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On the existence of the scattering operator

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1. Introduction

Scattering, in its most simple form, can be described as a process during which two elementary particles collide. The particles are assumed to be infinitely separated in space both at the beginning and at the end of the process. The state at each instant is given by a function $\psi \in L^2(E_m)$. The particles interact with a certain force V, which decreases to zero as the distance between the particles tends to infinity. With each initial state (at the time $t=-\infty$) is uniquely associated a final state (at the time $t=\infty$) by means of the Schrödinger equation

$$i\frac{\partial}{\partial t}\psi(t)=H_1\psi(t), \tag{1}$$

where H_1 is the total Hamiltonian operator. H_1 is defined as a self-adjoint extension of the operator H_0+V, where H_0 represents the kinetic energy and V the potential energy. H_0 and V are self-adjoint. (1) describes the time development

$$\psi(s)=e^{-iH_1(s-t)}\psi(t) \tag{2}$$

for a state ψ.

One introduces a time-dependent representation of the Hilbert space $L^2(E_m)$ so that $e^{itH_0}\psi$ represents the function $\psi \in L^2(E_m)$. In this representation, "the interaction picture", the unitary operator $U(s,t)=e^{iH_0 s}\,e^{-iH_1(s-t)}\,e^{-iH_0 t}$ takes a state at the time t to the corresponding state at the time s. The operator U has the following properties

$$\left.\begin{aligned} U(s,t)&=U(s,t')\,U(t',t),\\ U^*(s,t)&=U(t,s).\end{aligned}\right\} \tag{3}$$

Under the assumption that $\lim_{t\to\pm\infty}U(0,t)=U(0,\pm\infty)$ exist in some sense,one defines the scattering operator S as

$$S=U^*(0,\infty)\,U(0,-\infty). \tag{4}$$

Then S transfers an initial state to the corresponding final state.

In the formal scattering theory one assumes that the wave operators $U(0,\pm\infty)$ exist and that S is unitary. A rigorous mathematical theory for scattering was given in 1957 by J. M. Jauch [1]. Earlier K. O. Friedrichs [2] and J. M. Cook [3] gave sufficient conditions for the existence of $U(0, \pm\infty)$. Further contributions to the theory were later given by M. N. Hack [4], Jauch-Zinnes [5] and S. T. Kuroda [6]. As a sufficient condition for the existence of the strong limits

$$U(0, \pm\infty) = \lim_{t\to\pm\infty} e^{itH_1} e^{-itH_0} \tag{5}$$

with $H_0 = -\Delta$ and $H_1 = -\Delta + V(\mathbf{x})\cdot$ Kuroda [6] stated

$$V(\mathbf{x})(1+|\mathbf{x}|)^{-\frac{1}{2}m+1+\varepsilon} \in L^2(E_m), \quad \varepsilon > 0, \tag{6}$$

In theorem 1 we shall give an estimate sharper than that in (6).

Jauch [1] bases his theory of scattering on the following definition of a *simple scattering system:*

I. $U(0, \pm\infty)$ exist as strong limits in the whole of $L^2(E_m)$ with the ranges R_+ and R_-.
II. $R_+ = R_- = R$.
III. $M_\perp = R$.

M is defined as the subspace spanned by the eigenfunctions of H_1.

If condition I holds true, then condition II is equivalent with the requirement that $S = U^*(0,\infty)\, U(0, -\infty)$ be unitary. We shall show in theorem 2 that the conditions for a simple scattering system are not satisfied for a function $V(x)$ in E_1, which decreases as $1/|x|$ when $|x| \to \infty$.

2. On the existence of the strong limits $\lim_{t\to\pm\infty} e^{itH_1} e^{-itH_0}$

Let E_m be an m-dimensional euclidian space and H_1 the differential operator $-\Delta + V(\mathbf{x})\cdot$ in $L^2(E_m)$, where $\Delta = \sum_{i=1}^m \partial^2/(\partial x_i^2)$, and V is a multiplication operator such that $V(\mathbf{x})$ is real and measurable in E_m. The operator $H_0 = -\Delta$, with region of definition

$$D_{H_0} = \{u \,|\, |\mathbf{k}|^2 \hat{u}(\mathbf{k}) \in L^2(E_m)\}, \tag{1}$$

is self-adjoint.[1] ($\hat{u}(\mathbf{k})$ is the Fourier transform of $u(\mathbf{x})$.) The region of definition of V is the totality of all $u(\mathbf{x}) \in L^2(E_m)$ such that $V(\mathbf{x})\cdot u(\mathbf{x}) \in L^2(E_m)$. $V(\mathbf{x})$ is assumed to belong to $L^2(E_m)$ locally. $H_1 = H_0 + V$ is defined on a dense set, and is symmetric and real. Thus, it has at least one self-adjoint extension. Let H_1 be one of them. We set

$$U_t = e^{itH_1} e^{-itH_0}$$

and write

$$(U_t - U_s)u = i\int_s^t e^{itH_1} V e^{-itH_0} u\, dt,$$

from which it follows that

[1] Kato [7].

$$\| (U_t - U_s) u \| \leqslant \int_s^t \| V e^{-itH_0} u \| dt \tag{2}$$

since e^{itH_1} is unitary.

