ARKIV FÖR MATEMATIK Band 7 nr 22

1.67 05 2 Communicated 8 March 1967 by H. Cramér

Application of the Hellinger integrals to q-variate stationary stochastic processes

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Introduction

Let $(\mathbf{x}_k)_{-\infty}^{\infty}$ be a q-variate discrete parameter weakly stationary stochastic process (**SP**) with the spectral distribution measure $\mathbf{F}$ defined on $\mathcal{B}$ the Borel family of subsets of $(-\pi, \pi]$. It is known (8, Thm. 2] that for matrix-valued measures $\mathbf{M}$ and $\mathbf{N}$ the Hellinger integral $(\mathbf{M}, \mathbf{N})_{\mathbf{F}} = \int_{-\pi}^{\pi} (d\mathbf{M}\, d\mathbf{N}^*)/d\mathbf{F}$ (* = conjugate) may be defined in such a way that $\mathbf{H}_{2,\mathbf{F}}$ the space of all matrix-valued measures $\mathbf{M}$ for which $(\mathbf{M}, \mathbf{M})_{\mathbf{F}} = \int_{-\pi}^{\pi} (d\mathbf{M}\, d\mathbf{M}^*)/d\mathbf{F}$ exist becomes a Hilbert space under the inner product $\tau(\mathbf{M}, \mathbf{N})_{\mathbf{F}}$ (τ = trace). The significance of these integrals when $\mathbf{M}$ and $\mathbf{N}$ are complex-valued measures and $\mathbf{F}$ is a non-negative real-valued measure has been pointed out by H. Cramér [1, p. 487] and U. Grenander [2, p. 207] in relation to univariate **SP**'s. In this paper we will indicate the importance of our Hellinger integrals with regard to q-variate **SP**'s. In particular, we will obtain a natural extension of a certain result due to A. N. Kolmogorov [3, Thm. 24] which under a certain assumption was generalized by P. Masani [4, pp. 145–150].

Let K be any bounded subset of integers. K' will denote the complement of K in the set of integers. $\mathcal{M}_K$ and $\mathcal{M}_{K'}$ will denote the subspaces spanned by $\mathbf{x}_k$, $k \in K$ and $\mathbf{x}_k$, $k \in K'$ respectively, i.e., $\mathcal{M}_K = \mathfrak{S}\{\mathbf{x}_k, k \in K\}$ and $\mathcal{M}_{K'} = \mathfrak{S}\{\mathbf{x}_k, k \in K'\}$. $\mathcal{M}_\infty$ will denote $\mathfrak{S}\{\mathbf{x}_k, k \text{ an integer}\}$ and finally $\mathcal{N}_K$ will denote $\mathcal{M}_\infty \cap \mathcal{M}_{K'}^{\perp}$, where $\mathcal{M}_{K'}^{\perp}$ denotes the orthogonal complement of $\mathcal{M}_K$ in a fixed Hilbert space $\mathcal{H}^q$ containing the **SP** $(\mathbf{x}_k)_{-\infty}^{\infty}$.

Definition 1. *We say that* (a) *K is interpolable with respect to (w.r.t.) $(\mathbf{x}_k)_{-\infty}^{\infty}$ if $\mathcal{N}_K = \{\mathbf{0}\}$.*

(b) *$(\mathbf{x}_k)_{-\infty}^{\infty}$ is interpolable if each bounded subset K of integers is interpolable w.r.t. $(\mathbf{x}_k)_{-\infty}^{\infty}$.*

(c) *$(\mathbf{x}_k)_{-\infty}^{\infty}$ is minimal if for each k, $\{k\}$ is not interpolate w.r.t. $(\mathbf{x}_k)_{-\infty}^{\infty}$.*

It is easy to see that for any $\mathbf{x} \in \mathcal{N}_K$, $(\mathbf{x}, \mathbf{x}_k) = \mathbf{0}$ for all $k \in K'$. Thus the following definition makes sense.

