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Direct sum decompositions in Grothendieck categories

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Throughout this paper, $\mathcal{A}$ will denote a Grothendieck category, i.e. an abelian category with generators and exact direct limits. Our main theorem gives a sufficient condition for an object to decompose into a direct sum of indecomposable objects. This theorem will then be applied to obtain decompositions of injective objects in locally noetherian categories and of projective modules over perfect rings. Some applications will also be given to relative splitting problems, i.e. splitting by $\mathcal{E}$-proper subobjects where $\mathcal{E}$ is a proper class in the sense of relative homological algebra.

In a preliminary version of this paper (cited in [12]), $\mathcal{A}$ was assumed to be locally finitely generated. I am grateful to J.-E. Roos for pointing out that the results may be extended to AB 6 categories.

1. Main theorem

For the validity of the subsequent decomposition theorems it turns out to be essential that $\mathcal{A}$ should satisfy some condition of local finiteness. The following two axioms will be used for this purpose:

AB 6: Every object M is a sum of subobjects which are finitely generated relative to M.

AB 6 (ess): Every object $M \neq 0$ contains a subobject $\neq 0$ which is finitely generated relative to M.

Here a subobject L of M is called *finitely generated relative to M* if whenever $M=\sum M_i$ for a directed family $(M_i)_I$ of subobjects of M, there is an $i \in I$ such that $L \subset M_i$. The axiom AB 6 was first introduced in [6] in the following form:

For every M and every family $(M_j)_J$ of directed families of subobjects $(M_{j\alpha})_{A_j}$ of M, the canonical morphism

$$\varphi: \sum_{\alpha(j) \in \pi A_j} \left(\bigcap_{j \in J} M_{j\alpha}\right) \to \bigcap_{j \in J}\left(\sum_{\alpha \in A_j} M_{j\alpha}\right)$$

is an isomorphism.

It has been shown by Roos [12] that the two formulations of AB 6 are equivalent, and that AB 6 (ess) is equivalent to the condition that φ should always be an essential monomorphism.

We will consider the following condition for an object M of $\mathcal{A}$:

(C) Every union of an ascending chain of direct summands of M is a direct summand of M.

Lemma 1. *If M satisfies (C), then also every direct summand of M satisfies (C).*

Proof. Clear.

Theorem 1. *Suppose $\mathcal{A}$ satisfies AB 6 (ess). Every object M satisfying condition (C) is a direct sum of indecomposable objects.*

Proof. Let $(L_i)_I$ be a maximal family of indecomposable subobjects of M such that the sum $\sum L_i$ is direct and splits M. The existence of such families is guaranteed by condition (C) and axiom AB 5. Write $M = (\sum L_i) \oplus L$. Suppose $L \neq 0$. To reach a contradiction we only have to find an indecomposable direct summand $\neq 0$ of L. Choose a relatively finitely generated subobject $B \neq 0$ of L. Then there exists a direct summand L' of L which does not contain B and is maximal with these properties. For let $(K_\alpha)_A$ be any ascending chain of direct summands of L with $B \not\subset K_\alpha$. Then $L = (\bigcup K_\alpha) \oplus K'$ by Lemma 1, and $(K_\alpha \oplus K')_A$ is a directed family of subobjects of L. Hence $B \subset K_\alpha \oplus K'$ for some $\alpha \in A$, but then B must have non-zero projection on K' and it follows that $B \not\subset \bigcup K_\alpha$.

Write $L = L' \oplus E$. E is indecomposable, for if $E = F \oplus G$ with non-zero F and G, then $L = L' \oplus F \oplus G$ and either $B \not\subset L' \oplus F$ or $B \not\subset L' \oplus G$, which would contradict the maximality of L'.

2. Decomposition of injective or projective objects

Theorem 2. *Suppose every injective object of $\mathcal{A}$ is a direct sum of indecomposable objects. Then every direct sum of injective objects is injective.*

Proof. Consider any family $(E_i)_I$ of injective objects and let F denote the injective envelope of $\oplus_I E_i$. Then $F = \oplus_J F_j$ where F_j are indecomposable injective objects. Since $\oplus E_i$ is an essential subobject of F, it has a non-zero intersection with each F_j. Hence $B_j = F_j \cap (E_{i_1} \oplus \dots \oplus E_{i_n}) \neq 0$ for some $i_1, \dots, i_n$ (depending on j). But F_j is indecomposable, so $F_j = E(B_j) \subset E_{i_1} \oplus \dots \oplus E_{i_n}$. It follows that $\oplus E_i$ coincides with its injective envelope F.