We thus have

Lemma 1. *Suppose there exist a $c > 0$ such that*

$$\int_c^\infty \| V e^{-itH_0} u \| dt < \infty, \int_{-\infty}^{-c} \| V e^{itH_0} u \| dt < \infty$$

for all $u \in \mathfrak{L}$, where $\mathfrak{L}$ is a set, dense in $L^2(E_m)$. Then the strong limits $\lim_{t\to\pm\infty} e^{itH_1} e^{-itH_0} u$ exist for all $u \in L^2(E_m)$.

We shall need the following formula

$$e^{-itH_0} u(\mathbf{x}) = \frac{c_m}{|t|^{\frac{1}{2}m}} \int_{E_m} \exp\left(- \frac{i(\mathbf{x}-\mathbf{s})\cdot(\mathbf{x}-\mathbf{s})}{4t} \right) u(\mathbf{s})\, d\mathbf{s}, \tag{3}$$

where $$c_m = \frac{(1 + i \operatorname{signum}(t))^m}{2^{\frac{1}{2}3m} \pi^{\frac{1}{2}m}}.$$

Derivation of formula (3). Let $\hat{u}(\mathbf{y})$ be the Fourier transform of u. That is,

$$\hat{u}(\mathbf{y}) = \frac{1}{(2\pi)^{\frac{1}{2}m}} \int_{E_m} e^{-i\mathbf{x}\cdot\mathbf{y}} u(\mathbf{x})\, d\mathbf{x},$$

where the integration is over the whole of E_m and $\mathbf{x}\cdot\mathbf{y} = \sum_{i=1}^m x_i y_i$.

$$\begin{aligned} e^{it\Delta} u(\mathbf{x}) &= (2\pi)^{-\frac{1}{2}m} \int_{E_m} e^{i\mathbf{x}\cdot\mathbf{y}} e^{it\mathbf{y}\cdot\mathbf{y}} \hat{u}(\mathbf{y})\, d\mathbf{y} \\ &= \lim_{R\to\infty} (2\pi)^{-m} \int_{|y|\leqslant R} e^{i\mathbf{x}\cdot\mathbf{y}} e^{it\mathbf{y}\cdot\mathbf{y}} \int_{E_m} e^{-i\mathbf{y}\cdot\mathbf{s}} u(\mathbf{s})\, d\mathbf{s}\, d\mathbf{y} \\ &= (2\pi)^{-m} \int_{E_m} u(\mathbf{s}) \int_{E_m} e^{it\mathbf{y}\cdot\mathbf{y} + i\mathbf{y}(\mathbf{x}-\mathbf{s})} d\mathbf{y}\, d\mathbf{s}. \end{aligned} \tag{4}$$

$$\int_{E_m} e^{it\mathbf{y}\cdot\mathbf{y} + i\mathbf{y}(\mathbf{x}-\mathbf{s})} d\mathbf{y} = \prod_{k=1}^m \int_{-\infty}^{\infty} e^{ity_k^2 + iy_k(x_k - s_k)} dy_k. \tag{5}$$

For $t > 0$ one gets

$$\begin{aligned} \int_{-\infty}^{\infty} e^{ity_k^2 + iy_k(x_k - s_k)} dy_k &= \exp\left(- \frac{i(x_k - s_k)^2}{4t} \right) \int_{-\infty}^{\infty} e^{ity_k^2} dy_k \\ &= \exp\left(- \frac{i(x_k - s_k)^2}{4t} \right) \cdot \frac{1}{\sqrt{t}} \sqrt{\frac{\pi}{2}} (1 + i). \end{aligned} \tag{6}$$

(6), (5) and (4) leads to formula (3) for $t > 0$.

The case where $t < 0$ is treated analogously.

Theorem 1. *If* $|V(\mathbf{x})| \leqslant V^*(|\mathbf{x}|)$, *where* $V^*(|\mathbf{x}|) \in L^2(E_m)$ *locally, and there exist a number* $N > 0$ *such that* $V^*(|\mathbf{x}|)$ *is monotonically decreasing for* $|\mathbf{x}| \geqslant N$, *and* $V^*(|\mathbf{x}|) \in L^1(M, \infty)$, $0 < M < \infty$, *then the strong limits exist for all* $u \in L^2(E_m)$.

By virtue of lemma 1 it is sufficient to prove

$$\int_M^\infty \| V e^{-itH_0} u \| < \infty \quad \text{and} \quad \int_{-\infty}^{-M} \| V e^{-itH_0} u \| dt < \infty \tag{7}$$

for all u in a dense set.