Definition 2. (a) *For each $\mathbf{x} \in \mathcal{N}_{\mathbf{K}}$, we let*

$$\mathbf{P}_{\mathbf{x}}(e^{i\theta}) = \sum (\mathbf{x}, \mathbf{x}_k)\, e^{-ik\theta}.$$

(b) *We define the operator* $\mathbf{T}$ *on* $\mathcal{N}_K$ *into* $\mathbf{H}_{2,\mathbf{F}}$ *as follows: for each* $\mathbf{x} \in \mathcal{N}_K$

$$\mathbf{Tx} = \frac{1}{\sqrt{2\pi}} \mathbf{M}_{\mathbf{P}_\mathbf{x}},$$

where for any trig-polynomial P *with matrix coefficients the measure* $\mathbf{M}_\mathbf{P}$ *on* $\mathcal{B}$ *is given by* $\mathbf{M}_\mathbf{P}(B) = \int_B \mathbf{P}(e^{i\theta})\, d\theta$.

The important properties of $\mathbf{T}$ are given in the following theorem.

Theorem 1. (a) *Let* $\mathbf{x} \in \mathcal{N}_K$ *and* $\mathbf{\Psi}$ *be in* $\mathbf{L}_{2,\mathbf{F}}$ *such that* $\mathbf{V\Psi} = \mathbf{x}$, *where* $\mathbf{V}$ *is the isomorphism on* $\mathbf{L}_{2,\mathbf{F}}$ *onto* $\mathcal{M}_\infty$ *[7, p. 297]. Then for each* $B \in \mathcal{B}$, $\mathbf{M}_{\mathbf{P}_\mathbf{x}}(B) = \int_B \mathbf{\Psi}\, d\mathbf{F}$.

(b) $\mathbf{T}$ *is an isometry on* $\mathcal{N}_K$ *into* $\mathbf{H}_{2,\mathbf{F}}$. *In fact for all* $\mathbf{x}$ *and* $\mathbf{y}$ *in* $\mathcal{N}_K$

$$(\mathbf{x}, \mathbf{y}) = (\mathbf{Tx}, \mathbf{Ty})_\mathbf{F}.$$

(c) *The range of* $\mathbf{T}$ *is a closed subspace of the Hilbert space* $\mathbf{H}_{2,\mathbf{F}}$.

Proof. (a) Let $\mathbf{\Psi} \in \mathbf{L}_{2,\mathbf{F}}$ and $\mathbf{x} = \mathbf{V\Psi}$. Then by [7, p. 297]

$$(\mathbf{x}, \mathbf{x}_k) = (\mathbf{\Psi}, e^{-ik\theta})_\mathbf{F} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbf{\Psi}\, d\mathbf{F}\, e^{ik\theta}\, d\theta. \tag{1}$$

Also by the definition of $\mathbf{M}_{\mathbf{P}_\mathbf{x}}$,

$$\begin{aligned}\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik\theta}\, d\mathbf{M}_{\mathbf{P}_\mathbf{x}}(e^{i\theta}) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbf{P}_\mathbf{x}(e^{i\theta})\, e^{ik\theta}\, d\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \{\sum_{j \in \mathbf{K}} (\mathbf{x}, \mathbf{x}_j)\, e^{-ij\theta}\}\, e^{ik\theta}\, d\theta \\ &= \sum_{j \in K} \frac{1}{2\pi} \int_{-\pi}^{\pi} (\mathbf{x}, \mathbf{x}_j)\, e^{i(k-j)\theta}\, d\theta = (\mathbf{X}, \mathbf{X}_k).\end{aligned} \tag{2}$$

