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# High submodules and purity

By BO T. STENSTRÖM

## 1. Introduction

The $N$-high subgroups of an abelian group $G$ were defined by Irwin [5] as maximal subgroups having zero intersection with the given subgroup $N$ of $G$. In this note we extend some well-known relations between neat and $N$-high subgroups ([2], § 28 and [4], p. 327) to abelian categories and in particular to modules over general rings. As an application we will generalize a characterization of intersections of neat subgroups, due to Rangaswamy [7]. The term "high" will here be used in a sense more general than that it has in [5].

*Notation.* $\mathcal{A}$ is an abelian category in which every object $M$ has an injective envelope $E(M)$. For any subobject $L$ of $M$ we consider $E(L)$ as a well-defined subobject of $E(M)$.

## 2. High subobjects

Let $M$ be an object in $\mathcal{A}$ with a given subobject $K$. A subobject $L$ of $M$ is called *$K$-high* if $L \cap K = 0$ and $L$ is maximal with respect to this. $K$-high subobjects do exist for any $K$ ([3], p. 360). We obviously have

**Proposition 1.** *A subobject $L$ of $M$ is $K$-high if and only if the composed morphism $K \to M \to M/L$ is an essential monomorphism.*

**Corollary.** *If $L$ is $K$-high in $M$, then*

(i) *$L+K$ is essential in $M$.*
(ii) $E(M) = E(L) \oplus E(K)$.

The $K$-high subobjects of $M$ may be described in terms of injective envelopes, as was done in [5] and [6] for abelian torsion groups.

**Proposition 2.** *The $K$-high subobjects of $M$ are just the intersections of $M$ with complementary summands of $E(K)$ in $E(M)$.*

*Proof.* If $L$ is $K$-high, then $E(M) = E(L) \oplus E(K)$ by the corollary, and $L = E(L) \cap M$ since also $E(L) \cap M \cap K = 0$. Conversely, suppose $E(M) = E(K) \oplus H$. Then $H \cap M \cap K = 0$, and if $L$ is $K$-high in $M$ with $L \supset H \cap M$, then $E(L) \supset E(H \cap M) = H$. Clearly it follows that $E(L) = H$, and $H \cap M = E(L) \cap M = L$ is $K$-high.

**Proposition 3.** *Let $L$ be a subobject of $M$. The following statements are equivalent:*

(a) *$L$ is $K$-high for some $K$ in $M$.*
(b) *$L$ is the intersection of $M$ with a direct summand of $E(M)$.*
(c) *$L$ is essentially closed in $M$, i.e. $L=E(L)\cap M$.*
(d) *For every essential subobject $H$ of $M$ such that $L\subset H$, also $H/L\to M/L$ is essential.*

*Proof.* The proof of (a)$\Leftrightarrow$(b)$\Leftrightarrow$(c) is similar to the proof of prop. 2. (a)$\Rightarrow$(d): Let $H$ be an essential subobject of $M$ with $L\subset H$. Suppose $F$ is any subobject of $M$ with $F\supset L$ and $F/L\cap H/L=0$. This means that $F\cap H=L$, and we should show that $F=L$. If $L$ is $K$-high in $M$, then $F\cap K\cap H=L\cap K=0$ and hence $F\cap K=0$. By the maximality of $L$ it follows that $F=L$.

(d)$\Rightarrow$(a): Choose any $L$-high subobject $K$ of $M$. $K+L$ is an essential subobject of $M$ (cor. of prop. 1), so $K+L/L\to M/L$ is essential by hypothesis, and hence $L$ is $K$-high (prop. 1).

## 3. High sequences

A short exact sequence $0\to L\to M\to N\to 0$ is called *high* if it makes $L$ a $K$-high subobject of $M$ for some $K$.

**Proposition 4.** *The high sequences form a proper class.*

*Proof.* We have to verify the axioms for a proper class as given e.g. in [8]. Suppose $\alpha: L\to M$ and $\beta: M\to N$ are monomorphisms. We will make repeated use of prop. 3 (c) when verifying axioms P2 and P3, and of prop. 3 (d) when verifying P2* and P3*.
P2. If $\alpha$ and $\beta$ are high, then $\beta\alpha$ is high. For $L=E(L)\cap M=E(L)\cap E(M)\cap N=E(L)\cap N$.