Formula (3) gives

$$e^{-itH_0} u(\mathbf{x}) = \frac{c_m}{t^{\frac{1}{2}m}} \int_{E_m} \exp\left(-\frac{(\mathbf{x}-\mathbf{s})^2}{4t} i\right) u(\mathbf{s}) d\mathbf{s} = \frac{c_m}{t^{\frac{1}{2}m}} \exp\left(-\frac{\mathbf{x}^2 i}{4t}\right) \int_{E_m}$$
$$\times \exp\left(-\frac{\mathbf{s}^2}{4t} i + \frac{\mathbf{x}\cdot\mathbf{s}}{2t} i\right) u(\mathbf{s}) d\mathbf{s}.$$

For $t > M > 0$ we have

$$|e^{-itH_0} u(\mathbf{x})| = \frac{|c_m|}{t^{\frac{1}{2}m}} \left| \int_{E_m} \exp\left(-\frac{\mathbf{s}^2}{4t} i + \frac{\mathbf{x}\cdot\mathbf{s}}{2t} i\right) u(\mathbf{s}) d\mathbf{s} \right|$$

and

$$\int_M^\infty \| V(\mathbf{x}) e^{-itH_0} u(\mathbf{x}) \| dt = |c_m| \int_M^\infty \frac{dt}{t^{\frac{1}{2}m}} \left\{ \int_{-\infty}^\infty \cdots \right.$$
$$\left. \cdots \int_{-\infty}^\infty V^2(\mathbf{x}) \left| \int_{E_m} \exp\left(-\frac{\mathbf{s}^2}{4t} i + \frac{\mathbf{x}\cdot\mathbf{s}}{2t} i\right) u(\mathbf{s}) d\mathbf{s} \right|^2 d\mathbf{x} \right\}^{\frac{1}{2}}. \tag{8}$$

Definition. $\mathfrak{L} = \{u(\mathbf{s}) = u_1(s_1)\cdot u_2(s_2)\cdot \ldots \cdot u_m(s_m) \mid (d^n/ds_v^n) u_v(s_v) \in L^1(-\infty, \infty), n, v = 1, 2 \ldots m;$ *$u_v(s_v)$ has compact support and* $\hat{u}_v(0) = 0$ *for* $v = 1, 2 \ldots, m\}$.

One can show that the set $\mathfrak{L}$ is dense in $L^2(E_m)$. We shall verify (7) for a function $u(\mathbf{s}) \in \mathfrak{L}$.

Lemma 2. For the function

$$F(\mathbf{x}, t) = \left| \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty \exp\left(-\frac{\mathbf{s}\cdot\mathbf{s}}{4t} i + \frac{\mathbf{x}\cdot\mathbf{s}}{2t} i\right) u_1(s_1) u_2(s_2) \cdot \ldots \cdot u_m(s_m) ds_1 \ldots ds_m \right|,$$

where $u(\mathbf{s}) = \prod_{v=1}^m u_v(s_v) \in \mathfrak{L}$, we have

$$F(\mathbf{x}, t) = O\left(\prod_{v=1}^m \left|\frac{x_v}{t}\right|\right) + O(t^{-1}) \quad \text{for} \quad \left|\frac{x_v}{t}\right| \to 0 \quad \text{and} \quad t \to \infty, \tag{9}$$

$$F(\mathbf{x}, t) = O\left(\left|\frac{t}{x_v}\right|^m\right) \quad \text{for} \quad \left|\frac{t}{x_v}\right| \to 0, \quad v = 1, 2, \ldots, m, \quad t \geqslant 1. \tag{10}$$

Proof.

$$F(\mathbf{x}, t) = \left| \prod_{v=1}^m \int_{-A}^A \exp\left(\frac{x_v \cdot s_v}{2t} i\right) v_v(s_v) ds_v \right.$$
$$\left. + \int_{-A}^A \cdots \int_{-A}^A \left(\exp\left(-\frac{\mathbf{s}\cdot\mathbf{s}}{4t} i\right) - 1\right) \prod_{v=1}^m u_v(s_v) \exp\left(\frac{x_v \cdot s_v}{2t} i\right) ds_1 \ldots ds_m \right|. \tag{11}$$

Each factor in the first term is an entire function of the variable x_v/t. From $\hat{u}_v(0)=0$ it now follows that

$$\int_{-A}^{A}\exp\left(\frac{x_v\cdot s_v}{2t}i\right)u_v(s_v)\,ds_v=O\left(\frac{x_v}{t}\right)\quad\text{when}\quad\left|\frac{x_v}{t}\right|\to 0.$$

This takes care of the first term in (9). For the second term in (9), we use the inequality

$$\left|\exp\left(-\frac{\mathbf{s}\cdot\mathbf{s}}{4t}i\right)-1\right|\leqslant 2\sin\frac{\mathbf{s}\cdot\mathbf{s}}{8t}=O\left(\frac{1}{t}\right)$$

when $t\to\infty$ in (11). (10) is obtained by partial integration.

It is easily seen that $F(x,t)$ is bounded. Thus we have

$$F(\mathbf{x},t)\leqslant D. \tag{12}$$

Put $r=(\mathbf{x}\cdot\mathbf{x})^{\frac{1}{2}}$. From (9) and (10) it follows that

$$F(\mathbf{x},t)=O\left(\frac{r}{t}\right)^m+O(t^{-1})\quad\text{for}\quad\frac{r}{t}\to 0\quad\text{and}\quad t\to\infty.$$

$$F(\mathbf{x},t)=O\left(\frac{t}{r}\right)^m\quad\text{for}\quad\left(\frac{t}{r}\right)\to 0,\quad t\geqslant 1.$$

For convenience, in the remainder of this paper we shall reserve the letter c for real constants in the open interval $(0,\infty)$.