By (1) and (2), the measures $\int_B \mathbf{\Psi}\, d\mathbf{F}$ and $\int_B \mathbf{P}_\mathbf{x}(e^{i\theta})\, d\theta$ have the same Fourier-coefficients and hence for each $B \in \mathcal{B}$,

$$\mathbf{M}_{\mathbf{P}_\mathbf{x}}(B) = \int_B \mathbf{\Psi}\, d\mathbf{F}.$$

(b) Let $\mathbf{x}$ and $\mathbf{y}$ be in $\mathcal{N}_K$, and let $\mathbf{\Phi}$ and $\mathbf{\Psi}$ be in $\mathbf{L}_{2,\mathbf{F}}$ such that $\mathbf{V\Phi} = \mathbf{x}$ and $\mathbf{V\Psi} = \mathbf{y}$. Then by [8, Thm. 1]

$$2\pi(\mathbf{Tx}, \mathbf{Ty})_\mathbf{F} = (\mathbf{\Phi}, \mathbf{\Psi})_\mathbf{F}. \tag{3}$$

Also by [7, p. 297]

$$2\pi(\mathbf{x}, \mathbf{y}) = (\mathbf{\Phi}, \mathbf{\Psi})_\mathbf{F}. \tag{4}$$

From (3) and (4) (b) follows. (Q.E.D.)

(c) Since $\mathcal{N}_K$ is a closed subspace and since by (b) $\mathbf{T}$ is an isometry on $\mathcal{N}_K$ into $\mathbf{H}_{2,\mathbf{F}}$, therefore the range of $\mathbf{T}$ is a closed subspace of $\mathbf{H}_{2,\mathbf{F}}$. (Q.E.D.)

In the following theorem a characterization is given for the interpolability of a **SP**.

Theorem 2. *$(\mathbf{x}_k)_{-\infty}^{\infty}$ is interpolable iff for any trig-polynomial $\mathbf{P}$ with matrix coeffieients for which $\mathbf{M_P}$ is not a null-point in $\mathbf{H}_{2,\mathbf{F}}$, $(\mathbf{M_P}, \mathbf{M_P})$ is not Hellinger integrable w.r.t. $\mathbf{F}$.*

Proof. ($\Leftarrow$) If K is any bounded subset of integers, it is a consequence of Theorem 1 (b) that $\mathcal{N}_K = \{\mathbf{0}\}$; hence by definition 1 (a), K is interpolable w.r.t. $(\mathbf{x}_k)^{\infty}_{\infty}$. Since K is arbitrary it follows by definition 1 (b) that $(\mathbf{x}_k)_{-\infty}^{\infty}$ is interpolable.

($\Rightarrow$) Suppose there exists a trig-polynomial $\mathbf{P}$ with matrix coefficients for which $\mathbf{M_P}$ is not a null point in $\mathbf{H}_{2,\mathbf{F}}$ and $(\mathbf{M_P}, \mathbf{M_P})$ is Hellinger integrable w.r.t. $\mathbf{F}$. Hence by [8, Thm. 1 (c)], $\boldsymbol{\Phi} = (d\mathbf{M_P}/d\mu)(d\mathbf{F}/d\mu) \in \mathbf{L}_{2,\mathbf{F}}$, where μ is any σ-finite non-negative real-valued measure w.r.t. which $\mathbf{M_P}$ and $\mathbf{F}$ are a.c. If $\mathbf{x} \in \mathcal{M}_\infty$ such that $\mathbf{V}\boldsymbol{\Phi} = \mathbf{x}$, where $\mathbf{V}$ is as in Theorem 1, then by [7, p. 297] and (8, Thm. 2 (b)]

$$
\begin{aligned}
(\mathbf{x}, \mathbf{x}_k) &= \frac{1}{2\pi}\int_{\pi}^{\pi} \boldsymbol{\Phi}\, d\mathbf{F}\, e^{ik\theta} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta}\boldsymbol{\Phi}\, d\mathbf{F} \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta}(d\mathbf{M_P}/d\mu)\,(d\mathbf{F}/d\mu)^{-} d\mathbf{F} \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta}(d\mathbf{M_P}/d\mu)\,(dF/d\mu)^{-}\,(d\mathbf{F}/d\mu)\, d\mu \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta}(d\mathbf{M_P}/d\mu)\, d\mu \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta} d\,\mathbf{M_P} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta}\mathbf{P}(e^{i\theta})\, d\theta.
\end{aligned}
$$

