiom. We say that $\mathbf{x}_t(-\infty < t < \infty)$ is a S.P. on Φ if for each $\varphi \in \Phi$, $\mathbf{x}_t(\varphi)$ is a complex-valued random variable on a probability space (Ω, P) with mean zero and $\mathcal{E}|\mathbf{x}_t(\varphi)|^2$ finite. The process $\{\mathbf{x}_t\}(-\infty < t < +\infty)$ is called weakly stationary (or briefly, stationary) if for all $\varphi, \psi \in \Phi$ and arbitrary real numbers s, t and τ, we have $\mathcal{E}[\mathbf{x}_{t+\tau}(\varphi)\overline{\mathbf{x}_{s+\tau}(\psi)}] =$

$\mathcal{E}[\mathrm{x}_t(\varphi)\overline{\mathrm{x}_s(\psi)}]$. The covariance function $\mathcal{E}[\mathrm{x}_t(\varphi)\overline{\mathrm{x}_s(\psi)}]$ of the process depends on $t-s,\varphi,\psi$. It should be noted that the stationarity considered here is a temporal one and does not involve Φ. Nevertheless, it is sufficiently general and useful for our purpose since it includes as special cases many stationary random processes of practical interest. For instance, if Φ is a q-dimensional euclidean space and $\mathrm{x}_t(\varphi)$ is linear with respect to φ for each t, then the x_t-process can be regarded as a q-variate stationary process (see Yu. A. Rozanov [18]).

Associated with the x_t-process are the following spaces. (a) The past and present up to time t of the x_t-process, $L_2(\mathrm{x};t)$ is the subspace $\mathfrak{S}\{\mathrm{x}_\tau(\varphi),\varphi\in\Phi,\tau\leqslant t\}$ of $L_2(\Omega,P)$ generated by the random variables $\{\mathrm{x}_\tau(\varphi),\varphi\in\Phi,\tau\leqslant t\}$. (b) The remote past of the process $L_2(\mathrm{x};-\infty)=\bigcap_t L_2(\mathrm{x};t)$. (c) The space of the process $L_2(\mathrm{x})$ is the smallest subspace of $L_2(\Omega,P)$ containing $L_2(\mathrm{x};t)$ for each t.

The stationary S.P. considered will be assumed to satisfy the following condition.

$$\left.\begin{array}{ll}\text{(i)} & \text{If } \varphi_n\to\varphi \text{ then } \mathcal{E}\,|\mathrm{x}_t(\varphi_n)-\mathrm{x}_t(\varphi)|^2\to 0 \text{ for each } t,\\ \text{(ii)} & \text{For each } \varphi\in\Phi,\ |\mathrm{x}_t(\varphi)-\mathrm{x}_s(\varphi)|^2\to 0 \text{ as } s\to t,\\ \text{(iii)} & L_2(\mathrm{x},\,-\infty)=\{0\}.\end{array}\right\} \tag{5.1}$$

It has been proved by us ([12], Lemma 2.1) that, under (5.1), $L_2(\mathrm{x})$ is a separable Hilbert space. If we define the operator U_t from $L_2(\mathrm{x})$ to $L_2(\mathrm{x})$ by $U_t x_s(\varphi)=x_{s+t}(\varphi)$, $\varphi\in\Phi$ and s, t real, then U_t is a unitary operator for each t. Under condition 5.1(ii) and stationarity $\{U_t-\infty<t<+\infty\}$ is a strongly continuous group of unitary operators. We shall refer to this group as the unitary group of the stationary S.P. For $t\geqslant 0$, $U_t^*=U_{-t}$ is reduced by $L_2(\mathrm{x};0)$. If S_t denotes the restriction of U_t^* to $L_2(\mathrm{x};0)$ then clearly $\{S_t\}(t\geqslant 0)$ is a strongly continuous semi-group of isometries on $L_2(\mathrm{x};0)$ which we shall call the semi-group of isometries associated with the process $\{\mathrm{x}_t\}$. In what follows we shall write $\mathcal{Y}$ in place of $L_2(\mathrm{x};0)$. From (5.1)(iii), we have $\lim_{t\to+\infty} S_t(\mathcal{Y})=0$. The following lemma gives the relation of the infinitesimal generator iK of $\{U_t\}$ to iH, the infinitesimal generator of $\{S_t\}$.