For $F^2(\mathbf{x},t)$ the following inequalities hold

$$F^2(\mathbf{x},t)\leqslant c\left(\frac{r}{t}\right)^{2m}+ct^{-2}\quad\text{for}\quad t>r\quad\text{and}\quad t>1, \tag{13}$$

$$F^2(\mathbf{x},t)\leqslant c\left(\frac{t}{r}\right)^{2m}\quad\text{for}\quad r>t\geqslant 1. \tag{14}$$

We can set the lower limit of integration equal to $M=1$.

$$\int_1^\infty\frac{dt}{t^{\frac{1}{2}m}}\left\{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}V^2(\mathbf{x})F^2(\mathbf{x},t)\,d\mathbf{x}\right\}^{\frac{1}{2}}\leqslant$$

$$c\int_1^\infty\frac{dt}{\sqrt{t}}\left\{\int_0^t V^{*2}(r)\left(\frac{r}{t}\right)^{m-1}\left(c\left(\frac{r}{t}\right)^{2m}+ct^{-2}\right)dr\right.$$

$$\left.+c\int_t^\infty V^{*2}(r)\left(\frac{r}{t}\right)^{m-1}\left(\frac{r}{t}\right)^{-2m}dr\right\}^{\frac{1}{2}} \tag{15}$$

$$\leqslant c\int_1^\infty\frac{dt}{t^{3m/2}}\left\{\int_0^t V^{*2}(r)\,r^{3m-1}\,dr\right\}^{\frac{1}{2}}+ \tag{16}$$

$$+c\int_1^\infty \frac{dt}{t^{\frac{1}{2}m+1}}\left\{\int_0^t V^{*2}(r)r^{m-1}dr\right\}^{\frac{1}{2}} \tag{17}$$

$$+c\int_1^\infty \frac{dt}{\sqrt{t}}\left\{\int_t^\infty V^{*2}(r)\left(\frac{t}{r}\right)^{m+1}dr\right\}^{\frac{1}{2}}. \tag{18}$$

For the term (16) we have

$$\int_1^\infty \frac{dt}{t^{\frac{1}{2}3m}}\left\{\int_0^t V^{*2}(r)r^{3m-1}dr\right\}^{\frac{1}{2}} \leqslant \int_1^\infty \frac{dt}{t^{\frac{1}{2}3m}}\sum_{n=0}^\infty \left\{\int_{2^{-n-1}t}^{2^{-n}t} V^{*2}(r)r^{3m-1}dr\right\}^{\frac{1}{2}},$$

where $$\left\{\sum_{n=0}^\infty \int_{2^{-n-1}t}^{2^{-n}t} V^{*2}(r)r^{3m-1}dr\right\}^{\frac{1}{2}} \leqslant c\sum_{n=0}^\infty V^*(2^{-n-1}t)\,2^{-\frac{1}{2}3mn}t^{\frac{1}{2}3m},$$

thus $$\int_1^\infty \frac{dt}{t^{\frac{1}{2}3m}}\left\{\int_0^t V^{*2}(r)r^{3m-1}dr\right\}^{\frac{1}{2}} \leqslant c\sum_{n=0}^\infty 2^{-\frac{1}{2}3mn}\int_0^\infty V^*(2^{-n-1}t)\,dt$$

$$=c\sum_{n=0}^\infty 2^{-n(\frac{1}{2}3m-1)}\int_0^\infty V^*(r)\,dr<\infty.$$

For (17) we have

$$\int_1^\infty \frac{dt}{t^{\frac{1}{2}m+1}}\left\{\int_0^t V^{*2}(r)r^{m-1}dr\right\}^{\frac{1}{2}}<\infty$$

because $V^*(r)\in L^2(E_m)$ locally implies that $\int_0^1 V^{*2}(r)r^{m-1}dr<\infty$ and from $V^*(r)\in L^1(1,\infty)$ follows, for example, $\int_1^t V^{*2}(r)r^{m-1}dr=O(t^{m-\frac{3}{2}})$. For (18) we have

$$\int_1^\infty \frac{dt}{\sqrt{t}}\left\{\int_t^\infty V^{*2}(r)\left(\frac{t}{r}\right)^{m+1}dr\right\}^{\frac{1}{2}} = \int_1^\infty \frac{dt}{\sqrt{t}}\left\{\sum_{n=0}^\infty \int_{2^n t}^{2^{n+1}t} V^{*2}(r)\left(\frac{t}{r}\right)^{m+1}dr\right\}^{\frac{1}{2}}$$

$$\leqslant c\int_1^\infty \frac{dt}{\sqrt{t}}\sum_{n=0}^\infty 2^{-\frac{1}{2}n(m+1)}\left\{\int_{2^n t}^{2^{n+1}t} V^{*2}(r)\,dr\right\}^{\frac{1}{2}}$$

$$\leqslant c\sum_{n=0}^\infty 2^{-\frac{1}{2}n(m+1)}\int_1^\infty \frac{dt}{\sqrt{t}}\left\{\int_{2^n t}^{2^{n+1}t} V^{*2}(r)\,dr\right\}^{\frac{1}{2}}<\infty$$

because

$$\int_1^\infty \frac{dt}{\sqrt{t}}\left\{\int_{2^n t}^{2^{n+1}t} V^{*2}(r)\,dr\right\}^{\frac{1}{2}}<K \quad \text{for all} \quad n. \tag{19}$$

(19) follows from $V^*(r)\in L^1(1,\infty)$ and $V^*(r)\searrow 0$ for $r\geqslant N$, because

$$\int_1^\infty \frac{dt}{\sqrt{t}}\left\{\int_{2^n t}^{2^{n+1}t} V^{*2}(r)\,dr\right\}^{\frac{1}{2}} \leqslant \int_1^a + \int_a^\infty \frac{dt}{\sqrt{t}}\{V^{*2}(2^n t)(2^{n+1}-2^n)t\}^{\frac{1}{2}}<K,$$

where $a=\max(1, N2^{-n})$.