Let $\mathbf{P}(e^{i\theta}) = \sum_{j\in K}\mathbf{A}_{-j}e^{-ij\theta}$. Then

$$
\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta}\mathbf{P}(e^{i\theta})\, d\theta = \frac{1}{2\pi}\sum_{j\in K}\mathbf{A}_{-j}e^{i(k-j)\theta}d\theta = \begin{cases}\mathbf{A}_{-k}, & k \in K \\ 0 & k \notin K\end{cases}. \tag{2}
$$

By (1) and (2) we have that $(\mathbf{x}, \mathbf{x}_k) = \mathbf{0}$ if $k \notin K$, and hence $\mathbf{x} \in \mathcal{M}_{K'}^{\perp}$. But $\mathbf{x} \in \mathcal{M}_\infty$, therefore by definition of $\mathcal{N}_K$, $\mathbf{X} \in \mathcal{N}_K$. Now by Definition 2, (1) and (2),

$$
\mathbf{P_x} = \sum_{k\in K}(\mathbf{x}, \mathbf{x}_k)\, e^{-ik\theta} = \sum_{k\in K}\mathbf{A}_{-k}\, e^{-ik\theta} = \mathbf{P}.
$$

Hence $\mathbf{M_P} = \mathbf{M_{P_x}}$. It then follows by Theorem 1 (b) that $(\mathbf{x}, \mathbf{x}) = (\mathbf{Tx}, \mathbf{Tx})_\mathbf{F} = (\mathbf{M_P}, \mathbf{M_P})_\mathbf{F} \neq \mathbf{0}$. Hence $\mathcal{N}_K$ is not interpolable w.r.t. $(\mathbf{x}_k)_{-\infty}^{\infty}$. Consequently by Definition 1 (b), $(\mathbf{x}_k)_{-\infty}^{\infty}$ is not interpolable. (Q.E.D.)

The following theorem which is a consequence of Theorem 1 is a generalisation of results given by Masani [5, pp. 147 & 149].

Theorem 3. *Let* $\mathbf{z}_k$ *be the orthogonal projection of* $\mathbf{x}_k$ *onto the subspace* $\mathfrak{S}^{\perp}\{\mathbf{x}_n, n \neq k\}$, *and let* $\mathbf{y}_k = (\mathbf{z}_0, \mathbf{z}_0)^- \mathbf{z}_k$, *where* $(\mathbf{z}_0, \mathbf{z}_0)^-$ *is the generalized inverse of* $(\mathbf{z}_0, \mathbf{z}_0)$ *[6, p. 407]. Then*

(a) $$(\mathbf{z}_0, \mathbf{z}_0)^- = (\mathbf{y}_0, \mathbf{y}_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{d\mathbf{M_J}\, d\mathbf{M_J}}{d\mathbf{F}}, \quad (\mathbf{z}_0, \mathbf{z}_0) = \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{d\mathbf{M_J}\, d\mathbf{M_J}}{d\mathbf{F}}\right]^-,$$

where $\mathbf{J}$ *is the projection matrix on the space* C^q *of q-tuples of complex numbers onto the range of* $(\mathbf{z}_0, \mathbf{z}_0)$ *in the privileged basis of* C^q.

(b) $(\mathbf{x}_k)_{-\infty}^{\infty}$ *is minimal iff*

$$\int_{-\pi}^{\pi} \frac{d\mathbf{M_J}\, d\mathbf{M_J}}{d\mathbf{F}} \neq \mathbf{0}.$$

(c) $(\mathbf{y}_n)_{-\infty}^{\infty}$ *is a weakly stationary* **SP** *with the spectral distribution*

$$\frac{1}{2\pi}\int_{-\pi}^{\theta} \frac{d\mathbf{M_J}\, d\mathbf{M_J}}{d\mathbf{F}}.$$

