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Overconvergence of sequences of rational functions with sparse poles

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In this paper we consider Banach spaces S of functions holomorphic in $D^+ = \{z \,|\, |z| < 1\}$ which contain all functions $\{1/(z-b)\}$ with $b \in D^- = \{z \,|\, |z| > 1\}$. We show (Theorem 1) that under rather mild restrictions on S, a sequence $\{f_n\}$ of rational functions which converges in the norm of S, if the poles of all f_n are confined to a "sparse" subset E of D^- (here the sparseness criterion is determined by the particular space S, and we give it only in an implicit form), necessarily converges uniformly on compact subsets of $D^- \setminus E$. This it seems appropriate to call an "overconvergence" theorem, although it has a somewhat different character than other theorems bearing this designation, e.g. those due to Ostrowski [3] and to Walsh ([5], p. 77). In general the sequence $\{f_n\}$ will not converge in domains which intersect the unit circumference, and the limit functions in D^+ and D^- are not analytic continuations of one another. In Theorem 2 it is shown, with additional hypotheses on S, that when the closure of E does not include the entire unit circumference we get overconvergence in domains which intersect the circumference, and hence analytic continuability of the limit functions. These theorems are proved in § 1.

The present paper was motivated by a study of the paper [1] of Akutowicz and Carleson, from which we have adapted the method of proof of Theorem 2. Conversely, many of the theorems of [1] are deducible from our Theorem 2. In § 2 the relationship of our paper with [1] is discussed briefly. In § 3 some problems which invite further investigation are pointed out.

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1. The main theorems

We consider a Banach space S of functions $f(z)$ holomorphic in D^+, and satisfying the following conditions:

(1) $1 \in S$, and for every $|b| > 1$, $1/(z-b)$ is an element of S. The set of all functions $1/(z-b)$ with $|b| > 1$ is total in S.

(2) For every $|a| < 1$, the evaluation functional I_a defined by $I_a f = f(a)$ is bounded.

(3) $\lim\limits_{b \to \infty} \left\| \dfrac{1}{z-b} \right\| = 0.$

(4) For every $|b|>1$, multiplication by $1/(z-b)$ is a bounded operator M_b from S to S.

(5) $\|M_b\|$ is bounded on compact subsets of D^-.

Observe that most Banach spaces of analytic functions in D^+ previously studied (in particular the H^p spaces) satisfy these properties, as well as the extra ones to be introduced below in the hypotheses of Theorem 2.

Theorem 1. *Let E be any subset of D^- such that the set U of functions $1/(z-b)$ with $b \in E$ is not total in S. Let R denote the set of finite linear combinations of elements of U. Then, if f_n is a norm convergent sequence of elements of R, the sequence $f_n(z)$ converges uniformly on compact subsets of $D^- \setminus \bar{E}$.*

Remark. It would be easy to extend this result to allow also $|b|=1$, if the corresponding $1/(z-b)$ belong to S.

Proof of Theorem 1. Let $f \in R$, $\|f\|=1$. Write $f(z)=\sum_{k=1}^n p_k/(z-b_k)$, $b_k \in E$. Let G be a compact subset of $D^- \setminus \bar{E}$. To prove the theorem, it is sufficient to show that the maximum of $|f(z)|$ on G is bounded by a constant not depending upon the particular $f \in R$ of norm 1. Indeed, this then implies that the maximum of $|f_m(z)-f_n(z)|$ on G is less than a constant times $\|f_m-f_n\|$, which clearly implies the desired result.

Now, since $\{1/(z-b)\}$, $b \in E$ are not total there is a non-null linear functional $L \in S^*$ such that

$$\lambda(w)=L\left(\frac{1}{z-w}\right) \quad (\lambda \text{ is the "Borel transform" of } L)$$

vanishes on K. By (1), $\lambda(w) \not\equiv 0$. Also, it is easily deduced from our hypotheses that λ is holomorphic for $|w|>1$ and vanishes at ∞. Now fix a point $c \in D^- \setminus \bar{E}$ such that $\lambda(c) \neq 0$ and define

$$Q_c=\frac{LM_c}{\lambda(c)};$$

clearly Q_c is a bounded linear functional on S, and

$$\|Q_c\| \leqslant \frac{\|L\| \cdot \|M_c\|}{|\lambda(c)|}. \tag{6}$$

Now, for $b \in E$ we have

$$\lambda(c)Q_c\left(\frac{1}{z-b}\right)=L\left(\frac{1}{(z-b)(z-c)}\right)=\frac{1}{c-b}L\left(\frac{1}{z-c}-\frac{1}{z-b}\right)=\frac{\lambda(c)}{c-b}.$$