Lemma 5.1.

(1) *The infinitesimal generator of the unitary semi-group* $\{U_t^*, t\geqslant 0\}$ *is* $-iK$;

(2) $-iK$ *is reduced by the space* $L_2(\mathrm{x};0)$;

(3) iH *is the restriction of* $-iK$ *to* $L_2(\mathrm{x};0)$.

Proof of (1). By the definition of K we get that for every real t, $U_t=\exp(itK)$. From this it follows $U_t^*=\exp(-itK)$ for $t\geqslant 0$. Since $\{U_t^*, t\geqslant 0\}$ is a strongly continuous semi-group of unitary operators, from Theorem XII.6.1 ([6], p. 1243) it follows that $\{U_t^*\}$ has a unique infinitesimal generator iK_0 given by $U_t^*=\exp(itK_0)$. Hence $K_0=-K$.

Proof of (2) *and* (3). For each $t>0$ and $f\in\mathcal{Y}$, by the definition of S_t we have

$$t^{-1}[S_t-I]f=t^{-1}[U_t^*-I]f. \tag{5.2}$$

If $f\in\mathcal{D}_H$, then $\lim_{t\to 0} t^{-1}[U_t^*-I]f$ exists; i.e., $\mathcal{D}_H\subset\mathcal{D}_{-K}\cap\mathcal{Y}$. Also from (5.2) we get, by a similar argument, that $\mathcal{D}_{-K}\cap\mathcal{Y}\subset\mathcal{D}_H$. For each $f\in\mathcal{D}_{-K}\cap\mathcal{Y}$, $-iKf$ belongs to $\mathcal{Y}$ and equals iHf. Hence it follows that iH is the restriction of $-iK$ to $\mathcal{Y}$.

Let $W = c(H)$ and $V = c(K)$. Since $-K$ is reduced by $\mathcal{Y}$ and $c(-K) = V^{-1}$ it follows that V^{-1} is also reduced by $\mathcal{Y}$. Further, from (2) and (3) of Lemma 5.1, it is easy to see that $\mathcal{D}_W = \mathcal{D}_{V^{-1}} \cap \mathcal{Y}$ and $Wg = V^{-1}g$ for all $g \in \mathcal{D}_{V^{-1}} \cap \mathcal{Y}$. Hence we have

Corollary 5.1. (a) *V^{-1} is reduced by $L_2(\mathbf{x}; 0)$; and* (b) *W is the restriction of V^{-1} to $L_2(\mathbf{x}; 0)$.*

6. Representation of stationary S.P.'s: Multiplicity as generalization of rank

The rank of a discrete parameter q-variate stationary S.P. is defined as the rank of its $q \times q$ prediction error matrix [21]. This definition brings out the importance of this notion to prediction theory and to the development of the spectral theory of stationary S.P.'s. The definition of the rank of a q-variate, continuous parameter stationary S.P., however, is less direct. In this case, the rank is defined to be the rank of the associated discrete parameter process. Let $x_t(-\infty < t < \infty)$ be a continuous in quadratic mean, univariate stationary S.P., and let $\{U_t\}$ be its unitary group with infinitesimal generator iK. If V is the Cayley transform of K, the S.P. $\{V^n_{x_0}, n = 0, \pm 1, \ldots\}$ is called the associated discrete parameter process [15]. This definition extends easily to infinite dimensional stationary S.P.'s (see [12]). Using this extension we were able to show that the multiplicity of an infinite dimensional stationary S.P. is the proper generalization of rank.

In this section we rederive this result and also obtain a representation of the purely non-deterministic component of the S.P., basing ourselves on Theorem 2.1 and Theorem 6.2 below. The representation and multiplicity theory of continuous parameter stationary S.P.'s is thus put on an independent footing without any appeal to discrete parameter processes. For the proof of Theorem 6.2 we need the following result proved by us in [12].