(d) $(\mathbf{y}_k)_{-\infty}^{\infty}$ and $(\mathbf{x}_k)_{-\infty}^{\infty}$ *are biorthogonal, i.e.*,

$$(\mathbf{y}_m, \mathbf{x}_n) = \delta_{mn}\mathbf{J}.$$

Proof. (a) By Theorem 1, $(\mathbf{z}_0, \mathbf{z}_0) = (1/2\pi)(\mathbf{M}_{\mathbf{z}_0}, \mathbf{M}_{\mathbf{z}_0})_{\mathbf{F}}$, where for each $B \in \mathcal{B}$, $\mathbf{M}_{z_0}(B) = \int_B (\mathbf{z}_0, \mathbf{z}_0)\, d\theta$.

Hence

$$(\mathbf{z}_0, \mathbf{z}_0)^- = (\mathbf{z}_0, \mathbf{z}_0)^- (\mathbf{z}_0, \mathbf{z}_0)(\mathbf{z}_0, \mathbf{z}_0)^- = \frac{1}{2\pi}(\mathbf{z}_0, \mathbf{z}_0)^- (\mathbf{M}_{\mathbf{z}_0}, \mathbf{M}_{\mathbf{z}_0})_{\mathbf{F}} (\mathbf{z}_0, \mathbf{z}_0)^- = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{d\mathbf{M_J}\, d\mathbf{M_J}}{d\mathbf{F}}.$$

Consequently

$$(\mathbf{z}_0, \mathbf{z}_0)^- = (\mathbf{y}_0, \mathbf{y}_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{d\mathbf{M_J}\, d\mathbf{M_J}}{d\mathbf{F}} \quad \text{and} \quad (\mathbf{z}_0, \mathbf{z}_0) = \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{d\mathbf{M_J}\, d\mathbf{M_J}}{d\mathbf{F}}\right]^-.$$

(b) By (a), $$(\mathbf{z}_0, \mathbf{z}_0) = \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{d\mathbf{M_J}\, d\mathbf{M_J}}{d\mathbf{F}}\right]^-.$$

From this and Definition 1 (c), (b) follows.

(c) Obviously $(\mathbf{y}_k)_{-\infty}^{\infty}$ is weakly stationary. Hence by (a)

$$(\mathbf{y}_0, \mathbf{y}_0) = (\mathbf{z}_0, \mathbf{z}_0)^- = \frac{1}{2\pi}(\mathbf{M_J}, \mathbf{M_J})_{\mathbf{F}}.$$

It follows that the spectral distribution of

$$(\mathbf{y}_k)_{-\infty}^{\infty} \quad \text{is} \quad \frac{1}{2\pi}\int_{-\pi}^{\theta} \frac{d\mathbf{M_J}\, d\mathbf{M_J}}{d\mathbf{F}}.$$

(d) $$(\mathbf{y}_0, \mathbf{y}_0) = ((\mathbf{z}_0, \mathbf{z}_0)^- \mathbf{z}_0, \mathbf{x}_0) = (\mathbf{z}_0, \mathbf{z}_0)^- (\mathbf{Z}_0, \mathbf{X}_0) = (\mathbf{z}_0, \mathbf{z}_0)^- (\mathbf{z}_0, \mathbf{z}_0) = \mathbf{J}.$$

For $n \neq 0$, $\mathbf{z}_n \perp \mathfrak{S}(\mathbf{x}_k, k \neq n)$, therefore $(\mathbf{y}_n, \mathbf{x}_0) = \mathbf{0}$. Hence $(\mathbf{y}_m, \mathbf{x}_n) = \delta_{mn}\mathbf{J}$. (Q.E.D.)