Therefore, $Q_c(1/(z-b))=1/(c-b)$, hence

$$|f(c)|=\left|\sum_{k=1}^n \frac{p_k}{c-b_k}\right|=\left|\sum_{k=1}^n p_k Q_c\left(\frac{1}{z-b_k}\right)\right|=|Q_c f| \leqslant \|Q_c\|.$$

Now, we may surround G by a contour on which $\lambda(w)$ does not vanish, and which contains no points of E. By the last inequality, together with (6) and (5), we obtain a bound for f on the contour, and so by the maximum modulus theorem on G. This proves Theorem 1.

Theorem 2. *Suppose for some $p>1$ S satisfies, in addition to (1)–(5), the additional conditions*

$$\int_0^1\int_0^{2\pi}(\log^+\log^+\|I_a\|)^p\, r\,dr\,dt<\infty \quad (a=re^{it}), \tag{7}$$

$$\int_1^Q(\log^+\log^+N(x))^p\,dx<\infty\,(\text{for every } 1<Q<\infty), \tag{8}$$

where $N(x)$ satisfies $\|M_b\|<N(|b|)$ for $|b|>1$.

Then, under the hypotheses of Theorem 1, $\{f_n(z)\}$ converges uniformly on compact subsets of $Z\setminus\bar{E}(Z=\text{Riemann sphere})$.

Remark. Theorem 2 gives new information in the case that $\bar{E}$ does not include $|z|=1$, assuring analytic continuation of $\lim f_n$.

We require first a lemma.

Lemma. *Let $F(z)$ be holomorphic in D^+, and write $M(r)=\max_t|F(re^{it})|$. If for some $q<1$*

$$\int_q^1[\log^+\log^+M(r)]^p\,dr<\infty, \text{ where } p\geq 1, \tag{9}$$

then

$$\int_q^1\int_0^{2\pi}\left[\log^+\log^+\frac{1}{|F(re^{it})|}\right]^p r\,dr\,dt<\infty. \tag{10}$$

Proof. It is easy to reduce the general case to the case when $F(0)=1$, which we assume. Now

$$\log^+\frac{1}{|F(re^{it})|}\leq -\log|F(re^{it})|+\log^+M(r).$$

Integrating and recalling that $F(0)=1$, we get

$$\frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{1}{|F(re^{it})|}\,dt\leq\log^+M(r). \tag{11}$$

Let us first suppose that $p\leq 3/2$. Now, the function $(\log x)^p$ is concave for $x\geq e^{p-1}$ (and so for $|x|\geq 2$), hence using Jensen's convexity inequality

$$\frac{1}{2\pi}\int_0^{2\pi}\left[\log^+\log^+\frac{1}{|F(re^{it})|}\right]^p dt \leqslant \frac{1}{2\pi}\int_0^{2\pi}\left[\log\left(2+\log^+\frac{1}{|F(re^{it})|}\right)\right]^p dt$$

$$\leqslant \left[\log \frac{1}{2\pi}\int_0^{2\pi}\left(2+\log^+\frac{1}{|F(re^{it})|}\right)dt\right]^p, \text{ and this is, by (11),}$$

$$\leqslant [\log\,(2+\log^+ M(r))]^p \leqslant [\log 3 + \log^+\log^+ M(r)]^p$$

$$\leqslant 2^p[(\log 3)^p + (\log^+\log^+ M(r))^p]$$

and now, observing (9), (10) follows readily, The case $p>3/2$ involves a trivial modification of the argument, which we leave to the reader.

Remark. It is clear, by the change of variables $z=1/w$, that the lemma is true also under the hypothesis that F is holomorphic in D^-, where now the range of integration in (9) is from 1 to Q, and in (10) over the annulus $1<r<Q$. It is in this form that we shall actually employ the lemma.