P2*. If $\beta\alpha$ and $M/L\to N/L$ are high, then $\beta$ is high. For if $H$ is essential in $N$ and $M\subset H$, then $H/L\to N/L$ is essential and hence $H/M\simeq H/L/M/L\to N/L/M/L\simeq N/M$ is essential.

P3. If $\beta\alpha$ is high, then also $\alpha$ is. For if $L=E(L)\cap N$ then $L=E(L)\cap M$.

P3*. If $\beta$ is high, then also $M/L\to N/L$ is. For if $H/L$ is essential in $N/L$ and $H\supset M$, then $H$ is essential in $M$ and hence $H/L/M/L\simeq H/M\to N/M\simeq N/L/M/L$ is essential.

An object $M$ is called *simple* if it has no subobjects except 0 and $M$. Following the terminology of [8] (sec. 9.6), we call a short exact sequence *neat* if every simple object is a relative projective for it.

**Proposition 5.** *Every simple object is high-projective.*

*Proof.* Suppose $0\to L\to M\to P\to 0$ is a high sequence with $P$ simple. If $L$ is $K$-high in $M$, then $K\simeq P$ and $K+L=M$, so the sequence splits.

**Corollary.** *Every high sequence is neat.*

As a result in the reverse direction we have

**Proposition 6.** *Let $O$ be a class of objects such that for every $M\neq 0$ there exists a monomorphism $P\to M$ for some $P\neq 0$ in $O$. Then $\pi^{-1}(O)\subset\{$high sequences$\}$.*

*Proof.* Let $0 \to L \to M \to N \to 0$ be in $\pi^{-1}(O)$. Then also $0 \to L \to E(L) \cap M \to E(L) \cap M/L \to 0$ belongs to $\pi^{-1}(O)$ by axiom P3 for proper classes. We want to use prop. 3 and therefore assume $L \neq E(L) \cap M$. But then there exists a monomorphism $\varphi: P \to E(L) \cap M/L$ with $P \neq 0$ in $O$. We may lift $\varphi$ to a monomorphism $\bar{\varphi}: P \to E(L) \cap M$. But then $\operatorname{Im}\bar{\varphi} \cap L = 0$, which is impossible.

*Remark.* The notion of high subobjects may, of course, be dualized. We call a subobject $L$ of $M$ *K-low* if $K + L = M$ and $L$ is minimal with respect to this. $K$-low subobjects do not necessarily exist for all $K$, unless $\mathcal{A}$ has projective envelopes. But in any case one may verify that the class of low sequences is a proper class, and that every simple object is relatively injective for it.

## 4. High submodules

We now let $\mathcal{A}$ be the category of left modules over a ring $A$. Choosing $O$ in prop. 6 to be the class of cyclic modules, we obtain

**Proposition 7.** *Every pure sequence is high.*

The term "pure" is here used in the sense of [8] (sec. 9.3). When $A$ is commutative, we may take $O$ to be $O_p = \{A/I \mid I \text{ prime ideal or } (0)\}$ and apply [1] (§ 1), which gives

**Proposition 8.** *Let $A$ be a noetherian, commutative ring. Then every $O_p$-pure sequence is high.*

**Corollary.** *Let $A$ be a noetherian, commutative ring where every prime ideal $\neq 0$ is maximal. Then a short exact sequence is neat if and only if it is high.*

## 5. Intersections of neat submodules

In this section $A$ is assumed to be a noetherian, commutative ring with every prime ideal $\neq 0$ maximal. So $A$ is either a noetherian integral domain of Krull dimension 1 or an artinian ring (assume not a field). As was found above, the concepts of neat and high sequences coincide over $A$, and this fact makes it possible to extend a result on intersections of neat subgroups of an abelian group, proved by Rangaswamy [7], to modules over $A$.