According to (15) and (8) we thus have

$$\int_1^\infty \| V(\mathbf{x}) e^{-itH_0} u(\mathbf{x}) \| dt < \infty$$

for all $u \in \mathfrak{L}$. This implies that $U(0, \infty)$ exists.

The inequality $\int_{-\infty}^{-1} \| V(\mathbf{x}) e^{-itH_0} u(\mathbf{x}) \| dt < \infty$, which is proved analogously, implies the existence of the strong limit $U(0, -\infty)$.

This completes the proof of theorem 1.

3. On the conditions for a simple scattering system

Theorem 2. *There exists a function $u(x)$ belonging to $L^2(-\infty, \infty)$ and not identically zero such that*

$$\lim_{t\to\infty} (u, e^{itH_1} e^{-itH_0} v) = 0 \tag{1}$$

for all $v \in L^2(-\infty, \infty)$ if $H_1 = -\Delta + V(x)$, where

$$V(x) = \begin{cases} \dfrac{1}{|x|} & \text{for} \quad |x| \geqslant 1, \\ h(x), h(x) \in L^2(-1,1), h(x) \geqslant 0 \textit{ and is even for } |x| < 1. \end{cases}$$

For the particular choice of the function $V(x)$ made in theorem 2, one can easily see that the conditions I, II and III in section 1 for a simple scattering system are not fulfilled. (1) implies that $u(x) \perp R_+$. From III it then follows that $u(x) \in M$, where M is the subspace spanned by the eigenfunctions of H_1. But this yields a contradiction, because H_1 in theorem 2 has no eigenfunctions. This fact is seen by considering

$$(H_1 f, f) = \| f' \|^2 + \int_{-\infty}^{\infty} V(x) |f(x)|^2 dx > 0 \quad \text{for all} \quad f \in L^2(-\infty, \infty), f \not\equiv 0.$$

which shows that H_1 has no non-positive eigenvalues. Furthermore, it follows from the Sturmian separation theorem that all solutions of the differential equation

$$H_1 f = \lambda f$$

are oscillatory for $\lambda > 0$. Thus H_1 has no eigenfunctions.

For the proof of theorem 2 let us consider the differential equation

$$-y'' + V(x) y = \xi^2 y. \tag{2}$$

Lemma 1.[1] *There exists a solution $K(x, \xi)$ of (2) with the following properties:*

1°. $$K(x, \xi) = \exp\left(i \int_1^x \sqrt{\xi^2 - V(s)}\, ds \right) + R_1(x, \xi) \quad \textit{for} \quad x \geqslant 1,$$

$$|R_1(x, \xi)| \leqslant 2 \int_x^\infty \left| \frac{V'(s)}{V(s) - \xi^2} \right| ds < \frac{1}{x}.$$

[1] Essentially Bellman [8], pp. 50–54.

2°. $$K(x,\xi)=\exp\left(-i\int_x^{-1}\sqrt{\xi^2-V(s)}\,ds\right)+R_2(x,\xi)\quad \text{for}\quad x\leqslant -1,$$

$$|R_2(x,\xi)|\leqslant 2\int_{-\infty}^{x}\left|\frac{V'(s)}{V(s)-\xi^2}\right|ds<\frac{1}{|x|}.$$

3°. $$|K(x,\xi)|\leqslant C\quad \textit{for}\quad |x|\leqslant 1.$$

4°. $$K(x,\xi)=K(-x,\xi).$$

Set $u(x)=\int_2^3 K(x,\xi)d\xi$. Then $u(x)$ is an even function. The scalar product in (1) is then equal to zero for all t if v is an odd function, since $e^{itH_1}e^{-itH_0}v$ is odd if v is odd.

It is thus sufficient to prove (1) for all v in a dense subset of $L^2(0,\infty)$. Let us choose $\hat{v}_a=e^{-ax^2}$, $a>0$, which functions generate a dense subset of $L^2(0,\infty)$. We set $a=1$ to simplify the calculations. Then

$$e^{itH_1}u=\lim_{n\to\infty}\sum_{k=0}^{n}\frac{(itH_1)^k u(x)}{k!}=\lim_{n\to\infty}\int_2^3\sum_{k=0}^{n}\frac{(it\xi^2)^k}{k!}K(x,\xi)d\xi=\int_2^3 e^{it\xi^2}K(x,\xi)d\xi$$

and $$e^{-itH_0}v=(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty}e^{-it\eta^2+ix\eta}\hat{v}(\eta)d\eta\quad \text{with}\quad \hat{v}(\eta)=e^{-\eta^2}.$$

$$(u,e^{itH_1}e^{-itH_0}v)=(e^{-itH_1}u,e^{-itH_0}v)=\left(\int_2^3 e^{-it\xi^2}K(x,\xi)d\xi,(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty}e^{-it\eta^2+ix\eta-\eta^2}d\eta\right)$$

$$\int_{-\infty}^{\infty}e^{-it\eta^2+ix\eta-\eta^2}d\eta=\exp\left(-\frac{x^2}{4(1+it)}\right)\int_{-\infty}^{\infty}e^{-(1+it)\eta^2}d\eta=\frac{\sqrt{\pi}}{\sqrt{1+it}}\exp\left(-\frac{x^2}{4(1+it)}\right).$$