The following theorem generalizes results due to Gabriel [5], Matlis [9], Nouazé [10] and Papp [11]:

Theorem 3. *Let $\mathcal{A}$ satisfy AB 6. The following properties of $\mathcal{A}$ are equivalent:*

(a) *$\mathcal{A}$ is locally noetherian.*

(b) *Every direct sum of injective objects is injective.*

(c) *Every direct limit of injective objects is injective.*

(d) *Every injective object satisfies (C).*

(e) *Every injective object is a direct sum of indecomposable injective objects.*

Proof. (a) => (c) has been proved by Gabriel [5]. (c) => (d) is clear and (d) => (e) follows from Theorem 1. (e) => (b) is given by Theorem 2. Finally, (b) => (a) is obtained by an obvious modification of the proof given by Nouazé ([10], cf. [12]).

Let us now consider the more difficult dual problem of decomposing projective objects. We assume at once that $\mathcal{A}$ is the category of left modules over a ring A. Note that A satisfies condition (C) if and only if A has no infinite set of orthogonal idempotents. More particularly we have:

Lemma 2. *Every projective left module satisfies (C) if and only if A is a left perfect ring.*

Proof. If A is left perfect, then every direct limit of projective modules is projective, so every projective module satisfies (C). The converse follows from the proof of 5) => 6) in Theorem P of [1].

An application of our Theorem 1 immediately gives

Theorem 4. (*Eilenberg* [3]). *Every projective left module over a left perfect ring is a direct sum of indecomposable projective modules.*

3. Relative splitting

Let $\mathcal{E}$ be an allowable class of short exact sequences of $\mathcal{A}$, and suppose $\mathcal{E}$ contains all split sequences and $\mathcal{E}_m$ is closed under composition (we follow here the terminology of [8], ch. 9). Consider the following generalization of the condition (C):

$(C-\mathcal{E})$ Every union of an ascending chain of $\mathcal{E}$-allowable subobjects of M is $\mathcal{E}$-allowable in M.

We call M *$\mathcal{E}$-simple* if it does not contain any $\mathcal{E}$-allowable subobjects $\neq 0, M$. M is then of course indecomposable.

Theorem 5. *Let $\mathcal{A}$ satisfy $AB6$ (ess). Suppose M is an object in $\mathcal{A}$ which satisfies $(C-\mathcal{E})$ and is split by all its $\mathcal{E}$-allowable subojects. M is then a direct sum of $\mathcal{E}$-simple objects.*

Proof. M also satisfies condition (C), so M is a direct sum of indecomposable subobjects. These subobjects must be $\mathcal{E}$-simple, since they also are split by $\mathcal{E}$-allowable subobjects.

Our next result gives a necessary and sufficient condition for an object to be split by all its $\mathcal{E}$-allowable subobjects. Let $\mathcal{E}$ be a proper class. A monomorphism $\alpha: L\to M$ is *$\mathcal{E}$-essential* if every $\varphi: M\to N$, such that $\varphi\alpha\in\mathcal{E}_m$, is a monomorphism. If furthermore M is $\mathcal{E}$-injective, we call α an *$\mathcal{E}$-injective envelope* of L (cf. [13], sec. 4).

Lemma 3. *If α and β belong to $\mathcal{E}_m$ and $\beta\alpha$ is $\mathcal{E}$-essential, then also α and β are $\mathcal{E}$-essential.*

Proof. Easy.

Theorem 6. *Let $\mathcal{E}$ be a proper class such that all objects have $\mathcal{E}$-injective envelopes. M is split by all its $\mathcal{E}$-allowable subobjects if and only if M has no $\mathcal{E}$-essential subobjects $\neq M$.*

Proof. The necessity of the condition is clear, so we will prove its sufficiency. Let $\alpha: L \to M$ be any monomorphism in $\mathcal{E}_m$. Choose an $\mathcal{E}$-injective envelope $\mu: L \to E$ and extend μ to a morphism $\mu': M \to E$ with $\mu'\alpha = \mu$. Put $K = \operatorname{Ker} \mu'$. Then $K \cap L = 0$ and the composed morphism $L \to M \to M/K$ is $\mathcal{E}$-essential by Lemma 3. It now suffices to show that $\beta: K + L \to M$ is $\mathcal{E}$-essential. So suppose $\varphi: M \to N$ is a morphism such that $\varphi\beta \in \mathcal{E}_m$. Then also $L \to M/K \to N/K$ belongs to $\mathcal{E}_m$ by axiom P3* for proper classes ([13], sec. 2). If follows that $M/K \to N/K$ is a monomorphism and hence $\varphi: M \to N$ is a monomorphism.