Remark. Let $\int_{-\pi}^{\pi}(d\mathbf{M}_{\mathbf{I}}d\mathbf{M}_{\mathbf{I}})/d\mathbf{F}$ exists ($\mathbf{I}$ denotes the identity matrix of order q). Then by [8, Thm. 1 (c)], $\mathbf{\Phi}=(d\mathbf{M}_{\mathbf{I}}/d\mu)(d\mathbf{F}/d\mu)^{-}$ is in $\mathbf{L}_{2,\mathbf{F}}$, where μ is any σ-finite non-negative real-valued measure w.r.t. which $\mathbf{M}_{\mathbf{I}}$ and $\mathbf{F}$ are a.c. Let $\mathbf{x}\in\mathcal{M}_{\infty}$ be such that $\mathbf{x}=\mathbf{V}\mathbf{\Phi}$, where $\mathbf{V}$ is as in Theorem 1. Then by repeating the same argument used in the proof (1) in Theorem 2,

$$(\mathbf{x},\mathbf{x}_k)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathbf{I}\,e^{ik\theta}\,d\theta=\begin{cases}\mathbf{0} & \text{if} \quad k\neq 0\\ \mathbf{I} & \text{if} \quad k=0\end{cases}.$$

Therefore $\qquad \mathbf{x}\in\mathcal{N}_{\{0\}}$ and $(\mathbf{x},\mathbf{x})=\frac{1}{2\pi}(\mathbf{\Phi},\mathbf{\Phi})_{\mathbf{F}}=\frac{1}{2\pi}(\mathbf{M}_{\mathbf{I}},\mathbf{M}_{\mathbf{I}})_{\mathbf{F}}$.

Since $\mathbf{x}\in\mathcal{N}_{\{0\}}$, $\mathbf{x}=\mathbf{A}\mathbf{z}_0$. Consequently

$$\mathbf{A}(\mathbf{z}_0,\mathbf{z}_0)\,\mathbf{A}^*=\frac{1}{2\pi}(\mathbf{M}_{\mathbf{I}},\mathbf{M}_{\mathbf{I}})_{\mathbf{F}}).$$

Hence
$$\operatorname{rank}(\mathbf{M}_{\mathbf{I}},\mathbf{M}_{\mathbf{I}})_{\mathbf{F}}\leqslant\operatorname{rank}(\mathbf{z}_0,\mathbf{z}_0). \tag{1}$$

By Theorem 3 (a),

$$(\mathbf{z}_0,\mathbf{z}_0)=\frac{1}{2\pi}(\mathbf{z}_0,\mathbf{z}_0)(\mathbf{M}_{\mathbf{I}},\mathbf{M}_{\mathbf{I}})_{\mathbf{F}}(\mathbf{z}_0,\mathbf{z}_0).$$

Hence
$$\operatorname{rank}(\mathbf{z}_0,\mathbf{z}_0)\leqslant\operatorname{rank}(\mathbf{M}_{\mathbf{I}},\mathbf{M}_{\mathbf{I}})_{\mathbf{F}}. \tag{2}$$

By (1) and (2) we get $\operatorname{rank}(\mathbf{z}_0,\mathbf{z}_0)=\operatorname{rank}(\mathbf{M}_{\mathbf{I}},\mathbf{M}_{\mathbf{I}})_{\mathbf{F}}$. Consequently

$$\frac{1}{2\pi}(\mathbf{M}_{\mathbf{I}},\mathbf{M}_{\mathbf{I}})_{\mathbf{F}}=\frac{1}{2\pi}\mathbf{J}(\mathbf{M}_{\mathbf{I}},\mathbf{M}_{\mathbf{I}})_{\mathbf{F}}\,\mathbf{J}=\frac{1}{2\pi}(\mathbf{M}_{\mathbf{J}},\mathbf{M}_{\mathbf{J}})_{\mathbf{F}}. \tag{3}$$

The following result due to Masani [4, p. 149] is a consequence of this remark and Theorem 1.