We thus have

$$(u,e^{itH_1}e^{-itH_0}v)=\frac{1}{\sqrt{2}\sqrt{1-it}}\int_{-\infty}^{\infty}\exp\left(-\frac{x^2}{4(1+t^2)}-i\frac{tx^2}{4(1+t^2)}\right)\int_2^3 e^{-it\xi^2}K(x,\xi)d\xi dx. \tag{3}$$

We perform the x-integration separately over the three intervals $(-\infty,-1)$ $(-1,1)$ and $(1,\infty)$.

$$(u,e^{itH_1}e^{-itH_0}v)=\frac{1}{\sqrt{2}\sqrt{1-it}}\left(\int_{-\infty}^{-1}+\int_{-1}^{1}+\int_1^{\infty}\right)=I_1+I_2+I_3.$$

3° and (3) imply

$$|I_2|\leqslant\frac{c}{\sqrt{2}|1-it|^{\frac{1}{2}}}\int_{-1}^{1}\exp\left(-\frac{x^2}{4(1+t^2)}\right)dx\leqslant\frac{c'}{(1+t^2)^{\frac{1}{4}}}\to 0,\quad t\to\infty.$$

According to 1° and 2°, each of I_1 and I_3 can be divided into two integrals $I_1=I_1^{(1)}+I_1^{(2)}$ and $I_3=I_3^{(1)}+I_3^{(2)}$, where $I_1^{(2)}$ and $I_3^{(2)}$ correspond to the contributions from $R_2(x,\xi)$ and $R_1(x,\xi)$.

$I_1^{(2)}$ and $I_3^{(2)}$ tend to zero, as t tends to infinity.

$$|I_3^{(2)}| \leqslant \frac{c}{\sqrt{2}(1+t^2)^{\frac14}} \int_1^\infty \frac{\exp\left(-\dfrac{x^2}{4(1+t^2)}\right)}{x}\,dx = O\left(\frac{\log t}{\sqrt{t}}\right), \quad t \to \infty.$$

Set $x = yt$

$$\int_1^\infty \frac{\exp\left(-\dfrac{x^2}{4(1+t^2)}\right)}{x}\,dx = \int_{1/t}^\infty \frac{\exp\left(-\dfrac{y^2}{4(1+(1/t^2))}\right)}{y}\,dy \leqslant \int_{1/t}^1 \frac{dy}{y}$$

$$+ \int_1^\infty \exp\left(-\frac{y^2}{4(1+(1/t^2))}\right)dy = O(\log t), \, t \to \infty.$$

In the same way it is seen that

$$|I_1^{(2)}| \leqslant \frac{c}{\sqrt{2}(1+t^2)^{\frac14}} \int_{-\infty}^{-1} \frac{\exp\left(-\dfrac{x^2}{4(1+t^2)}\right)}{x}\,dx = O\left(\frac{\log t}{\sqrt{t}}\right), \quad t \to \infty.$$

It remains to show that $I_1^{(1)}$ and $I_3^{(1)}$ tend to zero as t tends to infinity.

$$I_3^{(1)} = \frac{1}{\sqrt{2}\sqrt{1-it}} \int_1^\infty \exp\left(-\frac{x^2}{4(1+t^2)} - i\frac{tx^2}{4(1+t^2)}\right) \int_2^3 \exp\left(-it\xi^2 + i\int_1^x \sqrt{\xi^2 - (1/s)}\,ds\right) d\xi\, dx.$$

Set $x = yt$. There is no essential restriction here in replacing $\dfrac{y^2}{4(1+t^{-2})}$ by $y^2/4$. Now we must estimate

$$\frac{t}{\sqrt{2}\sqrt{1-it}} \int_{1/t}^\infty \exp\left(-\frac{y^2}{4} - i\frac{ty^2}{4}\right) \int_2^3 \exp\left(-it\xi^2 + i\int_1^{yt} \sqrt{\xi^2 - \frac{1}{s}}\,ds\right) d\xi\, dy$$

$$= \frac{t}{\sqrt{2}\sqrt{1-it}} \left(\int_{1/t}^{4-\delta}\int_2^3 + \int_{4-\delta}^{4+\delta}\int_2^{2+\delta} + \int_{4-\delta}^{4+\delta}\int_{2+\delta}^3 + \int_{4+\delta}^{6-\delta}\int_2^{y/2-\delta/4} + \int_{4+\delta}^{6-\delta}\int_{y/2-\delta/4}^{y/2+\delta/4} \right.$$

$$\left. + \int_{4+\delta}^{6-\delta}\int_{y/2+\delta/4}^3 + \int_{6-\delta}^{6+\delta}\int_2^{3-\delta} + \int_{6-\delta}^{6+\delta}\int_{3-\delta}^3 + \int_{6+\delta}^\infty\int_2^3 \right). \tag{4}$$

We shall show that each integral in (4) tends to zero faster than $t^{-\frac12}$ as t tends to infinity. With $\delta = t^{-\frac13}$ it is seen that the 2nd and 8th integrals have absolute values which are less than $2\cdot\delta^2 = 2\cdot t^{-\frac23}$, since the integrands have absolute values less than 1.