Theorem 6.1. ([12], *Theorem 5.2*). *For each t, let $E(t)$ denote the projection operator from $L_2(\mathbf{x})$ onto $L_2(\mathbf{x}; t)$ (cf. Section 5). Then $\{E(t)\}$ $(-\infty < t < +\infty)$ is a resolution of the identity in $L_2(\mathbf{x})$ and its maximal spectral type ϱ has uniform multiplicity M.*

The fact that the multiplicity is uniform, is of great importance in the ensuing argument. For each $t \geqslant 0$, $\bar{E}(t) = E(0) - E(-t)$ and $\bar{E}(t) = 0$ for $t < 0$. For any $f \in L_2(\mathbf{x})$ and $-\infty < a < b \leqslant 0$, $\|E(a,b]E(0)f\|^2 = \|E(a,b]f\|^2 = \|\bar{E}[-b,-a)f\|^2$. Also for

$$0 \leqslant a \leqslant b < \infty, \quad \|E(a,b]f\|^2 = \|\bar{E}[-b,-a)f\|^2.$$

Therefore the spectral function $\bar{\varrho}_f$ of f with respect to $\{\bar{E}(t)\}$ can be regarded as the spectral function ϱ_f of $E(0)f$ with respect to $\{E(t)\}$; i.e., every spectral type $\bar{\varrho}$ of $\{\bar{E}(t)\}$ is a spectral type of $\{E(t)\}$. But ϱ is the maximal spectral type with respect to $\{E(t)\}$, so that $\bar{\varrho} < \varrho$. The multiplicity of ϱ being uniform by Theorem 6.1, we have $M =$ multiplicity of every spectral type $\bar{\varrho}$ with respect to $\{\bar{E}(t)\}$. In particular, M equals the multiplicity of the maximal spectral type $\bar{\varrho}$ of $\{\bar{E}(t)\}$. Hence from Theorem 2.1, $M = N$.

Theorem 6.2. *The multiplicity M of a weakly stationary S.P. satisfying* (5.1) *is equal to the dimension of the space $L_2(\mathbf{x}; 0) \ominus V^{-1} L_2(\mathbf{x}; 0)$ where $V = c(K)$, iK being the infinitesimal generator of the unitary group $\{U_t\}$ $(-\infty < t < +\infty)$ of the process.*

Proof. It has just been shown above that $M=N$. From Theorem 2.1, $N=\dim(R^{\perp})=\dim(L_2(\mathbf{x};0)\ominus WL_2(\mathbf{x};0))$ where W is as defined in Section 5. Corollary 5.1 shows that W is equal to the restriction of V^{-1} to $L_2(\mathbf{x};0)$. Therefore $M(=N)$ equals

$$\dim[L_2(\mathbf{x};0)\ominus V^{-1}L_2(\mathbf{x};0)].$$

If in the definition of a stationary process, Φ is q-dimensional and $\mathbf{x}_t$ is linear on Φ, then $\{\mathbf{x}_t-\infty<t<\infty\}$ is a q-variate process (see Rozanov [18]). Let us denote the associated discrete process by $\{\tilde{\mathbf{x}}_n\}(n=0,\pm 1,\dots)$. The "prediction error matrix" of $\{\tilde{\mathbf{x}}_n\}$ has rank equal to the dimension of $L_2(\tilde{\mathbf{x}};0)\ominus L_2(\tilde{\mathbf{x}};-1)$ where $L_2(\tilde{\mathbf{x}};n)=\mathfrak{S}\{\mathbf{x}_m(\varphi),\ \varphi\in\Phi, m=n,n-1,\dots\}$. But $L_2(\tilde{\mathbf{x}};0)=L_2(\mathbf{x};0)$ and $V^{-1}L_2(\mathbf{x};0)=L_2(\tilde{\mathbf{x}};-1)$. Thus the rank of the associated discrete process is equal to the $\dim[L_2(\mathbf{x};0)\ominus V^{-1}L_2(\mathbf{x};0)]$ which by Theorem 6.2 equals the multiplicity. Hence the multiplicity of the S.P. is, in reality, the generalization of rank.

Let us define for each k, $\xi_k(u)=-\xi^{(k)}(0,u]$ for $u\geqslant 0$, and $=U_u\xi^{(k)}(0,-u]$ for $u<0$ where $\{\xi^{(k)}(u),\ u\geqslant 0\}$ is as defined in Section 2 and $\{U_t\}$ is the unitary group of the S.P. With this definition $\{\xi_k(u),\ -\infty<u<+\infty\}$ $(k=1,2,\dots,N)$ are orthogonal processes with stationary orthogonal increments possessing the property $U_t\xi_k(u)=\xi_k(u-t)$ for all real t and u.