Corollary. (a) $(\mathbf{x}_k)_{-\infty}^{\infty}$ *is minimal and* $\operatorname{rank}(\mathbf{z}_0,\mathbf{z}_0)=q$ *iff for almost all* θ, $F'(e^{i\theta})$ *has an inverse and* $\int_{-\pi}^{\pi}(\mathbf{F}')^{-1}(e^{i\theta})\,d\theta$ *exists.*

(b) *If* $(\mathbf{x}_k)_{-\infty}^{\infty}$ *is minimal and* $\operatorname{rank}(\mathbf{z}_0,\mathbf{z}_0)=q$, *then*

$$(\mathbf{z}_0,\mathbf{z}_0)=\left\{\frac{1}{2\pi}\int_0^{2\pi}(\mathbf{F}')^{-1}(e^{i\theta})\,d\theta\right\}^{-1}.$$

Proof. Let $\mathbf{F}_a$ and $\mathbf{F}_s$ be the absolutely continuous and singular components of $\mathbf{F}$ w.r.t. Lebesgue measure on $(-\pi,\pi]$ [5, p. 18]. Then

(I) $$\mathbf{M}_{\mathbf{I}}\in\mathbf{H}_{2,\mathbf{F}} \quad\text{iff}\quad \mathbf{M}_{\mathbf{I}}\in\mathbf{H}_{2,\mathbf{F}_a},$$
$$\mathbf{M}_{\mathbf{I}}\in\mathbf{H}_{2,\mathbf{F}}\Rightarrow(\mathbf{M}_{\mathbf{I}},\mathbf{M}_{\mathbf{I}})_{\mathbf{F}}=(\mathbf{M}_{\mathbf{I}},\mathbf{M}_{\mathbf{I}})_{\mathbf{F}_a}.$$

We proceed to prove (I). Let μ be a σ-finite non-negative real-valued measure w.r.t. which $\mathbf{F}$ and $\mathbf{M}_{\mathbf{I}}$ are a.c. Let $\mathbf{M}_{\mathbf{I}}\in\mathbf{H}_{2,\mathbf{F}_a}$. Then

$$\mathbf{F}=\mathbf{F}_a+\mathbf{F}_s\Rightarrow\mathbf{F}\geqslant\mathbf{F}_a\Rightarrow(d\mathbf{F}/d\mu)\geqslant(d\mathbf{F}_a/d\mu)\ \text{a.e.}\ \mu.$$

Hence $$(d\mathbf{F}/d\mu)^{-} \leqslant (d\mathbf{F}_a/d\mu)^{-} \tag{1}$$

Since $\mathbf{M_I} \in \mathbf{H}_{2,\mathbf{F}_a}$ by (1) it follows that $\mathbf{M_I} \in \mathbf{H}_{2,\mathbf{F}}$. Moreover

$$(\mathbf{M_I}, \mathbf{M_I})_{\mathbf{F}} \leqslant (\mathbf{M_I}, \mathbf{M_I})_{\mathbf{F}_a}. \tag{2}$$

Since $\mathbf{M_I} \in \mathbf{H}_{2,\mathbf{F}}$ then by [8, Thm. 1 (c)] there exists a $\mathbf{\Psi} \in \mathbf{L}_{2,\mathbf{F}}$ such that for each $B \in \mathcal{B}$

$$\mathbf{M_I}(B) = \int_B \mathbf{\Psi}\, d\mathbf{F} = \int_B \mathbf{\Psi}\, d\mathbf{F}_a + \int_B \mathbf{\Psi}\, d\mathbf{F}_s. \tag{3}$$

Since $\mathbf{M_I}(B) = L(B)\,\mathbf{I}$, $L(B) =$ Lebesgue measure of B, from (3) it follows that for each $B \in \mathcal{B}$, $\int_B \mathbf{\Psi}\, d\mathbf{F}_s = 0$. Hence

$$\mathbf{M_I}(B) = \int_B \mathbf{\Psi}\, d\mathbf{F} = \int_B \mathbf{\Psi}\, d\mathbf{F}_a. \tag{4}$$