Proof of Theorem 2. We shall base the proof on a theorem of Beurling (see [2]) according to which, if we have a family of functions holomorphic in a domain H and there exists a function $J(z)$ such that $J(z)\geqslant |f(z)|$ for all f in the family, where for some $p>1$

$$\int_H\int [\log^+\log^+ J(re^{it})]^p\, r\,dr\,dt < \infty, \tag{12}$$

then the functions of the family are uniformly bounded on compact subsets of H. Again, we consider the family R_0 of all $f\in R$ of norm one. Note first that, for $a\in D^+$,

$$|f(a)|\leqslant \|I_a\|. \tag{13}$$

Now, when $|c|>1$, as we have seen

$$|f(c)|\leqslant \frac{A_1 N(|c|)}{|\lambda(c)|}, \quad \text{where } A_1=\|L\|. \tag{14}$$

In what follows $A_2, A_3, \ldots$ denote positive constants. Define $J(a)$ to be $\|I_a\|$ for $|a|<1$ and $J(c)=$ right-hand side of (14) for $|c|>1$. Clearly $|f(z)|\leqslant J(z)$ for $f\in R_0$. Moreover, for $|c|>1$,

$$\log^+\log^+ J(c)\leqslant A_2+\log^+\log^+ N(|c|)+\log^+\log^+\frac{1}{|\lambda(c)|},$$

therefore

$$(\log^+\log^+ J(c))^p \leqslant A_3\left[1+(\log^+\log^+ N(|c|))^p+\left(\log^+\log^+\frac{1}{|\lambda(c)|}\right)^p\right]. \tag{15}$$

Let now H be any bounded open subset of $Z\backslash\bar{E}$. All $f\in R_0$ are holomorphic on H.

Moreover (7) guarantees that the contribution to the integral (12) coming from $H \cap D^+$ is finite. Likewise the contribution coming from $H \cap D^-$ is finite, as we see from (15), together with (8), providing we can show, for any $Q > 1$

$$\int_0^{2\pi}\int_1^Q \left(\log^+ \log^+ \frac{1}{|\lambda(c)|}\right)^p r\,dr\,dt < \infty \quad (c = r\mathrm{e}^{it}). \tag{16}$$

But,
$$|\lambda(c)| = \left|L\left(\frac{1}{z-c}\right)\right| \leqslant \|L\| \cdot \|M_c\| \leqslant A_1 N(|c|).$$

Again observing (8), and applying the lemma (note the remark following its proof) we see that (16) holds. Therefore, the functions in R_0 are uniformly bounded on compact subsets of $Z \backslash \bar{E}$, and this implies Theorem 2.

2. The Akutowicz–Carleson minimum problem

In [1] the following type of problem is studied: One has a Banach space F of functions holomorphic in a domain D, and two given complex sequences $\{z_n\}$, $z_n \in D$ and $\{a_n\}$ such that *the interpolation problem* $f(z_n) = a_n$ *is underdetermined*, i.e. there exist two distinct functions (and hence infinitely many) in F which satisfy $f(z_n) = a_n$, $n = 1, 2, \dots$ Then, under suitable hypotheses about F, which are moreover sufficient to guarantee the existence of a unique solution of the interpolation problem *which has minimal norm* (for instance, uniform convexity of F guarantees this), it is shown in [1] that this minimal solution is analytically continuable across each boundary point of D which is not a limit point of $\{z_n\}$. Moreover, the analytic behaviour of the function thus continued on the whole Riemann sphere is obtained.

From Theorem 2 one may deduce many of the theorems in [1], notably those where F is a *Hilbert space*. We content ourselves with sketching the method of deduction in one typical case only. Let B denote the Hilbert space of analytic functions in D^+ normed by $\|f\|^2 = \int_0^1 \int_0^{2\pi} |f(r\mathrm{e}^{it})|^2 r\,dr\,dt$. Let $\{z_n\}$ be points of D^+, and a_n complex numbers such that the interpolation problem $f(z_n) = a_n (n = 1, 2, \dots)$ has more than one solution in B. Let $E = \{1/\bar{z}_n\}$, and suppose β is an arc of $|z| = 1$ disjoint from the closure of E. Then, it was shown in [1] that the (unique) solution f^* of the interpolation problem which has minimal norm is analytically continuable across β and into all of $Z \backslash \bar{E}$.