Let $M$ be an $A$-module. Denote by Soc $M$ its socle and by $\operatorname{Soc}_{\mathfrak{m}} M$ the homogenous component of Soc $M$ determined by the maximal ideal $\mathfrak{m}$. If $L$ is any submodule of $M$, $E(L) \cap M$ is a minimal neat submodule of $M$ containing $L$. We call it a *neat hull* of $L$ in $M$. Every neat submodule of $M$ containing $L$ also contains a neat hull of $L$ in $M$. If $N$ is a neat hull of $L$ in $M$, then $L$ contains the socle of $N$.

**Proposition 9.** *Let $L$ be a submodule of $M$ and put $O' = \{A/\mathfrak{m} \mid \mathfrak{m}$ maximal ideal with $\operatorname{Soc}_{\mathfrak{m}} M \subset L\}$. The following statements are equivalent:*

(a) *$L$ is an intersection of neat submodules of $M$.*
(b) *$L$ is the intersection of its neat hulls in $M$.*
(c) *$M/L$ has no simple submodules of type in $O'$.*
(d) *$L$ is $O$-pure in $M$ (where $O = O' \cup (0)$).*

*Proof.* The equivalence of (a) and (b) is obvious. The condition (c) means that $M/L$ is $\mathcal{O}$-torsion-free, so the equivalence of (c) and (d) follows from [8] (prop. 6.2).

(a) $\Rightarrow$ (c): Suppose $L = \cap N_i$ with $N_i$ neat in $M$. Then $M/L = M/\cap N_i \subset \Pi M/N_i$, which is $\mathcal{O}$-torsion-free since every $M/N_i$ is $\mathcal{O}$-torsion-free ([8], prop. 6.2).

(c) $\Rightarrow$ (a): For each $x \notin L$ we have to find a high submodule of $M$ containing $L$ but not $x$. If $Ax \cap L = 0$, this is trivial to do. So suppose $Ax \cap L \neq 0$ and put

$$I = \{a \in A \mid ax \in L\}.$$

$I$ is a product of primary ideals, and each primary ideal contains a power of its radical. Therefore we have $\mathfrak{m}_1 \cdot \ldots \cdot \mathfrak{m}_n x \subset L$, where $\mathfrak{m}_i$ are maximal ideals and $\mathfrak{m}_2 \cdot \ldots \cdot \mathfrak{m}_n x \not\subset L$. Choose a $y \in \mathfrak{m}_2 \cdot \ldots \cdot \mathfrak{m}_n x$ such that $y \notin L$; then $\mathfrak{m}_1 y \subset L$. It will be sufficient to look for a high submodule containing $L$ but not $y$. The hypothesis (c) implies that $A/\mathfrak{m}_1 \notin \mathcal{O}'$, so there is a $z \notin L$ with $\mathfrak{m}_1 z = 0$. Put $u = y + z$ and let $N$ be a neat hull of $L + Au$ in $M$. If $y \notin N$, then we are done. If $y \in N$, then $z \in \operatorname{Soc} N$ and hence $z \in L + Au$ by the remark just preceding the proposition. Write $z = v + au$ with $v \in L$ and note that $a \notin \mathfrak{m}_1$ since otherwise $z = v + ay \in L$. We get $(1-a)z = v + ay$, which gives $\mathfrak{m}_1(v + ay) = 0$. Since $v + ay \notin L$ and $\mathfrak{m}_1$ is maximal, we conclude that $L \cap A(v + ay) = 0$. Any $A(v + ay)$ – high submodule $N'$ containing $L$ will now do, because $y \notin N'$.

**Corollary.** *Let $A$ be a noetherian integral domain of dimension 1. Suppose $L$ is a torsion-free submodule of $M$. $L$ is then an intersection of neat submodules of $M$ if and only if $Ass(M/L) = Ass(M)$.*

*Proof.* $\operatorname{Ass}(M)$ denotes the set of prime ideals which are annihilators of elements in $M$. When $L$ is torsion-free, $\mathcal{O}' = \{A/\mathfrak{m} \mid \mathfrak{m} \notin \operatorname{Ass}(M)\}$ and the result follows.

*Dept. of Mathematics, Royal Institute of Technology, Stockholm 70, Sweden*

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