By (4) and [8, Lemma 3] we get

$$(\mathbf{M_I}, \mathbf{M_I})_{\mathbf{F}} = (\mathbf{\Psi}, \mathbf{\Psi})_{\mathbf{F}}, \;\; (\mathbf{M_I}, \mathbf{M_I})_{\mathbf{F}_a} = (\mathbf{\Psi}, \mathbf{\Psi})_{\mathbf{F}_a}. \tag{5}$$

We note that since $\mathbf{F}_a \leqslant \mathbf{F}$,

$$(\mathbf{\Psi}, \mathbf{\Psi})_{\mathbf{F}_a} \leqslant (\mathbf{\Psi}, \mathbf{\Psi})_{\mathbf{F}}. \tag{6}$$

Therefore by (2), (5) and (6) we obtain that if $\mathbf{M_I} \in \mathbf{H}_{2,\mathbf{F}_a}$ then $\mathbf{M_I} \in \mathbf{H}_{2,\mathbf{F}}$ and $(\mathbf{M_I}, \mathbf{M_I})_{\mathbf{F}_a} = (\mathbf{M_I}, \mathbf{M_I})_{\mathbf{F}}$. Conversely if $\mathbf{M_I} \in \mathbf{H}_{2,\mathbf{F}}$, then repeating the argument following (2), we conclude that $\mathbf{M_I} \in \mathbf{H}_{2,\mathbf{F}_a}$, and $(\mathbf{M_I}, \mathbf{M_I})_{\mathbf{F}} = (\mathbf{M_I}, \mathbf{M_I})_{\mathbf{F}_a}$. Hence (I) is proved.

(a) ($\Rightarrow$) Since rank $(\mathbf{z}_0, \mathbf{z}_0) = q$, $\mathbf{J} = \mathbf{I}$. Hence by Theorem 3 (a) and (I),

$$(\mathbf{z}_0, \mathbf{z}_0)^{-1} = \frac{1}{2\pi}(\mathbf{M_I}, \mathbf{M_I})_{\mathbf{F}} = \frac{1}{2\pi}(\mathbf{M_I}, \mathbf{M_I})_{\mathbf{F}_a} = \frac{1}{2\pi}\int_{-\pi}^{\pi} (\mathbf{F}')^{-}(e^{i\theta})\, d\theta. \tag{7}$$

Since rank $(\mathbf{z}_0, \mathbf{z}_0) = q$, $(\mathbf{x}_k)_{-\infty}^{\infty}$ is of full-rank. Hence rank $\mathbf{F}' = q$ a.e., and $(\mathbf{F}')^{-1}$ exists a.e. [4, p. 147]. From (7) it follows that $\int_{-\pi}^{\pi} (\mathbf{F}')^{-1}(e^{i\theta})\, d\theta$ exists.

($\Leftarrow$) By (I), $$\frac{1}{2\pi}(\mathbf{M_I}, \mathbf{M_I})_{\mathbf{F}} = \frac{1}{2\pi}(\mathbf{M_I}, \mathbf{M_I})_{\mathbf{F}_a} = \frac{1}{2\pi}\int_{-\pi}^{\pi} (\mathbf{F}')^{-1}(e^{i\theta})\, d\theta.$$

Hence from Theorem 3 (c) and previous remark (3) it follows that the spectral density of the SP $(\mathbf{y}_k)_{-\infty}^{\infty}$ is $(\mathbf{F}')^{-1}(e^{i\theta})$. $(\mathbf{y}_k)_{-\infty}^{\infty}$ is of full-rank, because $\int_{-\pi}^{\pi} \log\det \mathbf{F}'^{-1}(e^{i\theta})\, d\theta$ exists [4, p. 148]. Therefore rank $(\mathbf{z}_0, \mathbf{z}_0) =$ rank $(\mathbf{y}_0, \mathbf{y}_0) = q$, and hence by Definition 1 (c) $(\mathbf{x}_k)_{-\infty}^{\infty}$ is minimal.

(b) This is a special case of Theorem 3 (a). (Q.E.D.)

ACKNOWLEDGEMENT

This research was partially supported by National Science Foundation GP-7535.

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Tryckt den 29 december 1967

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