For the remaining integrals, except the 5th, we shall use the following lemma to estimate the integrals over ξ.

Lemma 2.[1] *If* $|f'(u)| \geqslant \lambda > 0$ *and* $f'(u)$ *is monotonic in the interval* (a, b) *then the following inequality holds*

$$\left| \int_a^b e^{if(u)}\, du \right| \leqslant \frac{4}{\lambda}.$$

Set

$$f(\xi) = -t\xi^2 + \int_1^{yt} \sqrt{\xi^2 - \frac{1}{s}}\, ds,$$

then we have the following lemma.

Lemma 3. $|f'(\xi)| > t^{\frac{2}{3}}/4$ *and* $f''(\xi) < 0$ *hold for* t *sufficiently large,*

$$y > \frac{1}{t} \quad \text{and} \quad |y - 2\xi| \geqslant \frac{t^{-\frac{1}{3}}}{2}.$$

Proof.

$$f'(\xi) = -2t\xi + \int_1^{yt} \left(1 - \frac{1}{\xi^2 s}\right)^{-\frac{1}{2}} ds$$

$$= -2t\xi + yt - 1 + \frac{1}{2\xi^2} \int_1^{yt} \frac{ds}{s} + \frac{3}{4\xi^4} \int_1^{yt} \frac{1}{s^2}\left(1 - \frac{\theta}{s\xi^2}\right)^{-5/2} ds$$

$$= t(y - 2\xi) + O(\log yt), \ (0 < \theta < 1)$$

$$|f'(\xi)| \geqslant t \cdot \frac{t^{-\frac{1}{3}}}{2} - |O(\log yt)| \geqslant \frac{t^{\frac{2}{3}}}{4}, \text{ for } t \text{ large enough,}$$

$$f''(\xi) = -2t - \int_1^{yt} \frac{1}{\xi^2 s}\left(1 - \frac{1}{\xi^2 s}\right)^{-\frac{3}{2}} ds < 0.$$

For all integrals in (4) we have $y > 1/t$, and for all except the 2nd, 5th and 8th the inequality $|y - 2\xi| \geqslant \frac{1}{2} t^{-\frac{1}{3}}$ holds. By lemma 2 we thus have for these integrals

$$\left| \int\int \right| \leqslant \int e^{-\frac{1}{4}y^2} dy \left| \int e^{if(\xi)} d\xi \right| dx \leqslant \frac{8}{t^{\frac{2}{3}}} \int e^{-\frac{1}{4}y^2} dy,$$

i.e.

$$\int\int = O(t^{-\frac{2}{3}}).$$

It remains to show that the 5th integral

$$I_5 = \int_{4+\delta}^{6-\delta} \int_{y/2-\delta/4}^{y/2+\delta/4}$$

tends to zero faster than $t^{-\frac{1}{2}}$ as t tends to infinity.

$$S = \int_{y/2-\delta/4}^{y/2+\delta/4} \exp\left(-it\xi^2 + i\int_1^{yt} \sqrt{\xi^2 - \frac{1}{s}}\, ds\right) d\xi$$

$$= \int_{J_1} \exp\left(-it\xi^2 + i\int_1^{yt} \left(\xi - \frac{1}{2\xi s} + R_1(\xi, s)\right) ds\right) d\xi$$

$$= e^{\frac{1}{4}ity^2} \int_{J_1} \exp\left(-it\left(\xi - \frac{y}{2}\right)^2 - \frac{i}{2\xi} \log yt - i\phi\right) d\xi,$$

[1] Van der Corput.

where
$$\phi(\xi) = -\int_1^{yt}\left\{\sqrt{\xi^2-\frac{1}{s}}-\xi+\frac{1}{2\xi s}\right\}ds+\xi.$$

$$S = e^{\frac{1}{4}iy^2}\int_{J_1}\exp\left(-it\left(\xi-\frac{y}{2}\right)^2-i\phi\right)\left(\exp\left(-\frac{i}{2\xi}\log yt\right)-\exp\left(-\frac{i}{2y}\log yt\right)\right)d\xi$$
$$+\,e^{\frac{1}{4}ity^2}\exp\left(\left(\frac{-i}{y}\right)\log yt\right)\int_{J_1}\exp\left(-it\left(\xi-\frac{y}{2}\right)^2-i\phi\right)d\xi = s_1+s_2.$$

$$|s_1| \leqslant \int_{J_1}\left|\exp\left(-\frac{i}{2\xi}\log yt\right)-\exp\left(-\frac{i}{y}\log yt\right)\right|d\xi \leqslant \left(\frac{\delta}{2}\right)^2\log yt = O\left(\frac{\log t}{t^{\frac{2}{3}}}\right),$$
$$t\to\infty \quad \text{for} \quad 4+\delta\leqslant y\leqslant 6-\delta.$$