Theorem 6.3. *Let $\{\mathbf{x}_t-\infty<t<+\infty\}$ be a stationary S.P. on a separable Hausdorff space Φ satisfying* (5.1) (i) (ii). *Then*

$$\mathbf{x}_t(\varphi)=\sum_{n=1}^{M}\int_0^\infty F_n(\varphi;u)\,d\xi_n(u-t)+x_t(\varphi),\quad \textit{where}$$

(i) *M is equal to the multiplicity of $\{\mathbf{x}_t\}$,*

(ii) *$\sum_{n=1}^{M}\int_0^\infty |F_n(\varphi;u)|^2\,du$ is finite for each φ and,*

(iii) *$\{\mathbf{z}_t,\ -\infty<t<+\infty\}$ is a deterministic stationary stochastic process on Φ*; i.e., $L_2(\mathbf{z};-\infty)=L_2(\mathbf{z})$

(iv) *For each (t,φ), (s,ψ): $\mathcal{E}[\mathbf{z}_t(\varphi),\overline{\mathbf{y}_s(\psi)}]=0$ with $\mathbf{y}_s(\psi)=\sum_{n=1}^{M}\int_0^\infty F_n(\psi;u)\,d\xi_n(u-s)$.*

Proof. From Theorems 2.1, 6.1 and 6.2 we get

$$\mathbf{x}_0(\varphi)=\sum_{n=1}^{M}\int_0^\infty F_n(\varphi;u)\,d\xi_n(u)+P_{L_2(\mathbf{x};-\infty)}\mathbf{x}_t(\varphi),$$

where M is the multiplicity of $\{\mathbf{x}_t,-\infty<t<\infty\}$. But for each t,

$$\mathbf{x}_t(\varphi)=U_t\mathbf{x}_0(\varphi)=\sum_{1}^{M}\int_0^\infty F_n(\varphi;u)\,d\xi_n(u-t)+P_{L_2(\mathbf{x};-\infty)}\mathbf{x}_t(\varphi)$$

by definition of ξ_n process and the fact that $U_t P_{L_2(\mathbf{x};-\infty)}=P_{L_2(\mathbf{x};-\infty)}U_t$. Let us define $\mathbf{z}_t(\varphi)=P_{L_2(\mathbf{x};-\infty)}\mathbf{x}_t(\varphi)$; then $\{\mathbf{z}_t,\ -\infty<t<+\infty\}$ is a stationary process on Φ and since $L_2(\mathbf{x};0)=L_2(\mathbf{y};0)\oplus L_2(\mathbf{z};0)=L_2(\mathbf{y};0)\oplus L_2(\mathbf{x};-\infty)$, $L_2(\mathbf{z};0)=L_2(\mathbf{x};-\infty)=L_2(\mathbf{z};-\infty)$.

As a corollary we obtain the following representation for finite-dimensional processes due to E. G. Gladyshev ([7], see also [17]). For univariate processes the corresponding result was first given by K. Karhunen [13] (also [9]).

Corollary 6.3. *Let* $[x_1(t), \ldots, x_q(t))]$ *be a continuous in q.m., weakly stationary q-variate process. Then*

$$x_i(t) = \sum_{n=1}^{M} \int_0^\infty F_{in}(u)\, d\xi_n(u-t) + z_i(t);$$

where (i) M *is the rank of the process,* (ii) $\sum_1^M \int_0^\infty |F_{in}(u)|^2\, du$ *is finite,* (iii) $[(z_1(t), \ldots, z_q(t))]$ *is a q-variate stationary process orthogonal to* $[(y_1(t), \ldots, y_q(t))]$ *where* $y_i(t) = \sum_{n=1}^M \int_0^\infty F_{in}(u)\, d\xi_n(u-t)$, $(i = 1, \ldots, q)$.

Theorem 6.3 and Corollary 6.3 were obtained by us in [12] (see also [11]). The method used there was an extension of Hanner's approach made possible by the application of the ideas of multiplicity theory. The proof given here is directly based on Theorem 6.2 and the modified version of Cooper's result given in Section 2. The essential unity of these two approaches is thus demonstrated.

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