To deduce this result from Theorem 2, observe that B has the reproducing kernel $K(z, \zeta) = 1/(1 - \bar{\zeta} z)^2$, and that the minimal interpolating function (as is known from general principles concerning Hilbert spaces with reproducing kernel) is that solution which is spanned by the functions $K(z, z_n)$ (this set is not total by the underdeterminedness hypothesis). Now, we can't yet apply Theorem 2 in this situation since the $K(z, z_n)$ have double poles. However, let H be the Hilbert space of functions $g(z) = \sum_0^\infty c_n z^n$ normed by $\|g\|^2 = \sum_0^\infty (n+1)|c_n|^2$. The correspondence of orthonormal bases

$$(n+1)^{\frac{1}{2}} z^n \leftrightarrow (n+1)^{-\frac{1}{2}} z^n$$

induces an isometry between B and H, under which $K(z, z_n)$ corresponds to $1/(1 - \bar{z}_n z)$ in H. The transform g^* of f^* is, by Theorem 2, continuable across β into

$Z\backslash\bar{E}$, and finally the corresponding continuability for f^* is readily inferred from the fact that f^* is the Hadamard product of g^* and $1/(1-z)^2$.

We wish to remark that throughout [1] the authors assumed that the closure of the set $\{z_n\}$ doesn't include the whole boundary of the domain. Our Theorem 1, together with the technique just employed (or, suitable modification of the discussion in [1]) enables one to assert something also about the case where the z_n cluster at all boundary points. For instance, returning to the example of the space B just discussed, if f_n^* denotes the unique solution of the *finite* interpolation problem

$$f(z_k) = a_k, \quad k = 1, 2, \dots, n$$

having least norm, then the f_n^* converge in norm to f^*: and what is more $\{f_n^*(z)\}$ converges uniformly on compact subsets of $D^-\backslash\bar{E}$ to a certain function g meromorphic in D^-. In other words, we still have overconvergence of the solutions of the corresponding minimization problems with finitely many interpolation conditions. Whether the functions f^* and g bear to one another some simple relation in the case when $\bar{E}$ includes the whole unit circumference (in which case f^* will in general be nowhere continuable), for instance whether they are connected by a functional equation, or exhibit some kind of matching boundary behaviour, seems worthy of investigation.

3. Concluding remarks and open questions

3.1. In Theorem 1 we showed that if $f_n \in R$ and $\|f_n - f\| \to 0$, then (under suitable hypotheses) $\{f_n(z)\}$ converges on compact subsets of $D^-\backslash\bar{E}$ to a certain function, say $\tilde{f}$, which is then meromorphic in D^- (indeed, the hypothesis of non-totality implies that E has no limit points in D^-). Moreover, as the proof of Theorem 1 showed, $f = 0$ implies $\tilde{f} = 0$, that is, f uniquely determines $\tilde{f}$.

It seems plausible that the converse is true, i.e. that $\tilde{f} = 0$ implies $f = 0$, but we have not been able to show this, not even under stronger hypotheses on S. We wish also to raise the following more general questions:

(i) If f is analytically continuable across some point of $|z| = 1$, must the continuation coincide with $\tilde{f}$? (The answer is *yes* in the special case $f = 0$, as we just remarked.)

(ii) If $\tilde{f}$ is analytically continuable across some point of $|z| = 1$, must the continuation coincide with f?

Of course, the answers to these questions are trivially yes under the hypotheses of Theorem 2, if $\bar{E}$ doesn't include the whole unit circumference, since then f and $\tilde{f}$ are analytic continuations of one another. These questions are suggested by the author's speculations about generalizing the notion of analytic continuation [4].

3.2. It seems of some interest to generalize the results of this paper to open sets more general than the unit disk. For smoothly bounded domains this is quite routine, but for general open sets serious technical difficulties appear, for instance in extending the lemma needed for Theorem 2.

In like manner, one could attempt to generalize the results to topological linear spaces (even the unit disk) of holomorphic functions other than normed spaces. Here again some new ideas are needed. That the overconvergence theorems are true for some interesting non-normed spaces may perhaps be considered plausible in view of the remark that they are true (rather trivially) for the topology of uniform convergence on compact sets. Here the non-totality hypothesis is equivalent to the finiteness of E.

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