As an example we let $\mathcal{E}$ be the class of all short exact sequences and obtain the following equivalent descriptions of *semi-simple* objects:

Theorem 7. *Let $\mathcal{A}$ satisfy AB 6 (ess). The following conditions are equivalent for an object M:*

(a) *M is split by all its subobjects.*

(b) *M has no essential subobjects $\neq M$.*

(c) *M is the sum of its simple subobjects.*

(d) *M is a direct sum of simple objects.*

Proof. (a) $\Leftrightarrow$ (b) follows from Theorem 6, (a) $\Rightarrow$ (d) follows from Theorem 5, (d) $\Rightarrow$ (c) is trivial, and (c) $\Rightarrow$ (a) is proved as in the case of modules [2].

4. Splitting by high subobjects

A short exact sequence $0 \to L \to M \to N \to 0$ is called *high* if $L = M \cap E(L)$, where $E(L)$ as usual denotes the injective envelope of L (cf. [14]). The high sequences form a proper class.

Lemma 4. *Let $\mathcal{A}$ satisfy AB 6. $\mathcal{A}$ is then locally noetherian if and only if every object of $\mathcal{A}$ satisfies $(C - High)$.*

Proof. Suppose $\mathcal{A}$ is locally noetherian and let (L_α) be any ascending chain of high subobjects of M. Then $\cup E(L_\alpha) = E(\cup L_\alpha)$ by Theorem 3, so $M \cap E(\cup L_\alpha) = \cup (M \cap E(L_\alpha)) = \cup L_\alpha$. For the converse one notes that conditions (C) and (C $-$ High) coincide for injective objects and Theorem 3 therefore is applicable.

Lemma 5. *M is high-simple if and only if it is coirreducible.*

Proof. Clear from Proposition 3 of [14].

Applying Theorem 5 we now obtain

Theorem 8. *Let $\mathcal{A}$ be locally noetherian and suppose M is split by all its high subobjects. M is then a direct sum of coirreducible objects.*

As an application of this theorem we will prove a decomposition result for quasi-injective objects. Recall that an object is *quasi-injective* if it is stable under all endomorphisms of its injective envelope. The following two lemmata have been proved for modules by Johnson and Wong ([7], Theorem 1.1), resp. by Faith and Utumi ([4], Corollary 2.2). Their proofs may easily be extended to any abelian category with injective envelopes.

Lemma 6. *M is quasi-injective if and only if every morphism $L \to M$, for any subobject L of M, may be extended to a morphism $M \to M$.*

Lemma 7. *Every quasi-injective object is split by its high subobjects.*

Theorem 9. *Let $\mathcal{A}$ be locally noetherian. Every quasi-injective object is a direct sum of coirreducible quasi-injective objects.*

5. Splitting by neat subobjects

A short exact sequence is called *neat* if every simple object is a relative projective for it. In particular every high sequence is neat ([14], Proposition 5).

Lemma 8. *Every object of $\mathcal{A}$ satisfies $(C-Neat)$ when $\mathcal{A}$ is locally noetherian.*

Proof. Let (L_α) be an ascending chain of neat subobjects of M. Every simple object S is noetherian, so every morphism $\varphi: S \to M/\cup L_\alpha$ may be factored through some M/L_α ([5], p. 358) and may therefore be lifted to M.

Lemma 9. *M is neat-simple if and only if M is coirreducible and all its quotient objects $\neq 0$, M have non-zero socles.*

Proof. If M is neat-simple, then it is also high-simple, hence coirreducible. If then $L \subset M$ and Soc $M/L = 0$, L is trivially neat in M.

Conversely, suppose M is coirreducible and $L \subset M$ with $L \neq 0$, M and Soc $M/L \neq 0$. Then there exists a non-zero morphism $\varphi: S \to M/L$ for some simple object S. If we could lift φ to a morphism $\bar{\varphi}: S \to M$, then we would have Im $\bar{\varphi} \cap L = 0$, which is impossible since L is essential in M. Hence L cannot be neat in M.

Theorem 5 now gives

Theorem 10. *Let $\mathcal{A}$ be locally noetherian and suppose M is split by all its neat subobjects. M is then a direct sum of coirreducible objects M_i such that all quotient objects $\neq 0$, M_i of M_i have non-zero socles.*

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