Thus we have
$$I_5 = o(t^{-\frac{1}{2}})+\int_{4+\delta}^{6-\delta}\exp\left(\frac{-y^2}{4}-\frac{i}{y}\log yt\right)\int_{J_1}\exp\left(-it\left(\xi-\frac{y}{2}\right)^2-i\phi\right)d\xi\,dy. \tag{5}$$

Lemma 4.
$$s_3 = \sqrt{t}\int_{J_1}\exp\left(-it\left(\xi-\frac{y}{2}\right)^2-i\phi\right)d\xi$$

converges uniformly to
$$g(y) = (1-i)\sqrt{\frac{\pi}{2}}\exp\left(-i\psi\left(\frac{y}{2}\right)\right),\ \psi(\xi) = -\int_1^{\infty}\left[\sqrt{\xi^2-\frac{1}{s}}-\xi+\frac{1}{2\xi s}\right]ds+\xi$$

when $t\to\infty$ *for* $4\leqslant y\leqslant 6$.

Proof. We can replace ϕ by ψ in the integral since the error thereby introduced is less than
$$\sqrt{t}\max_{4\leqslant y\leqslant 6}\int_{J_1}|e^{-i\phi}-e^{-i\psi}|\,d\xi \leqslant \sqrt{t}\,\tfrac{1}{2}t^{-\frac{1}{6}}\max_{4\leqslant y\leqslant 6}\int_{yt}^{\infty}\left[\sqrt{\xi^2-\frac{1}{s}}-\xi+\frac{1}{2\xi s}\right]ds = O(t^{-\frac{2}{3}})$$

Set $\sqrt{t}(\xi-\frac{1}{2}y)=\eta$ and let J_2 be the interval $|\eta|\leqslant\frac{1}{4}t^{\frac{1}{6}}$.
$$s_3 = \int_{J_2}e^{-i\eta^2}\exp\left(-i\psi\left(\frac{y}{2}+\frac{\eta}{t}\right)\right)d\eta = \exp\left(-i\psi\left(\frac{y}{2}\right)\right)\int_{J_2}e^{-i\eta^2}d\eta$$
$$+\int_{J_2}e^{-i\eta^2}\left(\exp\left(-i\psi\left(\frac{y}{2}+\frac{\eta}{\sqrt{t}}\right)\right)-\exp\left(-i\psi\left(\frac{y}{2}\right)\right)\right)d\eta.$$

The fist integral on the right side converges to $(1-i)\sqrt{\frac{1}{2}\pi}\exp(-i\psi(\frac{1}{2}y))$ when $t\to\infty$. The second integral tends to zero uniformly as $t\to\infty$ for $4\leqslant y\leqslant 6$ since it is in absolute value less than

$$\frac{t^{\frac{1}{6}}}{2} \max_{\substack{4 \leqslant y \leqslant 6 \\ |\eta| \leqslant t^{\frac{1}{6}}/4}} \left| \psi\left(\frac{y}{2} + \frac{\eta}{\sqrt{t}}\right) - \psi\left(\frac{y}{2}\right) \right| \leqslant \frac{t^{\frac{1}{2}}}{2} \cdot \frac{c \cdot t^{\frac{1}{6}}}{t^{\frac{1}{2}}}$$

$|\psi(\frac{1}{2}y + \eta/\sqrt{t}) - \psi(\frac{1}{2}y)| \leqslant c|\eta/\sqrt{t}|$ follows from $|\psi'| \leqslant c$. This proves the lemma.

The integral

$$\frac{1}{\sqrt{t}} \int_{4+\delta}^{4-\delta} \exp\left(-\frac{y^2}{4} - \frac{i}{y} \log yt\right) s_3 \, dy$$

in (5) is equal to

$$\frac{1}{\sqrt{t}} \int_{4+\delta}^{6-\delta} (s_3(y,t) - g(y)) \exp\left(-\frac{y^2}{4} - \frac{i}{y} \log yt\right) dy + \frac{1}{\sqrt{t}} \int_{4+\delta}^{6-\delta} h(y) \exp\left(\frac{-i}{y} \log t\right) dy, \qquad (6)$$

where

$$h(y) = \exp\left(\frac{-y^2}{4} - \frac{i}{y} \log y\right) g(y). \qquad (7)$$

The first integral in (6) is $o(t^{-\frac{1}{2}})$. This also holds for the second, since

$$\int_{4+\delta}^{6-\delta} h(y) \exp\left(\frac{-i}{y} \log t\right) dy = \int_{(6-\delta)^{-1}}^{(4+\delta)^{-1}} \frac{h(1/x)}{x^2} \exp(-ix \log t)$$

tends to zero, when $t \to \infty$, according to the Riemann-Lebesgue lemma, because, by (7), the integral

$$\int_{(6-\delta)^{-1}}^{(4+\delta)^{-1}} \left| \frac{h(1/x)}{x^2} \right| dx$$

exists. Thus $I_5 = o(t^{-\frac{1}{2}})$, $t \to \infty$ and $I_3^{(1)} \to 0$, $t \to \infty$. Analogously it is shown that $I_1^{(1)} \to 0$, $t \to \infty$. Finally, lemma 1 and lemma 2 imply that

$$u(x) = \int_2^3 K(x, \xi) \, d\xi \in L^2(-\infty, \infty),$$

which proves theorem 2.

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